New examples of minimal Lagrangian tori in the complex projective plane

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A new family of minimal Lagrangian tori in the complex projective plane is constructed. The family is characterized by its invariance by a one-parameter group of isometries of the complex projective plane.

1 Introduction.

Minimal surfaces of the complex projective plane $\mathbb{CP}^2$ have been studied during the last years from different points of view. So, in [6], they have been treated considering the corresponding liftings to the twistor space $SU(3)/T^3$ over $\mathbb{CP}^2$. Another point of view is given in [7], where they make an exhaustive study of them analyzing deeply the Kähler angle of the immersion.

But in both cases, the Lagrangian (or totally real) minimal surfaces of $\mathbb{CP}^2$ appear as an extremal case and there are not specific results in such family of surfaces. In fact, only two compact examples are known: the totally geodesic embedding of $\mathbb{HP}^2$ in $\mathbb{CP}^2$, the only one with genus zero, and the generalized Clifford torus (see [9]) which is the only flat minimal Lagrangian surface and which has been characterized ([10] and [12]) by a pinching of the Gauss curvature. In [6], it is stated the importance to find new compact examples (and hence of genus bigger than zero) of minimal Lagrangian surfaces of $\mathbb{CP}^2$.

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The main purpose of this paper is to construct new examples of Lagrangian minimal tori in $\mathbb{CP}^2$. Our work is inspired — among others — in the works of Bobenko [1] and Pinkall and Sterling [11].

Among all the compact minimal Lagrangian surfaces of $\mathbb{CP}^2$, our examples are characterized by their invariance by a one-parameter group of isometries of $\mathbb{CP}^2$ (Corollary 2). This invariance of the surface means that the affine Toda equation which satisfied the induced metric [2] is an ordinary differential equation (Proposition 1), which allows us to control the solutions of such equation in terms of elliptic functions. In the appendix, we integrate that equation and study some properties of its solutions which will be used along the paper.

The used method consists of lifting the minimal Lagrangian immersion in $\mathbb{CP}^2$ to an horizontal and minimal immersion in $S^5$ via the Hopf fibration of $S^5$ over $\mathbb{CP}^2$. The Frenet equations of the lifting are completely integrated and so we can give explicitly a two-parameter family of minimal Lagrangian immersions from $\mathbb{R}^2$ in $\mathbb{CP}^2$ (Theorem 3). Finally, in Theorem 4, we characterize in terms of certain rationality conditions those immersions which are doubly-periodic, obtaining in this way infinitely many new examples of minimal Lagrangian tori of $\mathbb{CP}^2$.

2 Preliminaries.

Let $\mathbb{CP}^2$ be the two-dimensional complex projective space endowed with the Fubini–Study metric $g$ of constant holomorphic sectional curvature 4. If $J$ denotes the complex structure on $\mathbb{CP}^2$, then the Kähler two-form $\Omega$ is given by $\Omega(X,Y) = g(X, JY)$, for any tangent vector fields $X$ and $Y$.

Let $\phi : \Sigma \longrightarrow \mathbb{CP}^2$ be an immersion of a surface $\Sigma$ in $\mathbb{CP}^2$. We also denote by $g$ the induced metric. $\phi$ is called Lagrangian (or totally real) if $\phi^* \Omega = 0$. This means that $J$ defines an isomorphism from the tangent bundle of $\Sigma$ to the normal bundle of $\phi$ and we can define a trilinear form $C$ on $\Sigma$ by

\begin{equation}
C(X, Y, Z) = g(\sigma(X, Y), JZ)
\end{equation}

where $\sigma$ denotes the second fundamental form of $\phi$ and $X$, $Y$, $Z$ are tangent vector fields to $\Sigma$. The most elementary property of $\phi$
is that $C$ is symmetric.

If $\Sigma$ is an orientable surface, we can define a cubic differential $\Theta$ on $\Sigma$ in the following way. If $z = x + iy$ is a local isothermal coordinate on $\Sigma$, then

\[(2) \quad \Theta(z) = f(z)dz^3, \text{ with } f(z) = 4C(\partial_z, \partial_z, \partial_z)\]

where $C$ is extended $C^\infty$-linearly to the complexified tangent bundle.

From now on we assume that $\phi$ is a minimal immersion. Then it is well-known, [4], that $\Theta$ is a holomorphic differential. Moreover, if in that local coordinate the induced metric is written as $g = e^{2u}|dz|^2$, with $|dz|^2$ the Euclidean metric, then using the Gauss equation of $\phi$ we get

\[(3) \quad |f|^2 = e^{6u}(|\sigma|^2)^2 = 2e^{6u}(1 - K)\]

where $K$ is the Gauss curvature of $\Sigma$ and $|\sigma|$ is the length of $\sigma$. So, $\sigma$ vanishes precisely at the isolated zeroes of $\Theta$. Such kind of points will be called geodesic points. We call $\Sigma' = \{p \in \Sigma / \sigma_p \neq 0\}$.

Now, $g^* = |\sigma|^{2/3}g$ is a metric on $\Sigma$ whose ramification points are the geodesic points of $\phi$. So $g^*$ is a regular metric on $\Sigma'$ and it is easy to prove (see [12]) that

\[(4) \quad \triangle \log |\sigma|^{1/3} = K\]

where $\triangle$ is the Laplacian of $g$. So $(\Sigma', g^*)$ is a flat surface.

We remark that if $\tilde{u}$ is the function on $\Sigma'$ defined by $\tilde{u} = -\log |\sigma|^{1/3}$, then using (4) and the Gauss equation we can see that $\tilde{u}$ satisfies the affine Toda equation

\[(5) \quad \Delta^* \tilde{u} + e^{2\tilde{u}} - \frac{e^{-4\tilde{u}}}{2} = 0\]

where $\Delta^*$ is the Laplacian of the metric $g^*$.

The most important result proved for this kind of immersions is the following (see [12], Theorem 7 and [10]).

