Stability problems on minimal Lagrangian submanifolds

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Let (M, ω, g) be a Kähler manifold of complex dimension n, Kähler form ω and metric g. We will denote by J the complex structure of M. An immersion $\Phi : \Sigma \to M$ of a manifold Σ of dimension n is called *lagrangian* if $\Phi^* \omega = 0$. This means that the complex structure J defines a bundle isomorphism

$$J : T\Sigma \to N\Sigma,$$

from the tangent bundle to the normal bundle to Σ. If H is the mean curvature vector of Φ, then we can define a 1-form on Σ by $\sigma_H = H \lrcorner \omega$. Then it is known that

$$d\sigma_H = \Phi^* \mathrm{Ric},$$

where Ric denotes the Ricci 2-form on M. In particular, if M is Einstein, and as Σ is lagrangian, we obtain that σ_H is a closed 1-form on Σ and it defines a cohomology class in $H^1_{dR}(\Sigma)$.

A *variation* of Φ is a smooth map

$$F : (-\varepsilon, \varepsilon) \times \Sigma \to M$$

satisfying

1. Each map $\Phi_t = F(t, -) : \Sigma \to M$ is an immersion,
2. $\Phi_0 = \Phi$.

Let $X = F_* (\frac{\partial}{\partial t})|_{t=0}$ denote the *variation vector field.* If dv_t denotes the volume form of the induced metric on Σ by Φ_t, then we consider the *volume functional* $V : (-\varepsilon, \varepsilon) \to \mathbb{R}$ given by

$$V(t) = \int_\Sigma dv_t.$$

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Then, it is well known that

$$V'(0) = -n \int_\Sigma g(X, H)\, dv_0,$$

H being the mean curvature vector of Φ. Φ is called a *minimal immersion* if its volume is critical for arbitrary compactly supported variations, i.e. $H = 0$.

A variation F of Φ is called a *Lagrangian variation* if for each $t \in (-\varepsilon, \varepsilon)$, $\Phi_t : \Sigma \to M$ is a lagrangian immersion. The lagrangian immersion Φ is called *lagrangian stationary* if its volume is critical for arbitrary compactly supported Lagrangian variations.

Proposition 1 *(Schoen-Wolfson, 2001) Let M be a Kähler-Einstein manifold and $\Phi : \Sigma \to M$ a Lagrangian immersion. Then Φ is lagrangian stationary if and only if Φ is minimal.*

Among the lagrangian variations, are particulary importants the *hamiltonian variations*. A variation with variation vector field X is called *hamiltonian*, if the 1-form on Σ

$$\alpha_X := \Phi^*(X \lrcorner\, \omega)$$

is exact, i.e. there exists a function $f \in C^\infty(\Sigma)$ such that $df = \alpha_X$. The lagrangian immersion Φ is called *Hamiltonian stationary* (*Hamiltonian minimal*) if its volume is critical for arbitrary compactly supported hamiltonian variations. This means that

$$0 = \int_\Sigma g(X, H) = \int_\Sigma g(df, \sigma_H) = \int_\Sigma g(f, \delta\sigma_H),$$

for all $f \in C_0^\infty(\Sigma)$, where δ is the adjoint operator of the differential d. Hence Φ is Hamiltonian stationary if and only if $\delta\sigma_H = 0$. This condition is equivalent to $\operatorname{div} JH = 0$, where div stands for the divergence operator. If M is Kähler-Einstein, σ_H is also closed, and then σ_H is a harmonic 1-form.

Let M be a Calabi-Yau manifold, i.e., a Kähler manifold of complex dimension n whose holonomy group is a subgroup of $SU(n)$. Then there exists a non-trivial parallel section of the canonical bundle of M, i.e. a parallel complex volume form Ω on M. It is known that $\Theta_\theta := \Re(e^{-i\theta}\Omega)$, $\theta \in [0, 2\pi[$ is a 1-parameter family of calibrations on M, called the *special lagrangian calibration with phase θ*. An immersion $\Phi : \Sigma \to M$ calibrated for the calibration Θ_θ, i.e. such that $\Phi^*(\Theta_\theta)$ is a volume form on Σ, is called *special Lagrangian with phase θ*.

