

A Willmore functional for compact surfaces in the complex projective plane

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1 Introduction

Amongst the global conformal invariants for compact surfaces Σ in a Riemannian manifold $(M, \langle \cdot, \cdot \rangle)$, perhaps the best known is the Willmore functional, defined by

$$W(\Sigma) = \int_{\Sigma} (|H|^2 + \bar{K}) dA,$$

where H denotes the mean curvature vector of the surface, \bar{K} the sectional curvature of M restricted to Σ and dA the canonical measure of the induced metric. This functional has been studied in depth when M is the Euclidean space \mathbb{R}^n (or the sphere \mathbb{S}^n or the hyperbolic space \mathbb{RH}^n , because W is invariant under conformal transformations of ambient space), to get lower bounds [K, LY, M, MoR, R, S] or to classify critical surfaces (the known Willmore surfaces) [B, BB, Mo, P].

In this paper the authors study the Willmore functional for compact surfaces in the complex projective plane \mathbb{CP}^2 (with constant holomorphic sectional curvature 4). If Σ is an orientable compact surface in \mathbb{CP}^2 , the Willmore functional is given by

$$W(\Sigma) = \int_{\Sigma} (|H|^2 + 1 + 3C^2) dA,$$

where C is the Kähler function. This functional splits (see Section 2) into $W = \frac{1}{2}(W^+ + W^-)$, where W^+ and W^- are also conformal invariant func-

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tionals (see Proposition 1) defined by

$$W^+(\phi) = \int_{\Sigma} (|H|^2 + 6C^2) dA, \quad W^-(\phi) = \int_{\Sigma} (|H|^2 + 2) dA.$$

These functionals W^{\pm} are closely related with the Penrose twistor bundles \mathcal{P}^{\pm} over \mathbb{CP}^2 (see Proposition 3), because twistor holomorphic surfaces, i.e., surfaces of \mathbb{CP}^2 whose twistor liftings are holomorphic, are critical surfaces for these functionals (in fact they are minimizers for W^{\pm}). As \mathbb{CP}^2 , with its canonical orientation, is a self-dual Riemannian manifold but not an anti-self-dual one, the twistor bundle \mathcal{P}^- is a complex manifold and the twistor bundle \mathcal{P}^+ is an almost complex but non complex manifold (see [AHS]). Then one can easily construct minimizers for W^- by taking algebraic curves in the complex manifold \mathcal{P}^- . However, since \mathcal{P}^+ is not a complex manifold, it is difficult to obtain, in a similar way, minimizers for W^+ .

The Euler–Lagrange equations for the functionals W^{\pm} , which can be obtained as in [W], say that the minimal surfaces of \mathbb{CP}^2 are critical for the functional W^- . It is also interesting to note that the functional W^- restricted to minimal surfaces is twice the area functional. Due to these considerations about W^{\pm} , the authors think that the Willmore functional W^- is the natural one to be studied for surfaces in \mathbb{CP}^2 . In Section 3, we obtain a lower bound for W^- which is the main result in the paper:

Let Σ be a compact surface of \mathbb{CP}^2 . Then

$$W^-(\Sigma) \geq 4\pi\mu - 2 \int_{\Sigma} |C| dA,$$

being μ its maximum multiplicity. The equality holds if and only if $\mu = 1$ and Σ is either a complex projective line or a totally geodesic real projective plane, or $\mu = 2$ and Σ is a Lagrangian Whitney sphere.

We wish to mention that this result could be extended to surfaces in the complex hyperbolic plane.

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2 The Willmore functional for surfaces in four manifolds

Let $(M, \langle \cdot, \cdot \rangle)$ be a four-dimensional oriented Riemannian manifold and $\phi : \Sigma \rightarrow M$ an immersion of an oriented compact surface Σ . The Willmore functional of this immersion is defined by

$$W(\phi) = \int_{\Sigma} (|H|^2 + \bar{K}) dA,$$

where H is the mean curvature of ϕ , dA the induced measure on Σ and $\bar{K} = \bar{R}(e_1, e_2, e_2, e_1)$, being \bar{R} the curvature of $\langle \cdot, \cdot \rangle$ and $\{e_1, e_2\}$ an orthonormal basis in Σ . If $T^\perp \Sigma$ is the normal bundle of ϕ , then we have the orthogonal decomposition

$$\phi^*TM = T\Sigma \oplus T^\perp \Sigma.$$

Let $\bar{\nabla}$ be the connection on ϕ^*TM induced by the Levi-Civita connection of TM and let $\bar{\nabla} = \nabla + \nabla^\perp$ be the corresponding decomposition. If $\{e_1, e_2, e_3, e_4\}$ is an oriented orthonormal local frame on ϕ^*TM such that $\{e_1, e_2\}$ is an oriented frame on $T\Sigma$, then we define the *normal curvature* K^\perp of the immersion ϕ by

$$K^\perp = R^\perp(e_1, e_2, e_3, e_4),$$

where R^\perp is the curvature tensor of the normal connection ∇^\perp . Also we will denote by \bar{K}^\perp the function on Σ given by

$$\bar{K}^\perp = \bar{R}(e_1, e_2, e_3, e_4).$$

The Euler characteristics of $T\Sigma$ and $T^\perp \Sigma$ are given respectively by

$$\chi = \frac{1}{2\pi} \int_{\Sigma} K dA \quad \text{and} \quad \chi^\perp = \frac{1}{2\pi} \int_{\Sigma} K^\perp dA.$$

We may decompose the Willmore functional in the following way

$$W(\phi) = \frac{1}{2} (W^+(\phi) + W^-(\phi)),$$

where

$$\begin{aligned} W^+(\phi) &= \int_{\Sigma} (|H|^2 + \bar{K} - \bar{K}^\perp) dA, \\ W^-(\phi) &= \int_{\Sigma} (|H|^2 + \bar{K} + \bar{K}^\perp) dA. \end{aligned}$$

Proposition 1 *The functionals W^+ and W^- are invariant under conformal changes of the metric \langle , \rangle on M .*

Proof: Let $\langle , \rangle_* = e^{2u} \langle , \rangle$ a metric on M conformal to \langle , \rangle , being $u : M \rightarrow \mathbb{R}$ a smooth function. Then it is well known that the second fundamental forms σ and σ_* of ϕ with respect to \langle , \rangle and \langle , \rangle_* are related by

$$\sigma_*(u, v) = \sigma(u, v) - \langle u, v \rangle \bar{\nabla} u^\perp,$$

where $^\perp$ means normal component. From here, it is an exercise to check that

$$\begin{aligned} (|H_*|^2 + \bar{K}_* - K_*) dA_* &= (|H|^2 + \bar{K} - K) dA, \\ (\bar{K}_*^\perp - K_*^\perp) dA_* &= (\bar{K}^\perp - K^\perp) dA \end{aligned}$$

where $*$ means the corresponding object for the metric \langle , \rangle_* . The above formulae prove the proposition taking into account the Gauss-Bonnet theorem and that the normal bundles of ϕ (with respect to both metrics) are isomorphic. q.e.d.