**Theorem 1** Let $\phi: \Sigma \longrightarrow \mathbb{CP}^2$ be a minimal Lagrangian immersion of a compact orientable surface.

a) If $\Sigma$ has genus zero, then $\phi$ is totally geodesic and it is the standard immersion of $S^2(1)$ into $\mathbb{CP}^2$. 
b) If the genus of \( \Sigma \) is nonzero and either \( K \geq 0 \) or \( K \leq 0 \), then \( \Sigma \) is flat and \( \phi \) is the generalized Clifford torus.

Remark 1. The generalized Clifford torus is the quotient by the Hopf fibration of the following isometric embedding

\[
S^1(1/\sqrt{3}) \times S^1(1/\sqrt{3}) \times S^1(1/\sqrt{3}) \hookrightarrow S^5(1) \subset \mathbb{C}^3.
\]

This example is well-known ([9]) and no more examples of minimal Lagrangian tori in \( \mathcal{H}P^2 \) are known.

3 Minimal Lagrangian surfaces invariant by a one-parameter group of isometries of \( \mathcal{H}P^2 \).

Let \( \phi : \Sigma \rightarrow \mathcal{H}P^2 \) be an immersion and \( \{F_t \mid t \in \mathbb{R}\} \) a one-parameter group of isometries of \( \mathcal{H}P^2 \). We say that \( \phi \) is invariant by \( \{F_t \} \) if for any \( p \in \Sigma \) there exists a positive real number \( \epsilon \) such that \( F_t(\phi(p)) \subset \phi(\Sigma) \) for \( |t| < \epsilon \).

We start this section proving the following result.

Proposition 1 Let \( \phi : \Sigma \rightarrow \mathcal{H}P^2 \) be a non-totally geodesic minimal Lagrangian immersion of an orientable surface invariant by a one-parameter group of isometries of \( \mathcal{H}P^2 \). Then, around each point, there is a conformal parametrization of \( \Sigma \), \( (U, z = x + iy) \), and a real number \( \alpha \in [0, 2\pi] \) such that the induced metric \( g \) and the holomorphic cubic differential \( \Theta \) are given by

\[
g = e^{2u(x)} |dz|^2 \quad \text{and} \quad \Theta(z) = \sqrt{2} e^{i\alpha} dz^3
\]

respectively, where \( u = u(x) \) is a solution of the O.D.E.

\[
u'' + e^{2u} - e^{-4u} = 0.
\]

In particular, \( \phi \) has no geodesic points.

Proof: Let \( \mathcal{Y} \) be the Killing vector field on \( \mathcal{H}P^2 \) associated to the one-parameter group of isometries \( \{F_t\} \). Then for any point \( p \in \Sigma, \mathcal{Y}_{\phi(p)} \in \phi_*(T_p\Sigma) \), where \( T_p\Sigma \) denotes the tangent plane of \( \Sigma \) at \( p \). So \( \mathcal{Y} \) induces a Killing vector field \( \mathcal{X} \) on \( (\Sigma, g) \) which is \( \phi \)-related to \( \mathcal{Y} \). If \( \{\varphi_t\} \) is the local one-parameter group of isometries of \( (\Sigma, g) \) associated to \( \mathcal{X} \), then it is not difficult to see that

\[
\varphi_t^* C = C
\]
where $C$ is the trilinear form defined in (1).

In particular, $|\sigma|^2$ is constant along the integral curves of $\mathcal{X}$ and hence $\mathcal{X}(|\sigma|^2) = 0$. Since $g^* = |\sigma|^{2/3}g$ in $\Sigma'$ (see section 2), this implies that $\mathcal{X}$ is also a Killing vector field on $(\Sigma', g^*)$.

Now we are going to see that $\{p \in \Sigma / \mathcal{X}_p \neq 0\} = \Sigma'$. In fact, if $p$ is a geodesic point of $\phi$, from (2) we get that $\varphi_t(p)$, $|t| < \epsilon$, are geodesic points too and as they are isolated, we obtain that $\varphi_t(p) = p$, which implies that $\mathcal{X}_p = 0$. Conversely, if $\mathcal{X}_p = 0$ and $p$ is not a geodesic point then $\varphi_t(p) = p$, for $t$ in a certain interval. Using this fact in (2) it is not difficult to prove that $(d\varphi_t)_p = Id$. Hence $\varphi_t = Id$ and then $F_t \circ \phi = \phi$ on a neighbourhood $W$ of $p$. This implies that $\phi$ is totally geodesic on $W$ which contradicts the assumptions. So the claim is established.

Since $\mathcal{X}$ has no zeroes on $\Sigma'$, using standard arguments we can prove that for any point $p \in \Sigma'$ there exists a local complex co-ordinate $(U, z = x + iy)$, $U \subset \Sigma'$ around $p$ such that $\mathcal{X} = \partial / \partial y$. Thus, from (2) we obtain that the holomorphic function $f(z)$ defined in (2) is constant on the curves $x =$constant and then it is constant on $U$. Hence there exist real numbers $\mu, \alpha, \mu > 0$, such that $\Theta(z) = \mu e^{ia} dz^3$. Now (3) says that $\mu^{2/3} = e^{2a}|\sigma|^{2/3}$ and as $\mathcal{X} = \partial / \partial y$ and $\mathcal{X}(|\sigma|^2) = 0$, we have that $u(z) = u(x)$. Also we obtain that $g^* = \mu^{2/3}|dz|^2$, which implies that $g^*(\mathcal{X}, \mathcal{X}) = \mu^{2/3}$ and so we get that $g^*(\mathcal{X}, \mathcal{X})$ is a constant function on $\Sigma'$ by the connectedness of $\Sigma'$. Normalizing we take $g^*(\mathcal{X}, \mathcal{X}) = \sqrt{2}$. Then $g^* = \sqrt{2}|dz|^2$, $u = \tilde{u} + (1/6)\log 2$ (see section 2) and from (5) we conclude that $u(x)$ satisfies the O.D.E. (1). We observe that the cubic differential $\Theta$ is written as $\Theta(z) = \sqrt{2}e^{i\alpha} dz^3$, with $\alpha \in [0, 2\pi]$, and note that $\mathcal{X}$ is a parallel vector field on $(\Sigma', g^*)$ because it is a Killing vector field of constant length on a flat surface.

