Non associative C^*-algebras revisited

Kaidi El Amin and Antonio Morales Campoy[a], and Angel Rodríguez Palacios*[b]

[a]Departamento de Algebra y Análisis Matemático,
Universidad de Almería, Facultad de Ciencias Experimentales
04120-Almería, Spain
elamin@ual.es and amorales@ual.es

[b]Departamento de Análisis Matemático, Facultad de Ciencias,
Universidad de Granada, 18071 Granada, Spain.
apalacio@goliat.ugr.es

To Professor Manuel Valdivia on the occasion of his seventieth birthday.

Abstract

We give a detailed survey of some recent developments of non-associative C^*-algebras. Moreover, we prove new results concerning multipliers and isometries of non-associative C^*-algebras.

MCS 2000 46K70, 46L70

1. Introduction

In this paper we are dealing with non-associative generalizations of C^*-algebras. In relation to this matter, a first question arises, namely how associativity can be removed in C^*-algebras. Since C^*-algebras are originally defined as certain algebras of operators on complex Hilbert spaces, it seems that they are "essentially" associative. However, fortunately, the abstract characterizations of (associative) C^*-algebras given by either Gelfand-Naimark or Vidav-Palmer theorems allows us to consider the working of such abstract systems of axioms in a general non-associative setting.

To be more precise, for a norm-unital complete normed (possibly non associative) complex algebra A, we consider the following conditions:

(VP) (**VIDAV-PALMER AXIOM**). $A = H(A, \mathbf{1}) + iH(A, \mathbf{1})$.

(GN) (**GELFAND-NAIMARK AXIOM**). *There is a conjugate-linear vector space involution $*$ on A satisfying $\mathbf{1}^* = \mathbf{1}$ and $\| a^*a \| = \| a \|^2$ for every a in A.*

*Partially supported by Junta de Andalucía grant FQM 0199

In both conditions, $\mathbf{1}$ denotes the unit of A, whereas, in *(VP)*, $H(A, \mathbf{1})$ stands for the closed real subspace of A consisting of those element h in A such that $f(h)$ belongs to $\mathbb{R}$ whenever f is a bounded linear functional on A satisfying $\| f \| = f(\mathbf{1}) = 1$.

If the norm-unital complete normed complex algebra A above is associative, then *(GN)* and *(VP)* are equivalent conditions, both providing nice characterizations of unital C^*-algebras (see for instance [10, Section 38]). In the general non-associative case we are considering, things begin to be funnier. Indeed, it is easily seen that *(GN)* implies *(VP)* (argue as in the proof of [10, Proposition 12.20]), but the converse implication is not true (take A equal to the Banach space of all 2×2-matrices over $\mathbb{C}$, regarded as operators on the two-dimensional complex Hilbert space, and endow A with the product $a \circ b := \frac{1}{2}(ab + ba)$).

The funny aspect of the non-associative consideration of Vidav-Palmer and Gelfand-Naimark axioms greatly increases thanks to the fact, which is explained in what follows, that Condition *(VP)* (respectively, *(GN)*) on a norm-unital complete normed complex algebra A implies that A is "nearly" (respectively, "very nearly") associative. To specify our last assertion, let us recall some elemental concepts of non-associative algebra. **Alternative algebras** are defined as those algebras A satisfying $a^2 b = a(ab)$ and $ba^2 = (ba)a$ for all a, b in A. By Artin's theorem [65, p. 29], an algebra A is alternative (if and) only if, for all a, b in A, the subalgebra of A generated by $\{a, b\}$ is associative. Following [65, p. 141], we define **non-commutative Jordan algebras** as those algebras A satisfying the **Jordan identity** $(ab)a^2 = a(ba^2)$ and the **flexibility** condition $(ab)a = a(ba)$. Non-commutative Jordan algebras are **power-associative** [65, p. 141] (i.e., all single-generated subalgebras are associative) and, as a consequence of Artin's theorem, alternative algebras are non-commutative Jordan algebras. For an element a in a non-commutative Jordan algebra A, we denote by U_a the mapping $b \to a(ab + ba) - a^2 b$ from A to A. In Definitions 1.1 and 1.2 immediately below we provide the algebraic notions just introduced with analytic robes.

Definition 1.1. By a **non-commutative JB^*-algebra** we mean a complete normed non-commutative Jordan complex algebra (say A) with a conjugate-linear algebra-involution $*$ satisfying

$$\| U_a(a^*) \| = \| a \|^3$$

for every a in A.

Definition 1.2. By an **alternative C^*-algebra** we mean a complete normed alternative complex algebra (say A) with a conjugate-linear algebra-involution $*$ satisfying

$$\| a^* a \| = \| a \|^2$$

for all a in A.