Now we are going to relate these two functionals with the two twistor liftings of the immersion. Given a point $x \in M$, let \mathcal{P}_x^\pm be the set of almost Hermitian structures J_x^\pm over $T_x M$ such that if $\Omega^\pm(u, v) = \langle J_x^\pm u, v \rangle$, then $\pm \Omega^\pm \wedge \Omega^\pm$ is the orientation induced on $T_x M$ from M . Then $\mathcal{P}^\pm = \cup_{x \in M} \mathcal{P}_x^\pm$ are \mathbb{CP}^1 -fiber bundles over M called the *twistor bundles* of M . We will represent by $\pi^\pm : \mathcal{P}^\pm \rightarrow M$ the projections. Atiyah, Hitchin and Singer [AHS] defined almost complex structures \mathcal{J}^\pm on \mathcal{P}^\pm and proved a central result saying that $(\mathcal{P}^+, \mathcal{J}^+)$ is a complex manifold if and only if (M, \langle , \rangle) is anti-self-dual, and $(\mathcal{P}^-, \mathcal{J}^-)$ is a complex manifold if and only if (M, \langle , \rangle) is self-dual.

We define two almost complex structures J^\pm on $\phi^* TM$ by

$$J^\pm(e_1) = e_2, J^\pm(e_3) = \pm e_4,$$

where $\{e_1, e_2, e_3, e_4\}$ an orthonormal local frame on $\phi^* TM$ such that $\{e_1, e_2\}$ is an oriented frame on $T\Sigma$. We remark that J^\pm , restricted to $T\Sigma$, is the complex structure on Σ compatible with the given orientation, and that, on $T^\perp \Sigma$, one has $\nabla^\perp J^\pm = 0$. These almost complex structures allow us to define the so called *twistor liftings* $\tilde{\phi}^\pm : \Sigma \rightarrow \mathcal{P}^\pm$ of the immersion ϕ by

$$\tilde{\phi}^\pm(p) = J_{\phi(p)}^\pm.$$

Although it is not explicitly stated, the following result was proved in [F].

Proposition 2 *Let $\phi : \Sigma \rightarrow M$ be an immersion from an oriented surface into an oriented four-dimensional Riemannian manifold M . Then the two following assertions are equivalent:*

- i) *The twistor liftings $\tilde{\phi}^\pm : (\Sigma, J^\pm) \rightarrow (\mathcal{P}^\pm, \mathcal{J}^\pm)$ of ϕ are holomorphic.*
- ii) *The second fundamental form σ of ϕ satisfies*

$$\sigma(u, v) = \sigma(J^\pm u, J^\pm v) - J^\pm \sigma(J^\pm u, v) - J^\pm \sigma(u, J^\pm v)$$

for any vectors $u, v \in T\Sigma$.

An immersion $\phi : \Sigma \rightarrow M$ satisfying one of the two equivalent conditions given in Proposition 2 will be called *twistor holomorphic with positive or negative spin*. As a consequence of Proposition 2, the twistor holomorphicity of ϕ does not depend on the chosen orientation on the surface Σ . Hence we can talk about twistor holomorphic immersions from an *orientable* surface into an oriented four-dimensional Riemannian manifold. Twistor holomorphic surfaces with positive or negative spin which are also minimal are called *superminimal surfaces with positive or negative spin* ([B, F, G]). The surfaces which are simultaneously twistor holomorphic with positive and negative spin are the umbilical ones.

We consider the two following bilinear forms σ^\pm on Σ valuated on $T^\perp \Sigma$

$$\sigma^\pm(u, v) = \sigma(u, v) - \sigma(J^\pm u, J^\pm v) + J^\pm \sigma(J^\pm u, v) + J^\pm \sigma(u, J^\pm v)$$

for vectors $u, v \in T\Sigma$. Then it is straightforward to check that

$$\begin{aligned} |H|^2 + \bar{K} - \bar{K}^\perp &= K - K^\perp + \frac{1}{16}|\sigma^+|^2, \\ |H|^2 + \bar{K} + \bar{K}^\perp &= K + K^\perp + \frac{1}{16}|\sigma^-|^2. \end{aligned}$$

Now Proposition 2 gives the following result, which relates the above functionals W^\pm with the twistor theory.

Proposition 3 *Let $\phi : \Sigma \rightarrow M$ be an immersion from an orientable compact surface into an oriented four-dimensional Riemannian manifold M . Then*

- i) $W^+(\phi) \geq 2\pi(\chi - \chi^\perp),$
- ii) $W^-(\phi) \geq 2\pi(\chi + \chi^\perp).$

Moreover the equality in i) holds if and only if ϕ is twistor holomorphic with positive spin, and the equality in ii) holds if and only if ϕ is twistor holomorphic with negative spin.