To finish the proof, we only have to see that $\Sigma' = \Sigma$. In fact, if $p$ is a geodesic point, then $\mathcal{X}_p = 0$. Let $\gamma : [0, \epsilon] \rightarrow \Sigma$ be a curve such that $\gamma(0) = p$ and $\gamma(0, \epsilon)$ is a geodesic of $(\Sigma', g^*)$ with $g^*(\gamma', \gamma') = \sqrt{2}$. If $h : (0, \epsilon) \rightarrow IR$ is the function given by $h(t) = \tilde{u}(\gamma(t)) + (1/6)\log 2$, then $\lim_{t \to 0} h(t) = \infty$. As $\mathcal{X}$ is parallel in $(\Sigma', g^*)$ and $\mathcal{X}_{\gamma(0)} = 0$, we have $g^*(\mathcal{X}_{\gamma(t)}, \gamma'(t)) = 0$. So we can check that $h''(t) = \sqrt{2}(\Delta^* \tilde{u})(\gamma(t))$. From (5) we get that $h(t)$ satisfies the O.D.E. (1). But (5) implies that the solutions of that equation are bounded, which contradicts that $\lim_{t \to 0} h(t) = \infty$. So there are
not geodesic points and we finish the proof. □

Now, in order to study the Frenet equations of a minimal Lagrangian immersion on \( \mathcal{CIP}^2 \), we are going to lift it to \( S^5(1) \) via the Hopf fibration \( \Pi : S^5(1) \rightarrow \mathcal{CIP}^2 \). We denote by \( \langle \cdot, \cdot \rangle \) the Euclidean metric on \( \mathcal{C}^3 \) as well as that induced on \( S^5(1) \). The complex structure on \( \mathcal{C}^3 \) that induces \( J \) on \( \mathcal{CIP}^2 \) via \( \Pi \) will be also denote by \( J \).

So, the vertical space of \( \Pi \) at \( p \in S^5(1) \) is \( V_p = \text{span}\{Jp\} \).

Let \( \phi : \Sigma \rightarrow \mathcal{CIP}^2 \) be a minimal Lagrangian immersion. Since locally we can lift \( \phi \) to \( S^5(1) \), let \( \psi : U \rightarrow S^5(1) \) be a lifting of \( \phi \) to \( S^5(1) \) on a simply-connected open subset \( U \) of \( \Sigma \). Let \( \omega \) be the 1-form on \( U \) given by \( \omega(v) = \langle \psi_\ast(v), J\psi \rangle \) with \( v \) tangent to \( U \). Then, it is easy to see that if \( v \) and \( w \) are tangent vectors to \( U \) then \( d\omega(v,w) = 2\langle \psi_\ast(w), J\psi_\ast(v) \rangle = 2g(\phi_\ast(w), J\phi_\ast(v)) = 0 \), because \( \phi \) is a Lagrangian immersion. So, there exists \( \eta \in C^\infty(U) \) such that \( d\eta = \omega \). Then \( \tilde{\psi} = e^{-i\eta}\psi \) is another lifting of \( \psi \) to \( S^5(1) \), and for any tangent vector \( v \) on \( U \) we have that

\[
\langle \tilde{\psi}_\ast(v), J\tilde{\psi} \rangle = -d\eta(v) + \omega(v) = 0.
\]

Thus \( \tilde{\psi} \) is an horizontal lifting of \( \phi \) to \( S^5(1) \). Renaming \( \tilde{\psi} \) as \( \psi \), we can assume that \( \psi \) is an horizontal lifting of \( \phi \) to \( S^5(1) \). In this case we obtain that \( g \) is also the induced metric on \( U \) by \( \psi \). Also \( \psi \) is a minimal immersion in \( S^5(1) \) and \( \psi \) is a totally real immersion in \( \mathcal{C}^3 \) with respect to the complex structure \( J \). If \( \hat{\psi} \) is another horizontal lifting to \( S^5(1) \), then it is easy to prove that there exists \( \theta \in IR \) such that \( \hat{\psi} = e^{i\theta}\psi \). So, up to rotations of \( S^5(1) \), the horizontal lifting of \( \phi \) to \( S^5(1) \) is unique.

Suppose now that \( \phi \) is not totally geodesic and invariant by a one-parameter group of isometries of \( \mathcal{CIP}^2 \). If \( z = x + iy \) is an isothermal coordinate on \( U \) given in Proposition 1, we consider the matrices

\[
X = \begin{pmatrix} \psi_x \\ \psi_y \end{pmatrix}, \quad A = \frac{1}{2} \begin{pmatrix} u' & 0 & 0 \\ 0 & 0 & e^{-2u} \\ 1 & 0 & 0 \end{pmatrix}, \quad \hat{A} = \frac{1}{2} \begin{pmatrix} 0 & 0 & -e^{2u} \\ 0 & u' & 0 \\ 0 & 1 & 0 \end{pmatrix}, \quad B = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & e^{-2u}e^{i\alpha} & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},
\]
where $\psi$ is the horizontal lifting of $\phi$ to $S^5(1)$.

Using standard arguments, the Frenet equations of the immersion $\psi$ can be written as

$$X_z = AX + BJX, \quad X_{\bar{z}} = \tilde{A}X + B^*JX,$$

where $B^*$ denotes the conjugate transpose of $B$. In this way, the integrability conditions for (3) are given by the differential equations

$$A_{\bar{z}} - \tilde{A}_z + [A, \tilde{A}] + [B^*, B] = 0,$$

$$B_{\bar{z}} - B^*_z + [A, B^*] + [B, \tilde{A}] = 0.$$

Now it is not difficult to see that the integrability conditions (4) are equivalent to the condition that $u = u(x)$ satisfied the O.D.E. (1).