Proposition 2 *(Harvey-Lawson, (1982)). Let $\Phi : \Sigma \to M$ an immersion of an n-dimensional orientable manifold Σ in a Calabi-Yau manifold M of complex dimension n. Then Φ is a minimal lagrangian immersion if and only if Φ is a special lagrangian immersion with phase θ for some $\theta \in [0, 2\pi[$.*

Now we are going to consider the second variation of the volume functional and from now on we assume that the lagrangian submanifold Σ is *compact*. Suppose that F is a variation of a *minimal* Lagrangian immersion $\Phi : \Sigma \to M$ whose variational vector fiel ξ is normal to Σ. Then it is well-known that

$$V''(0) = -\int_{\Sigma} g(L\xi, \xi)\, dv_0$$

where $L = \Delta^{\perp} + \mathcal{B} + \mathcal{R} : \Gamma(N\Sigma) \to \Gamma(N\Sigma)$ is the *Jacobi operator* given by:

$$\Delta^{\perp} = \sum_{i=1}^{n} \left\{ \nabla^{\perp}_{e_i} \nabla^{\perp}_{e_i} - \nabla^{\perp}_{\nabla_{e_i} e_i} \right\}, \quad \mathcal{B}(\xi) = \sum_{i=1}^{n} \sigma(A_{\xi} e_i, e_i),$$
$$\mathcal{R}(\xi) = \sum_{i=1}^{n} \bar{R}(\xi, e_i) e_i^{\perp},$$

being $\nabla^{\perp}$ the normal connection, σ the second fundamental form of Φ, A_{ξ} the Weingarten endomorphism associated to ξ, $\bar{R}$ the curvature operator of M and $\{e_1, \ldots, e_n\}$ an orthonormal reference tangent to Σ, where $\perp$ stands for normal component.

Let $Q(\xi) = -\int_M \langle L\xi, \xi\rangle\, dv_0$ be the quadratic form associated to the Jacobi operator L. We will represent by $\mathrm{Ind}(\phi)$ and $\mathrm{Nul}(\phi)$ the index and nullity of the quadratic form Q which are respectively the number of negative eigenvalues of L and the multiplicity of zero as an eigenvalue of L. We say that Φ is *stable* if $\mathrm{Ind}(\Phi) = 0$.

As Φ is Lagrangian, we consider the identification

$$\begin{aligned} \Gamma(N\Sigma) &\equiv \Omega^1(\Sigma) \\ \xi &\equiv \alpha, \end{aligned}$$

where $\alpha = \Phi^*(\xi \lrcorner\, \omega)$, and in general $\Omega^p(\Sigma)$ denotes the space of p-forms on Σ. As consequence the Jacobi operator becomes in a operator $L : \Omega^1(\Sigma) \to \Omega^1(\Sigma)$ given by (Oh,1990)

$$L(\alpha) = \Delta\alpha + S(\alpha)$$

where $\Delta = \delta\, d + d\, \delta$ is the Laplacian operator acting on 1-forms and $S(\alpha)(V) = Ricc(\alpha, V)$, for any vector field V tangent to Σ.

The Hodge decomposition

$$\Omega^1(\Sigma) = \mathcal{H}(\Sigma) \oplus dC^\infty(\Sigma) \oplus \delta\Omega^2(\Sigma),$$

allows to write in a unique way any 1-form α as $\alpha = \alpha_0 + df + \delta\beta$, with α_0 a harmonic 1-form, f a real function and β a 2-form on Σ. The space $\mathcal{H}(\Sigma)$ of harmonic 1-forms is the kernel of Δ, and its dimension is $\beta_1(\Sigma)$, the first Betti number of Σ. In the general case

Theorem 1 *(Harvey-Lawson (1982),Oh (1990), Takeuchi (1984)). Let $\Phi : \Sigma \to M$ be a minimal Lagrangian immersion of a compact manifold Σ.*

1. *If the first Chern class of M is negative, then Φ is stable and $Nul(\Phi) = 0$.*