Perhaps the first case to study the functionals W^\pm was when $(M, \langle \cdot, \cdot \rangle)$ is the four-dimensional Euclidean space, or equivalently when $(M, \langle \cdot, \cdot \rangle)$ is a sphere \mathbb{S}^4 with its standard metric of constant curvature one. In this case, if $\phi : \Sigma \rightarrow \mathbb{S}^4$ is an immersion of an orientable compact surface, then

$$W^+(\phi) = W^-(\phi) = W(\phi) = \int_{\Sigma} (|H|^2 + 1) dA,$$

and so these functionals coincide with the classical Willmore functional. Moreover, as \mathbb{S}^4 is self-dual and anti-self-dual, the twistor spaces $(\mathcal{P}^+, \mathcal{J}^+)$ and $(\mathcal{P}^-, \mathcal{J}^-)$ are complex manifolds, and it is well known that they are biholomorphic to \mathbb{CP}^3 with its standard complex structure. Also the twistor projections π^\pm are related by $\pi^- = A \circ \pi^+$, where A is the antipodal map on \mathbb{S}^4 , and hence the twistor holomorphic surfaces with negative spin of \mathbb{S}^4 are the images by the antipodal map of the twistor holomorphic surfaces with positive spin.

3 The functional W^- for surfaces in \mathbb{CP}^2

We consider the Hermitian product in \mathbb{C}^3

$$(z, w) = \sum_{i=1}^3 z_i \bar{w}_i,$$

for any $z, w \in \mathbb{C}^3$, where \bar{z} stands for the conjugate of z . Then $\Re(\cdot, \cdot)$ is the Euclidean metric on \mathbb{C}^3 . Let \mathbb{CP}^2 be the complex projective plane with its canonical Fubini–Study metric $\langle \cdot, \cdot \rangle$ of constant holomorphic sectional curvature 4. Then

$$\mathbb{CP}^2 = \{\Pi(z) = [z] \mid z = (z_1, z_2, z_3) \in \mathbb{C}^3 - \{0\}\}$$

where $\Pi : \mathbb{C}^3 - \{0\} \rightarrow \mathbb{CP}^2$ is the standard Hopf projection, which is a Riemannian submersion. The complex structure of \mathbb{C}^3 induces via Π the canonical complex structure J on \mathbb{CP}^2 . The Kähler two form Ω in \mathbb{CP}^2 is

defined by $\Omega(u, v) = \langle Ju, v \rangle$. We will consider \mathbb{CP}^2 with the orientation $\Omega \wedge \Omega$.

If $\phi : \Sigma \rightarrow \mathbb{CP}^2$ is an immersion of an oriented surface Σ , the *Kähler function* C on Σ is defined by $\phi^*\Omega = C dA$, where dA is the area form on Σ . We remark that only the sign of C depends on the orientation of Σ , so functions such that C^2 or $|C|$ are defined even when Σ is not orientable. It is clear that the Kähler function satisfies $-1 \leq C \leq 1$. The surfaces with $C = 1$, $C = -1$ and $C = 0$ are called respectively *holomorphic*, *anti-holomorphic* and *Lagrangian*. *Complex surface* will be synonym of either a holomorphic or anti-holomorphic surface.

There is a relation between J and the almost complex structures J^\pm on $\phi^*T\mathbb{CP}^2$ defined in Section 2 above, namely

$$(1) \quad (Jv)^\top = CJ^+v = CJ^-v, \quad (J\xi)^\perp = CJ^+\xi = -CJ^-\xi,$$

for any $v \in T\Sigma$ and $\xi \in T^\perp\Sigma$ and where \top and \perp stand for tangent and normal components. Using (1) and the well known expression of the curvature of the Fubini–Study metric, it is straightforward to check that the functionals W, W^\pm are given, in this case, by

$$W(\phi) = \int_{\Sigma} (|H|^2 + 1 + 3C^2) dA$$

$$W^+(\phi) = \int_{\Sigma} (|H|^2 + 6C^2) dA, \quad W^-(\phi) = \int_{\Sigma} (|H|^2 + 2) dA.$$

Differently from the case of the four–sphere, the two functionals W^+ and W^- are distinct, reflecting in this way the asymmetry of the Weyl tensor of \mathbb{CP}^2 . Consequently the corresponding twistor holomorphic minimizers are of a different nature. For instance, although Gauduchon [G] proved that the complex and minimal Lagrangian surfaces are the only superminimal surfaces with positive spin in \mathbb{CP}^2 , it seems difficult to find other minimizers for W^+ because $(\mathcal{P}^+, \mathcal{J}^+)$ is not a complex manifold. However (see [ES]) the twistor bundle $(\mathcal{P}^-, \mathcal{J}^-)$ is a complex manifold which can be endowed with a Riemannian metric becoming it in the following Einstein complex hypersurface of $\mathbb{CP}^2 \times \overline{\mathbb{CP}}^2$

$$\mathcal{P}^- \equiv \{([z], [w]) \in \mathbb{CP}^2 \times \overline{\mathbb{CP}}^2 \mid z^t \bar{w} = 0\}$$

and the corresponding projection is given by

$$([z], [w]) \in \mathcal{P}^- \longmapsto [\bar{z} \wedge \bar{w}] \in \mathbb{CP}^2.$$

Then the twistor lifting of the immersion ϕ is of the form

$$\tilde{\phi}^- = (T_1(\phi), T_2(\phi)) \quad \text{where } T_i(\phi) : \Sigma \rightarrow \mathbb{CP}^2, i = 1, 2.$$

When ϕ is twistor holomorphic with negative spin, the map $T_1(\phi)$ (respectively $T_2(\phi)$) is holomorphic (respectively antiholomorphic). So one can produce a lot of minimizers for W^- from suitable pairs of complex curves in \mathbb{CP}^2 . In that case, the degrees d_1 , d_2 and d of the three maps $T_1(\phi)$, $T_2(\phi)$ and ϕ and the functional W^- are related as follows

$$(2) \quad \int_{\Sigma} C dA = \pi d = \pi(d_2 - d_1) \quad W^-(\phi) = 2\pi(d_1 + d_2).$$

Amongst these minimizers, we would like to bring out the following 1-parameter family of surfaces $\phi_t : \mathbb{S}^2 \rightarrow \mathbb{CP}^2$, $t \in [0, \infty[$, defined by

$$\phi_t(x, y, z) = \Pi(x, y, z \cosh t + i \sinh t),$$

for any $(x, y, z) \in \mathbb{S}^2 = \{(x, y, z) \in \mathbb{R}^3 / x^2 + y^2 + z^2 = 1\}$. They are twistor holomorphic Lagrangian spheres with negative spin and $W^- = 8\pi$. These examples are called *Whitney spheres* and in [CU2] they were characterized as the only twistor holomorphic Lagrangian surfaces with negative spin. We remark that ϕ_0 is the totally geodesic immersion of \mathbb{S}^2 , which is a covering of the totally geodesic embedding of the real projective plane \mathbb{RP}^2 . For $t > 0$, ϕ_t is an embedding except at the poles of \mathbb{S}^2 where it has a double point. Notice that ϕ_0 is the only minimal surface in the family.