We can summarize this in the following result.

**Theorem 2** If $u : \mathbb{IR} \rightarrow \mathbb{IR}$ is a solution of equation (1) and if the initial conditions $\psi(z_0), \psi_z(z_0), \psi_{\bar{z}}(z_0)$ are compatible with the conditions $\psi = \psi, \psi_z = \psi_z$, then by integrating the equations (3) and projecting on $\mathbb{CP}^2$ via $\Pi$, we obtain a one-parameter family of minimal Lagrangian immersions invariant by a one-parameter group of isometries of $\mathbb{CP}^2$

$$\phi_\alpha : (\mathbb{IR}^2, e^{2u}|dz|^2) \rightarrow \mathbb{CP}^2,$$

such that $\Theta(z) = \sqrt{2}e^{i\alpha}dz^3$ is the holomorphic cubic form associated to $\phi_\alpha$.

**Proof:** We only need to prove that $\phi_\alpha$ is invariant by a one-parameter group of isometries of $\mathbb{CP}^2$. In fact, it is clear that $\phi_t(x, y) = (x, y + t)$ are isometries of $(\mathbb{IR}^2, e^{2u}|dz|^2)$. So, if $\psi_\alpha : \mathbb{IR}^2 \rightarrow S^5(1)$ is the horizontal lift of $\phi_\alpha$ to $S^5(1)$, then using (3) it follows that $\psi_\alpha \circ \phi_t$ and $\psi_\alpha$ satisfy the same Frenet equations and so there exists an isometry $F_t$ of $S^5(1)$ such that $F_t \circ J = J \circ F_t$ and $F_t \circ \psi_\alpha = \psi_\alpha \circ \phi_t$. In this way, via $\Pi$, $\{F_t / t \in \mathbb{IR}\}$ projects in a one-parameter group of isometries of $\mathbb{CP}^2$ and $\phi_\alpha$ is invariant by it. 

The solutions of equation (1) are known since they can be expressed in terms of certain class of elliptic functions (see appendix).
As they are periodic, we can consider —without loss of generality— only those solutions $u(x)$ such that $u'(0) = 0$. For any such a solution, there exists a positive real number $T$, such that

$$u(x + 2T) = u(x), \ u(-x) = u(x), \ u'(T) = 0. \tag{5}$$

So, $\hat{u}(x) = u(x + T)$ is also a solution of (1) with $\hat{u}'(0) = 0$. As $u(0)$ is positive if and only if $\hat{u}(0) = u(T)$ is negative (see appendix), then it is sufficient to consider the solutions of (1) with $u(0) \geq 0$.

This set of solutions can be parametrized by the interval $[3/2, +\infty[$ in the following way. Let $u(x)$ be a solution of (1) with $u(0) \geq 0$. Multiplication by $2u'$ in (1) and integration yields

$$u'^2 + e^{2u} + \frac{e^{-4u}}{2} = a = \text{undetermined constant}. \tag{6}$$

From (6) it is easy to see that $a \geq 3/2$ and the equality holds if and only if $u(x) \equiv 0$. Also, if $a > 3/2$, there are exactly two solutions of (1) with the same $a$. They are precisely $u$ and $\hat{u}$.

So, for any $a \in [3/2, +\infty[$, $u_a$ will denote the unique solution of equation (1) such that $u'(0) = 0$, $u(0) \geq 0$ and $e^{2u(0)} + e^{-4u(0)}/2 = a$.

We denote by $\{\phi_{a,\alpha}; \alpha \in [0,2\pi]\}$ the one-parameter family of immersions given in Theorem 2 associated to the solution of $u_a$ (1) and by $\psi_{a,\alpha}$ the corresponding horizontal lifting of $\phi_{a,\alpha}$ to $S^5(1)$. The maps $S_1, S_2 : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ given by $S_1(x, y) = (x, -y)$ and $S_2(x, y) = (-x, y)$ are isometries of $(\mathbb{R}^2, e^{2u(\cdot)}|dz|^2)$. Using (5) in (3) it is easy to prove that $\psi_{a,2\pi-\alpha}$ and $\psi_{a,\alpha} \circ S_1$ (resp. $\psi_{a,\pi-\alpha}$ and $\psi_{a,\alpha} \circ S_2$ with $\alpha \in [0, \pi]$) satisfy the same Frenet equations and following a similar reasoning like in the proof of Theorem 2, we get that $\phi_{a,2\pi-\alpha}$ and $\phi_{a,\alpha}$ (resp. $\phi_{a,\pi-\alpha}$ and $\phi_{a,\alpha}$ with $\alpha \in [0, \pi]$) are congruent immersions.

Also, in a similar way, $\phi_{3/2,\alpha}$ and $\phi_{3/2,0} \circ R_{\alpha/3}$ ($R_{\alpha/3}$ is a rotation in $\mathbb{R}^2$ of angle $\alpha/3$) are congruent immersions, because in this case $u_{3/2} \equiv 0$ and of course $R_{\alpha/3}$ is an isometry of $(\mathbb{R}^2, |dz|^2)$.

So, the set of Lagrangian immersions $\phi_{a,\alpha}$ given in Theorem 2 can be parametrized by the subset $\Gamma$ of $\mathbb{R}^2$ defined by

$$\Gamma = \left\{ \left( \frac{3}{2}, 0 \right) \right\} \cup \left[ \frac{3}{2}, \infty \right[ \times \left[ 0, \frac{\pi}{2} \right]. \tag{7}$$

Finally, as a consequence of Proposition 1, Theorem 2 and the above reasoning, we conclude the following result.
Corollary 1 Let $\phi : (\Sigma, g) \rightarrow \mathbb{CP}^2$ be a minimal Lagrangian immersion of an orientable and complete surface. If $\phi$ is invariant by a one-parameter group of isometries of $\mathbb{CP}^2$, then either $\phi$ is totally geodesic or the universal covering of $\phi$ is congruent to $\phi_{a,\alpha}$ with $(a, \alpha) \in \Gamma$.