2. *If the first Chern class of M is nonpositive, then Φ is stable.*

3. *If M is a Calabi-Yau manifold, then Φ is volume minimizing in its homology class.*

4. *If the first Chern class of M is positive and Φ is stable, then $H^1_{dR}(\Sigma) = 0$.*

5. *If M is a Hermitian symmetric space of compact type and Σ is a compact totally geodesic lagrangian submanifold embedded in M, then Σ is stable if and only if Σ is simply-connected.*

Now we are going to study stability problems when the ambient space is the complex projevtive space. Let $\mathbb{CP}^n$ be the complex projective space endowed with the Fubibi-Study metric of constant holomorphic sectional curvature 4. Then $\mathbb{CP}^n$ is a Kähler-Einstein manifold of complex dimension n with $Ricc = 2(n+1)\, g$. Hence the Jacobi operator L of a minimal Lagrangian immersion $\Phi : \Sigma \to \mathbb{CP}^n$ is given by

$$L : \Omega^1(\Sigma) \to \Omega^1(\Sigma), \quad L = \Delta + 2(n+1)\, I_d.$$

So $\mathrm{Ind}\,(\Phi)$ is the number of eigenvalues of Δ (counted with multiplicity) less than $2(n+1)$, and $\mathrm{Nul}\,(\Phi)$ is the multiplicity of $2(n+1)$ as a eigenvalue of Δ. As Δ commutes with d and δ, the positive eigenvalues of $\Delta : \Omega^1(\Sigma) \to \Omega^1(\Sigma)$

are the positive eigenvalues of $\Delta : C^\infty(\Sigma) \rightarrow C^\infty(\Sigma)$ joint with the positive eigenvalues of $\Delta : \Omega^2(\Sigma) \rightarrow \Omega^2(\Sigma)$. Hence

$$Ind\,(\Phi) = \beta_1(\Sigma) + Ind_0(\Sigma) + Ind_2(\Sigma),$$

where $\mathrm{Ind}_0\Sigma$ is the number of positive eigenvalues (counted with multiplicity) of $\Delta : C^\infty(\Sigma) \rightarrow C^\infty(\Sigma)$ less than $2(n+1)$ and $\mathrm{Ind}_2\Sigma$ is the number of positive eigenvalues (counted with multiplicity) of $\Delta : \Omega^2(\Sigma) \rightarrow \Omega^2(\Sigma)$ less than $2(n+1)$. As the variations associated to 1-forms in $dC^\infty(\Sigma)$ are Hamiltonian ones, we will called to $\mathrm{Ind}_0(\Sigma)$ the Hamiltonian index of Φ. It is clear that Φ is Hamiltonian stable if $\mathrm{Ind}_0(\Sigma) = 0$.

Considering the standard minimal embedding $\mathbb{CP}^n \subset \mathbb{S}^{n(n+2)-1}$ of the complex projective space into the $(n(n+2)-1)$-dimensional sphere of constant curvature $\frac{2(n+1)}{n}$, then $\Phi : \Sigma \rightarrow \mathbb{S}^{n(n+2)-1}$ defines a minimal immersion too. In particular $2(n+1)$ is an eigenvalue of $\Delta : C^\infty(\Sigma) \rightarrow C^\infty(\Sigma)$ and the coordinates of the position vector $\Phi : \Sigma \rightarrow \mathbb{R}^{n(n+2)}$ are eigenfunctions for this eigenvalue. Hence we can say that Φ *is Hamiltonian stable iff the first eigenvalue of* $\Delta : C^\infty(\Sigma) \rightarrow C^\infty(\Sigma)$ *is* $2(n+1)$. Using this fact, it is possible to stimate the nullity of a such minimal lgrangian submanifold. In fact

Theorem 2 *(Montiel, Urbano (2006), Urbano (2007)). Let $\Phi : \Sigma \rightarrow \mathbb{CP}^n$ be a minimal Lagrangian immersion of a compact manifold Σ. Then*

1. $$Nul\,\Phi \geq \frac{n(n+3)}{2},$$
   and the equality holds if and only if Φ is either the totally geodesic Lagrangian immersion of the unit sphere or the totally geodesic Lagrangian embedding of the real projective space.

