# Twistor holomorphic Lagrangian surfaces in the complex projective and hyperbolic planes

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Abstract: We completely classify all the twistor holomorphic Lagrangian immersions in the complex projective plane $\mathbb{CP}^2$, i.e. those Lagrangian immersions such that their twistor liftings to the twistor space over $\mathbb{CP}^2$ are holomorphic. This classification provides a one-parameter family of new examples of Lagrangian spheres in $\mathbb{CP}^2$.

Key words: *Twistor holomorphic immersions, Lagrangian surfaces.*

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## 1 Introduction.

A minimal (non complex) immersion of an orientable surface $\Sigma$ in the complex projective plane $\mathbb{CP}^2$ is called superminimal ([2], [3]) if the holomorphic cubic differential $\Theta$ on $\Sigma$, naturally associated to the immersion (see §2), vanishes identically. These immersions can be characterized using twistorial constructions by the fact that their twistor liftings to the twistor space $SU(3)/T^3$ over $\mathbb{CP}^2$ are horizontal and so, in particular, they are holomorphic (see [3], [6]).

Using these ideas, Chern and Wolfson and Eells and Salamon ([2], [3]) showed that the family of superminimal surfaces in $\mathbb{CP}^2$ is very big, proving that any compact Riemann surface can be conformally immersed in $\mathbb{CP}^2$ as

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a superminimal surface. Properties and other characterizations of superminimal surfaces in $\mathbb{CP}^2$ can be found in [3], [4] and [6].

Friedrich ([6]) generalized this notion introducing the concept of twistor holomorphic immersions in $\mathbb{CP}^2$ as those immersions such that their twistor liftings to $SU(3)/T^3$ are holomorphic. In [6] it is implicitly proved that these immersions verify that the cubic differential $\Theta$ (which in general is not holomorphic) vanishes identically. Some specifical results for this family of surfaces can be found in [5] and [6].

In this paper, we make a contribution to the study of this family by classifying completely the twistor holomorphic immersions in $\mathbb{CP}^2$ which are Lagrangian —or totally real— (Theorem 1). This classification provides a one-parameter family of twistor holomorphic Lagrangian immersions of the two-sphere in $\mathbb{CP}^2$ (Proposition 2). In this family, only the superminimal immersion (i.e. the totally geodesic Lagrangian immersion) is known and the others are —from a Riemannian point of view— new examples of Lagrangian spheres in $\mathbb{CP}^2$. A characterization of these examples among all the Lagrangian spheres of $\mathbb{CP}^2$ is given in Corollary 1.

The used method is to lift locally the immersion to an horizontal immersion into $S^5(1)$ via the Hopf fibration of $S^5(1)$ over $\mathbb{CP}^2$, and to integrate completely the Frenet equations of this lifting.

In the appendix, we make an analogous study for surfaces in the complex hyperbolic plane $\mathbb{CH}^2$. We omit there the proofs because they are quite similar to the ones in the projective case. We obtain a bigger family of examples (Proposition 3) and the complete classification of the twistor holomorphic Lagrangian immersions in $\mathbb{CH}^2$ is given in Theorem 2.

The case of the complex Euclidean plane was studied in [1].

# 2 The examples.

Let $\mathbb{CP}^2$ be the two-dimensional complex projective space endowed with the Fubini–Study metric $g$ of constant holomorphic sectional curvature 4. If $J$ denotes the complex structure of $\mathbb{CP}^2$, then the Kähler two-form $\Omega$ on $\mathbb{CP}^2$ is given by $\Omega(X,Y) = g(X, JY)$, for any tangent vector fields $X$ and $Y$.

Given a point $x \in \mathbb{CP}^2$, let $P_x$ be the set of almost Hermitian structures $J_x$ over $T_x\mathbb{CP}^2$ such that, if $\omega(u,v) = g(J_x u, v)$, then $-\omega \wedge \omega$ is the orientation induced on $T_x\mathbb{CP}^2$ from $\mathbb{CP}^2$.