4 New examples of minimal Lagrangian tori.

If $\psi_{a,\alpha}$ is the horizontal lifting of $\phi_{a,\alpha}$ to $S^5(1)$, $(a, \alpha) \in \Gamma$ (see section 3), then the Frenet equations (3) of $\psi_{a,\alpha}$ (that for the sake of simplicity we rename as $\psi$) can be rewritten as:

$$
\begin{align*}
\psi_{xx} &= u'\psi_x + \frac{\cos \alpha}{\sqrt{2e^2u}}J\psi_x - \frac{\sin \alpha}{\sqrt{2e^2u}}J\psi_y - e^{2u}\psi \\
\psi_{xy} &= u'\psi_y - \frac{\sin \alpha}{\sqrt{2e^2u}}J\psi_x - \frac{\cos \alpha}{\sqrt{2e^2u}}J\psi_y \\
\psi_{yy} &= -u'\psi_x - \frac{\cos \alpha}{\sqrt{2e^2u}}J\psi_x + \frac{\sin \alpha}{\sqrt{2e^2u}}J\psi_y - e^{2u}\psi
\end{align*}
$$

(1)

where $u$ is, in fact, the function $u_a$.

In the next result, we obtain explicitly the immersions $\psi_{a,\alpha}$, $(a, \alpha) \in \Gamma$.

Theorem 3 In suitable coordinate systems of $\mathbb{CP}^3$, the horizontal liftings $\psi_{a,\alpha} : \mathbb{R}^2 \rightarrow S^5(1)$ of the minimal Lagrangian immersions $\phi_{a,\alpha} : \mathbb{R}^2 \rightarrow \mathbb{CP}^2$, $(a, \alpha) \in \Gamma$ are given by

$$
\psi_{a,\alpha}(x, y) = \left(f_1(x)e^{i(G_1(x)+\sqrt{r_1}y)}, f_2(x)e^{i(G_2(x)-\sqrt{r_2}y)}, f_3(x)e^{i(G_3(x)-\sqrt{r_3}y)}\right)
$$

for $\alpha \neq \pi/2$, and

$$
\psi_{a,\pi/2}(x, y) = \left(f_1(0)dn(Kx/T)e^{i\sqrt{r_1}y}, f_2(0)cn(Kx/T)e^{-i\sqrt{r_2}y}, f_3(T)sn(Kx/T)e^{-i\sqrt{r_3}y}\right)
$$

where $-r_1 \leq -r_2 < -r_3 \leq 0$ are the roots of the third degree polynomial $r^3 + 2ar^2 + a^2r + (\sin^2 \alpha)/2$,

$$
G_j(x) = \frac{\cos \alpha}{\sqrt{2}} \int_0^x \frac{ds}{a - r_j - e^{2u_a}(s)}, \quad f_j(x) = \left(\frac{e^{2u_a(x)} + r_j - a}{3r_j - a}\right)^{\frac{1}{2}}.
$$
\( j = 1, 2, 3 \), \( \text{dn}, \text{cn} \) and \( \text{sn} \) are the elementary Jacobi elliptic functions and \( K \) is the complete elliptic integral of the first kind. (See appendix).

In particular, \( \phi_{a, \alpha} \), \((a, \alpha) \in \Gamma \), is invariant by the one-parameter group of isometries of \( \mathbb{C}P^2 \) induced, via \( \Pi \), by

\[
\{(e^{i\sqrt{r_j} t}, e^{-i\sqrt{r_i} t}, e^{-i\sqrt{r_3} t}) / t \in \mathbb{R}\}.
\]

Remark 2. From the appendix (Lemma 1 and (5.2)) it follows that \( \psi_{a,0}(x, y + 2\pi/\sqrt{a}) = \psi_{a,0}(x, y) \) and \( \psi_{a,\pi/2}(x + 4T, y) = \psi_{a,\pi/2}(x, y) \), \( \forall (x, y) \in \mathbb{R}^2 \).

Proof: As before, we will simply write \( \psi \) instead of \( \psi_{a,\alpha} \) and \( u \) instead of \( u_a \). From (1) we obtain

\[
(\psi_{yy} + au)_y = -\sin \alpha \sqrt{2} J\psi.
\]

Taking derivatives with respect to \( y \) in (2), we obtain that \( \psi(-, y) \) satisfies the following linear differential equation

\[
\frac{\partial^6 \psi}{\partial y^6} + 2a\frac{\partial^4 \psi}{\partial y^4} + a^2\frac{\partial^2 \psi}{\partial y^2} + \frac{\sin^2 \alpha}{2} \psi = 0.
\]

Solving (3) when \( \alpha \neq 0 \) and (2) when \( \alpha = 0 \), we get that

\[
\psi(x, y) = \sum_{j=1}^{3}\{\cos(\sqrt{r_j} y)C_j(x) + \sin(\sqrt{r_j} y)D_j(x)\},
\]

where \(-r_1 \leq -r_2 < -r_3 \leq 0\) are the roots of \( r^3 + 2ar^2 + a^2r + \sin^2 \alpha/2 \). As \(-r_1 < -a < -r_2\) when \( \alpha \neq 0 \) (see appendix, Lemma 1), using (2) in (4) we obtain that \( D_1 = JC_1 \) and \( D_i = -JC_i, i = 2, 3 \), which is trivial for \( \alpha = 0 \). So we get the following expression for the immersion \( \psi \)

\[
\psi(x, y) = \cos(\sqrt{r_1} y)C_1(x) + \sin(\sqrt{r_1} y)JC_1(x) + \sum_{i=2}^{3}\{\cos(\sqrt{r_i} y)C_i(x) - \sin(\sqrt{r_i} y)JC_i(x)\}.
\]

Using now in (5) that \( \psi \) is horizontal and conformal, it is straightforward to see that

\[
|C_j(x)|^2 = \frac{e^{2u(x)} + r_j - a}{3r_j - a}, \quad \langle C_k, C_l \rangle = \langle C_k, JC_l \rangle = 0,
\]
j, k, l = 1, 2, 3, k ≠ l. From the appendix (Lemma 2) it follows that
\( C_j(x) = 0 \) if and only if either \( \alpha = \pi/2, j = 2 \) and \( x = (2n + 1)T, n \in \mathbb{Z} \) or \( \alpha = \pi/2, j = 3 \) and \( x = 2nT, n \in \mathbb{Z} \), where \( 2T \) is the period of \( u(x) \).

