On twistor harmonic surfaces in the complex projective plane

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Abstract: We completely classify all the twistor harmonic (non minimal) Lagrangian immersions of compact surfaces in the complex projective plane $\mathbb{CP}^2$, i.e. those Lagrangian immersions such that their twistor lifts to the twistor space over $\mathbb{CP}^2$ are harmonic maps.

Key words: *Twistor harmonic immersions, Lagrangian surfaces.*

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1 Introduction.

Let $(\mathbb{CP}^2, g, J)$ be the complex projective plane endowed with the Fubini-Study metric g of constant holomorphic sectional curvature 4 and the standard complex structure J. The twistor space $\mathcal{Z}$ of $\mathbb{CP}^2$ can be identified with the flag manifold $SU(3)/SU(1)^3$ (see [6], [9]), which can be endowed with a structure of Einstein-Kaehler manifold. An interesting question is to study those orientable surfaces of $\mathbb{CP}^2$ whose twistor lifts to $\mathcal{Z}$ have a regular behaviour.

As a first example, we have the *superminimal* surfaces of $\mathbb{CP}^2$, which are defined as those ones whose twistor lifts are horizontal. This family can be characterized as those minimal surfaces with holomorphic twistor lift and it has been studied by several authors ([4], [6], [9]). Chern and Wolfson and Eells and Salamon proved that any compact Riemann surface can be conformally immersed in $\mathbb{CP}^2$ as a superminimal surface; in fact, there is a one-to-one correspondence between non-complex superminimal surfaces of $\mathbb{CP}^2$ and holomorphic curves of $\mathbb{CP}^2$. Friedrich ([8]) generalized this notion introducing the concept of *twistor holomorphic* surfaces of $\mathbb{CP}^2$ as those surfaces such that their twistor lifts to $\mathcal{Z}$ are holomorphic. Some results for this family of surfaces can be found in [7] and [8].

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This paper will deal with *twistor harmonic* surfaces of the complex projective plane, i.e. surfaces of $\mathbb{CP}^2$ such that their twistor lifts are harmonic maps. Besides the twistor holomorphic surfaces, all the minimal (non-superminimal) surfaces are twistor harmonic. In this paper we make a contribution to this problem working with a natural family of surfaces of $\mathbb{CP}^2$, the *Lagrangian or totally real surfaces.* In [3] the authors classified all the twistor holomorphic *Lagrangian* surfaces in the complex projective plane, obtaining a one-parameter family of examples of Lagrangian spheres in $\mathbb{CP}^2$. Among them, only the totally geodesic one was superminimal. In [2] the authors constructed a family of minimal Lagrangian (in particular non-superminimal) tori of $\mathbb{CP}^2$. Here we get the complete classification of all the compact twistor harmonic Lagrangian surfaces of $\mathbb{CP}^2$ such that are not minimal. Besides spheres mentioned above now we obtain the classification of the tori (Corollary 1), which are the only two possible compact topologies (Proposition 2).

In §2 we describe in detail the twistor space of $\mathbb{CP}^2$ and the twistor lifts of its Lagrangian surfaces, which allows us in Theorem 1 to characterize the twistor harmonic Lagrangian immersions by the *holomorphy* of a cubic differential Θ naturally associated to the surface ($\Theta \equiv 0$ means that our surface is twistor holomorphic). In terms of the mean curvature vector H, our surfaces can be characterize by the fact that JH is a conformal vector field tangent to the surface (Proposition 1). Considering the holomorphic vector field $\mathcal{X}$ associated to the conformal vector field JH and the holomorphic cubic differential Θ, we construct a holomorphic function F defined on the whole surface (Definition 1). As on the tori F is a non-null constant (Proposition 1), we classify locally our surfaces when F is a non-null constant (Theorem 2), obtaining a three-parameter family of examples. In Theorem 3 we get explicitly the immersions of Theorem 2 in terms of elliptic integrals and Jacobi's elliptic functions, which allows us in Theorem 4 to characterize in terms of certain rationality conditions those immersions which are doubly-periodic, obtaining in this way infinitely many examples of twistor harmonic Lagrangian tori in $\mathbb{CP}^2$.

2 The twistor space of the complex projective plane and the twistor lift.

Let $\mathbb{CP}^2$ be the two-dimensional complex projective space endowed with the Fubini–Study metric g of constant holomorphic sectional curvature 4. If J denotes the complex structure of $\mathbb{CP}^2$, then the Kähler two-form Ω on $\mathbb{CP}^2$ is given by $\Omega\,(X,Y) = g(X, JY)$, for any tangent vector fields X and Y.

Let $\Pi : \mathbb{S}^5(1) \longrightarrow \mathbb{CP}^2$ be the Hopf fibration. We denote by $\langle\,,\,\rangle$ the

Euclidean metric of $\mathbb{C}^3$ as well as that induces on $\mathbb{S}^5(1)$, which becomes Π in a Riemannian submersion. The complex structure of $\mathbb{C}^3$, which induces J on $\mathbb{CP}^2$ via Π, will be also denoted by J.

We describe now the twistor space of $\mathbb{CP}^2$. Let

$$M = \{(x, y) \in \mathbb{S}^5(1) \times \mathbb{S}^5(1) \,/\, \langle x, y\rangle = \langle x, Jy\rangle = 0\}.$$

Then M is a codimension two submanifold of $\mathbb{S}^5(1) \times \mathbb{S}^5(1)$. The natural action of $\mathbb{S}^1 \times \mathbb{S}^1$ on $\mathbb{S}^5(1) \times \mathbb{S}^5(1)$ can be restricted to M and the quotient space in this way obtained will be denoted by $\mathcal{Z}$. So we have a fibration $\Pi \times \Pi : M \longrightarrow \mathcal{Z}$ and $\mathcal{Z}$ can be given by

$$\mathcal{Z} = \{([z], [w]) \in \mathbb{CP}^2 \times \mathbb{CP}^2 \,/\, z_1\overline{w}_1 + z_2\overline{w}_2 + z_3\overline{w}_3 = 0\}.$$

Hence $\mathcal{Z}$ is a codimension two submanifold of $\mathbb{CP}^2 \times \mathbb{CP}^2$.

We can summarize the above in the following commutative diagram:

$$\begin{array}{ccc} M & \hookrightarrow & \mathbb{S}^5(1) \times \mathbb{S}^5(1) \\ \downarrow & & \downarrow \\ \mathcal{Z} & \hookrightarrow & \mathbb{CP}^2 \times \mathbb{CP}^2 \end{array}$$

On $\mathbb{CP}^2 \times \mathbb{CP}^2$, we consider now the complex structure $\mathbb{J}$ given by $\mathbb{J} = (J, -J)$. Then $(\mathbb{CP}^2 \times \mathbb{CP}^2, g \times g, \mathbb{J})$ is a Kaehler manifold and $\mathcal{Z}$ is a complex hypersurface of $\mathbb{CP}^2 \times \mathbb{CP}^2$. If $\widehat{g}$ is the induced metric on $\mathcal{Z}$, then $(\mathcal{Z}, \widehat{g}, \mathbb{J})$ is an Einstein-Kaehler manifold (cf. [12]).