Then $\mathcal{P} = \bigcup_{x \in \mathbb{CP}^2} P_x$ is a $\mathbb{CP}^1$-fiber bundle over $\mathbb{CP}^2$ called the *twistor bundle* of $\mathbb{CP}^2$.

In [3], $\mathcal{P}$ is identified with $SU(3)/T^3$, where $T^3 = SU(1) \times SU(1) \times SU(1)$ and $SU(n)$ is the special unitary group of order n.

Let $\phi : \Sigma \longrightarrow \mathbb{CP}^2$ be an immersion of an oriented surface $\Sigma$ in $\mathbb{CP}^2$. The tangent space $T_{\phi(p)}\mathbb{CP}^2$, $p \in \Sigma$, splits into the sum of $d\phi_p(T_p\Sigma)$ and the normal space $T_p^\perp\Sigma$ as oriented linear spaces. The *twistor lifting* $\widetilde{\phi} : \Sigma \longrightarrow \mathcal{P}$ of the immersion $\phi : \Sigma \longrightarrow \mathbb{CP}^2$ is defined in the following way:

$\widetilde{\phi}(p) : T_{\phi(p)}\mathbb{CP}^2 \longrightarrow T_{\phi(p)}\mathbb{CP}^2$ is the rotation around the angle $\pi/2$ in the positive (negative) direction on $T_p\Sigma$ (on $T_p^\perp\Sigma$).

Then $\phi$ is called *twistor holomorphic* if its twistor lifting $\widetilde{\phi}$ is holomorphic (see [6]).

If $\sigma$ is the second fundamental form of $\phi$, we can define a cubic differential $\Theta$ on $\Sigma$ by

$$\Theta(z) = f(z)dz^3, \;\; f(z) = 4g(\sigma(\partial_z, \partial_z), J\phi_*\partial_z),$$

where $z = x + iy$ is a local isothermal coordinate on $\Sigma$ such that the induced metric by $\phi$ on $\Sigma$, which we also represented by $g$, is written as $e^{2u}|dz|^2$ with $|dz|^2$ the Euclidean metric, and $\sigma$, $J$ and $g$ are extended $\mathbb{C}$-linearly to the complexified bundles.

If $\{e_1, e_2, e_3, e_4\}$ is an oriented orthonormal local reference on $\phi^*\mathbb{CP}^2$ such that $\{e_1, e_2\}$ is an oriented reference on $T\Sigma$, we will denote $\sigma_{i\,j}^{\alpha} = g(\sigma(e_i, e_j), e_\alpha)$, $i, j \in \{1, 2\}$, $\alpha \in \{3, 4\}$. The Kaehler function, $\mathcal{C} : \Sigma \longrightarrow \mathbb{R}$, is a smooth function defined by $\mathcal{C} = g(Je_1, e_2)$. We say that $p \in \Sigma$ is a *complex* point if $\mathcal{C}(p)^2 = 1$.

In [6], Proposition 2, it is proved that an immersion $\phi$ is twistor holomorphic if and only if

$$\begin{aligned} \sigma_{1\,1}^3 + 2\sigma_{1\,2}^4 - \sigma_{2\,2}^3 &= 0, \\ \sigma_{2\,2}^4 + 2\sigma_{1\,2}^3 - \sigma_{1\,1}^4 &= 0. \end{aligned}$$

Taking $e_1 = e^{-u}\partial_x$, $e_2 = e^{-u}\partial_y$, it is straightforward to check that

$$|f|^2 = \frac{e^{6u}}{4}(1 - \mathcal{C}^2)[(\sigma_{1\,1}^3 + 2\sigma_{1\,2}^4 - \sigma_{2\,2}^3)^2 + (\sigma_{2\,2}^4 + 2\sigma_{1\,2}^3 - \sigma_{1\,1}^4)^2].$$

So, it is proved that if $\phi$ is a twistor holomorphic immersion then $\Theta \equiv 0$ and if $\Theta \equiv 0$ and the interior of the set of complex points of the immersion $\phi$ is empty, then $\phi$ is twistor holomorphic.