Now, using (5) and (6) in the Frenet equations (1) of \( \psi \), it is not difficult to check that \( C_j(x), j = 1, 2, 3, \) satisfy the following O.D.E.

\[
(7) \quad (e^{2u} + r_j - a)C'_j(x) = u'e^{2u}C_j(x) - \frac{\cos \alpha}{\sqrt{2}} JC_j(x).
\]

If \( \alpha \neq \pi/2 \), then \( C_j(x) \) has no zeroes and by integrating (7) we arrive

\[
C_j(x) = f_j(x)e^{G_j(x)}, \quad j = 1, 2, 3,
\]

in the complex plane \( \Pi_j = \text{span}\{e_j = C_j(0)/|C_j(0)|, Je_j\} \). Putting (8) in (5) we conclude the result for \( \alpha \neq \pi/2 \).

If \( \alpha = \pi/2 \), from the appendix (Lemma 2) the equations (7) reduce to

\[
\begin{align*}
\text{dn}(Kx/T)C'_1(x) &= \frac{d}{dx}(\text{dn}(Kx/T))C_1(x) \\
\text{cn}(Kx/T)C'_2(x) &= \frac{d}{dx}(\text{cn}(Kx/T))C_2(x) \\
\text{sn}(Kx/T)C'_3(x) &= \frac{d}{dx}(\text{sn}(Kx/T))C_3(x).
\end{align*}
\]

Integration of (9) yields

\[
\begin{align*}
C_1(x) &= f_1(0)\text{dn}(Kx/T) \\
C_2(x) &= f_2(0)\text{cn}(Kx/T) \\
C_3(x) &= f_3(T)\text{sn}(Kx/T),
\end{align*}
\]

in the complex planes \( \Pi_j = \text{span}\{e_j = C_j(0)/|C_j(0)|, Je_j\}, j = 1, 2 \) and \( \Pi_3 = \text{span}\{e_3 = C_3(T)/|C_3(T)|, Je_3\} \). Putting the above expressions in (5) we conclude the result for the case \( \alpha = \pi/2 \). □

To establish which immersions \( \phi_{a,\alpha}, (a, \alpha) \in \Gamma \), are doubly periodic with respect to some lattice of \( \mathbb{R}^2 \), we consider the following subset \( \Gamma_0 \) of \( \Gamma \):

\[
\Gamma_0 = \{(a, \alpha) \in \Gamma / (r_3/r_2)^{1/2} \text{ and } F(a, \alpha) \text{ are rational numbers}\}
\]
where
\[ F(a, \alpha) = \frac{1}{2\pi} \left( (r_3/r_2)^{1/2}G_2(2T) - G_3(2T) \right), \]
2T is the period of \( u_a \) and \( r_j, G_j, j = 2, 3 \) were defined in Theorem 3.

We remark that using Lemma 1 and Lemma 3 (see appendix), we have that \( \Gamma_0 \) can be also defined in terms of the others \( r_j, G_j(2T), j \in \{1, 2, 3\} \).

We observe that \((3/2, 0) \in \Gamma_0\) trivially. Moreover, using Remark 2 and Lemmas 1 and 3, we obtain that the sets
\[ \{a \in [3/2, \infty[ / (a, 0) \in \Gamma_0\} \quad \text{and} \quad \{a \in [3/2, \infty[ / (a, \pi/2) \in \Gamma_0\} \]
are dense in \([3/2, \infty[\).

**Theorem 4** The immersion \( \phi_{a,\alpha} : \mathbb{R}^2 \longrightarrow \mathbb{C}P^2, (a, \alpha) \in \Gamma, \) is doubly-periodic with respect to some lattice of \( \mathbb{R}^2 \) if and only if \( (a, \alpha) \in \Gamma_0 \).

**Proof:** First we prove that the periodicities of \( \phi_{a,\alpha} \) and \( \psi_{a,\alpha} \) (its horizontal lifting to \( S^5 \)) are equivalent.

As \( \phi_{a,\alpha} = \Pi \circ \psi_{a,\alpha}, \) \( \Pi \) being the Hopf fibration, the periodicity of \( \psi_{a,\alpha} \) trivially implies the \( \phi_{a,\alpha} \)’s one.

Conversely, we suppose that \( \phi_{a,\alpha} \) is periodic. Then there exists a non-null vector \((\lambda, \mu) \in \mathbb{R}^2\) such that
\[ \phi_{a,\alpha}(x + \lambda, y + \mu) = \phi_{a,\alpha}(x, y) \]
\( \forall (x, y) \in \mathbb{R}^2. \) In particular, as \( \phi_{a,\alpha} \) is a conformal map, \( u_a(x + \lambda) = u_a(x), \) \( \forall x \in \mathbb{R}. \) So, from (5), there exists \( m \in \mathbb{Z} \) such that \( \lambda = 2mT. \)

Also, as \( \phi_{a,\alpha} = \Pi \circ \psi_{a,\alpha}, \) then (10) implies that there exists a smooth function \( \theta : \mathbb{R}^2 \longrightarrow \mathbb{R} \) such that
\[ \psi_{a,\alpha}(x + 2mT, y + \mu) = e^{i\theta(x,y)}\psi_{a,\alpha}(x, y) \]
\( \forall (x, y) \in \mathbb{R}^2. \) Taking derivatives in (11) and using that \( \psi_{a,\alpha} \) is also a conformal map we get that \( \theta \) is a constant function.

