Lagrangian surfaces
with conformal Maslov form

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1 Introduction.

Consider $(\mathbb{CP}^2, g, J)$ the complex projective plane endowed with the Fubini-Study metric g of constant holomorphic sectional curvature 4 and the standard complex structure J.

Let $\phi : \Sigma \longrightarrow \mathbb{CP}^2$ be a Lagrangian immersion from an orientable surface Σ in $\mathbb{CP}^2$. This means that if Ω is the Kaehler two form on $\mathbb{CP}^2$, then the pull-back of Ω by ϕ vanishes identically: $\phi^*\Omega \equiv 0$. So, the complex structure J defines a bundle isomorphism between the tangent bundle to Σ and the normal bundle to ϕ: $J : T\Sigma \equiv T \perp \Sigma$. The most elementary property of such an immersion is that the trilinear form on $T\Sigma$

$$(X, Y, Z) \to g(\sigma(X, Y), JZ), \tag{1}$$

where σ is the second fundamental form of ϕ, is totally symmetric.

We deal with the following

PROBLEM: Find natural geometric properties and (possible) classification of those Lagrangian immersions verifying these properties.

For this purpose, we are going to make use of a similar context in a certain sense, taking the *real* space-form of positive curvature as target manifold and considering 1-codimension. So we are trying to follow a parallelism between "surfaces in $\mathbb{S}^3$" and "Lagrangian surfaces in $\mathbb{CP}^2$".

2 "Umbilical" Lagrangian surfaces.

We begin with the simplest examples, the totally geodesic ones. It is well-known that the equator $\mathbb{S}^2$ is the only example in $\mathbb{S}^3$ and in our case there is also only one example, the standard immersion of $\mathbb{S}^2$ in $\mathbb{CP}^2$.

The second family of examples, from the point of view of the second fundamental form, would be the totally umbilical surfaces of $\mathbb{S}^3$, that is, those

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satisfying that

$$\sigma(X,Y) = g(X,Y)H,$$

H of course the mean curvature vector. Now, there is a one-parameter deformation from the equator $\mathbb{S}^2$ to a point-pole by embedded round two-dimensional spheres.

If we impose the umbilicity condition in the Lagrangian case, then

$$g(\sigma(X,Y), JZ) = g(X,Y)g(H, JZ)$$

and becouse of symmetry of (**??**), we need to symmetrize it; so we add $+g(Y,Z)g(H,JX)+ g(Z,X)g(H,JY)$. In this way, we deduce the property that plays the similar role to umbilicity in our context (putting the appropiate constant):

$$\sigma(X,Y) = \frac{1}{2}\left(g(X,Y)H + g(H,JY)JX + g(H,JX)JY\right),$$

for any vectors X and Y tangent to Σ. In the following theorem (cf. [CU3]) we classify the Lagrangian surfaces of the complex projective plane with second fundamental form in this way, obtaining *also here* a one-parameter family of Lagrangian spheres.

Theorem 1 *Let $\phi : \Sigma \longrightarrow \mathbb{CP}^2$ be a Lagrangian immersion of an orientable surface. The second fundamental form σ of ϕ is given by*

$$\sigma(X,Y) = \frac{1}{2}\left(g(X,Y)H + g(H,JY)JX + g(H,JX)JY\right)$$

for any vectors X and Y tangent to Σ if and only if $\phi(\Sigma)$ is an open set of $\phi_t(\mathbb{S}^2)$, for some
$t \in [0,+\infty[$, where $\phi_t : \mathbb{S}^2 \longrightarrow \mathbb{CP}^2$, $t \in [0,+\infty[$, is given by

$$\phi_t(x,y,z) = \Pi\left(\frac{1}{c_t^2 + s_t^2 z^2}\left(c_t x, s_t xz, c_t y, s_t yz, z, s_t c_t(1+z^2)\right)\right)$$

with $\Pi : \mathbb{S}^5(1) \longrightarrow \mathbb{CP}^2$ the Hopf's fibration, $c_t = \cosh t$ and $s_t = \sinh t$.

These examples have very interesting properties and it seems to be natural that they could be characterized of several forms, in the same way that the totally umbilical spheres of $\mathbb{S}^3$ (cf also [CU3]).

Properties and characterizations of ϕ_t, $t \in [0,+\infty[$:

(a) First, they provide a deformation from the totally geodesic example corresponding to $t = 0$, $\phi_0(x,y,z) = \Pi(x,0,y,0,z,0)$, that degenerates in a point when t goes to $+\infty$, $\phi_{+\infty}(x,y,z) = \Pi(0,0,0,0,0,1)$.

(b) From a topological point of view, these spheres have the best possible behaviour, because if $t \neq 0$, denoting by N and S the poles of $\mathbb{S}^2$, then $\phi_t : \mathbb{S}^2 - \{N,S\} \longrightarrow \mathbb{CP}^2$ is an embedding. At the poles of $\mathbb{S}^2$ they have a double point.

