Willmore surfaces of $\mathbb{R}^4$ and the Whitney sphere

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Abstract

We make a contribution to the study of Willmore surfaces in four-dimensional Euclidean space $\mathbb{R}^4$ by making use of the identification of $\mathbb{R}^4$ with two-dimensional complex Euclidean space $\mathbb{C}^2$.

We prove that the Whitney sphere is the only Willmore Lagrangian surface of genus zero in $\mathbb{R}^4$ and establish some existence and uniqueness results about Willmore Lagrangian tori in $\mathbb{R}^4 \equiv \mathbb{C}^2$.

1 Introduction

Given a compact surface Σ in $\mathbb{R}^n$, the Willmore functional is defined by

$$\mathcal{W}(\Sigma) = \int_\Sigma |H|^2 dA,$$

where H denotes the mean curvature vector of Σ and dA is the canonical measure of the induced metric. An absolute minimum for $\mathcal{W}$ is 4π, which is only attained for round spheres. When Σ is a torus, Willmore, [Wi], conjectured that the minimum is $2\pi^2$, and it is attained only by the Clifford torus. Although several partial answers have been obtained, (see [LY],[MoR],[R]), the conjecture still remains open. Simon, in [S], proved the existence of an embedded torus which minimizes the Willmore functional. Given that the Clifford torus is Lagrangian in $\mathbb{R}^4$, in [M], Minicozzi proved the existence of an embedded Lagrangian torus which minimize the Willmore functional in the smaller class of Lagrangian tori.

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Other authors are interested in the study of critical surfaces of the Willmore fuctional (the so called *Willmore surfaces*), and in the construction of examples of Willmore surfaces. In [B], Bryant initiated this study, classifying the Willmore surfaces of genus zero in $\mathbb{R}^3$ (or $\mathbb{S}^3$, because the Willmore fuctional as well as the Willmore surfaces are invariant under conformal transformations of ambient space). He proved that *any compact Willmore surface of genus zero in* $\mathbb{R}^3$ *is the compactification by an inversion of a complete minimal surface of* $\mathbb{R}^3$ *with finite total curvature and embedded planar ends.*

In [Pi], Pinkall constructed, using the Hopf fibration $\Pi : \mathbb{S}^3 \longrightarrow \mathbb{S}^2$ and elastic curves of $\mathbb{S}^2$, a wide family of Willmore flat tori in $\mathbb{S}^3$. Unlike the Bryant examples, these tori do not come from minimal surfaces of $\mathbb{R}^3$, or minimal surfaces of $\mathbb{S}^3$, or minimal surfaces of $\mathbb{H}^3$. It is also interesting to note that these tori, which are also Willmore surfaces in $\mathbb{R}^4$, are Lagrangian surfaces in $\mathbb{R}^4$.

Following Bryant's ideas, and using different approaches, Ejiri, [E], Montiel, [Mo], and Musso, [Mu], classified Willmore compact surfaces of genus zero in $\mathbb{R}^4$ (or $\mathbb{S}^4$). Besides the Willmore compact surfaces of genus zero coming from complete minimal surfaces of $\mathbb{R}^4$ with finite total curvature and planar ends, a new family of Willmore spheres also appeared. These can be described as projections under the Penrose twistor fibration $\mathbb{CP}^3 \longrightarrow \mathbb{S}^4$ of holomorphic curves of $\mathbb{CP}^3$.

In [CU], the authors studied the Willmore functional for Lagrangian compact surfaces Σ of genus zero of $\mathbb{R}^4$. As they cannot be embedded, using a result of Li and Yau, [LY], $\mathcal{W}(\Sigma) \geq 8\pi$. We characterized the equality proving that the Whitney sphere is the only Lagrangian sphere Σ with $\mathcal{W}(\Sigma) = 8\pi$. This example can be defined as the compactification of the generalized catenoid of $\mathbb{R}^4$. In particular the Whitney sphere is not only a Willmore surface but a minimum for the Willmore functional in its homotopy class. Using the classification of Ejiri, Montiel and Musso, and some results obtained by the authors in [CU], we prove, in Corollary 1, that

> *The Whitney sphere is the only Willmore Lagrangian surface of genus zero in* $\mathbb{R}^4$*.*

The rest of the paper deals with Willmore Lagrangian tori in $\mathbb{R}^4$. In some sense, the second part of this paper has been inspired by Pinkall's paper. As we pointed out before, these Hopf tori are Willmore Lagrangian tori in $\mathbb{R}^4$, and, as they lie in the unit sphere, are trivially invariant under an inversion centered at the origin. In particular, the property to be Lagrangian is kept under this inversion. This is our starting point to find new examples of Willmore tori. As a summary of Theorem 3:

> *We classify all the Willmore Lagrangian tori Σ in $\mathbb{R}^4$ such that their images under an inversion centred at a point outside Σ (which are Willmore tori again) are also Lagrangian surfaces.*

In the above classification, apart from the Willmore Hopf tori discovered by Pinkall, two new important families of examples appear.

The first one (see examples in B, paragraph 3) is constructed following Pinkall's idea. We use the $\mathbb{S}^1$-fibration $\hat{\Pi} : \mathbb{M}^3 \subset \mathbb{R}^4 \longrightarrow \mathbb{S}^1 \times \mathbb{R}$, where

$$\mathbb{M}^3 = \{(z, w) \in \mathbb{C}^2 - \{0\}/|z| = |w|\}$$

and $\hat{\Pi}(z, w) = (zw/|zw|, \log 2|zw|)$. The pull-back to $\mathbb{M}^3$ by $\hat{\Pi}$ of a closed curve α in $\mathbb{S}^1 \times \mathbb{R}$, is a Lagrangian torus T_α in $\mathbb{C}^2$. The Willmore functional of T_α is computed in terms of a functional $\mathcal{F}$ on the curve α (see Proposition 1), and T_α is a Willmore surface if and only if α is a critical point of $\mathcal{F}$. The corresponding Euler-Lagrange equation for $\mathcal{F}$ is computed in Proposition 2. Apart from the closed geodesics of $\mathbb{S}^1 \times \mathbb{R}$, in Proposition 3 we obtain two non-trivial families of closed curves in $\mathbb{S}^1 \times \mathbb{R}$ which are critical points for the functional $\mathcal{F}$. The correponding tori are characterized in Theorem 2 as *the only compact Willmore Lagrangian surfaces of $\mathbb{C}^2$ coming from minimal surfaces either the 4-dimensional sphere or the 4-dimensional hyperbolic space.* The first one means that these Willmore surfaces are minimal surfaces with respect to the spherical metric on $\mathbb{C}^2$. The second one means that the surface Σ decomposes into three parts $\Sigma = \Sigma_+ \cup \Sigma_0 \cup \Sigma_-$, lying respectively on the unit ball, on the unit sphere and outside the closed unit ball, such that $\Sigma_\pm$ are minimal surfaces in hyperbolic spaces, and Σ_0 is a set of totally geodesic points. So these examples are constructed in a similar way to the Willmore tori of $\mathbb{R}^3$ obtained by Babich and Bobenko in [BB].

The other family of Willmore surfaces appearing in Theorem 3, is a countable family of embedded Willmore tori (see examples in C, paragraph 3), which can be described in terms of a couple of circles (see Definition 1). For this family we are able to determine the areas as well as the values of the Willmore functional precisely (see Proposition 5), since we can control them very well by using elliptic functions. In fact, the o.d.e. that governs their local geometric behaviour is the sinh-Gordon equation.

2 Willmore surfaces and the Whitney sphere

Let $\mathbb{R}^4$ be four-dimensional Euclidean space and $\langle , \rangle$ the Euclidean metric on it. Given an immersion $\phi : \Sigma \longrightarrow \mathbb{R}^4$ from a compact surface Σ, the Willmore functional of ϕ is defined by

$$\mathcal{W}(\phi) = \int_\Sigma |H|^2 dA,$$

where H is the mean curvature vector of ϕ and dA denotes the canonical measure of the induced metric, which will be also denoted by $\langle , \rangle$. The critical immersions for this functional are called *Willmore surfaces*, and they are characterized by the equation (see [W])

$$\Delta H - 2|H|^2 H + \tilde{A}(H) = 0,$$

where Δ is the Laplacian on the normal bundle of ϕ and $\tilde{A}$ is the operator defined by

$$\tilde{A}(\xi) = \sum_{i=1}^{2} \sigma(e_i, A_\xi e_i),$$

being σ the second fundamental form of ϕ, A_ξ the Weingarten endomorphism associated to a normal vector ξ and $\{e_1, e_2\}$ an orthonormal basis of the tangent bundle of Σ.