Now, we are going to state the main result of our paper.

Theorem 4 *Let $\phi : \Sigma \rightarrow \mathbb{CP}^2$ be an immersion of a compact surface and μ its maximum multiplicity. Then*

$$\int_{\Sigma} (|H|^2 + 2 + 2|C|) dA \geq 4\pi\mu$$

and the equality holds if and only if $\mu = 1$ and $\phi(\Sigma)$ is either a complex projective line or a totally geodesic real projective plane, or $\mu = 2$ and $\phi(\Sigma)$ is a Whitney sphere.

Proof: Observe that, by considering if necessary the orientable twofold covering, it suffices to prove the theorem when the surface Σ is orientable. So, from now on, we will take Σ to be oriented.

As the maximum multiplicity of ϕ is μ , let $\{p_1, \dots, p_\mu\}$ be points of Σ such that $\phi(p_i) = [a] \in \mathbb{CP}^2$ for any $i = 1, \dots, \mu$. We define a function $f : \mathbb{CP}^2 \rightarrow \mathbb{R}$ by

$$f([z]) = \frac{|(z, a)|^2}{|z|^2|a|^2},$$

for any $[z] \in \mathbb{CP}^2$. Then $0 \leq f \leq 1$, and $f([z]) = 0$ if and only if $[z]$ is in the cut locus $\mathbb{CP}^2_{[a]}$ of the point $[a]$. Also, $f([z]) = 1$ if and only if $[z] = [a]$. So $\log(1 - f)$ is a well defined function on $\mathbb{CP}^2 - \{[a]\}$. In order to simplify the notation, we will consider that $|a| = 1$ and we will restrict the projection $\Pi : \mathbb{C}^3 \rightarrow \mathbb{CP}^2$ to the unit sphere $\mathbb{S}^5 \subset \mathbb{C}^3$. So, the function f will be nothing but

$$f([z]) = |(z, a)|^2,$$

for any $z \in \mathbb{S}^5$. First we compute the gradient of f . If v is any tangent vector to \mathbb{CP}^2 at $[z]$, then

$$v \cdot f = 2\Re(v^*, (z, a)a),$$

being v^* the horizontal lifting of v to $T_z\mathbb{S}^5$. So

$$(3) \quad (\bar{\nabla} f)_{[z]} = 2(d\Pi)_z((z, a)a - |(z, a)|^2 z),$$

for any $[z] \in \mathbb{CP}^2$. From this equality, we have

$$(4) \quad |\bar{\nabla} f|^2 = 4f(1 - f)$$

$$\langle \bar{\nabla} f, u \rangle \langle \bar{\nabla} f, v \rangle + \langle \bar{\nabla} f, Ju \rangle \langle \bar{\nabla} f, Jv \rangle = 4f\Re((u^*, a)(a, v^*))$$

for any vectors $u, v \in T_{[z]}\mathbb{CP}^2$. Now, taking derivatives in (3) and using that $\Pi : \mathbb{S}^5 \rightarrow \mathbb{CP}^2$ is a Riemannian submersion, one has that the Hessian of f is given by

$$(5) \quad (\bar{\nabla}^2 f)(u, v) = -2f\langle u, v \rangle + 2\Re((u^*, a)(a, v^*)),$$

for any vectors $u, v \in T_{[z]}\mathbb{CP}^2$.

Now, let us take the restriction $h = f \circ \phi$ of the function f to the surface Σ and consider the function $\log(1 - h)$ which is well defined on $\Sigma^* = \Sigma - \{p_1, \dots, p_\mu\}$. We want to compute its Laplacian. By decomposing

$$(6) \quad \bar{\nabla} f = \nabla h + \xi$$

in its tangential and normal components and taking into account (4) and (5), we deduce that

$$(7) \quad \begin{aligned} |\nabla h|^2 &= 4h(1-h) - |\xi|^2, \\ (\nabla^2 h)(u, v) &= -2h\langle u, v \rangle + 2\Re((u^*, a)(a, v^*)) + \langle \sigma(u, v), \xi \rangle, \end{aligned}$$

for any vectors u, v tangent to Σ^* . From these equations we obtain that

$$\Delta \log(1-h) = -\frac{2}{1-h} \sum_{i=1}^2 |(e_i^*, a)|^2 - \frac{2}{1-h} \langle H, \xi \rangle + \frac{|\xi|^2}{(1-h)^2},$$

where $\{e_1, e_2\}$ is a local orthonormal frame on $T\Sigma$. As

$$0 \leq \left| H - \frac{\xi}{1-h} \right|^2 = |H|^2 + \frac{|\xi|^2}{(1-h)^2} - \frac{2}{1-h} \langle H, \xi \rangle,$$

we obtain that

$$(8) \quad \Delta \log(1-h) \geq -|H|^2 - \frac{2}{1-h} \sum_{i=1}^2 |(e_i^*, a)|^2,$$

and the equality holds if and only if the mean curvature field H of Σ coincides with the normal field $\xi/(1-h)$ on the open set Σ^* . Now we are going to look for a suitable upper bound for the function $\sum_{i=1}^2 |(e_i^*, a)|^2$ appearing on the right side. This is why we want to prove the following algebraic lemma.