Since, for elements a, b in an alternative algebra, the equality $U_a(b) = aba$ holds, it is not difficult to realize that alternative C^*-algebras become particular examples of non-commutative JB^*-algebras. In fact alternative C^*-algebras are nothing but those non-commutative JB^*-algebras which are alternative [48, Proposition 1.3]. Now the behaviour of Vidav-Palmer and Gelfand-Naimark axioms in the non-associative setting are clarified by means of Theorems 1.3 and 1.4, respectively, which follow.

Theorem 1.3 ([54, Theorem 12]). *Norm-unital complete normed complex algebras fulfilling Vidav-Palmer axiom are nothing but unital non-commutative JB^*-algebras.*

Theorem 1.4 ([53, Theorem 14]). *Norm-unital complete normed complex algebras fulfilling Gelfand-Naimark axiom are nothing but unital alternative C^*-algebras.*

After Theorems 1.3 and 1.4 above, there is no doubt that both alternative C^*-algebras and non-commutative JB^*-algebras become reasonable non-associative generalizations (the second containing the former) of (possibly non unital) classical C^*-algebras.

The basic structure theory for non-commutative JB^*-algebras is concluded about 1984 (see [3], [11], [48], and [49]). In these papers a precise classification of certain prime non-commutative JB^*-algebras (the so-called "non-commutative JBW^*-factors") is obtained, and the fact that every non-commutative JB^*-algebra has a faithful family of the so-called "Type I" factor representations is proven. When these results specialize for classical C^*-algebras, Type I non-commutative JBW^*-factors are nothing but the (associative) W^*-factors consisting of all bounded linear operators on some complex Hilbert space, and, consequently, Type I factor representations are precisely irreducible representations on Hilbert spaces. Alternative C^*-algebras are specifically considered in [12] and [48], where it is shown that alternative W^*-factors are either associative or equal to the (essentially unique) alternative C^*-algebra $\mathbb{O}_{\mathbb{C}}$ of complex octonions. In fact, as noticed in [59, p. 103], it follows easily from [76, Theorem 9, p. 194] that every prime alternative C^*-algebra is either associative or equal to $\mathbb{O}_{\mathbb{C}}$.

In recent years we have revisited the theory of non-commutative JB^*-algebras and alternative C^*-algebras with the aim of refining some previously known facts, as well as of developing some previously unexplored aspects. Most results got in this goal appear in [39], [40], and [41]. In the present paper we review the main results obtained in the papers just quoted, and prove some new facts.

Section 2 deals with the theorem in [39] that the product p_A of every non-zero alternative C^*-algebra A is a vertex of the closed unit ball of the Banach space of all continuous bilinear mappings from $A \times A$ into A. We note that this result seems to be new even in the particular case that the alternative C^*-algebra A above is in fact associative. If A is only assumed to be a non-commutative JB^*-algebra, then it is easily seen that the above vertex property for p_A can fail. The question whether the vertex property for p_A characterizes alternative C^*-algebras A among non-commutative JB^*-algebras remains an open problem. In any case, if the vertex property for p_A is relaxed to the extreme point property, then the answer to the above question is negative.

In Section 3 we collect a classification of prime non-commutative JB^*-algebras, which generalizes that of non-commutative JBW^*-factors. According to the main result of [40], if A is a prime non-commutative JB^*-algebra, and if A is neither quadratic nor commutative, then there exists a prime C^*-algebra B, and a real number λ with $\frac{1}{2} < \lambda \leq 1$ such that $A = B$ as involutive Banach spaces, and the product of A is related to that of B (denoted by $\square$, say) by means of the equality $ab = \lambda a \square b + (1 - \lambda) b \square a$. We note that prime JB^*-algebras which are either quadratic or commutative are well-understood (see [49, Section 3] and the Zel'manov-type prime theorem for JB^*-algebras [26, Theorem 2.3], respectively).

Following [67, Definition 20.18], we say that a bounded domain Ω in a complex Banach space is **symmetric** if for each x in Ω there exists an involutive holomorphic mapping $\varphi : \Omega \to \Omega$ having x as an isolated fixed point. It is well-known that the open unit balls of C^*-algebras are bounded symmetric domains. It is also folklore that C^*-algebras have approximate units bounded by one. In Section 4 we review the result obtained in [41] asserting that the above two properties characterize C^*-algebras among complete normed associative complex algebras. The key tools in the proof are W. Kaup's materialization (up to biholomorphic equivalence) of bounded symmetric domains as open unit balls of JB^*-triples [43], the Braun-Kaup-Upmeier holomorphic characterization of the Banach spaces underlying unital JB^*-algebras [13], and the Vidav-Palmer theorem (both in its original form [8, Theorem 6.9] and in Moore's reformulation [9, Theorem 31.10]). Actually, applying the non-associative versions of the Vidav-Palmer and Moore's theorems (see Theorem 1.3 and [44], respectively), it is shown in [41] that a complete normed complex algebra is a non-commutative JB^*-algebra if and only if it has an approximate unit bounded by one, and its open unit ball is a bounded symmetric domain.