Let $p : \mathcal{Z} \longrightarrow \mathbb{CP}^2$ the projection given by

$$p([x], [y]) = [z]$$

where $\langle z, x\rangle = \langle z, Jx\rangle = 0$ and $\langle z, y\rangle = \langle z, Jy\rangle = 0$. Then $p : \mathcal{Z} \longrightarrow \mathbb{CP}^2$ is a $\mathbb{CP}^1$-fiber bundle on $\mathbb{CP}^2$ called the *twistor bundle* of $\mathbb{CP}^2$. $(\mathcal{Z}, \widehat{g}, \mathbb{J})$ is called then the *twistor space* of $\mathbb{CP}^2$.

On the other hand, let $\phi : \Sigma \longrightarrow \mathbb{CP}^2$ be an immersion of an oriented surface Σ in $\mathbb{CP}^2$. ϕ is called *Lagrangian* or *totally real* if $\phi^*\Omega = 0$. This means that J defines an isomorphism from the tangent bundle of Σ to the normal bundle of ϕ. We define a trilinear form on Σ by

$$(X, Y, Z) \mapsto g(\sigma(X, Y), JZ)$$

where X, Y, Z are tangent vector fields to Σ and σ is the second fundamental form of ϕ. The most elementary property of such a Lagrangian immersion ϕ is that the above trilinear form is symmetric.

Let us see now that *Lagrangian* immersions in $\mathbb{CP}^2$ are —locally— projections via the Hopf fibration Π of *horizontal* immersions in $\mathbb{S}^5(1)$.

Let $\psi : U \subset \Sigma \longrightarrow \mathbb{S}^5(1)$ be a local lift of ϕ to $\mathbb{S}^5(1)$, whith $(U, z = x + iy)$ a simply-connected isothermal coordinate system on Σ where the induced metric, also denoted by g, is given by $g = e^{2u}|dz|^2$. If ω is the 1-form on U defined by

$$\omega(v) = \langle \psi_*(v), J\psi \rangle,$$

then it is easy to see that if v and w are tangent vectors to Σ, then $d\omega(v, w) = 2\langle \psi_*(w), J\psi_*(v)\rangle = 2g(\phi_*(w), J\phi_*(v)) = 0$, because ϕ is a Lagrangian immersion. So, there exists $\eta \in C^\infty(U)$ such that $d\eta = \omega$. Then $\widetilde{\psi} = e^{-i\eta}\psi$ is another local lift of ϕ to $\mathbb{S}^5(1)$ and it is not difficult to obtain that

$$\langle \widetilde{\psi}_*(v), J\widetilde{\psi} \rangle = -d\eta(v) + \omega(v) = 0,$$

for any tangent vector v on Σ. Thus, $\widetilde{\psi}$ is an horizontal (local) lift of ϕ to $\mathbb{S}^5(1)$. Renaming finally $\widetilde{\psi}$ as ψ, we can assume from now on that ψ is horizontal. We observe that g is also the induced metric on Σ by ψ and ψ is a totally real immersion in $\mathbb{C}^3$ with respect to the complex structure J.

Let define $\psi^j, j = 1, 2$ by

$$\begin{aligned} \psi^1 &= \frac{e^{-u}}{\sqrt{2}} \left(\psi_x + J\psi_y\right), \\ \psi^2 &= \frac{e^{-u}}{\sqrt{2}} \left(\psi_x - J\psi_y\right). \end{aligned} \tag{1}$$

The properties of the horizontal local lift ψ of the Lagrangian immersion ϕ says us that

$$|\psi^j|^2 = 1,\ j = 1, 2;\ \langle \psi^1, \psi^2 \rangle = \langle \psi^1, J\psi^2 \rangle = 0.$$

Then (ψ^1, ψ^2) defines a map from U to M.

If η were another local lift on U of ϕ to $\mathbb{S}^5(1)$, then it is clear that there exists $\theta \in \mathbb{R}$ such that $\eta^j = e^{i\theta}\psi^j$, $j = 1, 2$, and so $(\eta^1, \eta^2) = e^{i\theta} \cdot (\psi^1, \psi^2)$, where $e^{i\theta} \cdot (z, w) = (e^{i\theta}z, e^{i\theta}w)$ for any z, w in $\mathbb{C}^3$. Hence $\Pi \times \Pi \circ (\eta^1, \eta^2) = \Pi \times \Pi \circ (\psi^1, \psi^2)$ as maps from U to $\mathcal{Z}$; in this way, there exists a well-defined map $\widetilde{\phi} : U \longrightarrow \mathcal{Z}$, which can be given locally by $\Pi \times \Pi \circ (\psi^1, \psi^2)$ and in such a way that (ψ^1, ψ^2) is a local lift of $\widetilde{\phi}$ to M. By a standard connection argument it is easy to see that $\widetilde{\phi}$ is globally defined on Σ. So we have constructed a map

$$\widetilde{\phi} : \Sigma \longrightarrow \mathcal{Z}$$

in the way above described. Since locally $\Pi \times \Pi \circ (\psi^1, \psi^2) = \widetilde{\phi}$ and $\langle \psi, \psi^j \rangle = \langle \psi, J\psi^j \rangle = 0$, $j = 1, 2$, we have that $p \circ \widetilde{\phi} = \phi$. Hence $\widetilde{\phi}$ is a (canonical) lift of ϕ to $\mathcal{Z}$ called the *twistor lift* of ϕ.

3 Twistor harmonic Lagrangian surfaces.

Let $\phi : \Sigma \longrightarrow \mathbb{CP}^2$ be a Lagrangian immersion of an oriented surface Σ. We define a cubic differential Θ on Σ by

$$\Theta(z) = f(z)dz^3, \text{ with } f(z) = 4g(\sigma(\partial_z, \partial_z), J\partial_z), \tag{2}$$

where $z = x + iy$ is a local isothermal coordinate on Σ and g, σ and J are extended $\mathbb{C}$-linearly to the complexified bundles.

The main purpose of this paragraph is to prove the following result:

Theorem 1 *Let $\phi : \Sigma \longrightarrow \mathbb{CP}^2$ be a Lagrangian immersion of an oriented surface Σ. The twistor lift of ϕ, $\widetilde{\phi} : (\Sigma, g) \longrightarrow (\mathcal{Z}, \widehat{g})$, is harmonic if and only if the cubic differential Θ is holomorphic.*

Remark 1 In [3] was pointed out that $\widetilde{\phi}$ is holomorphic, that is, ϕ is twistor holomorphic, if and only if $\Theta \equiv 0$.

Proof: Using the notation of §2, if $\psi : U \longrightarrow \mathbb{S}^5(1)$ is a local horizontal lift of ϕ to $\mathbb{S}^5(1)$, then

$$\widetilde{\psi} = (\psi^1, \psi^2) : U \longrightarrow M$$

is a local lift of $\widetilde{\phi}$ to M, where ψ^j, $j = 1, 2$, are given in (1).