Lemma 5 *Let \mathbb{C}^2 be the complex two-plane endowed with the usual Hermitian product (\cdot, \cdot) . Then*

$$|(\mathbf{v}, \mathbf{u})|^2 + |(\mathbf{w}, \mathbf{u})|^2 \leq 1 + |(\mathbf{v}, \mathbf{w})|$$

for all unit vectors $\mathbf{u}, \mathbf{v}, \mathbf{w} \in \mathbb{C}^2$ such that $\Re(\mathbf{v}, \mathbf{w}) = 0$. The equality holds if and only if one of the following three possibilities occurs:

1. The real plane spanned by \mathbf{v} and \mathbf{w} is Lagrangian, that is, $(\mathbf{v}, \mathbf{w}) = 0$.
2. The vectors $\mathbf{v} + i\mathbf{w}$ and \mathbf{u} span the same complex line and $\Im(\mathbf{v}, \mathbf{w}) > 0$.
3. The vectors $\mathbf{v} - i\mathbf{w}$ and \mathbf{u} span the same complex line and $\Im(\mathbf{v}, \mathbf{w}) < 0$.

Proof of the Lemma: Fix the vectors \mathbf{v} and \mathbf{w} and consider the smooth function $F : \mathbb{S}^3 \subset \mathbb{C}^2 \rightarrow \mathbb{R}$ given by

$$F(\mathbf{u}) = |(\mathbf{v}, \mathbf{u})|^2 + |(\mathbf{w}, \mathbf{u})|^2 \quad \text{for each } \mathbf{u} \in \mathbb{S}^3.$$

Notice that when the plane P spanned by \mathbf{v} and \mathbf{w} is Lagrangian, or equivalently, when $(\mathbf{v}, \mathbf{w}) = 0$, we have that

$$\mathbf{u} = (\mathbf{u}, \mathbf{v})\mathbf{v} + (\mathbf{u}, \mathbf{w})\mathbf{w} \quad \text{for any } \mathbf{u} \in \mathbb{C}^2$$

and, as a consequence,

$$F(\mathbf{u}) = |\mathbf{u}|^2 = 1 \quad \text{for all } \mathbf{u} \in \mathbb{S}^3.$$

Hence, in that case, F is identically one and the equality holds for every \mathbf{u} .

Suppose now that the plane P is not Lagrangian, that is, assume that $(\mathbf{v}, \mathbf{w}) \neq 0$ and choose $\mathbf{u}_0 \in \mathbb{S}^3$ such that $F(\mathbf{u}_0)$ is the maximum value of F . Then we have $(dF)_{\mathbf{u}_0} = 0$ which is equivalent to

$$(9) \quad (\mathbf{u}_0, \mathbf{v})\mathbf{v} + (\mathbf{u}_0, \mathbf{w})\mathbf{w} = F(\mathbf{u}_0)\mathbf{u}_0.$$

Taking Hermitian products by the vectors \mathbf{v} and \mathbf{w} , we obtain

$$(10) \quad \begin{aligned} (\mathbf{u}_0, \mathbf{w})(\mathbf{w}, \mathbf{v}) &= (F(\mathbf{u}_0) - 1)(\mathbf{u}_0, \mathbf{v}) \\ (\mathbf{u}_0, \mathbf{v})(\mathbf{v}, \mathbf{w}) &= (F(\mathbf{u}_0) - 1)(\mathbf{u}_0, \mathbf{w}). \end{aligned}$$

Hence, if either $(\mathbf{u}_0, \mathbf{v})$ or $(\mathbf{u}_0, \mathbf{w})$ were null, then both two would vanish and so $F(\mathbf{u}_0) = 0$. As F attains its maximum at \mathbf{u}_0 , we would have that F is identically zero, which is impossible. As a consequence, we deduce that the Hermitian products $(\mathbf{u}_0, \mathbf{v})$ and $(\mathbf{u}_0, \mathbf{w})$ do not vanish. In this case, the two equalities above give

$$F(\mathbf{u}_0) - 1 = -\frac{(\mathbf{u}_0, \mathbf{w})}{(\mathbf{u}_0, \mathbf{v})}(\mathbf{v}, \mathbf{w}) = \frac{(\mathbf{u}_0, \mathbf{v})}{(\mathbf{u}_0, \mathbf{w})}(\mathbf{v}, \mathbf{w})$$

and so $(\mathbf{u}_0, \mathbf{w}) = \pm i(\mathbf{u}_0, \mathbf{v})$. From (10) we have

$$F(\mathbf{u}_0) = 1 \pm i(\mathbf{v}, \mathbf{w}) \leq 1 + |(\mathbf{v}, \mathbf{w})|.$$

Then we have proved the claimed inequality. If the equality holds and we have $\Im(\mathbf{v}, \mathbf{w}) > 0$ then $(\mathbf{u}_0, \mathbf{w}) = i(\mathbf{u}_0, \mathbf{v})$ and, from (9), we obtain

$$F(\mathbf{u}_0)\mathbf{u}_0 = (\mathbf{u}_0, \mathbf{v})(\mathbf{v} + i\mathbf{w})$$

and the vectors $\mathbf{v} + i\mathbf{w}$ and \mathbf{u}_0 span the same complex line. When $\Im(\mathbf{v}, \mathbf{w}) < 0$, the proof is similar. q.e.d.

Now we will continue with the proof of the theorem. We apply this Lemma 5 on the right side of (8), where \mathbb{C}^2 means the horizontal subspace of the Hopf fibration, and we have

$$(11) \quad \Delta \log(1 - h) \geq -|H|^2 - 2 - 2|C|.$$

Let $B_{[a]}(\varepsilon)$ be the geodesic ball in \mathbb{CP}^2 centered at the point $[a]$ with radius $\arccos \sqrt{1 - \varepsilon^2}$, that is, the set

$$B_{[a]}(\varepsilon) = \{p \in \mathbb{CP}^2 \mid 1 - f(p) \leq \varepsilon^2\},$$

with ε small enough in order to $B_\varepsilon = \phi^{-1}(B_{[a]}(\varepsilon))$ will be the disjoint union of neighbourhoods B_i , $i = 1, \dots, \mu$ around p_i in Σ . Then the divergence theorem on the manifold $\Sigma - B$ says that

$$\int_{\Sigma - B} \Delta \log(1 - h) dA = - \sum_{i=1}^{\mu} \int_{\partial B_i} \frac{\langle \nabla h, \nu_i \rangle}{1 - h} ds,$$

where ν_i is the unit conormal of ∂B_i pointing to the interior of B_i . Since the function h attains its maximum value 1 at each p_i and h is constant along each ∂B_i , we have that

$$\nu_i = \frac{\nabla h}{|\nabla h|_{|\partial B_i}}.$$

So, combining these equalities with the integral equality above, we have

$$\int_{\Sigma - B} \Delta \log(1 - h) dA = - \sum_{i=1}^{\mu} \int_{\partial B_i} \frac{|\nabla h|}{1 - h} ds = - \sum_{i=1}^{\mu} \frac{1}{\varepsilon^2} \int_{\partial B_i} |\nabla h| ds.$$

As ε tends to zero, $|\nabla h|$ along ∂B_i approaches to $|\nabla f| = 2\varepsilon\sqrt{1 - \varepsilon^2}$ and the length of ∂B_i approaches to 2π radius $B_i = 2\pi \arccos \sqrt{1 - \varepsilon^2}$. Then, we obtain that

$$\int_{\Sigma} \Delta \log(1 - h) dA = -4\pi\mu.$$

This equality and (11) prove the inequality we were looking for.