Now, using Theorem 3 and the appendix (Lemma 3 and (5.2)) in (11) we obtain that
\[ e^{imG_1(2T)}e^{i\sqrt{\imath} \mu} = e^{i\theta} \]
\[ e^{imG_j(2T)}e^{-i\sqrt{\imath} \mu} = e^{i\theta}, \quad j = 2, 3, \]
if $\alpha \neq \pi/2$, and

$$e^{i\sqrt{r_j}\mu} = e^{i\theta},$$

$$(-1)^m e^{-i\sqrt{r_j}\mu} = e^{i\theta}, \; j = 2, 3,$$

if $\alpha = \pi/2$. Using Lemma 1 and Lemma 3 (see appendix) we get, in both cases, that $e^{3i\theta} = 1$ and then (11) implies that $\psi_{a,\alpha}(x + 6mT, y + 3\mu) = \psi_{a,\alpha}(x, y), \; \forall \; (x, y) \in IR^2$. So, $\psi_{a,\alpha}$ is also periodic with respect to the vector $(6mT, 3\mu)$.

Now, in order to prove the Theorem we have to see that $\psi_{a,\alpha}$ is doubly-periodic with respect to some lattice of $IR^2$ if and only if $(a, \alpha) \in \Gamma_0$. Suppose $\psi_{a,\alpha}$ is doubly-periodic with respect to the lattice spanned by $\{(2m_1T, \mu_1), (2m_2T, \mu_2)\}$ with $m_k \in Z, \; k = 1, 2$. Then it is easy to see that $\lambda = m_1\mu_2 - m_2\mu_1$ is a non-null real number such that $\psi_{a,\alpha}(x, y + \lambda) = \psi_{a,\alpha}(x, y), \; \forall \; (x, y) \in IR^2$. Then using Theorem 3 we get that $e^{i\sqrt{r_j}\lambda} = 1, \; j = 1, 2, 3$. In particular, $(r_3/r_2)^{1/2}$ is a rational number. If $\alpha = \pi/2$, this means that $(a, \pi/2) \in \Gamma_0$. If $\alpha \neq \pi/2$, without loss of generality, we suppose that $m_1 \neq 0$. Using again Theorem 3, the periodicity of $\psi_{a,\alpha}$ with respect to $(2m_1T, \mu_1)$ implies that $e^{im_1G_{a,\alpha}/(2T)} e^{-i\sqrt{r_j}\mu_1} = 1, \; j = 2, 3$, which means that

$$m_1G_j(2T) - \sqrt{r_j}\mu_1 \in 2\pi Z, \; j = 2, 3.$$  

As $m_1 \in Z^*$ and $(r_3/r_2)^{1/2}$ is a rational number, (12) implies that $F(a, \alpha)$ is also a rational number. So $(a, \alpha) \in \Gamma_0$.

Conversely, if $(a, \alpha) \in \Gamma_0$, let $m_i, n_i \in Z$ with $(m_i, n_i) = 1, \; i = 1, 2$ such that $(r_3/r_2)^{1/2} = m_1/n_1$ and $F(a, \alpha) = m_2/n_2$. Then it is easy to check that $\psi_{a,\alpha}$ is doubly-periodic with respect to the lattice spanned by $\{(2n_2T, n_2G_2(2T)/\sqrt{r_2}), (0, 2n_1\pi/\sqrt{r_2})\}$ (respectively $\{(4T, 0), (0, 2n_1\pi/\sqrt{r_2})\}$ when $\alpha \neq \pi/2$ (respectively, when $\alpha = \pi/2$). So we finish the proof.

Remark 3. We observe that the immersions $\psi_{a,\alpha} : IR^2 \longrightarrow S^5(1)$, $(a, \alpha) \in \Gamma_0$, are also doubly-periodic. So they define a family of minimal tori in $S^5(1)$, which are finite coverings of the corresponding minimal tori defined by the immersions $\phi_{a,\alpha} : IR^2 \longrightarrow ICIP^2$, $(a, \alpha) \in \Gamma_0$.

Corollary 2 Let $\phi : \Sigma \longrightarrow ICIP^2$ be a minimal Lagrangian immersion of a compact surface in the complex projective plane. If $\phi$ is
invariant by a one-parameter group of isometries of \( \mathbb{CP}^2 \), then either \( \phi \) is totally geodesic or the universal covering of \( \phi \) is congruent to \( \phi_{a,\alpha} \), with \( (a, \alpha) \in \Gamma_0 \).

## Appendix

In this section we first proceed to integrate the equation (1) following the techniques used in [8]. Making the change of variable \( y = e^{2u} \), from (6) we arrive at

\[
y'^2 + 4y^3 - 4ay^2 + 2 = 0
\]

If

\[
F(\varphi) = F(\varphi, p) = \int_0^\varphi \frac{d\theta}{\sqrt{1 - p^2 \sin^2 \theta}}, \quad 0 \leq p \leq 1,
\]

is the elliptic integral of the first kind, then denoting the inverse of \( F(\varphi) \) by \( \text{am}(x, p) = \varphi \), the elementary Jacobi elliptic functions are given by

\[
\begin{align*}
\text{sn} x &= \text{sn}(x, p) = \sin \varphi, \quad \text{cn} x &= \text{cn}(x, p) = \cos \varphi \\
\text{dn} x &= \text{dn}(x, p) = \sqrt{1 - p^2 \sin^2 \varphi}.
\end{align*}
\]

The basic properties of these functions are

\[
\begin{align*}
\text{sn}^2(x, p) + \text{cn}^2(x, p) &= 1 = p^2 \text{sn}^2(x, p) + \text{dn}^2(x, p) \\
\text{sn}(x + 2K) &= -\text{sn} x \quad \text{cn}(x + 2K) = -\text{cn} x \quad \text{dn}(x + 2K) = \text{dn} x
\end{align*}
\]

where \( K = F(\pi/2) \) is the complete elliptic integral of the first kind.