Sections 5 and 6 are devoted to prove new results. In Section 5 we introduce multipliers on non-commutative JB^*-algebras, and prove that the set $M(A)$ of all multipliers on a given non-commutative JB^*-algebra A becomes a new non-commutative JB^*-algebra. Actually, in a precise categorical sense, $M(A)$ is the largest non-commutative JB^*-algebra which contains A as a closed essential ideal (Theorem 5.6). We note that, if A is in fact an alternative C^*-algebra, then so is $M(A)$.

Section 6 deals with the non-associative discussion of the Kadison-Paterson-Sinclair theorem [47] asserting that surjective linear isometries between C^*-algebras are precisely the compositions of Jordan-$*$-isomorphisms (between the given algebras) with left multiplications by unitary elements in the multiplier C^*-algebra of the range algebra. In this direction we prove (see Propositions 6.3 and 6.8, and Theorem 6.7) that, for a non-commutative JB^*-algebra A, the following assertions are equivalent:

1. Left multiplications on A by unitary elements of $M(A)$ are isometries.
2. A is an alternative C^*-algebra.
3. For every non-commutative JB^*-algebra B, and every surjective linear isometry $F : B \to A$, there exists a Jordan-$*$-isomorphism $G : B \to A$, and a unitary element u in $M(A)$ satisfying $F(b) = uG(b)$ for all b in B.

Section 6 also contains a discussion of the question whether linearly isometric non-commutative JB^*-algebras are Jordan-$*$-isomorphic (see Theorem 6.10 and Corollary 6.12). A similar discussion in the particular unital case can be found in [13, Section 5]. Moreover, we prove that hermitian operators on a non-commutative JB^*-algebra A are nothing but those operators on A which can be expressed as the sum of a left multiplication by a self-adjoint element of $M(A)$, and a Jordan-derivation of A anticommuting with the JB^*-involution of A (Theorem 6.13).

The concluding section of the paper (Section 7) is devoted to notes and remarks.

We devote the last part of the present section to briefly review the relation between non-commutative JB^*-algebras and other close mathematical models. First we note that, by the power-associativity of non-commutative Jordan algebras, every self-adjoint element of a non-commutative JB^*-algebra A is contained in a commutative C^*-algebra. Analogously, by Artin's theorem, every element of an alternative C^*-algebra is contained in a C^*-algebra. Let us also note that, in questions and results concerning a given non-commutative JB^*-algebra A, we often can assume that A is commutative (called then simply a **JB^*-algebra**). The sentence just formulated merits some explanation. For every algebra A, let us denote by A^+ the algebra whose vector space is the one of A and whose product $\circ$ is defined by $a \circ b := \frac{1}{2}(ab + ba)$. With this convention of symbols, the fact is that, if A is a non-commutative JB^*-algebra, then A^+ becomes a JB^*-algebra under the norm and the involution of A. JB^*-algebras were introduced by I. Kaplansky, and studied first by J. D. M. Wright [69] (in the unital case) and M. A. Youngson [74] (in the general case). By the main results in those papers, JB^*-algebras are in a bijective categorical correspondence with the so-called JB-algebras. The correspondence is obtained by passing from each JB^*-algebra A to its self-adjoint part A_{sa}. **JB-algebras** are defined as those complete normed Jordan real algebras B satisfying $\| x \|^2 \leq \| x^2 + y^2 \|$ for all x, y in B. They were introduced by E. M. Alfsen, F. W. Shultz, and E. Stormer [2], and their basic theory is today nicely collected in [29]. Finally, let us shortly comment on the relation between non-commutative JB^*-algebras and JB^*-triples (see Section 4 for a definition). Every non-commutative JB^*-algebra is a JB^*-triple under a triple product naturally derived from its binary product and its JB^*-involution (see [13], [67], and [74]). As a partial converse, every JB^*-triple can be seen as a JB^*-subtriple of a suitable JB^*-algebra [27]. Moreover, alternative C^*-algebras have shown useful in the structure theory of JB^*-triples [31]. JB^*-triples were introduced by W. Kaup [42] in the search of an algebraic setting for the study of bounded symmetric domains in complex Banach spaces.

2. Geometric properties of the products of alternative C^*-algebras

As in the case of C^*-algebras, the algebraic structure of non-commutative JB^*-algebras is closely related to the geometry of the Banach spaces underlying them. Let us therefore begin our work by fixing notation and recalling some basic concepts in the setting of normed spaces.

Let X be a normed space. We denote by S_X, B_X, and X' the unit sphere, the closed unit ball, and the dual space, respectively, of X. $BL(X)$ will denote the normed algebra of all bounded linear operators on X, and I_X will stand for the identity operator on X. Each continuous bilinear mapping from $X \times X$ into X will be called a **product** on X. Each product f on X has a natural norm $\| f \|$ given by $\| f \| := \sup\{\| f(x, y) \| \ : \ x, y \in B_X\}$. We denote by $\Pi(X)$ the normed space of all products on X.