On the other hand, an immersion $\phi : \Sigma \longrightarrow \mathbb{CP}^2$ is called *Lagrangian* (or *totally real*) if $\phi^*\Omega = 0$, which is equivalent to that $\mathcal{C} = 0$. This means that $J$ defines an isomorphism from the tangent bundle of $\Sigma$ to the normal bundle of $\phi$. Since such an immersion has not complex points, we can conclude that for a Lagrangian immersion $\phi$,

*$\phi$ is twistor holomorphic if and only if $\Theta \equiv 0$.*

We define a trilinear form $C$ on $\Sigma$ by

$$C(X, Y, Z) = g(\sigma(X, Y), JZ)$$

where $X$, $Y$, $Z$ are tangent vector fields to $\Sigma$. The most elementary property of such a Lagrangian immersion $\phi$ is that $C$ is symmetric.

So, $\Theta$ can be written as $\Theta(z) = 4C(\partial_z, \partial_z, \partial_z)$. Moreover, it is easy to see that

$$|f|^2 = e^{6u}(|\sigma|^2 - 3|H|^2) = e^{6u}(|H|^2 + 2 - 2K), \tag{1}$$

where $K$ is the Gauss curvature of the induced metric and $|\sigma|$, $|H|$ are the lengths of the mean curvature vector and second fundamental form of $\phi$, respectively. Hence, we have the following result.

**Proposition 1** *Let $\phi : \Sigma \longrightarrow \mathbb{CP}^2$ be a Lagrangian immersion of an orientable surface $\Sigma$. Then*

$$|H|^2 + 2 \geq 2K, \tag{2}$$

*and the equality holds if and only if $\phi$ is twistor holomorphic.*

Now, let $\Pi : S^5(1) \longrightarrow \mathbb{CP}^2$ be the Hopf fibration. We denote by $\langle\, ,\, \rangle$ the Euclidean metric of $\mathbb{C}^3$ as well as that induces on $S^5(1)$, which becomes $\Pi$ in a Riemannian submersion. The complex structure of $\mathbb{C}^3$, which induces $J$ on $\mathbb{CP}^2$ via $\Pi$, will be also denoted by $J$.

Let us see that Lagrangian immersions in $\mathbb{CP}^2$ are —locally— projections by the Hopf fibration of horizontal immersions in $S^5(1)$.

Let $\psi : \Sigma \longrightarrow S^5(1)$ be an immersion. If $\omega$ is the 1-form on $\Sigma$ defined by

$$\omega(v) = \langle \psi_*(v), J\psi \rangle$$

with $v$ tangent to $\Sigma$, then it is easy to prove that $d\omega(v, w) = 2\langle \psi_*(w), J\psi_*(v) \rangle$ with $v$ and $w$ tangent vectors to $\Sigma$. If $\psi$ is an horizontal immersion, as the vertical space of $\Pi$ at $p \in S^5(1)$ is $V_p$ =span$\{Jp\}$, then $\omega = 0$. Then $\phi = \Pi \circ \psi : \Sigma \longrightarrow \mathbb{CP}^2$ is a Lagrangian immersion because

$$0 = d\omega(v, w) = \langle \psi_*(w), J\psi_*(v) \rangle = g(\phi_*(w), J\phi_*(v)).$$

Conversely, if $\phi : \Sigma \longrightarrow \mathbb{CP}^2$ is a Lagrangian immersion of a surface $\Sigma$, let $\psi : U \longrightarrow S^5(1)$ be a local lifting to $S^5(1)$, where $U$ is a simply-connected open subset in $\Sigma$. As $\phi$ is a Lagrangian immersion, then $d\omega = 0$ (see above) and hence there exists $\eta \in C^\infty(U)$ such that $d\eta = \omega$. Now, $\widetilde{\psi} = e^{-i\eta}\psi$ is another local lifting of $\phi$ to $S^5(1)$ and it is not difficult to obtain that

$$\langle \widetilde{\psi}_*(v), J\widetilde{\psi} \rangle = -d\eta(v) + \omega(v) = 0,$$

for any tangent vector $v$ on $\Sigma$. Thus $\widetilde{\psi}$ is an horizontal local lifting of $\phi$ to $S^5(1)$.