Now we will analyse the case of the equality. Let $\phi : \Sigma \rightarrow \mathbb{CP}^2$ be an immersion attaining it. *We want to show that ϕ must be twistor holomorphic with negative spin.* It is clear that the immersion ϕ maps the set $h^{-1}(\{0\})$ into the complex projective line $\mathbb{CP}^1_{[a]}$ and so we have that ϕ is totally geodesic (and then twistor holomorphic) and $C^2 = 1$ on its interior. Hence, it remains to study the immersion ϕ only on the open set

$$\Sigma_0 = \{p \in \Sigma^* \mid h(p) > 0\}.$$

Following the different possibilities of the equality case in Lemma 5, we are going to define four open subsets of Σ_0 and prove that ϕ is twistor holomorphic with negative spin on each one of them. We put

$$\begin{aligned} A_+ &= \{p \in \Sigma_0 \mid 1 > C(p) > 0\} & A_- &= \{p \in \Sigma_0 \mid -1 < C(p) < 0\} \\ A_0 &= \text{int} \{p \in \Sigma_0 \mid C(p) = 0\} & A_1 &= \text{int} \{p \in \Sigma_0 \mid C(p)^2 = 1\}. \end{aligned}$$

Let us see that ϕ is twistor holomorphic on A_+ . In fact, as $C > 0$, from Lemma 5 and the equality in (8), we have that $(a, e_1^* + ie_2^*) = 0$, where $\{e_1, e_2\}$ is a local positively oriented orthonormal frame tangent to Σ . Then using (3) we obtain

$$(12) \quad \langle \bar{\nabla} f, e_1 + Je_2 \rangle = 0 \quad \langle \bar{\nabla} f, Je_1 - e_2 \rangle = 0.$$

From (1) and (6), one sees that this implies

$$(13) \quad (1 - C)J^{-}\nabla h = (J\xi)^{\top} \quad (1 - C)|\nabla h|^2 = (1 + C)|\xi|^2,$$

where we have used that $C < 1$ to obtain the second equation. Now putting (13) and (1) in the obvious equalities

$$\langle J\nabla h, J\xi \rangle = 0 \quad \langle J\nabla h + J\xi, \nabla h + \xi \rangle = 0,$$

one obtains that

$$(J\nabla h)^{\perp} = (1 + C)J^{-}\xi.$$

This last equality and (13) give us

$$(14) \quad J\bar{\nabla} f = J^{-}\bar{\nabla} f$$

on the open set A_+ . Taking derivatives in (12) with respect to the two directions tangent to Σ , using the expressions (4) and (5) for the Hessian of f and taking into account (14), it is straightforward to check that

$$\begin{aligned}\langle \bar{\nabla} f, \sigma(e_1, e_1) - \sigma(e_2, e_2) + 2J\sigma(e_1, e_2) \rangle &= 0 \\ \langle J\bar{\nabla} f, \sigma(e_1, e_1) - \sigma(e_2, e_2) + 2J\sigma(e_1, e_2) \rangle &= 0.\end{aligned}$$

From (14) these equalities become

$$\begin{aligned}\langle \xi, \sigma(e_1, e_1) - \sigma(e_2, e_2) + 2J^-\sigma(e_1, e_2) \rangle &= 0 \\ \langle J\xi, \sigma(e_1, e_1) - \sigma(e_2, e_2) + 2J^-\sigma(e_1, e_2) \rangle &= 0.\end{aligned}$$

But, from (13) and the first equation in (4), we know that $\{\xi, J^-\xi\}$ form an orthogonal frame on A_+ . Then we get

$$\sigma(e_1, e_1) - \sigma(e_2, e_2) + 2J^-\sigma(e_1, e_2) = 0,$$

that is, ϕ is twistor holomorphic with negative spin (see Proposition 2).

An completely analogous reasoning proves that the immersion ϕ is also twistor holomorphic with negative spin on the second open set A_- .

Now we will work on the third open set A_0 , where ϕ is a Lagrangian immersion. Since we assume that ϕ attains the equality on our integral inequality, we have that inequality (8) is in fact an equality. Then the mean curvature of ϕ satisfies $H = \xi/(1-h)$. Now we are going to classify Lagrangian surfaces of \mathbb{CP}^2 whose mean curvature is given in the above way. First, from (5), (7) and using elementary properties of Lagrangian surfaces it follows

$$(15) \quad \langle \nabla_v J\xi, w \rangle = \langle \sigma(v, w), J\nabla h \rangle - 2\Re((v^*, a)(a, (Jw)^*)).$$