Now, let u be a norm-one element in the normed space X. The set of **states** of X relative to u, $D(X, u)$, is defined as the non empty, convex, and weak*-compact subset of X' given by

$$D(X, u) := \{\phi \in B_{X'} \ : \ \phi(u) = 1\} \ .$$

For x in X, the **numerical range** of x relative to u, $V(X, u, x)$, is given by

$$V(X, u, x) := \{\phi(x) \ : \ \phi \in D(X, u)\} \ .$$

We say that u is a **vertex** of B_X if the conditions $x \in X$ and $\phi(x) = 0$ for all ϕ in $D(X, u)$ imply $x = 0$. It is well-known and easy to see that the vertex property for u implies that

u is an extreme point of B_X. For x in X, we define the **numerical radius** of x relative to u, $v(X,u,x)$, by

$$v(X,u,x) := Max\{|\rho| \ : \ \rho \in V(X,u,x)\} .$$

The **numerical index** of X relative to u, $n(X,u)$, is the number given by

$$n(X,u) := Max\{r \geq 0 \ : \ r \parallel x \parallel \leq v(X,u,x) \text{ for all } x \text{ in } X\} .$$

We note that $0 \leq n(X,u) \leq 1$ and that the condition $n(X,u) > 0$ implies that u is a vertex of B_X. Note also that, if Y is a subspace of X containing u, then $n(Y,u) \geq n(X,u)$.

The study of the geometry of norm-unital complex Banach algebras at their units ([8], [9]) takes its first impetus from the celebrated Bohnenblust-Karlin theorem [7] asserting that the unit $\mathbf{1}$ of such an algebra A is a vertex of the closed unit ball of A. As observed in [8, pp. 33-34], the Bohnenblust-Karlin paper actually contains the stronger result that, for such an algebra A, the inequality $n(A,\mathbf{1}) \geq \frac{1}{e}$ holds.

Now let A be a (possibly non unital and/or non associative) complete normed complex algebra. Then the product p_A of A becomes a natural distinguished element of the Banach space $\Pi(A)$ of all products on the Banach space underlying A. Moreover, in most natural examples (for instance, if A has a norm-one unit or is a non-zero non-commutative JB^*-algebra), we have $\parallel p_A \parallel = 1$. In these cases one can naturally wonder if p_A is a vertex of the closed unit ball of $\Pi(A)$. Even if A is a non-commutative JB^*-algebra, the answer to the above question can be negative. Indeed, if B is a C^*-algebra which fails to be commutative, if λ is a real number with $0 < \lambda < 1$, and if we replace the product xy of B with the one $(x,y) \to \lambda xy + (1-\lambda)yx$, then we obtain a non-commutative JB^*-algebra (say A) whose product is not an extreme point (much less a vertex) of $B_{\Pi(A)}$. With $\lambda = \frac{1}{2}$ in the above construction, we even obtain a (commutative) JB^*-algebra with such a pathology. However, in the case that A is in fact an alternative C^*-algebra, the answer to the question we are considering is more than affirmative. Precisely, we have the following theorem.

Theorem 2.1 ([39, Theorem 2.5]). *Let A be a non zero alternative C^*-algebra. Then $n(\Pi(A), p_A)$ is equal to 1 or $\frac{1}{2}$ depending on whether or not A is commutative.*

In order to provide the reader with a sketch of proof of Theorem 2.1, we comment on the background needed in such a proof, putting special emphasis in those results which will be applied later in the present paper. Among them, the more important one is the following.

Theorem 2.2 ([48, Theorem 1.7]). *Let A be a non-commutative JB^*-algebra. Then the bidual A'' of A becomes naturally a unital non-commutative JB^*-algebra containing A as a $*$-invariant subalgebra. Moreover A'' satisfies all multilinear identities satisfied by A.*

In fact, in the proof of Theorem 2.1 we only need the next straightforward consequence of Theorem 2.2.

Corollary 2.3 ([48, Corollary 1.9]). *If A is an alternative C^*-algebra, then A'' becomes naturally a unital alternative C^*-algebra containing A as a $*$-invariant subalgebra.*

It follows from Theorem 2.2 (respectively, Corollary 2.3) that the unital hull A_1 of a non-commutative JB^*- (respectively, alternative C^*-) algebra A can be seen as a non-commutative JB^*- (respectively, alternative C^*-) algebra for suitable norm and involution extending those of A [48, Corollary 1.10]. In fact we have had to refine this result by proving the following lemma (compare [10, Lemma 12.19 and its proof]).