Moreover, if $\widehat{\psi}$ is another horizontal local lifting to $S^5(1)$, then it can be proved that there exists $\theta \in \mathbb{R}$ such that $\widehat{\psi} = e^{i\theta}\psi$. So, up to rotations of $S^5(1)$, the horizontal local lifting $\psi$ of $\phi$ to $S^5(1)$ is unique and we observe that $g$ is also the induced metric on $\Sigma$ by $\psi$ and $\psi$ is a totally real immersion in $\mathbb{C}^3$ with respect to the complex structure $J$.

Next we are going to define a 1-parameter family of Lagrangian spheres immersed in $\mathbb{CP}^2$, which is a deformation by Lagrangian immersions of the totally geodesic Lagrangian immersion of $S^2(1)$ in $\mathbb{CP}^2$.

Let $S^2 = \{(x, y, z) \in \mathbb{R}^3;\ x^2 + y^2 + z^2 = 1\}$ be the two-dimensional unit sphere. For $t \in [0, \infty[$, we define $\psi_t : S^2 \longrightarrow S^5(1)$ by

$$\psi_t(x, y, z) = \frac{1}{c_t^2 + s_t^2 z^2} \Big( c_t x, s_t xz, c_t y, s_t yz, z, s_t c_t (1 + z^2) \Big)$$

with $c_t = \cosh t$, $s_t = \sinh t$ and where $\mathbb{C}^3$ is identified to $\mathbb{R}^6$ by $\mathbb{C} \equiv \mathbb{R} \oplus i\mathbb{R}$. It is easy to check that $\psi_t$ is an horizontal embedding.

**Proposition 2** *For* $t \in [0, \infty[$, $\phi_t = \Pi \circ \psi_t : S^2 \longrightarrow \mathbb{CP}^2$ *satisfies the following properties:*

(a) $\phi_t$ *is a twistor holomorphic Lagrangian immersion.*

(b) *If* $t \neq 0$, *denoting by* $N$ *and* $S$ *the poles of* $S^2$, *then* $\phi_t : S^2 - \{N, S\} \longrightarrow \mathbb{CP}^2$ *is an embedding.*

(c) *The area of* $(S^2, \phi_t^* g)$ *is* $8\pi \arctan(\tanh t)/\sinh 2t$.

(d) *The Gauss curvature* $K_t$ *of* $(S^2, \phi_t^* g)$ *verifies* $1 \leq K_t \leq 1 + 2\sinh^2 t$. *Moreover,* $K_t(x, y, z) = 1$ *(respectively* $K_t(x, y, z) = 1 + 2\sinh^2 t$*) if and only if* $z = \pm 1$ *(respectively* $z = 0$*).*

(e) $\phi_t$ *is invariant by the one-parameter group of holomorphic isometries of* $\mathbb{C}P^2$ *induced, via* $\Pi$*, by:*

$$\left\{ \begin{pmatrix} \cos s & -\sin s & 0 \\ \sin s & \cos s & 0 \\ 0 & 0 & 1 \end{pmatrix} ; \; s \in \mathbb{R} \right\}.$$

The proof is straightforward using that the stereographic projections from the poles of $S^2$ give conformal parametrizations of $\phi_t$.

We observe that $\phi_0$ is the totally geodesic Lagrangian immersion of $S^2$ in $\mathbb{C}P^2$. From (1), it is the only superminimal Lagrangian surface of $\mathbb{C}P^2$.

# 3 Statement of the result.

In the following result, we are going to classify completely the twistor holomorphic Lagrangian surfaces of $\mathbb{C}P^2$.