The vertical subspace of $\Pi \times \Pi : M \longrightarrow \mathcal{Z}$ at a point $(x, y) \in M$ is given by

$$V_{(x,y)} = \text{span}\{(Jx, 0), (0, Jy)\},$$

and the normal space of M in $\mathbb{S}^5(1) \times \mathbb{S}^5(1)$ in $(x, y) \in M$ is

$$\text{span}\{(y, x), \mathbb{J}(y, x)\}.$$

So, it is easy to check that the horizontal subspace of $\Pi \times \Pi : M \longrightarrow \mathcal{Z}$ in $\widetilde{\psi}(z)$ is

$$H_{\widetilde{\psi}(z)} = \text{span}\{E_i, \mathbb{J}E_i;\ i = 1, 2, 3\},$$

where

$$E_1 = (\psi, 0),\ E_2 = (0, \psi),\ E_3 = (\psi^2, -\psi^1).$$

Now, if $\widehat{\nabla}$ denotes the induced connection on $\widetilde{\phi}^* T\mathcal{Z}$ of the Levi-Civita connection of $(\mathcal{Z}, \hat{g})$, then the tension field of $\widetilde{\phi}$ is given —up to a conformal factor— by

$$\tau(\widetilde{\phi}) = \widehat{\nabla}_{\partial_{\bar{z}}} \widetilde{\phi}_*(\partial_z),$$

so that $\widetilde{\phi}$ is a harmonic map if and only if $\widehat{\nabla}_{\partial_{\overline{z}}}\widetilde{\phi}_*(\partial_z) \equiv 0$.

We proceed to compute the horizontal lift of $\tau(\widetilde{\phi})$ to the bundle $\widetilde{\psi}^*TM$, which will be denoted by $\tau(\widetilde{\phi})^*$. Using the fundamental equations of the Riemannian submersion $\Pi : (M, g \times g) \longrightarrow (\mathcal{Z}, \hat{g})$ (cf. [11]), it is not difficult to verify that

$$\tau(\widetilde{\phi})^* = (\widetilde{\psi}_{z\overline{z}})^H - 2\Re\left(\widetilde{\nabla}_{\partial_{\overline{z}}}\widetilde{\psi}_z^V\right)^H, \tag{3}$$

where $\widetilde{\nabla}$ is the induced connection on $\widetilde{\psi}^*TM$ of the Levi-Civita connection of M, $\Re$ denotes real part and $(\,)^H$, $(\,)^V$ are the horizontal and vertical components respectively.

Using complex coordinates and (1), ψ^1 and ψ^2 can be written as

$$\begin{aligned} \psi^1 &= \frac{e^{-u}}{\sqrt{2}}\left(\psi_z + \psi_{\overline{z}} + i(J\psi_z - J\psi_{\overline{z}})\right), \\ \psi^2 &= \frac{e^{-u}}{\sqrt{2}}\left(\psi_z + \psi_{\overline{z}} - i(J\psi_z - J\psi_{\overline{z}})\right). \end{aligned} \tag{4}$$

In order to compute $\tau(\widetilde{\phi})^*$, it is interesting to make use of the Frenet equations of ψ,

$$\begin{aligned} \psi_{zz} &= 2u_z\psi_z + \frac{e^{2u}\overline{k}}{2}J\psi_z + \frac{e^{-2u}f}{2}J\psi_{\overline{z}}, \\ \psi_{z\overline{z}} &= \frac{e^{2u}}{2}\left(kJ\psi_z + \overline{k}J\psi_{\overline{z}}\right) - \frac{e^{2u}}{2}\psi, \end{aligned} \tag{5}$$

where f was defined in (2) and

$$k(z) = 4e^{-4u}\langle\psi_{z\overline{z}}, J\psi_{\overline{z}}\rangle. \tag{6}$$

Using then (5) in (4), we have:

$$\widetilde{\psi}_z^V = (e^{2u}\overline{k}/2 + iu_z)(J\psi^1, 0) + (e^{2u}\overline{k}/2 - iu_z)(0, J\psi^2),$$

and from this it follows that

$$\left(\widetilde{\nabla}_{\partial_{\overline{z}}}\widetilde{\psi}_z^V\right)^H = (e^{2u}\overline{k}/2 + iu_z)((J\psi^1)^H_{\overline{z}}, 0) + (e^{2u}\overline{k}/2 - iu_z)(0, (J\psi^2)^H_{\overline{z}}).$$

Thus, from (3), we obtain:

$$\begin{aligned} \tau(\widetilde{\phi})^* = \big((\psi^1_{z\overline{z}})^H &- 2\Re(e^{2u}\overline{k}/2 + iu_z)(J\psi^1)^H_{\overline{z}}, \\ (\psi^2_{z\overline{z}})^H &- 2\Re(e^{2u}\overline{k}/2 - iu_z)(J\psi^2)^H_{\overline{z}}\big). \end{aligned} \tag{7}$$

From (7), using (4) and (5), it is straightforward to conclude that

$$\begin{aligned}\langle\tau(\widetilde{\phi})^*, E_i\rangle = \langle\tau(\widetilde{\phi})^*, \mathbb{J}E_i\rangle = 0,\ i = 1, 2,\\ \langle\tau(\widetilde{\phi})^*, E_3\rangle = \frac{e^{-2u}}{2}\Im(f_{\overline{z}} - e^{4u}k_{\overline{z}}),\\ \langle\tau(\widetilde{\phi})^*, \mathbb{J}E_3\rangle = \frac{e^{2u}}{2}\Re(f_{\overline{z}} + e^{4u}k_{\overline{z}}),\end{aligned}$$

where $\Im$ denotes imaginary part. But as $f(z)$ (see (2)) is also given by $f(z) = 4\langle\psi_{zz}, J\psi_z\rangle$, from (5) and (6) it is easy to check that

$$e^{4u}k_{\overline{z}} = \overline{f}_z, \tag{8}$$

and so $\Re(e^{4u}k_{\overline{z}}) = \Re(f_{\overline{z}})$ and $\Im(e^{4u}k_{\overline{z}}) = -\Im(f_{\overline{z}})$. Finally, from (7) we get:

$$\tau(\widetilde{\phi})^* = e^{-2u}\left(\Im(f_{\overline{z}})(\psi^2, -\psi^1) + \Re(f_{\overline{z}})\mathbb{J}(\psi^2, -\psi^1)\right).$$

Hence, $\widetilde{\phi}$ is harmonic, that is $\tau(\widetilde{\phi}) = 0$, if and only if $f_{\overline{z}} = 0$ (i.e. Θ is holomorphic).

From now on, we will call *twistor harmonic* to those immersions whose twistor lifts are harmonic maps. In this way, Theorem 1 says that twistor harmonic Lagrangian surfaces in $\mathbb{CP}^2$ are characterized by the holomorphy of the cubic differential Θ given in (2).

We now seek to express this property in terms of the mean curvature vector H of a Lagrangian immersion $\phi : \Sigma \longrightarrow \mathbb{CP}^2$.

If $\mathcal{L}$ is the Lie derivative in Σ, which has been extended $\mathbb{C}$-linearly to the complexified tangent bundle, then:

$$(\mathcal{L}_{JH}\, g)(\partial_z, \partial_z) = -2g([JH, \partial_z], \partial_z) = -2g(\nabla^{\perp}_{\partial_z} H, J\partial_z).$$

But $\sigma(\partial_z, \partial_{\overline{z}}) = (e^{2u}/2)H$, and then, using the Codazzi equation of ϕ, we have

$$\nabla^{\perp}_{\partial_z} H = 2e^{-2u}(\nabla\sigma)(\partial_{\overline{z}}, \partial_z, \partial_z),$$

being $\nabla\sigma$ the covariant derivative of σ. So, we conclude that

$$(\mathcal{L}_{JH}\, g)(\partial_z, \partial_z) = -4e^{-2u}g((\nabla\sigma)(\partial_{\overline{z}}, \partial_z, \partial_z), J\partial_z) = -e^{-2u}f_{\overline{z}}.$$

Thus, Θ is holomorphic (i.e. $f_{\overline{z}} = 0$) if and only if $(\mathcal{L}_{JH}\, g)(\partial_z, \partial_z) = 0$. Since

$$\begin{aligned}(\mathcal{L}_{JH}\, g)(\partial_z, \partial_z) = (1/4)\left((\mathcal{L}_{JH}\, g)(\partial_x, \partial_x) - \right.\\ \left. -(\mathcal{L}_{JH}\, g)(\partial_y, \partial_y)\right) - (i/2)(\mathcal{L}_{JH}\, g)(\partial_x, \partial_y),\end{aligned}$$

and taking into account Theorem 1, we have proved the following result:

Proposition 1 *Let $\phi : \Sigma \longrightarrow \mathbb{CP}^2$ be a Lagrangian immersion of an oriented surface Σ. Then ϕ is a twistor harmonic immersion if and only if JH is a conformal vector field.*

Remark 2 In particular, Lagrangian surfaces with parallel mean curvature vector are twistor harmonic.