2. *If $n = 2$ and Φ is not totally geodesic, then*
   $$Nul\,\Phi \geq 6,$$
   and the equality holds if and only if Φ is the Lagrangian Clifford torus.

With respect to the index we have the following general result.

Theorem 3 *(Lawson (19), Urbano (1993)) Let $\Phi : \Sigma \rightarrow \mathbb{CP}^n$ be a minimal Lagrangian immersion of a compact manifold Σ. Then the first eigenvalue μ_1 of $\Delta : \mathcal{H}(\Sigma) \oplus \delta\Omega^2(\Sigma) \rightarrow \mathcal{H}(\Sigma) \oplus \delta\Omega^2(\Sigma)$ satisfies*

$$\mu_1 \leq 2(n-1),$$

and the equality holds if and only if Φ is either the totally geodesic Lagrangian immersion of the unit sphere or the totally geodesic Lagrangian embedding of the real projective space.

In particular Φ is always unstable.

In order to understang the next result, we need to give some preliminaries. If $\Pi : \mathbb{S}^{2n+1} \to \mathbb{CP}^n$ is the Hopf fibration, then the canonical bundle $K_{\mathbb{CP}^n}$ can be identified with the bundle $\mathbb{S}^{2n+1}/\mathbb{Z}_{n+1}$. Using symplectic topology, it is easy to prove that if $\Phi : \Sigma \to \mathbb{CP}^n$ is a minimal Lagrangian immersion, then Φ has a horizontal lift (also called Legendrian) to $\mathbb{S}^{2n+1}/\mathbb{Z}_{n+1}$ if Σ is orientable and a horizontal lift to $\mathbb{S}^{2n+1}/\mathbb{Z}_{2(n+1)}$ if Σ is non-orientable. These mean that a $(n+1) : 1$-covering of Σ admits a Legendrian lift to $\mathbb{S}^{2n+1}$ when Σ is orientable and that a $(2n+2) : 1$-covering of Σ admits a Legendrian lift to $\mathbb{S}^{2n+1}$ when Σ is non-orientable. In this setting, we define *the level of* Φ as the smallest number $q|2(n+1)$ such that Φ admits a Legendrian lift to $\mathbb{S}^{2n+1}/\mathbb{Z}_q$. In particular the level of Φ is 1 when Φ admits a Legendrian lift to $\mathbb{S}^{2n+1}$ and the level is 2 when Φ admits a Legendrian lift to $\mathbb{RP}^{2n+1} = \mathbb{S}^{2n+1}/\mathbb{Z}_2$.

Theorem 4 *(Montiel, Urbano (2006)) Let $\Phi : \Sigma \to \mathbb{CP}^n$ be a minimal Lagrangian immersion of a compact manifold Σ. Then,*

1. *If the level of Φ is 1, then*

$$Ind_0\,(\Phi) \geq n+1,$$

and the equality holds if and only if Φ is the totally geodesic Lagrangian immersion of the unit n-sphere. If Φ is not totally geodesic, then $Ind_0(\Phi) \geq 2(n+1)$. Also

$$Ind\,(\Phi) \geq \frac{(n+1)(n+2)}{2},$$

and the equality holds if and only if Φ is the totally geodesic Lagrangian immersion of the unit n-sphere.

2. *If the level of Φ is 2, then*

$$Ind\,(\Phi) \geq \frac{n(n+1)}{2},$$

and the equality holds if and only if Φ is the totally geodesic Lagrangian embedding of the n-dimensional real projective space.

It is known that if $\Phi : \Sigma \to \mathbb{CP}^n$ is a minimal Lagrangian immersion of a compact manifold and $H^1(\Sigma, \mathbb{Z}_{2(n+1)}) = 0$ then the level of Σ is 1. So ,using the above result, *if* $\mathit{Ind}_0\,\Phi < n+1$ *then* $H^1(\Sigma, \mathbb{Z}_{2(n+1)}) \neq 0$, *and in particular* $H_1(\Sigma, \mathbb{Z}) \neq 0$.