(c) As a direct consequence of Theorem 1, we obtain for Lagrangian surfaces of $\mathbb{CP}^2$ an inequality involving the Gauss curvature K of Σ and the length of H, and characterize our examples as the only ones that reach the equality. Concretely, let $\phi : \Sigma \longrightarrow \mathbb{CP}^2$ be a Lagrangian immersion of an orientable surface Σ. Then:

$$|H|^2 + 2 \geq 2K,$$

and the equality holds if and only if $\phi(\Sigma)$ is an open set of $\phi_t(\mathbb{S}^2)$, for some $t \in [0, +\infty[$.

B.Y.Chen [Ch] have obtained the corresponding inequality in high dimension and also have characterized the equality using warped products.

(d) Finally, as a corollary, we obtain a lower bound for the Willmore functional of a Lagrangian sphere in $\mathbb{CP}^2$ and our examples are characterized precisely for reaching the equality. That is, let $\phi : \Sigma \longrightarrow \mathbb{CP}^2$ be a Lagrangian immersion of a sphere Σ. Then:

$$\int_\Sigma (|H|^2 + 2) dA \geq 8\pi,$$

and the equality holds if and only if ϕ is congruent to ϕ_t for some $t \in [0, \infty[$.

3 "Constant mean curvature" Lagrangian surfaces.

Following the analogy between Lagrangian surfaces of $\mathbb{CP}^2$ and surfaces of $\mathbb{S}^3$, the round spheres are the basic examples of surfaces in $\mathbb{S}^3$ with constant mean curvature.

Let's try to translate this property to our situation. Our examples ϕ_t have not parallel mean curvature vector, property which is generally taken as a version on higher codimension of the notion of constant mean curvature. Then, what does the mean curvature vector H satisfy in the family ϕ_t? We prove that they verify an intrinsic geometric property: the tangent vector field JH is a conformal vector field in the surface (moreover, it is a closed vector field).

So, we will take this property as the Lagrangian version of the concept of constant mean curvature surfaces. As the dual form of JH is the Maslov form ϖ of the Lagrangian immersion, we will refer from now on these submanifolds as *Lagrangian submanifolds with conformal Maslov form.*

To avoid any suspect about if this is the correct notion, we are able to obtain the Lagrangian version of a famous Hopf's result (cf. [C]).

Theorem 2 *Let $\phi : \Sigma \longrightarrow \mathbb{CP}^2$ be a Lagrangian immersion of a* sphere Σ *with conformal Maslov form. Then ϕ is congruent to ϕ_t for some $t \in [0, \infty[$.*

Hence, we can state our

PURPOSE: Study Lagrangian surfaces of the complex projective plane with conformal Maslov form.

Of course this is a big family: it contains the minimal ones, those with parallel mean curvature vector and the spheres ϕ_t. What do we

GET: The complete classification of the *compact non-minimal* ones,

as we will see below. What about the minimal case? As usual, it is of a different nature of the "constant mean curvature" case. However, we have obtained (cf. [CU2]) a two-parameter family of new examples of minimal Lagrangian tori in $\mathbb{CP}^2$ characterized by their invariability by a one-parameter group of holomorphic isometries of $\mathbb{CP}^2$.

Before trying to get our purpose, let me comment that we have also studied the similar problem in the complex Euclidean plane $\mathbb{C}^2$ (cf. [CU1]) and A.Ros and F.Urbano (cf. [RU]) have made it in $\mathbb{C}^n$.

First of all, we have an interesting characterization of the Lagrangian surfaces of $\mathbb{CP}^2$ with conformal Maslov form in terms of twistor theory.

The twistor space $\mathcal{Z}$ of $\mathbb{CP}^2$ can be identified with the flag manifold $SU(3)/SU(1)^3$, wich can be endowed with a structure of Kaehler-Einstein metric: $(\mathcal{Z}, \hat{g}, \mathbb{J})$.

It is an interesting fact that *Lagrangian* immersions in $\mathbb{CP}^2$ are —at least locally— projections via the Hopf's fibration Π of *horizontal* immersions in $\mathbb{S}^5(1)$. Then the *twistor lift* of a Lagrangian immersion $\phi : \Sigma \longrightarrow \mathbb{CP}^2$ is the globally-defined map $\widetilde{\phi} : \Sigma \longrightarrow \mathcal{Z}$ given locally by the flag $\widetilde{\phi} = (\psi, [\psi, \psi_z])$, where $\psi : U \subset \Sigma \longrightarrow \mathbb{S}^5(1) \subset \mathbb{C}^3$ is a local horizontal lift of ϕ. So in [CU4] we prove the following

Theorem 3 *Let $\phi : \Sigma \longrightarrow \mathbb{CP}^2$ be a Lagrangian immersion of an orientable surface Σ and $\widetilde{\phi} : \Sigma \longrightarrow \mathcal{Z}$ its twistor lift to the twistor space $\mathcal{Z}$ of $\mathbb{CP}^2$. Then:*

(i) *ϕ is a "twistor harmonic" immersion (i.e. $\widetilde{\phi}$ is a harmonic map) if and only if the Maslov form of ϕ is conformal.*

(ii) *ϕ is a "twistor holomorphic" immersion (i.e. $\widetilde{\phi}$ is a holomorphic map) if and only if $\phi(\Sigma)$ is an open set of $\phi_t(\mathbb{S}^2)$, for some $t \in [0, +\infty[$.*

In particular, the spheres ϕ_t have this other nice characterization: the holomorphicity of their twistor lifts.