For a twistor harmonic Lagrangian immersion $\phi : \Sigma \longrightarrow \mathbb{CP}^2$, we can also consider the *holomorphic* vector field canonically associated to the *conformal* vector field JH (Proposition 1), that is:

$$\mathcal{X} = -2e^{-2u} g(JH, \partial_{\overline{z}})\partial_z,$$

with $z = x + iy$ a local isothermal coordinate on Σ, where the induced metric is written as $g = e^{2u}|dz|^2$. It is not difficult to check that $\mathcal{X}$ can be written as

$$\mathcal{X}(z) = k(z)\partial_z,$$

where k was defined in (6).

We remark that (8) says that f is holomorphic if and only if k is holomorphic.

So, for such a twistor harmonic Lagrangian immersion, $\phi : \Sigma \longrightarrow \mathbb{CP}^2$, we have two holomorphic objects, $\Theta(z) = f(z)dz^3$ and $\mathcal{X}(z) = k(z)\partial_z$, whose lengths are given by

$$|f(z)|^2 = e^{6u}(|\sigma|^2 - 3|H|^2), \ |k(z)|^2 = e^{-2u}|H|^2. \tag{9}$$

Moreover, it is interesting for us to consider the globally-defined complex function given in the following definition.

Definition 1 *Associated to a twistor harmonic Lagrangian immersion, $\phi : \Sigma \longrightarrow \mathbb{CP}^2$, it is defined the* holomorphic *function*

$$F : \Sigma \longrightarrow \mathbb{C}$$

given locally by

$$F(z) = f(z)k(z)^3 = 32e^{-6u} g(\sigma(\partial_z, \partial_z), J\partial_z) g(H, J\partial_{\overline{z}})^3,$$

where $z = x + iy$ is a local isothermal coordinate such that the induced metric is written as $g = e^{2u}|dz|^2$.

From (9) and the Gauss equation of ϕ, it is easy to obtain that

$$|F(z)|^2 = |H|^6(|\sigma|^2 - 3|H|^2). \tag{10}$$

A first consequence of this is stated by the following restrictions in the compact case.

Proposition 2 *Let $\phi : \Sigma \longrightarrow \mathbb{CP}^2$ be a twistor harmonic Lagrangian immersion of a compact oriented surface Σ with genus γ. Then:*

(a) *If $\gamma \geq 2$, ϕ is minimal.*

(b) *If $\gamma = 0$, ϕ is twistor holomorphic.*

(c) *If $\gamma = 1$, either ϕ is minimal or the function F of the Definition 1 is a non-null constant.*

Proof: If $\gamma \geq 2$, it is well-known that there are not non null holomorphic vector fields on the surface; so $\mathcal{X} \equiv 0$ and so $H \equiv 0$ from (9).

If Σ is a sphere, every holomorphic differential is necessarily null on Σ; then $\Theta \equiv 0$, so that ϕ is twistor holomorphic (see Remark 1).

When Σ is a torus, then the holomorphic function F is constant and $\Theta \equiv 0$ or Θ has no zeroes on Σ. But if $\Theta \equiv 0$, then $|\sigma|^2 = 3|H|^2$ from (9), and using this in the Gauss equation of ϕ we obtain that $K = 1 + 2|H|^2 - |\sigma|^2/2 = 1 + |H|^2/2$. Then the Gauss-Bonnet theorem says that this is impossible. Hence Θ has no zeroes. Thus, from (10) we conclude the result.

Remark 3 In [3] we completely classified all the twistor holomorphic Lagrangian immersions in the complex projective plane, obtaining a one-parameter family of explicit examples of Lagrangian spheres in $\mathbb{CP}^2$.

In [2], we studied a particular family of new explicit examples of minimal Lagrangian tori in $\mathbb{CP}^2$, characterized by its invariability by a one-parameter group of holomorphic isometries of $\mathbb{CP}^2$.

Our next purpose is to classify all the twistor harmonic (non minimal) Lagrangian tori in the complex projective plane, which verify that F (Definition 1) is a non-null constant (see Proposition 2(c)). For this, we are going to study locally those twistor harmonic Lagrangian surfaces satisfying this last property.

Theorem 2 *Associated to each solution $u : \mathbb{R} \longrightarrow \mathbb{R}$ to the o.d.e.*

$$u'' + e^{2u} + \lambda^2 \sinh 4u = 0, \ \lambda \in (0, +\infty), \tag{11}$$

there exists a one-parameter family of twistor harmonic Lagrangian immersions

$$\phi_\alpha : (\mathbb{R}^2, e^{2u}|dz|^2) \longrightarrow \mathbb{CP}^2, \ \alpha \in [0, \pi],$$

such that the function F associated to ϕ_α is the constant $\lambda^4 e^{i\alpha}$.

Conversely, every twistor harmonic Lagrangian immersion with F (see Definition 1) a non-null constant is locally congruent to some of the latter examples.

Moreover, the immersion ϕ_α has parallel mean curvature vector if and only if u is constant.

Proof: We prove the result analyzing locally a twistor harmonic Lagrangian immersion $\phi : \Sigma \longrightarrow \mathbb{CP}^2$ of an oriented surface Σ. Following the preceeding notation, we consider a local isothermal coordinate system $(U, z = x + iy)$ in Σ, where the induced metric is given by $g = e^{2u}|dz|^2$. We have three holomorphic objets globally defined in Σ, $\Theta(z) = f(z)dz^3$, $\mathcal{X}(z) = k(z)\partial_z$ and $F(z) = f(z)k(z)^3$.

Firstly, we have that $(e^{2u}k)_z = 2g(\nabla^{\perp}_{\partial_z} H, J\partial_{\overline{z}})$, using (6) and the definition of $\mathcal{X}$. As JH is a closed vector field on Σ (in fact, its dual 1-form is —up to constants— the well-known Maslov form), it is easy to conclude from the latter equality that

$$\Im((e^{2u}k)_z) = 0. \tag{12}$$

If we suppose from now on that *the function F is a non-null constant*, in particular k has no zeroes and we can normalize it as $k(z) \equiv \eta > 0$ and it is clear that we can write also $f(z) \equiv \mu e^{i\alpha}$, $\mu > 0$, $\alpha \in [0, 2\pi[$.