So taking derivatives in $JH = J\xi/(1-h)$ and using (4), (5) and (15) it follows

$$\begin{aligned}\langle \nabla_v JH, w \rangle &= \frac{1}{1-h} \langle \sigma(v, w), J\nabla h \rangle + \frac{1+h}{2h(1-h)^2} \langle \nabla h, v \rangle \langle J\xi, w \rangle \\ &\quad - \frac{1}{2h(1-h)} \langle \nabla h, w \rangle \langle J\xi, v \rangle,\end{aligned}$$

for any v, w tangent to Σ on A_0 . As JH is a closed vector field on A_0 , the first term is symmetric. As the second fundamental form is also symmetric, we obtain that the other terms are symmetric too, and so

$$(16) \quad dh \wedge \alpha = 0,$$

where α is the 1-form on Σ given by

$$\alpha(v) = \langle v, J\xi \rangle.$$

Now we are going to prove that there exists a vector field X tangent to A_0 and functions a and b on A_0 with $a^2 + b^2 = h$ such that

$$(17) \quad \bar{\nabla} f = aX + bJX.$$

So in particular this vector field X will verify $|X|^2 = 4(1 - h)$. In fact, let $D = \{p \in A_0 / dh_p = 0\}$ and $E = \{p \in A_0 / \alpha_p = 0\}$. If $D = A_0$ or $E = A_0$ the claim is trivial. Otherwise, D and E are proper closed subsets of A_0 . Now, $dh \wedge \alpha = 0$ says that on $A_0 - D$ we can write $\alpha = \lambda dh$ for certain smooth function λ . Taking $X = \frac{\sqrt{1+\lambda^2}}{\sqrt{h}} \nabla h$, then on $A_0 - D$ we have $\bar{\nabla} f = aX + bJX$ for certain smooth functions a and b on $A_0 - D$ satisfying $a^2 + b^2 = h$. Making a similar reasoning with E we write on $A_0 - E$, $\bar{\nabla} f = a'X' + b'JX'$ with $a'^2 + b'^2 = h$. It is clear that, on the non-empty subset $A_0 - (D \cup E)$, we can take $X' = X$, $a' = a$ and $b' = b$. So we prove the existence of such X satisfying (17).

Taking derivative in (17) with respect to a vector v , decomposing in tangent and normal components and using (5), (7) and (17) we obtain

$$\begin{aligned} -2hv + \frac{1}{2}\langle v, X \rangle X &= \langle \nabla a, v \rangle X + a\nabla_v X - bA_{JX}v \\ \langle \nabla b, v \rangle X + aA_{JX}v + b\nabla_v X &= 0. \end{aligned}$$

From these equations it is easy to obtain that

$$\sigma(X, X) = 2\rho JX, \quad \sigma(X, V) = 2bJV,$$

for certain function ρ , where V is any orthogonal vector field to X . From here

$$2H = \frac{\rho + b}{2(1 - h)} JX + cJV,$$

for certain function c . But $\xi = bJX$, and then

$$H = \frac{b}{1 - h} JX.$$

So we get that $\rho = 3b$ and $c = 0$. This in particular means that $|\sigma|^2 = 3|H|^2$. The Gauss equation implies that $|H|^2 + 2 = 2K$, and since our surface is

Lagrangian, $K^\perp = K$. So finally we get that $|H|^2 + 2 = K + K^\perp$, which means that our surface is twistor holomorphic with negative spin on the set A_0 that we were working on.

It only remains to show that the immersion ϕ is twistor holomorphic with negative spin on the fourth open set A_1 . As $C^2 = 1$ on this set, we have that the immersion is complex and so minimal. Since we have that the equality holds on (8) we obtain $\xi = 0$ on A_1 , that is, the field $\bar{\nabla}f$ is tangent to Σ on this set. From (4) and (5), it follows that $\bar{\nabla}_u \bar{\nabla}f$ is tangent to Σ for all $u \in T\Sigma$ on A_1 . This means that $\sigma(u, \bar{\nabla}f) = 0$ for each u and then the immersion is totally geodesic, and so twistor holomorphic, on A_1 .

We have just finished to prove that our immersion ϕ attaining the equality is twistor holomorphic with negative spin. Hence (see Proposition 2) the twistor lifting $\tilde{\phi}^- = (T_1, T_2)$ is holomorphic.

If the Kähler function C changes sign, the sets A_+ and A_- are both non empty. But, as we have already pointed out, we have $(a, e_1^* + ie_2^*) = 0$ on A_+ and $(a, e_1^* - ie_2^*) = 0$ on A_- , where $\{e_1, e_2\}$ is a local positively oriented orthonormal frame. This means (see [ES], (12.4) in page 634) that $T_1(A_+) \subset \mathbb{CP}_{[a]}^1 \subset \mathbb{CP}^2$ and $T_2(A_-) \subset \mathbb{CP}_{[a]}^1 \subset \mathbb{CP}^2$. So, since T_1 and T_2 are complex, $T_i(\Sigma) \subset \mathbb{CP}_{[a]}^1$ for $i = 1, 2$, which implies that ϕ is identically $[a]$. This is impossible and so the function C does not change sign.

First, suppose that $0 \leq C \leq 1$ on Σ . If the interior of the set $\{p \in \Sigma \mid C(p) = 1\}$ is non empty, the immersion ϕ is complex and twistor holomorphic, and so totally geodesic, on that interior. As $\tilde{\phi}^-$ is holomorphic, this implies that ϕ is totally geodesic on the whole Σ and $\phi(\Sigma)$ is a complex projective line. Then, we can assume that the set $\{p \in \Sigma \mid 0 \leq C(p) < 1\}$ is dense in the surface. Let us see that, in this situation, C must vanish. In fact, if there is a point with $C(p) > 0$, then the set $\{p \in \Sigma \mid 0 < C(p) < 1\}$ is non empty. But then, the reasoning above implies that $(a, e_1^* + ie_2^*) = 0$ on Σ . Hence, equation (14) is valid on the whole surface Σ and, so, using (6), $J\xi = J^-\xi$ at the critical points of the function h . From (1) and as we know that $C \geq 0$, we deduce that $\xi = 0$ at those points. As conclusion, $\bar{\nabla}f = 0$ at the critical points of h . Hence, from (4), we have that the critical values of h are 0 and 1 and so they are global extrema for h . On the other hand, we know that h has exactly μ maxima and that $h = 0$ is equivalent to $T_2 = [a]$ because $(T_1, a) \equiv 0$. Then, as T_2 is antiholomorphic and non constant, h is a Morse function. Applying the Morse inequalities to the function h we conclude that Σ has genus zero and $\mu = 1$. Now, as $C \geq 0$ and our immer-

sion attains the equality, from (2), it follows that $d_1 \leq d_2 = \mu = 1$. As ϕ cannot be holomorphic, one has $d_1 = 1$ and so, again from (2), $C \equiv 0$, which contradicts our assumption. But the opposite of this assumption is the fact that the immersion is Lagrangian.