On the other hand, let $\mathbb{C}^2$ be two-dimensional complex Euclidean plane. We denote by $\langle , \rangle$ the Euclidean metric in $\mathbb{C}^2 \equiv \mathbb{R}^4$ and by J a canonical complex structure. The symplectic form Ω is defined by $\Omega(v, w) = \langle Jv, w \rangle$ for vectors v and w.

An immersion $\phi : \Sigma \longrightarrow \mathbb{C}^2$ of a surface Σ is called *Lagrangian* if any of the following equivalent conditions are satisfied:

(i) $\phi^*\Omega = 0$,

(ii) $J(T\Sigma) = N\Sigma$, where $T\Sigma$ and $N\Sigma$ are the tangent and normal bundles to Σ.

¿From this definition, J defines an isometry between $T\Sigma$ and $N\Sigma$ which commutes with the tangent and normal connections. Also the second fundamental form σ of ϕ and the Weingarten endomorphism A_ξ associated to a normal vector ξ are related by

$$\sigma(v, w) = JA_{Jv}w.$$

So, $\langle \sigma(v,w), Jz\rangle$ defines a fully symmetric trilinear form on $T\Sigma$. If H denotes the mean curvature vector of ϕ, the above properties joint to the Codazzi equation imply that JH is a *closed vector field* on Σ.

Using the mentioned properties of a Lagrangian immersion, the above Willmore equation can be written for Lagrangian immersions in the following way

$$\Delta H - (K + 2|H|^2)H + 2\sigma(JH, JH) = 0, \tag{1}$$

where K is the Gauss curvature of Σ.

Now we describe a result which will be use along the paper. In fact all the Willmore tori constructed in the paper belong to the following family of Lagrangian tori.

THEOREM A [RU] *Let $\phi : \Sigma \longrightarrow \mathbb{C}^2$ be a Lagrangian immersion of a torus Σ. If X is a closed and conformal vector field on Σ such that $\sigma(X,X) = \rho JX$ for certain function ρ, then the universal covering of ϕ, $\tilde{\phi} : \mathbb{R}^2 \longrightarrow \mathbb{C}^2$ is given by*

$$\tilde{\phi}(t,s) = \gamma(t)\tilde{\beta}(s),$$

where $\gamma : \mathbb{R} \longrightarrow \mathbb{C}^*$ is a regular curve, and $\tilde{\beta} : \mathbb{R} \longrightarrow \mathbb{S}^3 \subset \mathbb{C}^2$ is a regular curve in the unit sphere, which is horizontal with respect to the Hopf fibration $\Pi : \mathbb{S}^3 \longrightarrow \mathbb{S}^2$. This means that $\tilde{\beta}$ is a horizontal lift of a regular curve in $\mathbb{S}^2$.

If $\phi : \Sigma \longrightarrow \mathbb{C}^2$ is a Lagrangian immersion of a genus zero compact surface Σ, it is well known that ϕ is not an embedding, and so, using a result of Li and Yau, [LY], $\mathcal{W}(\phi) \geq 8\pi$. In [CU], Corollary 5, the authors characterized the equality in the above inequality, proving that *the only compact Lagrangian surface of genus zero with $\mathcal{W}(\phi) = 8\pi$ is the Whitney sphere*, which is defined by

$$(x,y,z) \in \mathbb{S}^2 \mapsto \frac{1}{1+z^2}(x(1+iz), y(1+iz)) \in \mathbb{C}^2,$$

being $\mathbb{S}^2 = \{(x,y,z) \in \mathbb{R}^3 / x^2 + y^2 + z^2 = 1\}$. This example is not only a Willmore surface, but a minimum for the Willmore functional in its homotopy class. It is obtained by compactification of a complex curve in $\mathbb{C}^2$ of finite total curvature -4π described by Hoffman and Osserman in [HO], which can be written as

$$z \in \mathbb{C}^* \mapsto (z, \frac{1}{z}) \in \mathbb{C}^2.$$

Now we are going to characterize this example, proving that the Whitney sphere is the only compact Willmore Lagrangian surface in $\mathbb{C}^2$ coming from a minimal surface of $\mathbb{C}^2$.

Theorem 1 *Let* $\phi : \Sigma \longrightarrow \mathbb{C}^2$ *be a Lagrangian immersion of a compact surface* Σ. ϕ *is the compactification by an inversion of a complete minimal surface of finite total curvature of* $\mathbb{C}^2$ *if and only if* ϕ *is the Whitney sphere.*

Proof: Suppose that ϕ is the compactification by an inversion of a complete minimal immersion with finite total curvature $\hat{\phi} : \hat{\Sigma} \longrightarrow \mathbb{C}^2$, where $\hat{\Sigma}$ is Σ punctured at a finite number of points. There is no restriction if we take the inversion F as $F : \mathbb{C}^2 \cup \{\infty\} \longrightarrow \mathbb{C}^2 \cup \{\infty\}$ defined by $F(p) = p/|p|^2$. So, as $\hat{\phi} = \phi/|\phi|^2$, a very easy computation of the mean curvature vector of $\hat{\phi}$ joint to the fact that $\hat{\phi}$ is a minimal immersion, says that

$$0 = H + 2|\phi|^{-2}\phi^{\perp}, \tag{2}$$

on $\hat{\Sigma}$, where H is the mean curvature vector of ϕ and $\perp$ means normal component to ϕ. Deriving (2) and using elementary properties of Lagrangian immersions, we obtain

$$0 = \langle \nabla_v JH, w\rangle - 4|\phi|^{-4}\langle d\phi(v), \phi\rangle\langle d\phi(w), J\phi\rangle + 2|\phi|^{-2}\langle \sigma(v, w), J\phi\rangle,$$

where ∇ is the Levi-Civita connection of the induced metric on Σ. As JH is a closed vector field on Σ, the first term is symmetric. Also, since the third term is symmetric, we get that the second term is symmetric too, and so

$$\delta \wedge \hat{\delta} = 0, \tag{3}$$

where δ and $\hat{\delta}$ are the 1-forms on $\hat{\Sigma}$ given by

$$\delta(v) = \langle d\phi(v), \phi\rangle \quad \text{and } \hat{\delta}(v) = \langle d\phi(v), J\phi\rangle.$$

We include the following Lemma because the next step in the proof will be used in other parts of the paper.

Lemma 1 *Let* $\phi : M \longrightarrow \mathbb{C}^2 - \{0\}$ *be a Lagrangian immersion of a connected surface* M. *If* δ *and* $\hat{\delta}$ *are the* 1*-forms on* M *defined by*

$$\delta(v) = \langle d\phi(v), \phi\rangle \quad \text{and } \hat{\delta}(v) = \langle d\phi(v), J\phi\rangle.$$

and $\delta \wedge \hat{\delta} = 0$, then there exist a closed and conformal vector field X on M and smooth functions a and b on M such that ϕ is given by

$$\phi = aX + bJX, \quad \text{with} \quad a^2 + b^2 = 1,$$

where $2a$ is the divergence of X: $divX$. Moreover, the second fundamental form σ of ϕ satisfies

$$\sigma(X, X) = \rho JX, \quad \sigma(X, V) = -bJV,$$

where V is any orthogonal vector field to X and ρ a smooth function on M.

Proof: Let $A = \{p \in M;\ \delta_p = 0\}$ and $B = \{p \in M;\ \hat{\delta}_p = 0\}$. If $A = M$, then $\phi = \phi^\perp$. So the Lemma follows taking $X = J\phi^\perp$, $a = 0$ and $b = 1$. In this case X is a parallel vector field. If $B = M$, then $\phi = \phi^\top$ and the Lemma follows taking $X = \phi^\top$, $a = 1$ and $b = 0$.

Let us assume now that A and B are proper subsets of M. Then, as $|\phi|$ has no zeroes, we have two disjoint closed subsets A and B of M, such that $M/(A \cup B)$ is a non empty open subset of M. Now, $\delta \wedge \hat{\delta} = 0$ says that on M/A, we can writte $\hat{\delta} = f\delta$ for certain smooth function f. Taking $X = \sqrt{1 + f^2}\,\phi^\top$, our immersion ϕ is given on M/A by $\phi = aX + bJX$, where a and b are smooth functions on M/A satisfying $a^2 + b^2 = 1$. Making a similar reasoning with B, we write ϕ on M/B as $\phi = a'X' + b'JX'$ with $a'^2 + b'^2 = 1$. It is clear that, on the non-empty subset $M/(A \cup B)$, we can take $X' = X$, $a' = a$ and $b' = b$. In this way, we have obtained a vector field X without zeroes and functions a and b defined on the whole M, such that

$$\phi = aX + bJX, \quad \text{with} \quad a^2 + b^2 = 1.$$

The rest of the Lemma can be found in [RU], Theorem 1, or can be check easily. This finishes its proof.$\square$

We continue with the proof of the Theorem. As $\phi^\perp = bJX$ and $|\phi|^2 = |X|^2$, from (2) we get that

$$H = -2|X|^{-2}bJX. \tag{4}$$

But, using the properties of the second fundamental form given in Lemma 1, the mean curvature vector is also given by

$$2H = |X|^{-2}(\rho - b)JX + cJv,$$

for a certain function c. Using (4), it follows that $\rho + 3b = 0$ and $c = 0$. Then it is easy to check that the Gauss curvature K of ϕ satisfies $|H|^2 = 2K$. From [CU], Corollary 3, or [RU], Corollary 3, this means that our surface is either the Whitney sphere or totally geodesic. The compactness of our surface proves the Theorem.$\square$

Corollary 1 *The Whitney sphere is the only Willmore Lagrangian sphere of* $\mathbb{C}^2$.