Then (12) says that $u_y = 0$, and so $u(x, y) = u(x)$; using now (9) and the Gauss equation of ϕ in the equality $\triangle_0 u = -e^{2u}K$, with $\triangle_0$ the Laplacian of the Euclidean metric $|dz|^2$, we obtain that u satisfies the o.d.e.

$$u'' + e^{2u} + \frac{\eta^2 e^{4u} - \mu^2 e^{-4u}}{2} = 0. \tag{13}$$

But we can reduce the two constant η and μ to only one λ, because if $u(x)$ is a solution to (13) then $u((\eta/\mu)^{1/4}x) + \log(\eta/\mu)/4$ is a solution to

$$u'' + e^{2u} + \lambda^2 \sinh 4u = 0, \tag{14}$$

with $\lambda = (\eta^3\mu)^{1/4}$. In particular, $F(z) = \eta^3\mu e^{i\alpha} = \lambda^4 e^{i\alpha}$. In this way we have completed a local description of the twistor harmonic Lagrangian surfaces verifying that F is a non-null constant.

In order to study the integrability conditions of the Frenet equations of a local horizontal lift ψ of such an immersion ϕ to $\mathbb{S}^5(1)$ (see (5) in §3), we can rewrite them as

$$X_z = AX + BJX, \qquad X_{\overline{z}} = \widehat{A}X + B^*JX, \tag{15}$$

where

$$X = \begin{pmatrix} \psi_z \\ \psi_{\overline{z}} \\ \psi \end{pmatrix}, \quad A = \begin{pmatrix} u' & 0 & 0 \\ 0 & 0 & -e^{2u}/2 \\ 1 & 0 & 0 \end{pmatrix},$$

$$\widehat{A} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & u' & 0 \\ 0 & 1 & 0 \end{pmatrix}, \quad B = \frac{\lambda e^{2u}}{2} \begin{pmatrix} 1 & e^{i\alpha}e^{-4u} & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

denoting B^* the conjugate transpose of B. In this way, the integrability conditions for (14) are given by the differential equations

$$\begin{aligned} A_{\overline{z}} - \widehat{A}_z + \left[A, \widehat{A}\right] + [B^*, B] &= 0, \\ B_{\overline{z}} - B^*_z + [A, B^*] + \left[B, \widehat{A}\right] &= 0. \end{aligned} \tag{16}$$

Now it is not difficult to see that (16) are equivalent to the condition that $u = u(x)$ satisfied the o.d.e. (14).

We also notice that it is sufficient to consider $\alpha \in [0, \pi]$ because if one takes the immersion corresponding to $-\alpha$ and write its Frenet equations in the coordinate $\overline{z} = x - iy$, one obtains exactly the same Frenet equations.

Finally, using the definition of the holomorphic vector field $\mathcal{X}$ (see §3), this is written as $\mathcal{X}(z) = \lambda\partial_z$ in the above coordinate system. Then JH can be written as $JH = -\lambda\partial_x$ and it is not difficult to obtain that $\nabla^{\perp}_V H = -\lambda u' JV$, for any tangent vector V at $(x, y) \in \mathbb{R}^2$. Thus, the immersion ϕ_α has parallel mean curvature vector if and only if u is constant.

We comment that in general the function F is of exponential type around each point where H does not vanish, but in this case we do not control the solutions to the corresponding o.d.e. of u.

4 The classification.

According to the Theorem 2, we need to control the solutions of (11) to obtain qualitative properties of the immersions given there. The solutions of equation (11) are known since they can be expressed in terms of a certain class of elliptic functions. As they are periodic, we can consider —without loss of generality— only those solutions $u(x)$ such that $u'(0) = 0$. In the Appendix, we integrate explicitely the problem

$$u'' + e^{2u} + \lambda^2 \sinh 4u = 0,\ \lambda \in (0, +\infty);\ a = e^{2u(0)},\ u'(0) = 0. \tag{17}$$

Nevertheless, we collect in the following lemma some useful properties of the solutions to (17).

Lemma 1 *Let $u = u(x, a)$ be the solution of (17). Then:*

(i) *If $a_0(\lambda)$ is the unique positive solution of the equation $\lambda^2 a^4 + 2a^3 - \lambda^2 = 0$, then $(1/2)\log a_0(\lambda)$ is the unique constant solution of (17).*

(ii) *There exists $T > 0$, depending on λ and the initial condition a, verifying:*

$$u(x + 2T) = u(x) = u(-x),\ u(T - x) = u(T + x),\ \forall\, x \in \mathbb{R}.$$

(iii) $u'(x) = 0 \Leftrightarrow x = nT$, $n \in \mathbb{Z}$

(iv) *For each $b < a_0(\lambda)$, there exists an unique $a > a_0(\lambda)$, such that $u(x,b) = u(x+T,a)$. So there is no restriction if we only consider those solutions of (17) with initial condition $a \geq a_0(\lambda)$.*

We will denote by $\phi^\lambda_{a,\alpha} = \Pi \circ \psi^\lambda_{a,\alpha}$ the three-parameter family of immersions ϕ_α of complete surfaces given in Theorem 2 associated to the solution $u(x,a)$ of (17). Observe that $\lambda > 0$ determines the equation of (17), the initial condition $a \geq a_0(\lambda)$ fixes the solution of (17) and α ranges in $[0,\pi]$.

So, the set of twistor harmonic Lagrangian immersions $\phi^\lambda_{a,\alpha}$, $\lambda > 0$, can be parametrized by the subset $\Lambda(\lambda)$, $\lambda > 0$, of $\mathbb{R}^2$ defined by

$$\Lambda(\lambda) = [a_0(\lambda), +\infty[\times[0,\pi],\ \lambda \in (0,+\infty).$$

We notice that the family $\phi^\lambda_{a_0(\lambda),\alpha}$ of twistor harmonic immersions corresponding to the constant solution to (17) (see Lemma 1(i)), all of them with parallel mean curvature vector (see Theorem 2), is in fact a two-parameter family. They are just the universal coverings of the Lagrangian tori of Naitoh and Takeuchi (cf. [10]). So, we pay our attention to the immersions $\phi^\lambda_{a,\alpha} = \Pi \circ \psi^\lambda_{a,\alpha}$, with

$$(a,\alpha) \in \Gamma(\lambda) =]a_0(\lambda), +\infty[\times[0,\pi],\ \lambda \in (0,+\infty).$$

The following step consists of integrating the Frenet equations of the lifts $\psi^\lambda_{a,\alpha}$ (that for the sake of simplicity we call ψ), which can be rewritten from (15) as:

$$\psi_{xx} = u'\psi_x + \tfrac{\lambda}{2}(3e^{2u} + \cos\alpha\, e^{-2u})J\psi_x -$$

$$-\tfrac{\lambda}{2}\sin\alpha\, e^{-2u}J\psi_y - e^{2u}\psi,$$

$$\psi_{xy} = u'\psi_y - \tfrac{\lambda}{2}\sin\alpha\, e^{-2u}J\psi_x + \tfrac{\lambda}{2}(e^{2u} - \cos\alpha\, e^{-2u})J\psi_y,$$

$$\psi_{yy} = -u'\psi_x + \tfrac{\lambda}{2}(e^{2u} - \cos\alpha\, e^{-2u})J\psi_x +$$

$$+\tfrac{\lambda}{2}\sin\alpha\, e^{-2u}J\psi_y - e^{2u}\psi.$$

In the next result, we obtain explicitly the immersions $\psi^\lambda_{a,\alpha}$, $(a,\alpha) \in \Gamma(\lambda)$, $\lambda \in (0,+\infty)$.