**Theorem 1** *Let* $\phi : \Sigma \longrightarrow \mathbb{C}P^2$ *be a twistor holomorphic immersion of an orientable surface* $\Sigma$*. Then* $\phi$ *is Lagrangian if and only if* $\phi(\Sigma)$ *is an open set of* $\phi_t(S^2)$ *for some* $t \in [0, \infty[$.

**Corollary 1** *Let* $\phi : \Sigma \longrightarrow \mathbb{C}P^2$ *be a Lagrangian immersion of a sphere* $\Sigma$*. Then*

$$\int_\Sigma (|H|^2 + 2) dA \geq 8\pi,$$

*and the equality holds if and only if* $\phi$ *is congruent to* $\phi_t$ *for some* $t \in [0, \infty[$.

*Proof of the Theorem:* From Proposition 2, we have only to prove the necessary condition. In this way, from Proposition 1 we suppose that $\Theta \equiv 0$. Then we define a complex vector field $\chi$ on $\Sigma$ by

$$\chi(z) = h(z)\partial_z, \text{ with } h(z) = 4e^{-4u} C(\partial_{\bar{z}}, \partial_z, \partial_{\bar{z}})$$

where $z = x + iy$ is a local isothermal coordinate on $\Sigma$ and $g = e^{2u}|dz|^2$.

Now, using the symmetry of $C$ and that $\Theta \equiv 0$, it is not difficult to prove that

$$h_{\overline{z}} = 0 \text{ and } \Im(h_z + 2u_z h) = 0. \tag{3}$$

In particular, $\chi$ defines a holomorphic vector field on $\Sigma$. So, either $\chi$ vanishes identically or it has isolated zeroes. In the first case, as $|h|^2 = e^{-2u}|H|^2$, from (1) and the Gauss equation it follows

that $\phi$ is totally geodesic and then $\phi(\Sigma)$ is an open set of $\phi_0(S^2)$.

If $\chi$ has only isolated zeroes, we consider

$$\Sigma' = \{p \in \Sigma;\ \chi(p) \neq 0\}$$

which is connected too. For any point $p \in \Sigma'$, let $(U, z = x + iy)$ be a local isothermal coordinate centred at $p$, where $U$ is a simply-connected open subset of $\Sigma'$. As $h(z)$ has not zeroes on $U$, we can normalize to get $h(z) \equiv 1$. From (3) we conclude that $u_y = 0$ and then $u(z) = u(x)$.

Using now the Gauss equation in the equality $\triangle_0 u = -e^{2u}K$, with $\triangle_0$ the Laplacian of the Euclidean metric $|dz|^2$, we see that $u$ satisfies the O.D.E.

$$u'' + e^{2u} + \frac{e^{4u}}{2} = 0. \tag{4}$$

Without loss of generality, we can consider only those solutions of (4) with $u'(0) = 0$. In this case, the general solution to (4) is

$$u(x) = \frac{1}{2}\log\frac{2\sinh^2 2t}{1 + \cosh 2t \cosh(2x \sinh 2t)} \tag{5}$$

which satisfies $u'(0) = 0$ and $e^{u(0)} = 2\sinh t$.

So there exists $t \in ]0, \infty[$ such that $u(x)$ is given in (5).

If $\psi : \widehat{U} \longrightarrow S^5(1)$, $\widehat{U} \subset U$, is the horizontal local lifting of $\phi$ to $S^5(1)$ (see §2), using that $\psi$ is also a conformal immersion, it is not difficult to see that the Frenet equations of $\psi$ are given by

$$\begin{aligned}
\psi_{xx} &= u'\psi_x + \frac{3e^{2u}}{2}J\psi_x - e^{2u}\psi, \\
\psi_{xy} &= u'\psi_y + \frac{e^{2u}}{2}J\psi_y, \\
\psi_{yy} &= -u'\psi_x + \frac{e^{2u}}{2}J\psi_x - e^{2u}\psi.
\end{aligned} \tag{6}$$