Lemma 2.4 ([39, Lemma 2.3]). *Let A be a non-commutative JB^*-algebra. For x in A_1, let T_x denote the operator on A defined by $T_x(a) := xa$. Then A_1, endowed with the unique conjugate-linear algebra involution extending that of A and the norm $\| \, . \, \|$ given by $\| x \| := \| T_x \|$ for all x in A_1, is a non-commutative JB^*-algebra containing A isometrically.*

Theorem 2.2 (respectively, Corollary 2.3) gives rise naturally to the so-called **non-commutative** JBW^***-** (respectively, **alternative** W^***-) algebras**, namely non-commutative JB^*- (respectively, alternative C^*-) algebras which are dual Banach spaces. The fact that *the product of every non-commutative JBW^*-algebra is separately w^*-continuous* [48, Theorem 3.5] will be often applied along this paper. For instance, such a fact, together with Theorem 2.2, yields easily Lemma 2.4 as well as the result that, *if A is a non-commutative JB^*-algebra, and if a is an element of A, then a belongs to the norm-closure of aB_A* [39, Lemma 2.4].

Another background result applied in the proof of Theorem 2.1 is a non-associative generalization of [19, Theorem 1] asserting that, *if A is a non zero non-commutative JB^*-algebra with a unit $\mathbf{1}$, then $n(A, \mathbf{1})$ is equal to 1 or $\frac{1}{2}$ depending on whether or not A is associative and commutative* [53, Theorem 26] (see also [33, Theorem 4]). Since commutative alternative complex algebras are associative [76, Corollary 7.1.2], it follows from the above that, *if A is a non zero alternative C^*-algebra with a unit $\mathbf{1}$, then $n(A, \mathbf{1})$ is equal to 1 or $\frac{1}{2}$ depending on whether or not A is commutative.*

Before to formally attack a sketch of proof of Theorem 2.1, let us note that, given a unital alternative C^*-algebra A, unitary elements of A are defined verbatim as in the associative particular case, that left multiplications on A by unitary elements of A are surjective linear isometries (a consequence of [65, p. 38]), and that, easily (see for instance [12, Theorem 2.10]), the Russo-Dye-Palmer equalities

$$B_A = \overline{co}\{u : u \text{ unitary in } A\} = \overline{co}\{e^{ih} : h \in A_{sa}\}$$

hold for A. Here $\overline{co}$ means closed convex hull and, to be brief, we have written e^{ih} instead of $\exp(ih)$.

Sketch of proof of Theorem 2.1.- Given a set E and a normed algebra B, let us denote by $B(E, B)$ the normed algebra of all bounded functions from E into B (with point-wise operations and the supremum norm). Now, for the non-zero alternative C^*-algebra A, let us consider the chain of linear mappings

$$A_1 \xrightarrow{F_1} BL(A) \xrightarrow{F_2} \Pi(A) \xrightarrow{F_3} \Pi(A'') \xrightarrow{F_4} B((A'')_{sa} \times (A'')_{sa}, A''),$$

where $F_1(z) := T_z$ for every z in A_1, $F_2(T)(a, b) := T(ab)$ for every T in $BL(A)$ and all a, b in A, $F_3(f) := f'''$ (the third Arens transpose of f [4]) for every f in $\Pi(A)$, and $F_4(g)(h,k) := e^{-ih}(g(e^{ih}, e^{ik})e^{-ik})$ for every g in $\Pi(A'')$ and all h,k in $(A'')_{sa}$. It follows easily from the information collected above that, for $i = 1,...,4$, F_i is a linear isometry. Moreover, we have $F_1(\mathbf{1}) = I_A$, $F_2(I_A) = p_A$, $F_3(p_A) = p_{A''}$, and $F_4(p_{A''}) = \mathbb{I}$, where $\mathbb{I}$ denotes the constant mapping equal to the unit of A'' on $(A'')_{sa} \times (A'')_{sa}$. Let δ denote either 1 or $\frac{1}{2}$ depending on whether or not A is commutative. Since A_1 and $B((A'')_{sa} \times (A'')_{sa}, A'')$ are alternative C^*-algebras with units $\mathbf{1}$ and $\mathbb{I}$, respectively, and they are commutative if and only if A is, it follows

$$\delta = n(A_1, \mathbf{1}) \geq n(BL(A), I_A) \geq n(\Pi(A), p_A)$$

$$\geq n(\Pi(A''), P_{A''}) \geq n(B((A'')_{sa} \times (A'')_{sa}, A''), \mathbb{I}) = \delta \ . \blacksquare$$