4 The method.

Next we start describing the method (see [CU4] for the details) which allows us to get the announced classification (see §3).

Associated to a (non neccesarily Lagrangian) immersion $\phi : \Sigma \longrightarrow \mathbb{CP}^2$, we define a cubic differential Θ on Σ, by

$$\Theta(z) = f(z)dz^3, \text{ with } f(z) = 4g(\sigma(\partial_z, \partial_z), J\phi_*\partial_z),$$

where $z = x + iy$ is a local isothermal coordinate on Σ and σ, J and g are extended $\mathbb{C}$-linearly to the complexified bundles.

Chern and Wolfson (cf [CW]) observed that Θ is holomorphic when Σ is a minimal surface of $\mathbb{CP}^2$. In the Lagrangian case, we obtain a new analityc characterization of our family

$$\varpi \text{ conformal} \Leftrightarrow \Theta \text{ holomorphic}.$$

We also prove that ϕ is twistor holomorphic (see Theorem 3) if and only if Θ vanishes identically.

But we can also consider the *holomorphic* vector field $\mathcal{X}$ canonically associated to the *conformal* vector field JH,

$$\mathcal{X}(z) = k(z)dz^3, \text{ with } k(z) = g(JH, \partial_{\overline{z}})/2g(\partial_z, \partial_{\overline{z}}).$$

Then we remark that we have two holomorphic objects associated to a Lagrangian immersion with conformal Maslov form.

Moreover, it is interesting for us to consider the globally-defined complex function given in the following

Definition 1 *Associated to a Lagrangian immersion* $\phi : \Sigma \longrightarrow \mathbb{CP}^2$ *with* conformal Maslov form, *it is defined the* holomorphic *function*

$$F : \Sigma \longrightarrow \mathbb{C}$$

given locally by

$$F(z) = f(z)k(z)^3 = e^{-6u} g(\sigma(\partial_z, \partial_z), J\partial_z) g(H, J\partial_{\overline{z}})^3,$$

where $z = x + iy$ *is a local isothermal coordinate such that the induced metric is written as* $g = e^{2u}|dz|^2$.

It is easy to compute the length of F obtaining

$$|F(z)|^2 = |H|^6(|\sigma|^2 - 3|H|^2). \tag{2}$$

A first consequence of this is stated by the following restrictions in the compact case.

Proposition 1 *Let* $\phi : \Sigma \longrightarrow \mathbb{CP}^2$ *be a Lagrangian immersion with conformal Maslov form of a* compact *oriented surface* Σ. *Then:*

(a) *If genus(Σ)≥ 2, ϕ is minimal.*

(b) *If genus(Σ)$= 0$, ϕ is twistor holomorphic.*

(c) *If genus(Σ)$= 1$, either ϕ is minimal or the function F of the Definition 1 is a non-null constant.*

Proof: If genus(Σ)≥ 2 then $\mathcal{X} \equiv 0$ and so $H = 0$. If Σ is a sphere neccesarily $\Theta \equiv 0$ and ϕ is twistor holomorphic (as we comment before). And when Σ is a torus, then F is constant and $\Theta \equiv 0$ or Θ has no zeroes on Σ. We must eliminate the first possibility to conclude the proof. But if $\Theta \equiv 0$, then $|\sigma|^2 = 3|H|^2$ and using the Gauss equation of ϕ we obtain that $K = 1+|H|^2/2$. The Gauss-Bonnet theorem says then that this is impossible.

Once classified the spheres (see Theorems 3 and 1), our purpose is to classify all the Lagrangian non-minimal tori with conformal Maslov form in $\mathbb{CP}^2$ and since they verify that F is a non-null constant, we are going to study locally those immersions verifying this property. The following theorem summerizes it.

Theorem 4 *Associated to each solution $u:\mathbb{R} \longrightarrow \mathbb{R}$ to the o.d.e.*

$$u'' + e^{2u} + \lambda^2 \sinh e^{4u} = 0, \ \lambda \in (0, +\infty),$$

there exists a one-parameter family of Lagrangian immersions with conformal Maslov form

$$\phi_\alpha : (\mathbb{R}^2, e^{2u}|dz|^2) \longrightarrow \mathbb{CP}^2, \ \alpha \in [0, \pi],$$

such that the function F associated to ϕ_α is the constant $\lambda^4 e^{i\alpha}$.

Conversely, every Lagrangian immersion with conformal Maslov form and F (Definition 1) a non-null constant is locally congruent to some of the latter examples.

Moreover, the immersion ϕ_α has parallel mean curvature vector if and only if u is constant.