Proof: We use the following result independently obtained by different ways by Ejiri, Montiel and Musso. Also, see [ES] for a general reference about Penrose twistor spaces.

THEOREM B ([E],[Mo],[Mu]) *Let $\psi : \Sigma \longrightarrow \mathbb{S}^4$ be a Willmore immersion of a sphere Σ. Then either the twistor lift of ψ (or the twistor lift of its antipodal $-\psi$) to the Penrose twistor space $\mathbb{CP}^3$ of $\mathbb{S}^4$ is a holomorphic curve or ψ is the compactification by the inverse of the stereographic projection of a complete minimal surface of $\mathbb{R}^4$ with finite total curvature and planar embedded ends.*

In order to use this result, we are going to rewrite it when the ambient space is $\mathbb{R}^4$.

Let G_2^2 be the Grassmannian of oriented 2-plane in $\mathbb{R}^4$ and $\nu : \Sigma \longrightarrow G_2^2$ the Gauss map of an immersion $\phi : \Sigma \longrightarrow \mathbb{R}^4$ from an oriented surface Σ. It is well known that $G_2^2 = \mathbb{S}_1^2(\frac{1}{\sqrt{2}}) \times \mathbb{S}_2^2(\frac{1}{\sqrt{2}})$ and for $i = 1, 2$, let $\nu_i : \Sigma \longrightarrow \mathbb{S}^2(\frac{1}{\sqrt{2}})$ be the i-th component of ν.

The two twistor spaces over $\mathbb{R}^4$ can be identified with $\mathbb{R}^4 \times \mathbb{S}_i^2$, $i = 1, 2$, with the natural projections, see [ES]. The twistor lifts $\tilde{\phi}_i$ of ϕ to $\mathbb{R}^4 \times \mathbb{S}_i^2$ are given by $\tilde{\phi}_i = \phi \times \nu_i$, $i = 1, 2$.

As the Penrose twistorial constructions (as well as the Willmore surfaces) are invariant under conformal transformations, taking the stereographic projection as a conformal transformation from $\mathbb{S}^4$ punctured at a point onto $\mathbb{R}^4$, we have that the Theorem B can be rewritten as follows:

Let $\phi : \Sigma \longrightarrow \mathbb{R}^4$ be a Willmore immersion of a sphere Σ. Then either one of the components ν_i, $i = 1, 2$, of the Gauss map ν of ϕ is holomorphic or ϕ is the compactification by an inversion of a complete surface of $\mathbb{R}^4$ with finite total curvature and embedded planar ends.

If the first happens, then Corollary 4 in [CU] says that the surface is the Whitney sphere. If the latter happens, Theorem 1 finishes the proof.□

3 Examples of Willmore Lagrangian tori

In this paragraph, we will construct new examples of Willmore tori in $\mathbb{R}^4$, which will be characterized in section 4. In order to make selfcontained the paper, we start describing the family of Hopf tori constructed by Pinkall in [Pi].

Examples A. Willmore Hopf tori.

Let $\Pi : \mathbb{S}^3 \longrightarrow \mathbb{S}^2$ be the Hopf fibration, which is defined by

$$\Pi(z, w) = (2z\overline{w}, |z|^2 - |w|^2),$$

where $\mathbb{S}^3 = \{(z, w) \in \mathbb{C}^2 / |z|^2 + |w|^2 = 1\}$. Let β be a closed curved in $\mathbb{S}^2$ and $\Sigma = \Pi^{-1}(\beta)$. Then Σ is a torus in $\mathbb{S}^3$ called a *Hopf torus*. In [Pi], Pinkall studied these Hopf tori, and among other things, he proved that *"The Hopf torus $\Sigma = \Pi^{-1}(\beta)$ is a Willmore surface of $\mathbb{C}^2$ (or a Willmore surface of $\mathbb{S}^3$) if and only if β is an elastica in $\mathbb{S}^2$"*. This means that β is a critical point for the functional

$$\int_0^L (k^2 + 1) ds,$$

where s is the arc parameter of β, k its curvature and L its length. The Euler-Lagrange equation of this functional, $2k'' + k^3 + 4k = 0$, was obtained by Langer and Singer in [LS], who studied deeply this kind of curves, describing the closed elasticae of $\mathbb{S}^2$ precisely. The corresponding tori will be referred as *Willmore Hopf tori*. Finally, we remark that these Willmore Hopf tori are Lagrangian surfaces in $\mathbb{C}^2$.

Examples B.

Let

$$\mathbb{M}^3 = \{(z, w) \in \mathbb{C}^2 - \{0\} / |z| = |w|\}.$$

It is easy to check that $\mathbb{M}^3$ is a minimal hypersurface of $\mathbb{C}^2$. We consider the action of $\mathbb{S}^1$ on $\mathbb{M}^3$ defined by

$$e^{i\theta}(z, w) = (e^{i\theta} z, e^{-i\theta} w).$$

This action defines a $\mathbb{S}^1$-submersion $\hat{\Pi} : \mathbb{M}^3 \longrightarrow \mathbb{S}^1 \times \mathbb{R}$ on the right circular cylinder given by

$$\hat{\Pi}(z, w) = (\frac{zw}{|zw|}, \log 2|zw|).$$

We define a connection on this bundle by assigning to each $x \in \mathbb{M}^3$ the subspace of $T_x\mathbb{M}^3$ orthogonal to the fibre of $\hat{\Pi}$ through x. So we can talk about horizontal lifts of curves of $\mathbb{S}^1 \times \mathbb{R}$. The horizontal lift of a curve of $\mathbb{S}^1 \times \mathbb{R}$ to $\mathbb{M}^3$ is unique up to isometries of $\mathbb{M}^3$ of type $\{\mathrm{diag}(e^{ia}, e^{-ia}), a \in \mathbb{R}\}$. Thus, given a curve

$$\alpha = (\alpha_1, \alpha_2, \alpha_3) : \mathbb{R} \longrightarrow \mathbb{S}^1 \times \mathbb{R},$$

its horizontal lift to $\mathbb{M}^3$ can be chosen by

$$t \in \mathbb{R} \mapsto \frac{1}{\sqrt{2}}(\gamma(t), \gamma(t)) \in \mathbb{M}^3,$$

where γ is given by $\gamma^2 = e^{\alpha_3}(\alpha_1, \alpha_2)$. We remark that α is parametrized by $|\alpha'(t)| = 2$ if and only if γ is parametrized by $|\gamma'(t)| = |\gamma(t)|$.

Now, if α is a closed curve in $\mathbb{S}^1 \times \mathbb{R}$, $\hat{\Pi}^{-1}(\alpha)$ is a torus of $\mathbb{M}^3 \subset \mathbb{C}^2$ and its universal covering is defined by

$$\phi(t, s) = \frac{\gamma(t)}{\sqrt{2}}(e^{is}, e^{-is}),$$

where $\gamma^2 = e^{\alpha_3}(\alpha_1, \alpha_2)$.

In the following Proposition, we are going to study properties of these tori.

Proposition 1 *Let $\alpha = (\alpha_1, \alpha_2, \alpha_3) : [0, L/2] \longrightarrow \mathbb{S}^1 \times \mathbb{R}$ be a closed curve of length L parametrized by $|\alpha'(t)| = 2$. Then:*

(i) *the torus $T_\alpha = \hat{\Pi}^{-1}(\alpha)$ is a Lagrangian surface of $\mathbb{C}^2$;*

(ii) *the image of T_α under the inversion of $\mathbb{C}^2$, $p \mapsto p/|p|^2$, is $T_{\hat{\alpha}}$ where $\hat{\alpha} = (\alpha_1, \alpha_2, -\alpha_3)$;*

(iii) *the torus T_α is a Willmore surface of $\mathbb{C}^2$ if and only if α is a critical point for the functional*

$$\mathcal{F}(\alpha) = \int_0^{L/2} (k^2 + 1 - \frac{{\alpha_3'}^2}{4}) dt$$

where k denotes the curvature of the cylindric curve α.