Among the Lagrangian submanifolds with parallel second fundamental form of $\mathbb{CP}^n$ obtained by Naitoh and Takeuchi [NT], Amarzaya and Ohnita determined those which are Hamiltonian stable, proving the following result.

Theorem 5 *The compact minimal Lagrangian submanifolds embedded in* $\mathbb{CP}^n$ *of the following list:*

1. $SU(p)/\mathbb{Z}_p, \quad n = p^2 - 1,$
2. $SU(p)/SO(p)\mathbb{Z}_p, \quad n = (p-1)(p+2)/2,$
3. $SU(2p)/Sp(p)\mathbb{Z}_{2p}, \quad n = (p-1)(2p+1),$
4. $E_6/F_4\mathbb{Z}_3, \quad n = 26,$

are Hamiltonian stables.

Now we consider the particular case $n = 2$, in which we can get better results.

If $\Phi : \Sigma \to \mathbb{CP}^2$ is a minimal Lagrangian immersion of a compact orientable surface Σ, then the star operator $\star : C^\infty(\Sigma) \to \Omega^2(\Sigma)$ tell us that the eigenvalues of Δ acting on $C^\infty(\Sigma)$ or on $\Omega^2(\Sigma)$ are the same, and so $\mathrm{Ind}_0(\Sigma) = \mathrm{Ind}_2(\Sigma)$. Hence, if g is the genus of Σ, we have

$$\mathrm{Ind}\,(\Sigma) = 2g + 2\mathrm{Ind}_0(\Sigma).$$

If $\Phi : \Sigma \to \mathbb{CP}^2$ is a minimal Lagrangian immersion of a compact nonorientable surface Σ, we consider $\Phi \circ \pi : \tilde{\Sigma} \to \mathbb{CP}^2$ the corresponding minimal Lagrangian immersion of its 2:1 orientable covering $\tilde{\Sigma}$). If $\tau : \tilde{\Sigma} \to \tilde{\Sigma}$ is the change of sheet involution, then the spaces of forms on $\tilde{\Sigma}$ can be decompose in the following way:

$$\Omega^i(\tilde{\Sigma}) = \Omega^i_+(\tilde{\Sigma}) \oplus \Omega^i_-(\tilde{\Sigma}), \quad i = 0, 1, 2,$$

where

$$\Omega^i_\pm(\tilde{\Sigma}) = \{\alpha \in \Omega^i(\tilde{\Sigma}) \,/\, \tau^*\alpha = \pm\alpha\}.$$

Also the space of harmonic 1-forms on $\tilde{\Sigma}$ decomposes into two subspaces $\mathcal{H}(\tilde{\Sigma}) = \mathcal{H}_+(\tilde{\Sigma}) \oplus \mathcal{H}_-(\tilde{\Sigma})$, where again $\mathcal{H}_\pm(\tilde{\Sigma}) = \{\alpha \in \mathcal{H}(\tilde{\Sigma}) \,/\, \tau^*\alpha = \pm\alpha\}$. In this way we obtain

$$\Omega^1_\pm(\tilde{\Sigma}) = \mathcal{H}_\pm(\tilde{\Sigma}) \oplus d\,\Omega^0_\pm(\tilde{\Sigma}) \oplus \delta\,\Omega^2_\pm(\tilde{\Sigma}).$$

As $\pi \circ \tau = \pi$, the map $\alpha \in \Omega^i(\Sigma) \mapsto \pi^*\alpha \in \Omega^i(\tilde{\Sigma})$ allows to identify $\mathcal{H}(\Sigma) \equiv \mathcal{H}_+(\tilde{\Sigma})$ and $\Omega^i(\Sigma) \equiv \Omega^i_+(\tilde{\Sigma})$, $i = 0, 1, 2$. Also, as Σ is nonorientable, $\star\tau^* = -\tau^*\star$, and so $\star$ identifies $\Omega^0_-(\tilde{\Sigma}) \equiv \Omega^2_+(\tilde{\Sigma})$. Hence we obtain the identification

$$\begin{array}{rcl} \Omega^2(\Sigma) & \equiv & \Omega^0_-(\tilde{\Sigma}) \\ \alpha & \equiv & f \end{array}$$

where $\pi^*\alpha = f\omega_0$, being ω_0 the volume 2-form on $\tilde{\Sigma}$.