Theorem 3 *In suitable coordinate systems of $\mathbb{C}^3$, the inmersions $\psi^\lambda_{a,\alpha} : (\mathbb{R}^2, e^{2u(x)}|dz|^2) \longrightarrow \mathbb{S}^5(1)$, $(a,\alpha) \in \Gamma(\lambda)$, $\lambda \in (0.+\infty)$, are given by*

$$\psi^\lambda_{a,\alpha}(x,y) = \Big(f_1(x)e^{i(G_1(x)+\sqrt{r_1}y)}, f_2(x)e^{i(G_2(x)-\sqrt{r_2}y)}, f_3(x)e^{i(G_3(x)-\sqrt{r_3}y)}\Big)$$

with

$$f_j(x) = \left(\frac{e^{2u(x)} + r_j - L}{3r_j - L}\right)^{\frac{1}{2}} \text{ and } G_j(x) = \frac{\lambda}{2}\int_0^x \frac{e^{4u(s)} - \cos\alpha}{e^{2u(s)} + r_j - L} ds,$$

$j = 1, 2, 3$, *being*

$$L = a + \frac{\lambda^2}{4}\left(a^2 + \frac{1}{a^2}\right) - \frac{\lambda^2 \cos\alpha}{2},$$

and where $-r_1 \leq -r_2 < -r_3 \leq 0$, $r_j = r_j(\lambda, a, \alpha)$, $j = 1, 2, 3$, *are the roots of the third degree polynomial* $r^3 + 2Lr^2 + L^2 r + (\lambda^2 \sin^2\alpha)/4$.

Consequently, the immersions $\phi^\lambda_{a,\alpha}$, $(a, \alpha) \in \Gamma(\lambda)$, $\lambda \in (0, +\infty)$, *are invariant by the one-parameter group of holomorphic isometries of* $\mathbb{CP}^2$ *induced, via* Π, *by*

$$\{\mathrm{diag}(e^{i\sqrt{r_1}t}, e^{-i\sqrt{r_2}t}, e^{-i\sqrt{r_3}t}) \,/\, t \in \mathbb{R}\}.$$

Moreover, if $\alpha = 0, \pi$:

$$\psi^\lambda_{a,\alpha}(x, y + 2\pi/\sqrt{L}) = \psi^\lambda_{a,\alpha}(x, y), \ \forall\, (x, y) \in \mathbb{R}^2,$$

where now $L = a + \lambda^2(a^2 + 1/a^2)/4 \mp \lambda^2/2$, *respectively according to* $\alpha = 0$ *or* $\alpha = \pi$.

Proof: Multiplication by $2u'$ in the equation of (17) and integration yields to

$$(u')^2 + e^{2u} + \frac{\lambda^2 \cosh 4u}{2} = A, \tag{18}$$

where A is a constant determined by the initial conditions of (17). In this case:

$$A = a + \frac{\lambda^2}{4}\left(a^2 + \frac{1}{a^2}\right). \tag{19}$$

From (18) we obtain:

$$(\psi_{yy} + L\psi)_y = -\frac{\lambda \sin\alpha}{2} J\psi, \tag{20}$$

where

$$L = A - \frac{\lambda^2 \cos\alpha}{2}.$$

Studying L as a function of λ, it is easy to conclude that $L > 0$ for $(a, \alpha) \in \Gamma(\lambda)$, $\lambda \in (0, +\infty)$.

Taking derivatives with respect to y in (21), we get that $\psi(-,y)$ satisfies the following linear differential equation

$$\frac{\partial^6\psi}{\partial y^6}+2L\frac{\partial^4\psi}{\partial y^4}+L^2\frac{\partial^2\psi}{\partial y^2}+\frac{\lambda^2\sin^2\alpha}{4}\psi=0. \tag{21}$$

Solving (22) when $\alpha\neq 0,\pi$ and (21) when $\alpha=0,\pi$, we obtain that

$$\psi(x,y)=\sum_{j=1}^{3}\{\cos(\sqrt{r_j}y)C_j(x)+\sin(\sqrt{r_j}y)D_j(x)\}, \tag{22}$$

for certain curves $C_j(x)$, $D_j(x)$, $j=1,2,3$ in $\mathbb{C}^3$, where $-r_1\leq -r_2<-r_3\leq 0$ are the roots of $r^3+2Lr^2+L^2r+\lambda^2\sin^2\alpha/4$. As $-r_1<-L<-r_2$ when $\alpha\neq 0,\pi$ (see Appendix, Lemma 2), using (21) in (23) we obtain that $D_1=JC_1$ y $D_i=-JC_i$, $i=2,3$, which is trivial for $\alpha=0,\pi$. So we get the following expression for the immersion ψ:

$$\begin{aligned}\psi(x,y) &= \cos(\sqrt{r_1}y)C_1(x)+\sin(\sqrt{r_1}y)JC_1(x) \\ &+ \sum_{i=2}^{3}\{\cos(\sqrt{r_i}y)C_i(x)-\sin(\sqrt{r_i}y)JC_i(x)\}.\end{aligned} \tag{23}$$

Using now in (24) that ψ is horizontal and conformal, it is not difficult to see that (see Lemma 2 in the Appendix)

$$|C_j(x)|^2=\frac{e^{2u(x)}+r_j-L}{3r_j-L},\ \langle C_k,C_l\rangle=\langle C_k,JC_l\rangle=0, \tag{24}$$

$j,k,l=1,2,3$, $k\neq l$. Now, using (24) and (25) in the Frenet equations (18) of ψ, it is straightforward to check that $C_j(x)$, $j=1,2,3$, satisfy the following o.d.e.

$$\begin{aligned}(e^{2u(x)}+r_j-L)C_j'(x) &= \\ = u'(x)e^{2u(x)}C_j(x)+\frac{\lambda}{2}\left(e^{4u(x)}-\cos\alpha\right)JC_j(x).\end{aligned} \tag{25}$$

From Lemma 3 in the Appendix, we deduce that $C_j(x)$ has no zeroes except in two particular cases. Integrating then (26) we get

$$C_j(x)=f_j(x)e^{iG_j(x)},\ j=1,2,3, \tag{26}$$

in the complex plane $\Pi_j=\mathrm{span}\{e_j=C_j(0)/|C_j(0)|,Je_j\}$, $j=1,2,3$. Putting (27) in (24) we conclude the result in the general case. The expressions so obtained are also valid in the two particular cases mentioned above (see Remark 4 in Appendix).

To establish which immersions $\phi^{\lambda}_{a,\alpha}$, $(a,\alpha)\in\Gamma(\lambda)$, $\lambda\in(0,+\infty)$, are doubly periodic with respect to some lattice of $\mathbb{R}^2$, and consequently define immersions of a torus, we consider the following subset $\Gamma_*(\lambda)$ of $\Gamma(\lambda)$:

$$\Gamma_*(\lambda)=\{(a,\alpha)\in\Gamma(\lambda)\,/\,\frac{1}{\sqrt{r_2}}\left(\sqrt{r_3},\frac{\Delta}{2\pi}\right)\in\mathbb{Q}\times\mathbb{Q}\},\ \lambda\in(0,+\infty),$$

with $\mathbb{Q}$ the field of rational numbers and Δ given by

$$\Delta=\begin{vmatrix}1 & G_1(2T) & \sqrt{r_1}\\ 1 & G_2(2T) & -\sqrt{r_2}\\ 1 & G_3(2T) & -\sqrt{r_3}\end{vmatrix}, \tag{27}$$

where $2T$ is the period of $u(x,a)$ (see Lemma 1(ii)) and r_j, $G_j(x)$, $j=1,2,3$ were defined in Theorem 3. It is not difficult to check that Δ is non-null making use of Lemma 2 in the Appendix.