Remark 1 It is suitable to point out that the curvature k and the component α_3 of a curve α in the cylinder parametrized by $|\alpha'| = 2$ are related by
$$2k = (\arcsin(\alpha_3'/2))'.$$

Proof: We recall that the universal covering of the torus T_α is given by
$$\phi(t,s) = \frac{\gamma(t)}{\sqrt{2}}(e^{is}, e^{-is}),$$
where $\gamma^2 = e^{\alpha_3}(\alpha_1, \alpha_2)$. We observe that as α is parametrized by $|\alpha'(t)| = 2$, then $|\gamma'(t)| = |\gamma(t)|$ and then the induced metric by ϕ is $|\gamma|^2(dt^2 + ds^2)$. In particular, ϕ is a conformal immersion.

Now (i) and (ii) are easy to check.

To prove (iii), we start computing the mean curvature vector H of ϕ. It is straigthforward to see that
$$2H = (\frac{k_\gamma}{|\gamma|} + \frac{\langle \gamma', J\gamma \rangle}{|\gamma|^4})J\phi_t,$$
where k_γ is the curvature of γ.

Hence the Willmore functional is given by
$$\mathcal{W}(T_\alpha) = \int_{\mathcal{F}} \frac{|\gamma|^2}{4}(k_\gamma + \frac{\langle \gamma', J\gamma \rangle}{|\gamma|^3})^2 dtds = \frac{\pi}{2}\int_0^{L/2}(|\gamma|k_\gamma + \frac{\langle \gamma', J\gamma \rangle}{|\gamma|^2})^2 dt,$$
being F a fundamental region for the covering $\mathbb{R}^2 \longrightarrow T_\alpha$.

Now, it is easy to check that the curvature of α is

(5)
$$k = \frac{1}{2}(|\gamma|k_\gamma - \frac{\langle \gamma', J\gamma \rangle}{|\gamma|^2}).$$

So, the Willmore functional is written as
$$\mathcal{W}(T_\alpha) = 2\pi \int_0^{L/2}(k + \frac{\langle \gamma', J\gamma \rangle}{|\gamma|^2})^2 dt.$$

Taking into account that the Laplacian of α_3 is
$$\alpha_3'' = -4k\frac{\langle \gamma', J\gamma \rangle}{|\gamma|^2},$$

and that our curve α is closed, the Willmore functional can be written as

$$\mathcal{W}(T_\alpha) = 2\pi \int_0^{L/2} (k^2 + \frac{\langle \gamma', J\gamma\rangle^2}{|\gamma|^4})dt.$$

Remenbering now that α is parametrized by $|\alpha'(t)| = 2$, it is easy to see that

$$|\gamma|^{-4}\langle \gamma', J\gamma\rangle^2 = 1 - \frac{\alpha_3'^{\,2}}{4},$$

and so $\mathcal{W}(T_\alpha) = 2\pi\mathcal{F}(\alpha)$.

On the other hand, the mean curvature vectors H and H' of T_α in $\mathbb{C}^2$ and $\mathbb{M}^3$ respectively, satisfy $H = H'$, and then

$$\mathcal{W}(T_\alpha) = \int_{T_\alpha} |H|^2 dA = \int_{T_\alpha} |H'|^2 dA.$$

Hence, the principle of symmetric criticality, [P], can be applied here. In conclusion, T_α is critical for the Willmore functional if and only if α is critical for the functional $\mathcal{F}$. This proves (iii).$\square$

In the following result, we compute the Euler-Lagrange equation for the fuctional $\mathcal{F}$ introduced in Proposition 1,(iii).

Proposition 2 *Let $\alpha = (\alpha_1, \alpha_2, \alpha_3) : [0, L/2] \longrightarrow \mathbb{S}^1 \times \mathbb{R}$ be a closed curve of lenght L parametrized by $|\alpha'(t)| = 2$. Then, α is a critical point for the functional*

$$\mathcal{F}(\alpha) = \int_0^{L/2} (k^2 + 1 - \frac{\alpha_3'^{\,2}}{4})dt$$

if and only if the curvature k of α satisfies

$$k'' + 2k^3 + 2k(1 - \frac{3\alpha_3'^{\,2}}{4}) = 0. \tag{6}$$

Proof: In [LS], Langer and Singer deduced the Euler-Lagrange equation for the functional

$$\mathcal{F}_1(\alpha) = \int_0^{L/2} (k^2 + 1)dt$$

and they called *elastica* to a curve which is a critical point for this functional. Using their result and the fact that the cylinder is a flat surface, we have that the first derivative of the functional $\mathcal{F}_1(\alpha)$ is given by

$$\delta\mathcal{F}_1(\alpha) = \int_0^{L/2} (\frac{k''}{2} + k^3 - k)\langle \tilde{\alpha}', \delta\alpha\rangle dt,$$

where $\delta\alpha$ stands for the variation vector field, and $\tilde{\alpha} = (-\alpha_2, \alpha_1, \alpha_3)$.

Following their notation, it can be proved that the first derivative for the functional

$$\mathcal{F}_2(\alpha) = \frac{1}{4}\int_0^{L/2} {\alpha_3'}^2 dt$$

is given by

$$\delta\mathcal{F}_2(\alpha) = \int_0^{L/2} k(\frac{3{\alpha_3'}^2}{4} - 2)\langle\tilde{\alpha}', \delta\alpha\rangle dt.$$

This is enough to finish the proof.□

The parallels of the right circular cylinder, that is,

$$\alpha_\mu(t) = (\cos 2t, \sin 2t, \mu), \ \ \mu \in \mathbb{R},$$

are trivially solution of (6). The corresponding Willmore tori T_{α_μ} are, up to dilations in $\mathbb{C}^2$, the Clifford torus, which corresponds with $\mu = 0$. In the next Proposition, two non trivial families of closed curves, which are critical points for the functional $\mathcal{F}$, are constructed.

Proposition 3 **(i)** *Let* $\alpha_\lambda : \mathbb{R} \longrightarrow \mathbb{S}^1 \times \mathbb{R}$, $\lambda \leq 1$, *be the curve given by*

$$\alpha_\lambda(t) = (e^{2i\lambda \int_0^t \cosh^2 u(s) ds}, 2u(t)),$$

where $u(t)$ *is the solution to the o.d.e.* $u'^2 + \lambda^2 \cosh^4 u = 1$ *satisfying* $u'(0) = 0$, *and*

(ii) *let* $\alpha_\nu : \mathbb{R} \longrightarrow \mathbb{S}^1 \times \mathbb{R}$, $\nu > 0$, *be the curve given by*

$$\alpha_\nu(t) = (e^{2i\nu \int_0^t \sinh^2 u(s) ds}, 2u(t)),$$

where $u(t)$ *is the solution to the o.d.e.* $u'^2 + \nu^2 \sinh^4 u = 1$ *satisfying* $u'(0) = 0$.

Then:

(a) *there exist countably many* λ*'s (respectively countably many* ν*'s) such that* α_λ *(respectively* α_ν*) is a closed curve;*

(b) α_λ *and* α_ν *are critical points of the functional* $\mathcal{F}$ *defined in Proposition 2, i.e. the curvatures of* α_λ *and* α_ν *verify equation (6).*

Proof: Firstly, we note that the above parametrizations of α_λ and α_ν satisfy that $|\alpha_\lambda'(t)| = |\alpha_\nu'(t)| = 2$. Thus it is an exercise to check that the curvatures of α_λ and $\alpha_n u$ are given by $k_\lambda = \lambda \sinh u \cosh u$ and $k_\nu = \nu \sinh u \cosh u$ satisfy (6) in both cases.

On the other hand, we claim that $u(t)$ is always a periodic function. This can be proved, for example, using that the orbits $t \mapsto (u(t), u'(t))$ are closed curves. By another reasoning, if we write $y = \tanh u$ then y satisfies the differential equation $y'^2 = (1-y^2)^2 - \lambda^2$ in case (i) (resp. $y'^2 = (1-y^2)^2 - \nu^2 y^4$ in case (ii)) that can be solved by standard techniques in terms of elliptic functions (see [D]). This let us to conclude that there exist $T = T(\lambda)$ and $T = T(\nu)$ such that $u(t+2T) = u(t) = u(-t)$, $\forall t \in \mathbb{R}$.