Analogously, if $-1 \leq C \leq 0$ on Σ , we would obtain that either $\phi(\Sigma)$ is a complex projective line or the immersion ϕ is Lagrangian.

Finally, using the main result in [CU2], we finish the proof. q.e.d.

Corollary 6 *Let Σ a compact surface immersed in \mathbb{CP}^2 with maximum multiplicity μ . The following facts occur:*

- i) $W^-(\Sigma) \geq 2\pi\mu$ and the equality holds if and only if Σ is a complex projective line. In particular, if Σ is minimal, then $\text{area}(\Sigma) \geq \pi\mu$ and the equality is attained only by the complex projective line.
- ii) If Σ is a Lagrangian surface, then $W^-(\Sigma) \geq 4\pi\mu$ and the equality characterises the totally geodesic real projective plane and the Whitney spheres. In particular, if Σ is also minimal, then $\text{area}(\Sigma) \geq 2\pi\mu$ and the equality holds if and only if the surface is a totally geodesic real projective plane or two-sphere.

Remark 1 We consider the following holomorphic and anti-holomorphic maps $T_i : \mathbb{C} \cup \{\infty\} \rightarrow \mathbb{CP}^2$, $i = 1, 2$ given by

$$T_1(z) = \Pi(1, z, 0) \quad T_2(z) = \Pi(\bar{z}(1 + \bar{z}), -(1 + \bar{z}), \bar{z}).$$

Then the corresponding twistor holomorphic immersion with negative spin $\phi : \mathbb{C} \cup \{\infty\} \rightarrow \mathbb{CP}^2$ is

$$\phi(z) = \Pi(-|z|^2, z, (1 + z)(1 + |z|^2)).$$

It is easy to check that ϕ is regular and that is embedded except at $z = 0, \infty$ where ϕ has a double point. So $\mu = 2$. As the degrees of T_1 and T_2 are 1 and 2, from (2), $W^-(\phi) = 6\pi$. This example shows that even, in the family of non complex compact surfaces of \mathbb{CP}^2 , the claim ii) in Corollary 6 is not true.

Remark 2 The totally geodesic surfaces and the Whitney spheres of \mathbb{CP}^2 have the property that their mean curvature vectors are given by

$$H = \frac{\bar{\nabla} f^\perp}{1 - f},$$

where $f([z]) = |(z, a)|^2$. The geometric meaning of this property is the following. Let $M = \mathbb{CP}^2 - \{[a]\}$, and g the metric on M conformal to the Fubini-Study metric defined by

$$g = \frac{1}{(1-f)^2} \langle \cdot, \cdot \rangle.$$

Then it is not difficult to see that (M, g) is a complete Riemannian manifold (with one end) and with zero scalar curvature. If $\phi : \Sigma \rightarrow \mathbb{CP}^2$ is an immersion with $\{p_1, \dots, p_\mu\} = \phi^{-1}([a])$, then the mean curvatures vectors \hat{H} and H of ϕ with respect to the metric induced by g and $\langle \cdot, \cdot \rangle$ are related (see proof of Proposition 1) by

$$\frac{\hat{H}}{(1-f)^2} = H - \frac{\bar{\nabla} f^\perp}{1-f}.$$

So the condition $H = \bar{\nabla} f^\perp / (1-f)$ means that the surface $\Sigma - \{p_1, \dots, p_\mu\}$ is minimal in (M, g) .

Remark 3 To end this paper we would like to remark something about the functional W^- acting on tori. When you consider the Willmore functional on compact surfaces of \mathbb{R}^4 , there is a very famous conjecture, due to Willmore, which says that the Willmore functional on tori is bounded below by $2\pi^2$ and the Clifford torus is the only torus which achieves this minimum. Since the Clifford torus is Lagrangian, Minicozzi [M] studied this problem in the smaller class of Lagrangian tori.

In our case, we will also call *Clifford torus* to the following torus T embedded in \mathbb{CP}^2

$$T = \{\Pi(z) \in \mathbb{CP}^2 / |z_i|^2 = \frac{1}{3}, i = 1, 2, 3\}.$$

It is easy to check that T is a minimal Lagrangian torus with area $4\pi^2/3\sqrt{3}$. So its Willmore functional is $W^-(T) = 8\pi^2/3\sqrt{3}$.

For complex tori of \mathbb{CP}^2 , the Willmore functional W^- take the value $2\pi d$, where d is the degree of the torus. So, as d must be non smaller than 3, we obtain that in this family $W^- \geq 6\pi > 8\pi^2/3\sqrt{3}$.

For twistor holomorphic tori in \mathbb{CP}^2 with negative spin, (2) implies $W^- = 2\pi(d_1 + d_2) \geq 8\pi$.

Very recently, Kenmotsu and Zhou have proved in [KZ] that the tori in \mathbb{CP}^2 with non zero parallel mean curvature vector are Lagrangian and flat, and then, up to isometries, they can be parametrized by $T_{r_1, r_2, r_3} / r_1 \geq r_2 \geq r_3 > 0, r_1^2 + r_2^2 + r_3^2 = 1$, where

$$T_{r_1, r_2, r_3} = \{\Pi(z) \in \mathbb{CP}^2 / |z_i|^2 = r_i^2, i = 1, 2, 3\}.$$

It is clear that $W^-(T_{r_1, r_2, r_3}) \geq 8\pi^2 r_1 r_2 r_3 \geq 8\pi^2 / 3\sqrt{3}$, and the equality is only achieved by the Clifford torus.

Also, in [CU1], Castro and Urbano classified minimal Lagrangian tori of \mathbb{CP}^2 invariant under a 1-parameter group of holomorphic isometries. This family of tori is described in terms of elliptic functions and it is possible to check that the Willmore functional W^- on these tori satisfy $W^- \geq 8\pi^2 / 3\sqrt{3}$, with equality only for the Clifford torus.

These considerations make reasonable the following conjecture:

The Clifford torus achieves the minimum of the Willmore functional W^- either amongst all tori in \mathbb{CP}^2 (or amongst all Lagrangian tori in \mathbb{CP}^2).

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