From (5) and (6) it is easy to get that $\psi(-,y)$ satisfies the following linear differential equation

$$\frac{\partial^3\psi}{\partial y^3}+\sinh^2 2t\frac{\partial\psi}{\partial y}=0$$

and so

$$\psi(x,y)=\cos(y\sinh 2t)C_1(x)+\sin(y\sinh 2t)C_2(x)+C_3(x), \tag{7}$$

for certain curves $C_j(x)$, $j=1,2,3$, in $\mathbb{C}^3$. Using in (7) that $\psi$ is an horizontal and conformal immersion, it is straightforward to obtain that

$$\begin{aligned} |C_i(x)|^2 &= \frac{e^{2u(x)}}{\sinh^2 2t},\ i=1,2,\\ |C_3(x)|^2 &= \frac{\sinh^2 2t-e^{2u(x)}}{\sinh^2 2t},\\ \langle C_i,C_j\rangle=\langle C_i,JC_j\rangle &= 0,\ i,j=1,2,3,\ i\neq j. \end{aligned} \tag{8}$$

Now, using (7) in the Frenet equations (6) it is not difficult to check that $C_j(x)$, $j=1,2,3$, satisfy the following O.D.E.

$$\begin{aligned} C_i' &= u'C_i+\frac{e^{2u}}{2}JC_i,\ i=1,2,\\ C_3' &= \frac{u'e^{2u}}{e^{2u}-\sinh^2 2t}C_3+\frac{e^{4u}}{2(e^{2u}-\sinh^2 2t)}JC_3. \end{aligned}$$

The integration of these, using (8), leads to

$$C_i(x)=\frac{e^{u(x)}}{\sinh 2t}e^{iG_(x)/2},\ i=1,2, \tag{9}$$

in the complex plane $\Pi_i=span\{e_i=C_i(0)/|C_i(0)|,Je_i\}$, $i=1,2$, where $G(x)=\int_0^x e^{2u(s)}ds$, and

$$C_3(x)=\frac{\sqrt{\sinh^2 2t-e^{2u(x)}}}{\sinh 2t}e^{i(\widehat{G}(x)+\pi)/2}, \tag{10}$$

in the complex plane $\Pi_3=span\{e_3=-JC_3(0)/|C_3(0)|,Je_3\}$, where $\widehat{G}(x)=\int_0^x(e^{4u(s)}/(e^{2u(s)}-\sinh^2 2t))ds$.

Then, from (5), (9) and (10), a straightforward computation proves that

$$C_i(x) = \frac{2\,(c_t \cosh(2s_tc_tx) + is_t \sinh(2s_tc_tx))}{1 + (c_t^2 + s_t^2)\cosh(4s_tc_tx)},\ i = 1,2,$$

and

$$C_3(x) = \frac{2s_tc_t \cosh(4s_tc_tx) - i\sinh(4s_tc_tx)}{1 + (c_t^2 + s_t^2)\cosh(4s_tc_tx)}$$

in the respective planes $\Pi_j$, $j = 1,2,3$, with $s_t = \sinh t$ and $c_t = \cosh t$. Putting them in (7) we arrive at

$$\begin{aligned} \psi(x,y) = \frac{1}{1 + (c_t^2 + s_t^2)\cosh(4s_tc_tx)} & \quad (2c_t \cosh(2s_tc_tx)\cos(2s_tc_ty), \\ 2s_t \sinh(2s_tc_tx)\cos(2s_tc_ty) & \ ,\ \ 2c_t \cosh(2s_tc_tx)\sin(2s_tc_ty), \\ 2s_t \sinh(2s_tc_tx)\sin(2s_tc_ty) & \ ,\ \ \sinh(4s_tc_tx)\ ,\ 2s_tc_t \cosh(4s_tc_tx)). \end{aligned}$$

Therefore, since $\psi(x, y+\pi/s_tc_t) = \psi(x,y)$, $\psi$ induces an immersion from an open set of $\mathbb{C}^*$ into $S^5(1)$ via the exponential map $e^{2s_tc_t\,z}$. By the stereographic projection, it is easy to see that the above immersion corresponds to the immersion $\psi_t$ and so $\psi(\widehat{U})$ is open in $\psi_t(S^2 - \{N,S\})$ with $N$ and $S$ the poles of $S^2$.