To determine which α_λ's and α_ν's are closed curves, we start calling $B(t) = 2\lambda \int_0^t \cosh^2 u(s)\, ds$ in case (i) (resp. $B(t) = 2\nu \int_0^t \sinh^2 u(s)\, ds$ in case (ii)). Then for any $m \in \mathbb{Z}$, $B(t + 2mT) = B(t) + 2mB(T)$ from the above properties of $u(t)$. Hence we must consider the rationality condition $B(T)/\pi \in \mathbb{Q}$. So we need quantitative information about $B(T)$ that we are going to obtain from the explicit expression of $B(T)$ in terms of the Heuman-lambda function (see [BF]).

We denote by sn, cn and dn the elementary Jacobian elliptic functions (see [BF]). In case (i) we arrive at

$$\cosh^2 u(t) = \frac{\mathrm{dn}^2(r\,t,k)}{\lambda(1+k^2\mathrm{sn}^2(r\,t,k))},$$

where $r = \sqrt{\lambda+1}$ and $k^2 = (1-\lambda)/(1+\lambda)$. Then, using formula 410.04 of [BF], we compute that $B(T)/\pi = \Lambda_0(\pi/4, k)$, which increases monotonically from $1/2$ to $\sqrt{2}/2$ as λ increases from 0 to 1.

In case (ii) we must distinguish another two cases: If $\nu \leq 1$ we arrive at

$$\sinh^2 u(t) = \frac{\mathrm{cn}^2(r\,t,k)}{\nu(1+\mathrm{sn}^2(r\,t,k))},$$

where $r = \sqrt{\nu+1}$ and $k^2 = (1-\nu)/(1+\nu)$. Then, using formula 411.03 of [BF], we compute that $B(T)/\pi = (1 - \Lambda_0(\pi/4, k))$, which decreases monotonically from $1/2$ to $1 - \sqrt{2}/2)$ as ν increases from 0 to 1.

If $\nu \geq 1$ we arrive at

$$\sinh^2 u(t) = \frac{\mathrm{cn}^2(r\,t,k)}{\nu + \mathrm{sn}^2(r\,t,k)},$$

where $r = \sqrt{2\nu}$ and $k^2 = (\nu - 1)/(2\nu)$. Then, using again formula 411.03 of [BF], we compute that $B(T)/\pi = 1 - \Lambda_0(\sin^{-1}\sqrt{\nu/(\nu+1)}, k)$, which decreases monotonically from $1 - \sqrt{2}/2$ to 0 as ν increases from 1 to ∞.$\square$

Now we are able to characterize the corresponding Willmore tori associated to the curves given in Proposition 3. In Theorem 1 we studied the Willmore Lagrangian compact surfaces of $\mathbb{C}^2$ coming from minimal surfaces of $\mathbb{C}^2$. Similarly, in the next Theorem, we will describe the Willmore Lagrangian compact surfaces of $\mathbb{C}^2$ coming from minimal surfaces of the sphere or of the hyperbolic space.

Theorem 2 *Let $\phi : \Sigma \longrightarrow \mathbb{C}^2$ be a Lagrangian immersion of an orientable compact surface Σ.*

(i) *If $\phi : \Sigma \longrightarrow \mathbb{C}^2$ is a minimal immersion with respect to the spherical metric on $\mathbb{C}^2$, $g = 4(1+|p|^2)^{-2}\langle,\rangle$, then, up to dilations, ϕ is congruent to some Willmore torus T_α, where α is one of the closed curves given in Proposition 3,(i).*

(ii) *If $\tilde{\Sigma} = \{p \in \Sigma / |\phi(p)| \neq 1\}$ and $\phi : \tilde{\Sigma} \longrightarrow \mathbb{C}^2 - \mathbb{S}^3$ is a minimal immersion with respect to the hyperbolic metric on $\mathbb{C}^2 - \mathbb{S}^3$, $g = 4(1 - |p|^2)^{-2}\langle,\rangle$, then, up to dilations, ϕ is congruent to some Willmore torus T_α, where α is one of the closed curves given in Proposition 3,(ii).*

Remark 2 $(\mathbb{C}^2, 4(1+|p|^2)^{-2}\langle,\rangle)$ is isometric to $\mathbb{S}^4$ punctured at a point with its canonical metric of constant curvature 1.

The bounded connected component of $\mathbb{C}^2 - \mathbb{S}^3$ with the metric $4(1 - |p|^2)^{-2}\langle,\rangle$ is the 4-dimensional hyperbolic space of constant curvature -1, and the other connected component is isometric, by an inversion, to the 4-dimensional hyperbolic space of constant curvature -1 punctured at a point. So this second family of tori is constructed using a similar method to the used by Babich and Bobenko in [BB].

Proof of (i): As a minimal surface of $\mathbb{S}^4$ is a Willmore surface and this property is kept under conformal transformations, our immersion ϕ is a Willmore immersion in $\mathbb{C}^2$. Also, computing the mean curvature vector of ϕ with respect to the spherical metric and using the minimality of ϕ we get

$$0 = H + \frac{2}{1+|\phi|^2}\,\phi^\perp, \tag{7}$$

where H is the mean curvature vector of ϕ and $\perp$ means normal component to ϕ.

Now we will proceed in a very similar way as in the proof of Theorem 1. Deriving (7), we obtain

$$0 = \langle \nabla_v JH, w \rangle - \frac{4}{(1+|\phi|^2)^2} \langle d\phi(v), \phi \rangle \langle d\phi(w), J\phi \rangle + \frac{2}{1+|\phi|^2} \langle \sigma(v,w), J\phi \rangle,$$

where ∇ is the Levi-Civita connection of the metric $\phi^* \langle , \rangle$. Making the same reasoning as in the proof of Theorem 1, the second term of the above expression is symmetric, and so

$$\delta \wedge \hat{\delta} = 0,$$

where δ and $\hat{\delta}$ are the 1-forms on Σ given by

$$\delta(v) = \langle d\phi(v), \phi \rangle \quad \text{and } \hat{\delta}(v) = \langle d\phi(v), J\phi \rangle.$$

¿From Lemma 1, we conclude that there exist a closed and conformal vector field X on Σ, without zeroes, and functions a and b defined on the whole Σ such that

$$\phi = aX + bJX, \quad a^2 + b^2 = 1.$$

In particular, as Σ admits a conformal vector field whithout zeroes, Σ is a torus. Since $\phi^\perp = bJX$ and $|\phi| = |X|$, from (7) we get

$$H = -2(1+|X|^2)^{-2} bJX. \tag{8}$$

But, using the properties of the second fundamental form of ϕ given in Lemma 1, we also obtain that

$$2H = |X|^{-2}(\rho - b)JX + cJV,$$

for a certain function c, where V is a unit vector field orthogonal to X. Comparing this expression of H with the given in (8) we have that

$$(1+|X|^2)\rho = (1-3|X|^2)b, \quad c = 0. \tag{9}$$

Deriving $\phi = aX + bJX$, using Lemma 1 and (9), it is easy to check that the gradient of b satisfies

$$\nabla b = \frac{a\, b(|X|^2 - 1)}{|X|^2(|X|^2+1)} X.$$

But $\nabla |X|^2 = 2aX$ because X is a closed and conformal vector field. Thus the gradient of the function $b|X|^2(1+|X|^2)^{-2}$ is zero and that function is a constant μ. Hence from (9) we get that

$$b = \mu \frac{(1+|X|^2)^2}{|X|^2}, \quad \rho = \mu \frac{(1+|X|^2)(1-3|X|^2)}{|X|^2}.$$

As $2a = divX$ and $b = 4\mu \cosh^2(\log |X|)$, the equation $a^2 + b^2 = 1$ becomes in the following o.d.e.

$$(10) \qquad (\frac{divX}{2})^2 + 16\mu^2 \cosh^4(\log |X|) = 1.$$

On the other hand, from Theorem A, the universal covering of ϕ, $\tilde{\phi}$: $\mathbb{R}^2 \longrightarrow \mathbb{C}^2$, is given by

$$\tilde{\phi}(t,s) = \gamma(t)\tilde{\beta}(s),$$

where $\gamma : \mathbb{R} \longrightarrow \mathbb{C}^*$ is a regular curve and $\tilde{\beta} : \mathbb{R} \longrightarrow \mathbb{S}^3$ is the horizontal lift by the Hopf fibration of a regular curve β in $\mathbb{S}^2$. As $c = 0$, the curvature of β is zero and so β is a great circle in $\mathbb{S}^2$. Then our immersion can be written as

$$\tilde{\phi}(t,s) = \frac{\gamma(t)}{\sqrt{2}}(e^{is}, e^{-is}).$$

Hence the immersion $\tilde{\phi}$ is exactly $\hat{\Pi}^{-1}(\alpha)$ where α is the cylindric curve $\alpha = (\gamma^2/|\gamma|^2, \log |\gamma|^2)$. If $\tilde{X}$ is the corresponding closed and conformal vector field on $\mathbb{R}^2$ and u the function on $\mathbb{R}^2$ defined by $e^u = |\tilde{X}| = |\gamma|$, then equation (10) is rewritten as

$$(11) \qquad u'^2 + 16\mu^2 \cosh^4 u = 1.$$

Moreover, as $\tilde{X} = \tilde{\phi}_t$, the equation $\tilde{\phi} = a\tilde{X} + bJ\tilde{X}$ on the points $(t,0)$ becomes in $\gamma' = (a-ib)\gamma$, and so γ is given by

$$\gamma(t) = e^{\int_0^t (a-ib)(r)dr} = \frac{e^{u(t)}}{e^{u(0)}} e^{4\mu i \int_0^t \cosh^2 u(r)dr}.$$

As a consequence, the curve α is written as

$$\alpha(t) = (e^{8\mu i \int_0^t \cosh^2 u(r)dr}, 2u(t)),$$

where u(t) is a solution of equation (11). So our curve corresponds to the given in Proposition 3,(i) with $\lambda = 4\mu$.