From now on we write simply G_j instead $G_j(2T)$, $j=1,2,3$, for simplicity.

If $\alpha=0$ or $\alpha=\pi$, then $r_3=0$ and $r_1=r_2=L$ (see Lemma 2(a)–(c) in the Appendix) and the conditions defining $\Gamma_*(\lambda)$ become a few easier. It is expected, although complicated to prove, that the sets $\{a\,/\,(a,0)\in\Gamma_*(\lambda)\}$ and $\{a\,/\,(a,\pi)\in\Gamma_*(\lambda)\}$, $\lambda>0$, are dense in the interval $]a_0(\lambda),+\infty[$.

Theorem 4 *The immersion $\phi^{\lambda}_{a,\alpha}:\mathbb{R}^2\longrightarrow\mathbb{CP}^2$, $(a,\alpha)\in\Gamma(\lambda)$, $\lambda\in(0,+\infty)$, is doubly-periodic with respect to some lattice of $\mathbb{R}^2$ if and only if $(a,\alpha)\in\Gamma_*(\lambda)$, $\lambda\in(0,+\infty)$.*

Proof: We will use the notation $\phi_{a,\alpha}\equiv\phi^{\lambda}_{a,\alpha}$ and $\psi_{a,\alpha}\equiv\psi^{\lambda}_{a,\alpha}$.

First, we characterize the periodicity of $\phi_{a,\alpha}$. If we suppose that $\phi_{a,\alpha}$ is periodic, then there exists a non-null vector $(\mu,\eta)\in\mathbb{R}^2$ such that

$$\phi_{a,\alpha}(x+\mu,y+\eta)=\phi_{a,\alpha}(x,y) \tag{28}$$

$\forall\,(x,y)\in\mathbb{R}^2$. In particular, as $\phi_{a,\alpha}$ is a conformal map, $u(x+\mu)=u(x)$, $\forall\,x\in\mathbb{R}$. So, from Lemma 1(ii), there exists $m\in\mathbb{Z}$ such that $\mu=2mT$.

Also, as $\phi_{a,\alpha}=\Pi\circ\psi_{a,\alpha}$, then (29) implies that there exists a smooth function $\theta:\mathbb{R}^2\longrightarrow\mathbb{R}$ such that

$$\psi_{a,\alpha}(x+2mT,y+\eta)=e^{i\theta(x,y)}\psi_{a,\alpha}(x,y) \tag{29}$$

$\forall\,(x,y)\in\mathbb{R}^2$. Taking derivatives in (30) and using that $\psi_{a,\alpha}$ is also a conformal map we get that θ is a constant function, which we will denote by θ too.

Using Lemma 1(ii) it is easy to see that the functions $G_j(x)$ verify

$$G_j(x+2mT)=G_j(x)+mG_j(2T),\ m\in\mathbb{Z},\ j=1,2,3. \tag{30}$$

Now, using Theorem 3, (31) and Lemma 2(v) of the Appendix in (30) we obtain that

$$\begin{aligned} e^{i\theta}e^{imG_1}e^{i\sqrt{r_1}\eta} &= 1, \\ e^{i\theta}e^{imG_i}e^{-i\sqrt{r_i}\eta} &= 1,\ i=2,3. \end{aligned}$$

From here we can conclude that $\phi_{a,\alpha}$ *is periodic with respect to* $(2mT,\eta)$ *if and only if there exists* $(m,\eta)\in\mathbb{Z}\times\mathbb{R}-\{(0,0)\}$ *and* $\theta\in\mathbb{R}$ *solving the following linear equations*

$$\begin{aligned} \theta+G_1m+\sqrt{r_1}\eta &= 2\pi h_1, \\ \theta+G_2m-\sqrt{r_2}\eta &= 2\pi h_2, \\ \theta+G_3m-\sqrt{r_3}\eta &= 2\pi h_3, \end{aligned} \tag{31}$$

for certain $h_1,h_2,h_3\in\mathbb{Z}$*, fixed but arbitrary.*

Firstly, (32) has only one solution because Δ (see (28)) is just the determinant of the coefficient matrix of (32). In particular:

$$m=\frac{2\pi}{\Delta}\begin{vmatrix} 1 & h_1 & \sqrt{r_1} \\ 1 & h_2 & -\sqrt{r_2} \\ 1 & h_3 & -\sqrt{r_3} \end{vmatrix}. \tag{32}$$

Now we can prove the result. Suppose $\phi_{a,\alpha}$ is doubly-periodic with respect to the lattice spanned by $\{(2m_1T,\eta_1),(2m_2T,\eta_2)\}$ with $m_k\in\mathbb{Z}$, $k=1,2$. Then it is easy to see that $\xi=m_1\eta_2-m_2\eta_1$ is a non-null real number such that $\phi_{a,\alpha}(x,y+\xi)=\phi_{a,\alpha}(x,y)$, $\forall\,(x,y)\in\mathbb{R}^2$. Then using the explicit expressions given in Theorem 3 it is not difficult to conclude that $(r_3/r_2)^{1/2}$ is a rational number.

We now have two solutions in $\mathbb{Z}$, m_1 and m_2 for m, no both zero, that put in (33) yield to $2\pi\sqrt{r_2}/\Delta$ is also a rational number. Thus we have proved that $(a,\alpha)\in\Gamma_*(\lambda)$, $\lambda\in(0,+\infty)$.

Conversely, if $(a,\alpha)\in\Gamma_*(\lambda)$, $\lambda>0$, let $m_i,n_i\in\mathbb{Z}$ with $(m_i,n_i)=1$, $i=1,2$ such that $(r_3/r_2)^{1/2}=q=m_1/n_1$ and $\Delta/(2\pi\sqrt{r_2})=\hat{q}=m_2/n_2$. Then it is straightforward to check that $\phi_{a,\alpha}$ is doubly-periodic respect to the lattice spanned by $\{(0,2n_1\pi/\sqrt{r_2}),(2\pi s,\eta_0)\}$, where s is given by $(\hat{q}-3q)/(2+q)=r/s$, with $(r,s)=1$, and η_0 is the corresponding solution to η in (32) when we take $h_1=s$, $h_2=-s$ and $h_3=r$. This finishes the proof.

We conclude with the hoped result of classification.

Corollary 1 *Let* $\phi:\Sigma\longrightarrow\mathbb{CP}^2$ *be a twistor harmonic Lagrangian immersion of a torus in the complex projective plane. Then* ϕ *is minimal or the universal covering of* ϕ *is congruent to* $\phi^{\lambda}_{a,\alpha}$ *with* $(a,\alpha)\in\Gamma_*(\lambda)$, $\lambda\in(0,+\infty)$.