Since $\Sigma'$ is connected, we obtain that $\Pi \circ \psi(\Sigma') = \phi(\Sigma')$ is open in $\phi_t(S^2 - \{N,S\})$. Hence, $\phi(\Sigma)$ is open in $\phi_t(S^2)$. $\square$

# 4 Appendix.

Similar results to those obtained in $\mathbb{CP}^2$ can be stated for surfaces of the complex hyperbolic plane $\mathbb{CH}^2$. As the method used to prove these is the same that we used for $\mathbb{CP}^2$, we will only show the results.

In this case, new families of examples of twistor holomorphic Lagrangian surfaces appear because the O.D.E. (4) has now the following form

$$u'' - e^{2u} + \frac{e^{4u}}{2} = 0, \tag{11}$$

giving different kinds of solutions depending on the initial conditions. In fact, the solution $u(x)$ of (11) satisfying $u'(0)=0$ and $e^{2u(0)}=a$ is given by

$$e^{2u(x)} = \begin{cases} \frac{a(a-4)}{-2+(a-2)\cosh(\sqrt{a(a-4)}x)} & \text{if } a>4, \\ \\ \frac{a(4-a)}{2-(a-2)\cos(\sqrt{a(4-a)}x)} & \text{if } a<4, \\ \\ \frac{4}{1+4x^2} & \text{if } a=4. \end{cases}$$

Let $\mathbb{C}H^2$ be the complex hyperbolic plane with the Bergmann metric $g$ of constant holomorphic sectional curvature -4. Then $\mathbb{C}H^2$ can be identified with $H_1^5(-1)/S^1$ via the Hopf fibration $\Pi : H_1^5(-1) \longrightarrow \mathbb{C}H^2$, where $H_1^5(-1) = \{z \in \mathbb{C}^3;\ (z,z) = -1\}$ is the anti-De Sitter space, where $(\ ,\ )$ is the Hermitian form

$$(z,w) = z_1\overline{w_1} + z_2\overline{w_2} - z_3\overline{w_3},$$

for any $z, w \in \mathbb{C}^3$. Then $\langle z,w\rangle = \Re(z,w)$ induces a Lorentz metric on $H_1^5(-1)$ of constant curvature -1.

For $t \in ]0,\infty[$, we define $\widehat{\psi}_t : S^2 \longrightarrow H_1^5(-1)$ by

$$\widehat{\psi}_t(x,y,z) = \frac{1}{s_t^2 + c_t^2 z^2}\left(s_t x, c_t xz, s_t y, c_t yz, z, -s_t c_t(1+z^2)\right),$$

with $s_t = \sinh t$ and $c_t = \cosh t$ as before.

For $s \in [0,\pi/4[$, we define $\Psi_s : \mathbb{C}^* \longrightarrow H_1^5(-1)$ by

$$\Psi_s(z) = \frac{1}{|w(z)|^2}\left(\cos^2 s\, z^2 - \sin^2 s\, \overline{z}^2, \frac{|z|^2-1}{\sqrt{2}}w(z), \frac{|z|^2+1}{\sqrt{2}}w(z)\right),$$

where $w(z) = \cos s\, z + \sin s\, \overline{z}$.