Proof of (ii) By a similar reason to (i), $\tilde{\Sigma}$ satisfies equation (1). Using that this equation (1) is elliptic and following a similar reasoning to the used by Babich and Bobenko in the proof of Theorem 4 of [BB], it follows that on $\Sigma_0 = \phi^{-1}(\mathbb{S}^3)$ equation (1) is also satisfied. So, our surface Σ is a Willmore surface of $\mathbb{C}^2$. Following again the proof of Theorem 4 of [BB], it can be proved in a similar way that Σ_0 is a set of umbilical points and then, by analiticity, Σ_0 has empty interior.

Also, computing the mean curvature vector of ϕ with respect to the hyperbolic metric and using the minimality of ϕ we get on $\tilde{\Sigma}$

$$H = \frac{2}{(1 - |\phi|^2)}\phi^{\perp},$$

where H is the mean curvature vector of ϕ and $\perp$ means normal component.

Making the same reasoning as in (i), we get that the 1-forms δ and $\hat{\delta}$ (which are defined on the whole Σ) satisfy on $\tilde{\Sigma}$, $\delta \wedge \hat{\delta} = 0$. As Σ_0 has empty interior, then by continuity $\delta \wedge \hat{\delta} = 0$ on the whole Σ.

Repeating the reasoning made in the proof of (i), we get that $\phi = aX + bJX$, where X is a closed and conformal vector field on Σ and $a^2 + b^2 = 1$. So Σ must be a torus. Also, the functions b and ρ are given in this case by

$$b = \mu\frac{(1 - |X|^2)^2}{|X|^2}, \quad \rho = \mu\frac{(1 - |X|^2)(1 + 3|X|^2)}{|X|^2},$$

on $\tilde{\Sigma}$, and by continuity on the whole Σ. In particular Σ_0 is a set of geodesic points.

Using the same arguments of (i), if $\tilde{\phi} : \mathbb{R}^2 \longrightarrow \mathbb{C}^2$ is the universal covering of ϕ, then $\tilde{\phi}$ is $\hat{\Pi}^{-1}(\alpha)$ where α is the cylindric curve

$$\alpha(t) = (e^{8\mu i \int_0^t \sinh^2 u(r) dr}, 2u(t)),$$

being u a solution of the equation

$$u'^2 + 16\mu^2 \sinh^4 u = 1. \tag{12}$$

So our curve corresponds to the given in Proposition 3,(ii) with $\nu = 4\lambda$.□

Examples C.

If we consider the curve $\tilde{\beta} : \mathbb{R} \longrightarrow \mathbb{S}^3$ given by

$$\tilde{\beta}(s) = \left(\cos\psi \, e^{-i\tan\psi \, s}, \sin\psi \, e^{i\cot\psi \, s}\right),$$

with $\psi \in (0, \pi/2)$, then it is clear that $\beta = \Pi \circ \tilde{\beta}$ is the parallel of $\mathbb{S}^2$ of latitude $\pi/2 - 2\psi$, where $\Pi : \mathbb{S}^3 \longrightarrow \mathbb{S}^2$ is the Hopf fibration. Also we have that $\langle \tilde{\beta}', J\tilde{\beta} \rangle = 0$. This just means that $\tilde{\beta}$ is a horizontal lift, by the Hopf fibration, of the parallel β in $\mathbb{S}^2$. It is unique up to rotations in $\mathbb{S}^3$. Here s is the arc parameter for $\tilde{\beta}$ and we note that $|\beta'(s)| = 2$.

We observe that $\tilde{\beta}$ is a closed curve in $\mathbb{S}^3$ if and only if $\tan^2\psi \in \mathbb{Q}$.

Definition 1 *For any $m, n \in \mathbb{N}$, $0 < n < m$, $(n, m) = 1$, let*

$$\phi_{m,n} : \mathbb{R}^2 \longrightarrow \mathbb{C}^2$$

be the immersion defined by

$$\phi_{m,n}(t, s) = \gamma(t)\tilde{\beta}(s)$$

where γ is the regular curve in $\mathbb{C}^*$ parametrized by $|\gamma(t)| = |\gamma'(t)|$ such that γ^2 is the circle centred at $(\frac{m+n}{m-n}, 0)$ with radius $\frac{2\sqrt{mn}}{m-n}$ and $\tilde{\beta}$ is the horizontal lift of the parallel in $\mathbb{S}^2$ of latitude $\arcsin \frac{m-n}{m+n}$. *According to the above expression,*

$$\tilde{\beta}(s) = \frac{1}{\sqrt{m+n}} \left(\sqrt{m}\, e^{-i\sqrt{n/m}\, s}, \sqrt{n}\, e^{i\sqrt{m/n}\, s}\right).$$

Thanks to the properties of periodicity that we are going to prove in the next Proposition, we can introduce a new family of Willmore Lagrangian tori.

Proposition 4 *The immersions $\phi_{m,n} : \mathbb{R}^2 \longrightarrow \mathbb{C}^2$, described in Definition 1, are doubly periodic with respect to the rectangular lattice $\Lambda_{m,n}$ in the (t, s)-plane generated by the vectors $(4\sqrt{mn}K/(\sqrt{m} + \sqrt{n})^2, 0)$ and $(0, 2\sqrt{mn}\pi)$, where K is the complete elliptic integral of first kind with modulus $8\sqrt{mn}(m+n)/(\sqrt{m} + \sqrt{n})^4$ (see [BF]).*

The tori $\mathcal{T}_{m,n} = \mathbb{R}^2/\Lambda_{m,n}$ are Willmore Lagrangian embedded surfaces in $\mathbb{C}^2$.

Proof: Firstly, it is easy to check that our immersions $\phi_{m,n}$ are conformal, since the induced metric by $\phi_{m,n}$ is written as $e^{2u(t)}(dt^2+ds^2)$, where $e^{2u(t)}=|\gamma(t)|^2$. For our purposes we need the explicit expression of γ. This neccesarily involves the use of Jacobian elliptic functions (see [BF]).

CLAIM: The curve γ determined in Definition 1 is given by

$$\gamma(t)=e^{u(t)+\frac{(m-n)i}{2\sqrt{mn}}\int_0^t \sinh 2u(x)\,dx},$$

where $u=u(t)$ is the solution to the sinh-Gordon equation

$$u''+\frac{(m-n)^2}{4mn}\sinh 4u=0$$

with initial conditions $u'(0)=0$, $\sinh 2u(0)=2\sqrt{mn}/(m-n)$.

In fact, in order to simplify the notation, let $a=(\sqrt{m}+\sqrt{n})^2/(m-n)$ and $b=(\sqrt{m}+\sqrt{n})^2/2\sqrt{mn}$.

We easily obtain that $|\gamma|=|\gamma'|$ and that the curvature of $\alpha=\gamma^2$ is constant, exactly $(m-n)/2\sqrt{mn}$. Moreover, we also need the explicit expression for $u=u(t)$ which is given by $e^{2u(t)}=a\,\mathrm{dn}(b\,t)$, where dn is the "delta amplitude" Jacobian elliptic function (see [BF]) with modulus $1-1/a^4$ ($a=e^{2u(0)}$). Now, it is not difficult to deduce that the farest and nearest points from the origin to the circle α are $\alpha(0)=(a,0)$ and $\alpha(K/b)=(1/a,0)$. The middle point of them is just $(\cosh 2u(0),0)=(\frac{(m+n)}{m-n},0)$, the center of α. So the claim is true.