The **normed space numerical index**, $N(X)$, of a non-zero normed space X is defined by the equality $N(X) := n(BL(X), I_X)$. The above argument clarifies the proof of Huruya's theorem [32] that, *if A is a non zero C^*-algebra, then $N(A)$ is equal to* 1 *or* $\frac{1}{2}$ *depending on whether or not A is commutative*, and generalizes Huruya's result to the setting of alternative C^*-algebras. In fact, with methods rather similar to those in the proof of Theorem 2.1, we have been able to prove the stronger result that, *if A is a non zero non-commutative JB^*-algebra, then $N(A)$ is equal to* 1 *or* $\frac{1}{2}$ *depending on whether or not A is associative and commutative* [39, Proposition 2.6]. This result was already formulated in [33, Theorem 5] as a direct consequence of Theorem 2.2, the particular case of such a result for unital non-commutative JB^*-algebras [53, Corollary 33], and the claim in [21] that, for every normed space X, the equality $N(X') = N(X)$ holds. However, as a matter of fact, the proof of the claim in [21] never appeared, and the question if for an arbitrary normed space X the equality $N(X') = N(X)$ holds remains an open problem among people interested in the field. In view of this open problem, we investigated about the normed space numerical indexes of preduals of non-commutative JBW^*-algebras, and proved that, *if A is a non zero non-commutative JBW^*-algebra (with predual denoted by A_*), then $N(A_*)$ is equal to* 1 *or* $\frac{1}{2}$ *depending on whether or not A is associative and commutative* [39, Proposition 2.8].

In relation to Theorem 2.1, we conjecture that, if A is a non-commutative JB^*-algebra such that p_A is a vertex of $B_{\Pi(A)}$, then A is an alternative C^*-algebra. We know that, if in the above conjecture we relax the condition that p_A is a vertex of $B_{\Pi(A)}$ to the one that p_A is an extreme point of $B_{\Pi(A)}$, then the answer is negative [39, Example 3.2].

3. Prime non-commutative JB^*-algebras

By a **non-commutative JBW^*-factor** we mean a prime non-commutative JBW^*-algebra. A non-commutative JBW^*-factor is said to be of **Type I** if the closed unit ball of its predual has some extreme point (compare [49, Theorem 1.11]). As we commented in Section 1, one of the main results in the structure theory of non-commutative JB^*-algebras is the following.

Theorem 3.1 ([49, Theorem 2.7]). *Type I non-commutative JBW^*-factors are either commutative, quadratic, or of the form $BL(H)^{(\lambda)}$ for some complex Hilbert space H and some $\frac{1}{2} < \lambda \leq 1$.*

We recall that, according to [65, pp. 49-50], an algebra A over a field $\mathbb{F}$ is called **quadratic** over $\mathbb{F}$ if it has a unit $\mathbf{1}, A \neq \mathbb{F}\mathbf{1}$, and, for each a in A, there are elements $t(a)$ and $n(a)$ of $\mathbb{F}$ such that $a^2 - t(a)a + n(a)\mathbf{1} = 0$. We also recall that, if A is a non-commutative JB^*-algebra, and if λ is a real number with $0 \leq \lambda \leq 1$, then the involutive Banach space of A, endowed with the product $(a, b) \to \lambda ab + (1 - \lambda)ba$, becomes a non-commutative JB^*-algebra which is usually denoted by $A^{(\lambda)}$.

In [40] we obtain a reasonable generalization of Theorem 3.1, which reads as follows.

Theorem 3.2 ([40, Theorem 4]). *Prime non-commutative JB^*-algebras are either commutative, quadratic, or of the form $C^{(\lambda)}$ for some prime C^*-algebra C and some $\frac{1}{2} < \lambda \leq 1$.*

In fact we derived Theorem 3.2 from Theorem 3.1 and the fact that *every JB^*-algebra has a faithful family of Type I factor representations* [49, Corollary 1.13]. In what follows we provide the reader with an outline of the argument. First we recall that a **factor representation** of a given non-commutative JB^*-algebra A is a w^*-dense range *-homomorphism from A into some non-commutative JBW^*-factor. For convenience, let us say that a factor representation $\varphi : A \to B$ is commutative, quadratic, or quasi-associative whenever the non-commutative JBW^*-factor B is commutative, quadratic, or of the form $B^{(\lambda)}$ for some W^*-factor B and some $\frac{1}{2} < \lambda \leq 1$, respectively. Now, if the non-commutative JB^*-algebra A is prime, then it follows easily from the information collected above that at least one of the following families of factor representations of A is faithful:

1. The family of all commutative Type I factor representations of A.

2. The family of all quadratic Type I factor representations of A.

3. The family of all quasi-associative Type I factor representations of A.

Since clearly A is commutative whenever the family in (1) is faithful, the unique remaining problem is to show that, if the family in (2) (respectively, (3)) is faithful, then A is quadratic (respectively, of the form $C^{(\lambda)}$ for some prime C^*-algebra C and some $\frac{1}{2} < \lambda \leq 1$). To overcome this obstacle we replaced algebraic ultraproducts with Banach ultraproducts [30] in an argument of E. Zel'manov [75] in his determination of prime nondegenerate Jordan triples of Clifford type, to obtain the proposition which follows. We note that, if $\{A_i\}_{i \in I}$ is a family of non-commutative JB^*-algebras, and if $\mathcal{U}$ is an ultrafilter on I, then the Banach ultraproduct $(A_i)_{\mathcal{U}}$ is a non-commutative JB^*-algebra in a natural way.