Finally, we define $\eta : \mathbb{C} \longrightarrow H_1^5(-1)$ by

$$\eta(x,y) = \\ \tfrac{1}{1+4x^2}\left(2y, 4xy, 2x - 4x^3 - 4xy^2, 6x^2 + 2y^2, 1 + 6x^2 + 2y^2, 4x^3 + 4xy^2\right).$$

**Proposition 3** (a) *For $t \in ]0,\infty[$, $\widehat{\phi}_t = \Pi \circ \widehat{\psi}_t : S^2 \longrightarrow \mathbb{C}H^2$ satisfies the following properties:*

(a1) $\widehat{\phi}_t$ *is a twistor holomorphic Lagrangian immersion.*

(a2) *The area of* $(S^2, \widehat{\phi}_t^* g)$ *is* $8\pi \arctan(\coth t)/\sinh 2t$.

(a3) *The Gauss curvature* $K_t$ *of* $(S^2, \widehat{\phi}_t^* g)$ *verifies* $-1 \leq K_t \leq -1 + 2\cosh^2 t$. *Moreover,* $K_t(x,y,z) = -1$ *(respectively* $K_t(x,y,z) = -1 + 2\cosh^2 t$*) if and only if* $z = \pm 1$ *(respectively* $z = 0$*).*

(a4) *On* $z > 0$, $\widehat{\phi}_0$ *is well defined and it is the totally geodesic Lagrangian embedding of* $\mathbb{RH}^2(-1)$ *into* $\mathbb{CH}^2$.

(b) *For* $s \in [0, \pi/4[$, $\Phi_s = \Pi \circ \Psi_s : \mathbb{C}^* \longrightarrow \mathbb{CH}^2$ *satisfies the following properties:*

(b1) $\Phi_s$ *is a conformal twistor holomorphic Lagrangian embedding.*

(b2) $\Phi_s^* g$ *is a complete metric.*

(b3) *The Gauss curvature* $K_s$ *of* $(\mathbb{C}^*, \Phi_s^* g)$ *verifies* $-\sin 2s \leq K_s \leq \sin 2s$. *Moreover,* $K_s(z) = -\sin 2s$ *(respectively* $K_s(z) = \sin 2s$*) if and only if* $\Im(z) = 0$, $\Re(z) < 0$ *(respectively* $\Im(z) = 0$, $\Re(z) > 0$*).*

(b4) $\int_{\mathbb{C}^*} K_s \, dA_s = 0$.

(b5) *On* $\Re(z) > 0$, $\Phi_{\pi/4}$ *is well defined and it is the totally geodesic Lagrangian embedding of* $\mathbb{RH}^2(-1)$ *into* $\mathbb{CH}^2$.

(c) $\Upsilon = \Pi \circ \eta : \mathbb{C} \longrightarrow \mathbb{CH}^2$ *satisfies the following properties:*

(c1) $\Upsilon$ *is a conformal twistor holomorphic Lagrangian embedding.*

(c2) $\Upsilon^* g$ *is a complete metric.*

(c3) *The Gauss curvature of* $(\mathbb{C}, \Upsilon^* g)$ *verifies* $-1 < K \leq 1$.

(c4) $\int_{\mathbb{C}} K \, dA = 0$.

**Theorem 2** (i) *Let* $\phi : \Sigma \longrightarrow \mathbb{CH}^2$ *be a twistor holomorphic immersion of an orientable surface* $\Sigma$. *Then* $\phi$ *is Lagrangian if and only if eiher* $\phi$ *is totally geodesic or* $\phi(\Sigma)$ *is an open set of* $\widehat{\phi}_t(S^2)$, $t \in ]0, \infty[$, $\Phi_s(\mathbb{C}^*)$, $s \in [0, \pi/4[$ *or* $\Upsilon(\mathbb{C})$.

(ii) *If* $\phi : \Sigma \longrightarrow \mathbb{CH}^2$ *is a Lagrangian immersion of a closed surface of genus* $\gamma$, *then*

$$\int_\Sigma (|H|^2 - 2) dA \geq 8\pi(1 - \gamma),$$

*and the equality holds if and only if $\phi$ is congruent to $\widehat{\phi}_t$ for some $t \in ]0, \infty[$.*

## References.


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