As Σ is nonorientable, the eigenvalues of $\Delta : \Omega^2(\Sigma) \to \Omega^2(\Sigma)$ are positives, and hence, taking into account the above remarks, *Ind*$_2(\Sigma)$ *is the number of eigenvalues (counted with multiplicity) of* $\Delta : \Omega^0_-(\tilde{\Sigma}) \to \Omega^0_-(\tilde{\Sigma})$ *less than* 6. Also, as *Ind*$_0(\Sigma)$ *is the number of positive eigenvalues (counted with multiplicity) of* $\Delta : \Omega^0_+(\tilde{\Sigma}) \to \Omega^0_+(\tilde{\Sigma})$ *less than* 6, we obtain that

$$\mathrm{Ind}_0(\Sigma) + \mathrm{Ind}_1(\Sigma) = \mathrm{Ind}_0(\tilde{\Sigma}), \tag{1}$$

and hence from (3)

$$2\mathrm{Ind}(\Sigma) = \mathrm{Ind}(\tilde{\Sigma}).$$

Theorem 6 *(Urbano (1993), (2007)) Let* $\Phi : \Sigma \to \mathbb{CP}^2$ *be a minimal Lagrangian immersion of a compact orientable surface of genus* g*.*

1. *If* Φ *is Hamiltonian stable and* $g \leq 4$*, then* Φ *is an embedding and* $\Phi(\Sigma)$ *is the Clifford torus.*

2. *Ind*$(\Phi) \geq 2$ *and the equality holds if and only if* Φ *is an embedding and* $\Phi(\Sigma)$ *is the Clifford torus.*

Theorem 7 *(Urbano (2007)) Let* $\Phi : \Sigma \to \mathbb{CP}^2$ *be a minimal Lagrangian immersion of a compact nonorientable surface with Euler characteristic* $\chi(\Sigma)$*.*

1. *If* Φ *is Hamiltonian stable and* $\chi(\Sigma) \geq -1$*, then* Φ *is an embedding and* $\Phi(\Sigma)$ *is the totally geodesic real projective plane.*

2. *Ind*$(\Sigma) \geq 3$ *and the equality holds if and only if* Φ *is an embedding and* $\Phi(\Sigma)$ *is the totally geodesic real projective plane.*

Now we consider as Kähler-Einstein surface the complex quadric $\mathbb{S}^2\times\mathbb{S}^2$ which is, joint to $\mathbb{CP}^2$, the only compact Hermitian symmetric space of complex dimension 2. In [CU], Castro and Urbano have studied in depth its minimal Lagrangian surfaces, with special emphasis in its stability.

Theorem 8 *Let* $\Phi : \Sigma \to \mathbb{S}^2 \times \mathbb{S}^2$ *be a minimal Lagrangian immersion of a compact surface* Σ*. Then*

1. *If* Σ *is stable, then* $\Phi(\Sigma)$ *is the totally geodesic Lagrangian sphere* $\mathbf{M}_0$.
2. *If* Σ *is Hamiltonian stable and* Σ *is orientable with genus* $g \leq 2$*, then* Φ *is an embedding and* $\Phi(\Sigma)$ *is either the totally geodesic sphere* $\mathbf{M}_0$ *or the totally geodesic torus* $\mathbf{T}$.
3. *If* Σ *is a Hamiltonian stable Klein bottle, then* Φ *is an embedding and* $\Phi(\Sigma)$ *is the Klein bottle* $\mathbf{B}$.
4. *If* Σ *is unstable, then Ind*$(\Sigma) \geq 2$ *and the equality holds if and only if* Φ *is an embedding and* $\Phi(\Sigma)$ *is the totally geodesic torus* $\mathbf{T}$.

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