If we write $F(t)=\int_0^t e^{2u(x)}dx$ and $G(t)=\int_0^t e^{-2u(x)}dx$, we can get that $F(t)=\arcsin(\mathrm{sn}(b\,t))$, if $t\in(-K/b,K/b)$ and $G(t)=\arccos(\mathrm{cd}(b\,t))$, if $t\in(0,2K/b)$, where sn and cd are Jacobian elliptic functions (see [BF]). ¿From here it is easy to prove that $\int_0^{t+2K/b}\sinh 2u(x)\,dx=\int_0^t \sinh 2u(x)\,dx$ and then $\gamma(t+2K/b)=\gamma(t)$, $\forall t\in\mathbb{R}$.

Looking at Definition 1, it is clear that $\tilde{\beta}(s+2\sqrt{mn}\pi)=\tilde{\beta}(s)$, $\forall s\in\mathbb{R}$.

On the other hand, $\phi_{m,n}$ is a Lagrangian immersion because the curve $\tilde{\beta}$ is horizontal.

To prove that $\phi:\mathcal{T}_{m,n}\longrightarrow\mathbb{C}^2$ is an embedding, it is enough to notice that $\gamma(t)$, $t\in[0,2K/b]$, and $\tilde{\beta}(s)$, $s\in[0,2\sqrt{mn}\pi]$, are simple closed curves in $\mathbb{C}^*$ and $\mathbb{S}^3$ respectively.

Finally, since we know $\phi_{m,n}$ explicitly, it is a very long but straightforward computation to check that the equation (1) is satisfied for the tori $\mathcal{T}_{m,n}$. $\square$

Proposition 5 *The Willmore Lagrangian tori* $\mathcal{T}_{m,n} = \mathbb{R}^2/\Lambda_{m,n}$ *introduced in Proposition 4 satisfy the following properties:*

(i) *the area* A *of the torus* $\mathcal{T}_{m,n}$ *is*

$$A = \frac{4mn}{m-n}\pi^2;$$

(ii) *the Willmore functional* W *of the torus* $\mathcal{T}_{m,n}$ *is*

$$\mathcal{W} = 2\pi\left((\sqrt{m}+\sqrt{n})^2 E + \frac{(m-n)^2}{(\sqrt{m}+\sqrt{n})^2}K\right),$$

where K *and* E *(depending on* m *and* n*) are the complete elliptic integrals of first and second kind with modulus* $8\sqrt{mn}(m+n)/(\sqrt{m}+\sqrt{n})^4$*, respectively (see [BF]).*

(iii) *the torus* $\mathcal{T}_{m,n}$ *is invariant under the inversion of* $\mathbb{R}^4$ *given by* $p \mapsto p/|p|^2$.

Proof: We follow the same notation as in the proof of Proposition 4. Using that the measure of the induced metric of the torus is given by $dA = e^{2u(t)}dt\,ds$ and that $F(2K/b) = 2\sqrt{mn}\pi/(m-n)$, we obtain the formula of the area A.

For the Willmore's one we first need to check that $|H|^2 = k_\alpha^2 e^{2u} + k_\beta^2 e^{-2u}$. Taking into account that $\int_0^{2K/b} e^{4u(t)}dt = (a^2/b)\int_0^{2K} \mathrm{dn}^2 w\,dw = (2a^2/b)E$ and that $k_\beta = (m-n)/(2\sqrt{mn})$, we finally get the expression given for $\mathcal{W}$.

We first observe that $\phi/|\phi|^2 = (\gamma/|\gamma|^2)\tilde{\beta}$. But γ^2 is a circle with center $C = ((m+n)/(m-n), 0)$ and radius $R = 2\sqrt{mn}/(m-n)$. Then $|C|^2 - R^2 = 1$ and this just implies that it remains invariant under the inversion.□

4 Classification

In this paragraph, we are going to characterize the families of Willmore tori defined in the last section among the compact Willmore surfaces of $\mathbb{C}^2$. We will assume that the Willmore surface is Lagrangian, and also that its image under some inversion, which is a Willmore surface again, is in addition a Lagrangian surface.

Theorem 3 *Let $\phi : \Sigma \longrightarrow \mathbb{C}^2$ be a Willmore immersion of an orientable compact surface Σ such that ϕ and its image under some inversion of $\mathbb{C}^2$, centered at a point of $\mathbb{C}^2 - \{\phi(\Sigma)\}$, are both Lagrangian immersions. Then ϕ is congruent, up to dilations of $\mathbb{C}^2$, either to a Willmore Hopf torus, or to some T_α where α is a closed curve in the right circular cylinder, critical for the functional $\mathcal{F}$ (see Propositions 1 and 2), or to some $\mathcal{T}_{m,n}$, $m,n \in \mathbb{N}$, $n < m$, $(m,n) = 1$ (see Proposition 4).*

Proof: Up to a translation we can assume that the inversion of the hypothesis of the Theorem is $F : \mathbb{C}^2 \cup \{\infty\} \longrightarrow \mathbb{C}^2 \cup \{\infty\}$ defined by $F(p) = p/|p|^2$. Let

$$\psi = F \circ \phi = \frac{\phi}{|\phi|^2} : \Sigma \longrightarrow \mathbb{C}^2,$$

which is also a Lagrangian immersion by the assumptions. If δ and $\hat{\delta}$ are the 1-forms on Σ given by

$$\delta(v) = \langle d\phi(v), \phi\rangle \qquad \text{and } \hat{\delta}(v) = \langle d\phi(v), J\phi\rangle,$$

then it is easy to see that $\psi^*\Omega = (-2/|\phi|^6)(\delta \wedge \hat{\delta})$; as ψ is also Lagrangian, we obtain that

$$\delta \wedge \hat{\delta} = 0. \tag{13}$$

¿From Lemma 1 and Theorem A, we get that the universal covering of ϕ, which will be also noted by ϕ, $\phi : \mathbb{R}^2 \longrightarrow \mathbb{C}^2$, is given by

$$\phi(t,s) = \gamma(t)\tilde{\beta}(s),$$

where $\gamma : \mathbb{R} \longrightarrow \mathbb{C}^*$ is a regular curve and $\tilde{\beta} : \mathbb{R} \longrightarrow \mathbb{S}^3 \subset \mathbb{C}^2$ is a regular curve in the unit sphere, which is a horizontal lift (with respect to the Hopf fibration) of a regular curve β in the sphere $\mathbb{S}^2$.

¿From now on, we reparametrized γ by $|\gamma'(t)| = |\gamma(t)|$ and $\tilde{\beta}$ by $|\tilde{\beta}'(s)| = 1$. Then the induced metric is given by $|\gamma|^2(dt^2 \times ds^2)$, and in particular ϕ is a conformal immersion.

Our first purpose is to write down in a suitable way the equation of a Willmore surface (equation (1)) in terms of γ and β.

It is straightforward to see that the second fundamental form σ of ϕ is given by

$$\begin{aligned}\sigma(\partial_t,\partial_t) &= |\gamma|k_\gamma J\phi_t,\\ \sigma(\partial_s,\partial_s) &= \frac{\langle\gamma',J\gamma\rangle}{|\gamma|^2}J\phi_t+2k_\beta J\phi_s,\end{aligned}$$

where k_γ and k_β are the curvatures of the planar curve γ and the spherical curve β respectively. So, the mean curvature vector H of ϕ is given by

$$2H=(\frac{k_\gamma}{|\gamma|}+\frac{\langle\gamma',J\gamma\rangle}{|\gamma|^4})J\phi_t+\frac{2k_\beta}{|\gamma|^2}J\phi_s.$$

It will be useful for our purpose to introduce the curve $\hat{\alpha}$ in $\mathbb{C}^*$ defined by $\hat{\alpha}(t)=\gamma^2(t)$. We observe that the parametrization of $\hat{\alpha}$ satisfies $|\hat{\alpha}'|=2|\hat{\alpha}|$. Their curvatures are related by

$$2k_{\hat{\alpha}}=\frac{k_\gamma}{|\gamma|}+\frac{\langle\gamma',J\gamma\rangle}{|\gamma|^4}.$$

Thus the mean curvature vector is

$$H=k_{\hat{\alpha}}J\phi_t+\frac{k_\beta}{|\hat{\alpha}|}J\phi_s.$$

Now, a very long but really straightforward computation says that ϕ is a Willmore surface, i.e. ϕ satisfies equation (1), if and only if the following equations hold:

$$\begin{aligned}&|\hat{\alpha}|\,k''_{\hat{\alpha}}+|\hat{\alpha}|'\,k'_{\hat{\alpha}}+\frac{\langle\hat{\alpha}',J\hat{\alpha}\rangle^2}{|\hat{\alpha}|^3}\,k_{\hat{\alpha}}-3\langle\hat{\alpha}',J\hat{\alpha}\rangle\,k^2_{\hat{\alpha}}+2|\hat{\alpha}|^3\,k^3_{\hat{\alpha}}\\ &\qquad-(\log|\hat{\alpha}|)'\,\dot{k}_\beta+2(\frac{\langle\hat{\alpha}',J\hat{\alpha}\rangle}{2|\hat{\alpha}|^2}-|\hat{\alpha}|k_{\hat{\alpha}})\,k^2_\beta=0,\\ &\qquad\ddot{k}_\beta+2k^3_\beta-2k_\beta(|\hat{\alpha}|^2k^2_{\hat{\alpha}}-\frac{\langle\hat{\alpha}',J\hat{\alpha}\rangle}{|\hat{\alpha}|}k_{\hat{\alpha}})=0,\end{aligned}\tag{14}$$

where $'$ (respectively $\dot{}$) means derivative with respect to t (respectively s).