Proposition 3.3 ([40, Proposition 2]). *Let A be a prime non-commutative JB^*-algebra, I a non-empty set, and, for each i in I, let φ_i be a $*$-homomorphism from A into a non-commutative JB^*-algebra A_i. Assume that $\cap_{i \in I} Ker(\varphi_i) = 0$. Then there exists an ultrafilter $\mathcal{U}$ on I such that the $*$-homomorphism $\varphi : x \to (\varphi_i(x))$ from A to $(A_i)_{\mathcal{U}}$ is injective.*

When in the above proposition the family $\{\varphi_i\}_{i \in I}$ actually consists of quadratic (respectively, quasi-associative) factor representations of the prime non-commutative JB^*-algebra A, it is not difficult to see that the non-commutative JB^*-algebra $(A_i)_{\mathcal{U}}$ is quadratic (respectively, of the form $C^{(\lambda)}$ for some prime C^*-algebra C and some $\frac{1}{2} < \lambda \leq 1$), so that, with some additional effort, it follows from the proposition that A is quadratic (respectively, of the form $C^{(\lambda)}$ for some prime C^*-algebra C and some $\frac{1}{2} < \lambda \leq 1$), thus concluding the proof of Theorem 3.2.

In relation to Theorem 3.2, we note that prime JB^*-algebras which are either quadratic or commutative are well-understood. Quadratic prime non-commutative JB^*-algebras have been precisely described in [49, Section 3]. According to that description, they are in fact Type I non-commutative JBW^*-factors. Commutative prime JB^*-algebras are classified in the Zel'manov-type theorem for such algebras [26, Theorem 2.3].

We recall that a **W^*-algebra** is a C^*-algebra which is a dual Banach space, and that a **W^*-factor** is a prime W^*-algebra. The next result follows directly from Theorem 3.2.

Corollary 3.4 ([3] [11]). *Non-commutative JBW^*-factors are either commutative, quadratic, or of the form $B^{(\lambda)}$ for some W^*-factor B and some real number λ with $\frac{1}{2} < \lambda \leq 1$.*

For (commutative) JBW^*-factors, the reader is referred to [26, Proposition 1.1]. According to Theorem 3.1, for non-commutative JBW^*-factors of Type I, the W^*-factor B arising in the above Corollary is equal to the algebra $BL(H)$ of all bounded linear operators on some complex Hilbert space H. This result follows from Corollary 3.4 and the fact that the algebras of the form $BL(H)$, with H a complex Hilbert space, are the unique W^*-factors of Type I [29, Proposition 7.5.2]. A classification of (commutative) JBW^*-factors of Type I can be obtained from the categorical correspondence between JBW-algebras and JBW^*-algebras [22] and the structure theorem for JBW-factors of Type I [29, Corollary 5.3.7, and Theorems 5.3.8, 6.1.8, and 7.5.11]. The precise formulation of such a classification can be found in [40, Proposition 6].

A normed algebra A is called *topologically simple* if $A^2 \neq 0$ and the unique closed ideals of A are $\{0\}$ and A. Since topologically simple normed algebras are prime, the following corollary follows with minor effort from Theorem 3.2.

Corollary 3.5 ([40, Corollary 7]). *Topologically simple non-commutative JB^*-algebras are either commutative, quadratic, or of the form $B^{(\lambda)}$ for some topologically simple C^*-algebra B and some real number λ with $\frac{1}{2} < \lambda \leq 1$.*

We note that every quadratic prime JB^*-algebra is algebraically (hence topologically) simple. For topologically simple (commutative) JB^*-algebras, the reader is referred to [26, Corollary 3.1].

4. Holomorphic characterization of non-commutative JB^*-algebras

An **approximate unit** of a normed algebra A is a net $\{b_\lambda\}_{\lambda\in\Lambda}$ in A satisfying

$$\lim\{ab_\lambda\}_{\lambda\in\Lambda} = a \text{ and } \lim\{b_\lambda a\}_{\lambda\in\Lambda} = a$$

for every a in A. If A is a non-commutative JB^*-algebra, the self-adjoint part A_{sa} of A (regarded as a closed real subalgebra of A^+) is a JB-algebra [29, Proposition 3.8.2]. In this way, the self-adjoint part of any non-commutative JB^*-algebra A is endowed with the order induced by the positive cone $\{a^2 \ : \ a \in A_{sa}\}$ [29, Section 3.3]. The following proposition is proved in [68]. We include here the proof because reference [68] is not easily available. We recall that *every JB-algebra has an increasing approximate unit consisting of positive elements with norm* ≤ 1 [29, Proposition 3.5.4].

Proposition 4.1. *Every non-commutative JB^*-algebra has an increasing approximate unit consisting of positive elements with norm* ≤ 1.