Returning to equation (1), we can rewrite it in the form

\[
y'^2 + 4(y - a_1)(y - a_2)(y + a_3) = 0
\]

where

\[
a_1 = e^{2u(0)}, \quad a_2 = \frac{1 + \sqrt{1 + 8a_1^2}}{4a_1^2}, \quad a_3 = \frac{\sqrt{1 + 8a_1^2} - 1}{4a_1^2}
\]
and its solution when \( u(0) \geq 0 \) is given by

\[
y = e^{2u(x)} = a_1(1 - q^2 \text{sn}^2(rx, p))
\]

where

\[
p^2 = \frac{a_1 - a_2}{a_1 + a_3}, \quad q^2 = \frac{a_1 - a_2}{a_1}, \quad r = \sqrt{a_1 + a_3}
\]

(for background on the solutions of such equations, see [5]).

Thus, there exists a positive real number \( T \) (\( T = K/r \)), such that (5) holds. We remark that \( a_2 = e^{2u(T)} \).

We noticed in section 3 that \( \hat{u}(x) = u(x + T) \) is also a solution of (1) with \( \hat{u}'(0) = 0 \). We observe that \( \hat{u}(0) = u(T) = \frac{1}{2} \log a_2 \), and then \( u(0) \) is positive if and only if \( \hat{u}(0) \) is negative.

The following lemmas are used in the proofs of Theorem 3 and Theorem 4 in section 4.

The proof of Lemma 1 can be obtained using standard arguments.

**Lemma 1** The cubic polynomial \( r^3 + 2ar^2 + a^2r + (\sin^2 \alpha)/2 \), with \((a, \alpha) \in \Gamma\), has three nonpositive real roots \( -r_i \), \( i = 1, 2, 3 \), satisfying:

- (a) \( r_1 \geq a \) and the equality holds if and only if \( \alpha = 0 \).
- (b) \( 1/(2a_2^2) \leq r_2 \leq a \) and the first equality (respectively the second one) holds if and only if \( \alpha = \pi/2 \) (respectively \( \alpha = 0 \)).
- (c) \( 0 \leq r_3 \leq 1/(2a_2^2) \) and the first equality (respectively the second one) holds if and only if \( \alpha = 0 \) (respectively \( \alpha = \pi/2 \)).
- (d) \( r_3 < a/3 < r_2 \).
- (e) \( \sqrt{r_1} = \sqrt{r_2} + \sqrt{r_3} \).

**Lemma 2** A real number \( x \) is a zero of the function \( e^{2u} + r_j - a \), \( j = 1, 2, 3 \), \((a, \alpha) \in \Gamma\) if and only if either \( \alpha = \pi/2 \), \( j = 2 \) and \( x = (2n + 1)T \), \( n \in \mathbb{Z} \) or \( \alpha = \pi/2 \), \( j = 3 \) and \( x = 2nT \), \( n \in \mathbb{Z} \).

Moreover, when \( \alpha = \pi/2 \),

\[
\begin{align*}
e^{2u(x)} + r_1 - a &= (a_1 + a_3)dn^2(Kx/T), \\
e^{2u(x)} + r_2 - a &= (a_1 - a_2)cn^2(Kx/T), \\
e^{2u(x)} + r_3 - a &= (a_2 - a_1)sn^2(Kx/T).
\end{align*}
\]
Proof: Lemma 1 says that $e^{2u} + r_1 - a$ is a positive function. Also from (3) and (4) we have that

$$a_2 = e^{2u(2n+1)T} \leq e^{2u(x)} \leq e^{2u(2nT)} = a_1.$$ 

Now, Lemma 1 and (6) prove the first part of Lemma 2. The expressions given for the case $\alpha = \pi/2$ can be obtained using (5.2), (3) and (4). □

**Lemma 3** The functions $G_j$ defined in Theorem 3 verify the following properties:

(a) $G_j(x + 2mT) = G_j(x) + mG_j(2T), m \in \mathbb{Z}, j = 1, 2, 3$.

(b) $\sum_{j=1}^{3} G_j(2T) = 0$.

(c) When $\alpha = 0$, $G_3(2T)/2\pi = (1/\sqrt{2\pi}) \int_0^{2T} e^{-2u(x)}dx = \Lambda_0(\xi, p)$, where $\Lambda_0$ is the Heumann-lambda function (see for instance [3]) and

$$\xi = \arcsin \sqrt{\frac{q^2 - p^2}{q^2(1 - p^2)}}.$$ 

As $a$ increases from $3/2$ to $\infty$ (so $p$ increases from 0 to 1), $G_3(2T)/2\pi$ decreases monotonically from $1/\sqrt{3}$ to $1/2$.

Proof: (a) is clear from the periodicity of the function $u_a$. (b) follows using that

$$G_1(x) + G_2(x) + G_3(x) = -\arctan \left( \frac{\sqrt{2}u'(x)e^{2u(x)}}{\cos \alpha} \right)$$

for $\alpha \neq \pi/2$.

The first equality in (c) is a particular case of (b). To prove the second one, we use (3) to obtain

$$\frac{1}{a_1} \int_0^{2T} e^{-2u(s)}ds = \frac{1}{a_1} \int_0^{2K/r} \frac{ds}{1 - q^2\text{sn}^2(rs)} = \frac{1}{a_1r} \int_0^{2K} \frac{dw}{1 - q^2\text{sn}^2(w)}.$$ 

Now, as $0 < p^2 < q^2 < 1$, using formulas 400.01 and 413.01 of [3] and (4) we obtain the second equality of (c).
The analytic behaviour of $G_3(2T)/2\pi$ can then be deduced from formulas 151.01, 710.11 and 730.04 of [3].

References.


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