In order to manipulate the above equations, we consider on the cylinder $\mathbb{S}^1\times\mathbb{R}$ the curve $\alpha=(\alpha_1,\alpha_2,\alpha_3)$ defined by

$$\alpha=(\hat{\alpha}/|\hat{\alpha}|,\log|\hat{\alpha}|).$$

We observe that $|\alpha'| = 2$. Its curvature k_α can be easily computed, obtaining that

$$k_\alpha = |\hat{\alpha}| k_{\hat{\alpha}} - \frac{\langle \hat{\alpha}', J\hat{\alpha}\rangle}{2|\hat{\alpha}|^2}. \tag{15}$$

Using (15), the equations (14) can be changed in the following easier equations:

$$\begin{aligned} & k_\alpha'' + 2k_\alpha^3 + 2k_\alpha\Big(1 - \frac{3{\alpha_3'}^2}{4} - k_\beta^2\Big) - \alpha_3' \, \dot{k}_\beta = 0, \\ & \ddot{k}_\beta + 2k_\beta^3 - 2k_\beta\Big(k_\alpha^2 - 1 + \frac{{\alpha_3'}^2}{4}\Big) = 0. \end{aligned} \tag{16}$$

Now, deriving the second equation of (16) with respect to t we obtain that

$$k_\beta\Big(k_\alpha^2 - 1 + \frac{{\alpha_3'}^2}{4}\Big)' = 0.$$

So either $k_\beta \equiv 0$ or $k_\alpha^2 - 1 + \frac{{\alpha_3'}^2}{4}$ is a constant ρ.

Case 1. Let us assume $k_\beta \equiv 0$.

Then β is a great circle in $\mathbb{S}^2$ and, up to rotations, its horizontal lift to $\mathbb{S}^3$ is given by $\tilde{\beta}(s) = (1/\sqrt{2})(e^{is}, e^{-is})$. So our original immersion ϕ is given by

$$\phi(t,s) = \frac{\gamma(t)}{\sqrt{2}}(e^{is}, e^{-is}).$$

With the notation of Examples B, it is clear that $\phi(\mathbb{R}^2)$ lies in M^3 and using the fibration $\hat{\Pi}$, it is also clear that ϕ and the universal covering of $\hat{\Pi}^{-1}(\alpha)$ are the same. The equations (16) become in

$$k_\alpha'' + 2k_\alpha^3 + 2k_\alpha\Big(1 - \frac{3{\alpha_3'}^2}{4}\Big) = 0.$$

This equation is just equation (6) in Proposition 2. Hence our torus, up to dilations of $\mathbb{C}^2$, is one of the given in Examples B.

Case 2. Suppose now

$$k_\alpha^2 - 1 + \frac{{\alpha_3'}^2}{4} \equiv \rho. \tag{17}$$

Deriving (17) with respect to t, we obtain that

$$k_\alpha k'_\alpha + \frac{\alpha'_3\alpha''_3}{4} = 0. \tag{18}$$

But deriving (15) we get

$$k'_\alpha - \alpha'_3\frac{\langle\hat{\alpha}', J\hat{\alpha}\rangle}{2|\hat{\alpha}|^2} = |\hat{\alpha}|k'_{\hat{\alpha}}. \tag{19}$$

Using that

$$\alpha''_3 = -2k_\alpha|\hat{\alpha}|^{-2}\langle\hat{\alpha}', J\hat{\alpha}\rangle, \tag{20}$$

the equations (18) and (19) become in

$$|\hat{\alpha}|k_\alpha k'_{\hat{\alpha}} = 0.$$

¿From here, as $\hat{\alpha}$ is a curve in $\mathbb{C}^*$, it is clear that either $k_\alpha = 0$ or $k_{\hat{\alpha}}$ is constant.

Case 2.1. If $k_\alpha = 0$, α must be a closed geodesic in $\mathbb{S}^1 \times \mathbb{R}$ because our surface is a torus. Necessarely this implies that γ must be a circle centered at the origin of $\mathbb{C}^2$. Hence, up to dilations of $\mathbb{C}^2$, we can assume that γ is a circle of radius one. In this case, our torus is $\Pi^{-1}(\beta)$, where $\Pi : \mathbb{S}^3 \longrightarrow \mathbb{S}^2$ is the Hopf fibration, and using the second equation of (16), β is a curve in $\mathbb{S}^2$ satisfying

$$\ddot{k}_\beta + 2k_\beta^3 + 2k_\beta = 0.$$

In this case our torus is one of the Willmore Hopf tori described by Pinkall in [Pi]. We observe that this equation corresponds exactly with the one given in Example A taking into account that the arc parameter for β is $2s$.

Case 2.2. The last case to study is when $k_{\hat{\alpha}}$ is constant. In this case, from (19) it follows

$$k'_\alpha = \alpha'_3\frac{\langle\hat{\alpha}', J\hat{\alpha}\rangle}{2|\hat{\alpha}|^2}.$$

Deriving this equation and using (15) and (20), we get

$$k''_\alpha = 2({\alpha'_3}^2 - 2)k_\alpha.$$

Putting this equation, joint to (17) in the equations (16) we obtain

$$\begin{aligned} \alpha'_3\dot{k}_\beta + 2k_\alpha(k_\beta^2 - \rho) &= 0 \\ \ddot{k}_\beta + 2k_\beta(k_\beta^2 - \rho) &= 0. \end{aligned} \tag{21}$$

Deriving the first equation of (21) with respect to s and using again both equations of (21), we conclude

$$(k_\alpha^2 + \frac{{\alpha_3'}^2}{4})\, k_\beta\, (k_\beta^2 - \rho) = 0.$$

But from (17) $k_\alpha^2 + {\alpha_3'}^2/4 = 1 + \rho$. If $1 + \rho = 0$, then $k_\alpha = 0$ and α_3 is constant. Hence our surface is up to dilations the Clifford torus, which is the simplest Willmore Hopf torus. So we can assume that $1 + \rho > 0$. As in this case k_β is non null, the above equation says that $k_\beta^2 = \rho$. So the curve β is a small circle of $\mathbb{S}^2$.

On the other hand, since $k_{\hat\alpha}$ is constant and our surface compact, $\hat\alpha$ must be a circle in $\mathbb{C}^*$, say of radius R and center C. So $|\hat\alpha - C| = R$ and $k_{\hat\alpha} = 1/R$. From here we arrive at

$$|\hat\alpha|^2 + |C|^2 - 2\langle\hat\alpha, C\rangle = R^2 \tag{22}$$

$$\hat\alpha' = \frac{2|\hat\alpha|}{R} J(\hat\alpha - C).$$

Taking into account that $k_\beta^2 = \rho$ and $k_{\hat\alpha} = 1/R$, using (15), (17) and the two equations of (22), we can conclude

$$R^2(1 + k_\beta^2) = |C|^2. \tag{23}$$

Finally, as our surface is compact, the horizontal lift $\tilde\beta$ to $\mathbb{S}^3$ of our small circle β of $\mathbb{S}^2$ must be a closed curve. Looking at Examples C in paragraph 3 we have that there exist positive integer numbers m, n with $n < m$ and $(m, n) = 1$ such that $k_\beta = (m - n)/2\sqrt{mn}$. It is not a restriction to take the center C of $\hat\alpha$ like $C = (a, 0)$. We obtain from (23) that $a^2 = R^2(m+n)^2/(4mn)$. This relation is invariant under homotethies of $\mathbb{C}$ and so, up to dilations on $\mathbb{C}^2$, we can take C and a like in the Examples C. So in this last case our torus is congruent with some torus $T_{m,n}$ described in Examples C, paragraph 3.$\square$

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