5 Appendix.

In this section we first proceed to integrate the problem (17). Making the change of variable $y = e^{2u}$, from (19) we arrive at

$$(y')^2 + \lambda^2 y^4 + 4y^3 - 4Ay^2 + \lambda^2 = 0. \tag{33}$$

If

$$F(\varphi) = F(\varphi, p) = \int_0^{\varphi} \frac{d\theta}{\sqrt{1 - p^2 \sin^2 \theta}}, \ 0 \le p \le 1,$$

is the elliptic integral of the first kind, then denoting the inverse of $F(\varphi)$ by $\mathrm{am}(x, p) = \varphi$, the elementary Jacobi elliptic functions are given by

$$\mathrm{sn}\, x = \mathrm{sn}(x, p) = \sin \varphi, \quad \mathrm{cn}\, x = \mathrm{cn}(x, p) = \cos \varphi$$

and

$$\mathrm{dn}\, x = \mathrm{dn}(x, p) = \sqrt{1 - p^2 \sin^2 \varphi}.$$

The basic properties of these functions (cf. [1]) are

$$\begin{gathered} \mathrm{sn}^2(x, p) + \mathrm{cn}^2(x, p) = 1 = p^2 \mathrm{sn}^2(x, p) + \mathrm{dn}^2(x, p), \\ \mathrm{sn}(x + 2K) = -\mathrm{sn}\, x, \quad \mathrm{cn}(x + 2K) = -\mathrm{cn}\, x, \\ \mathrm{dn}(x + 2K) = \mathrm{dn}\, x, \end{gathered}$$

where $K = F(\pi/2)$ is the complete elliptic integral of the first kind.

Returning to equation (34), we can rewrite it in the form

$$(y')^2 + \lambda^2 (y - a)(y - a_1)(y + a_2)(y + a_3) = 0$$

where $-a_3 < -a_2 < 0 < a_1 (\le a)$ are the roots of the cubic $\lambda^2 y^3 + (\lambda^2 a + 4)y^2 - (\lambda^2/a^2)y - \lambda^2/a = 0$; its solution when $a \ge a_0(\lambda)$ (see Lemma 1(i)) is given by

$$y = e^{2u(x)} = a \frac{1 - q^2 \mathrm{sn}^2(rx, p)}{1 + s^2 \mathrm{sn}^2(rx, p)}, \tag{34}$$

with

$$\begin{gathered} p^2 = \frac{(a - a_1)(a_3 - a_2)}{(a + a_2)(a_1 + a_3)}, \ q^2 = \frac{a_3(a - a_1)}{a(a_1 + a_3)}, \\ r = (\lambda/2)\sqrt{(a + a_2)(a_1 + a_3)}, \ s^2 = \frac{a - a_1}{a_1 + a_3} \end{gathered} \tag{35}$$

(for background on the solutions of such equations, see [5]).

Thus, there exists a positive real number T ($T = K/r$), such that Lemma 1 holds. We remark that $a_1 = e^{2u(T)}$. If $a > a_0(\lambda)$,

$$a_1 \leq e^{2u(x)} \leq a, \tag{36}$$

$\forall\, x \in \mathbb{R}$, and the first equality holds (resp. the second one) if and only if $x = (2k+1)T$, $k \in \mathbb{Z}$ (resp. $x = 2kT$, $k \in \mathbb{Z}$).

We observe that, following the notation of Lemma 1, $u(0, a_1) = u(T, a) = (1/2)\log a_1$, and then it is not difficult to check that $u(0, a) \geq (1/2)\log a_0(\lambda) \Leftrightarrow u(0, a_1) \leq (1/2)\log a_0(\lambda)$.

The following lemmas are used in the proofs of Theorem 3 and Theorem 4 in §4.

Lemma 2 *The cubic polynomial $r^3 + 2Lr^2 + L^2 r + \lambda^2 \sin^2\alpha/4$, with $(a, \alpha) \in \Gamma(\lambda)$, $\lambda \in (0, +\infty)$, has three non positive real roots $-r_j = -r_j(\lambda, a, \alpha)$, $j = 1, 2, 3$, satisfying:*

(a) *$L \leq r_1 < 4L/3$, and the equality holds if and only if $\alpha = 0, \pi$.*

(b) *$L - \hat{a} \leq r_2 \leq L$, with $\hat{a}$ the unique positive real number such that $A(a) = A(\hat{a})$, $\hat{a} < a$ (see (20)), and the first equality holds (resp. the second one) if and only if $\cos\alpha = \hat{a}^2$ (resp. $\alpha = 0, \pi$).*

(c) *$0 \leq r_3 \leq L - a$, and the first equality holds (resp. the second one) if and only if $\alpha = 0, \pi$ (resp. $a^2 \leq 1$ and $\cos\alpha = a^2$).*

(d) *$r_3 < L/3 < r_2$,*

(e) *$\sqrt{r_1} = \sqrt{r_2} + \sqrt{r_3}$.*

The proof of the lemma is a straightforward exercise.

Lemma 3 *A real number x is a zero of the function $e^{2u} + r_j - L$, $j = 1, 2, 3$, $(a, \alpha) \in \Gamma(\lambda)$, $\lambda \in (0, +\infty)$, if and only if either $j = 2$, $\cos\alpha = \hat{a}^2$ and $x = (2k+1)T$, $k \in \mathbb{Z}$, or $j = 3$, $a^2 \leq 1$, $\cos\alpha = a^2$ and $x = 2kT$, $k \in \mathbb{Z}$.*

Moreover, when $\cos\alpha = \hat{a}^2$:

$$e^{2u(x)} + r_2 - L = \frac{(a - \hat{a})\mathrm{cn}^2(rx)}{1 + s^2\mathrm{sn}^2(rx)};$$

and when $\cos\alpha = a^2$:

$$e^{2u(x)} + r_3 - L = \frac{(a + a_3)(\hat{a} - a)\mathrm{sn}^2(rx)}{(\hat{a} + a_3)(1 + s^2\mathrm{sn}^2(rx))},$$

Proof: Lemma 2 says that $e^{2u(x)} + r_1 - L$ is a positive function. We remark that $a_1 = \hat{a}$. From (38), we deduce that

$$\hat{a} + r_i - L \leq e^{2u(x)} + r_i - L \leq a + r_i - L,\ i = 2, 3,$$

and the first equality holds (resp. the second one) if and only if $x = (2k+1)T$, $k \in \mathbb{Z}$ (resp. $x = 2kT$, $k \in \mathbb{Z}$). Now, using Lemma 2(b)–(c) in the latter inequalities it is proved the first part of this Lemma.

For the exppressions given in the cases $\cos\alpha = \hat{a}^2$ and $\cos\alpha = a^2$ we must make use of (35), (36) and (37).

Remark 4 The integration of (26) in the particular cases $\cos\alpha = \hat{a}^2$ and $\cos\alpha = a^2$ for $C_2(x)$ and $C_3(x)$ yields to

$$\sqrt{\frac{a-\hat{a}}{2L-3\hat{a}}}\,\frac{\mathrm{cn}(rx)}{\sqrt{1+s^2\mathrm{sn}^2(r_\lambda x)}}\,e^{\frac{\lambda i}{2}\int_0^x (e^{2u(s)}+\hat{a})ds},$$

and

$$\sqrt{\frac{(a+a_3)(a-\hat{a})}{(\hat{a}+a_3)(2L-3\hat{a})}}\,\frac{\mathrm{sn}(rx)}{\sqrt{1+s^2\mathrm{sn}^2(rx)}}\,e^{\frac{\lambda i}{2}\int_0^x (e^{2u(s)}+a)ds},$$

respectively; both expressions are in agreement with $f_i(x)e^{iG_i(x)}$, $i = 2, 3$ in these cases.

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