Proof. Let A be a non-commutative JB^*-algebra, and let $\{b_\lambda\}_{\lambda\in\Lambda}$ be an increasing approximate unit of the JB-algebra A_{sa} consisting of positive elements with norm ≤ 1. We are proving that $\{b_\lambda\}_{\lambda\in\Lambda}$ is in fact an approximate unit of A. Since $\{b_\lambda\}_{\lambda\in\Lambda}$ is clearly an approximate unit of A^+, it is enough to show that $\lim\{[a,b_\lambda]\}_{\lambda\in\Lambda}=0$ for every a in A. Here $[.,.]$ denotes the usual commutator on A. But, keeping in mind that the commutator is a derivation of A^+ in each of its variables [65, p. 146], for a in A we obtain

$$\lim\{[a^2,b_\lambda]\}_{\lambda\in\Lambda}=2\lim\{a\circ[a,b_\lambda]\}_{\lambda\in\Lambda}=2\lim\{[a,a\circ b_\lambda]\}_{\lambda\in\Lambda}=0\ .$$

Now the proof is concluded by applying the well-known fact that every non-commutative JB^*-algebra is the linear hull of the set of squares of its elements. ∎

In [41] we rediscover the above result as a consequence of the following remarkable inequality for non-commutative JB^*-algebras.

Theorem 4.2 ([41, Theorem 1.3]). *Let A be a non-commutative JB^*-algebra, and a be in A_{sa}. Then, for all b in A we have*

$$\| [a,b] \|^2 \leq 16 \| b \| \| a^2\circ b - a\circ(a\circ b) \| .$$

The proof of Theorem 4.2 above involves the whole theory of Type I factor representations of non-commutative JB^*-algebras outlined in Section 3. If one is only interested in the specialization of Theorem 4.2 in the case that A is an alternative C^*-algebra, then the proof is much easier. Indeed, in such a case the argument given in [41, Lemma 1.1] for classical C^*-algebras works verbatim.

Together with Proposition 4.1, the following result becomes of special interest for the matter we are developing in the present section.

Proposition 4.3. *The open unit ball of every non-commutative JB^*-algebra is a bounded symmetric domain.*

The proof of the above proposition consists of the facts that non-commutative JB^*-algebras are JB^*-triples in a natural way ([13], [74]) and that *open unit balls of JB^*-triples are bounded symmetric domains* [42]. We recall that a JB^***-triple** is a complex Banach space J with a continuous triple product $\{...\}:J\times J\times J\to J$ which is linear and symmetric in the outer variables, and conjugate-linear in the middle variable, and satisfies:

1. For all x in J, the mapping $y\to\{xxy\}$ from J to J is a hermitian operator on J and has nonnegative spectrum.

2. The **main identity**

$$\{ab\{xyz\}\} = \{\{abx\}yz\} - \{x\{bay\}z\} + \{xy\{abz\}\}$$

holds for all a, b, x, y, z in J.

3. $\| \{xxx\} \| = \| x \|^3$ for every x in J.

Concerning Condition (1) above, we also recall that a bounded linear operator T on a complex Banach space X is said to be **hermitian** if it belongs to $H(BL(X), I_X)$ (equivalently, if $\| \exp(irT) \| = 1$ for every r in $\mathbb{R}$ [10, Corollary 10.13]).

For a vector space E, let $L(E)$ denote the associative algebra of all linear mappings from E to E, and for a non-commutative Jordan algebra A, let $(a, b) \to U_{a,b}$ be the unique symmetric bilinear mapping from $A \times A$ to $L(A)$ satisfying $U_{a,a} = U_a$ for every a in A. Now we can specify the result in [13] and [74] pointed out above. Indeed, *every non-commutative JB^*-algebra A is a JB^*-triple under the triple product $\{...\}$ defined by $\{abc\} := U_{a,c}(b^*)$ for all a, b, c in A.*

In [41] we prove that the properties given by Propositions 4.1 and 4.3 characterize non-commutative JB^*-algebras among complete normed complex algebras. This is emphasized in the theorem which follows.

Theorem 4.4 ([41, Theorem 3.3]). *Let A be a complete normed complex algebra. Then A is a non-commutative JB^*-algebra (for some involution $*$) if (and only if) A has an approximate unit bounded by one and the open unit ball of A is a bounded symmetric domain.*

Many old and new auxiliary results have been needed to prove Theorem 4.4 above. Concerning new ones, we make to stand out for the moment Lemma 4.5 which follows. We begin by recalling some concepts taken from [13, p. 285]. Let X be a normed space, u an element in X, and Q a subset of X. We define the **tangent cone to Q at u**, $T_u(Q)$, as the set of all $x \in X$ such that

$$x = \lim \frac{x_n - u}{t_n}$$

for some sequence $\{x_n\}$ in Q with $\lim\{x_n\} = u$ and some sequence $\{t_n\}$ of positive real numbers. When X is complex, the **holomorphic tangent cone to Q at u**, $\hat{T}_u(Q)$, is defined as

$$\hat{T}_u(Q) := \cap_{\lambda \in \mathbb{C} \setminus \{0\}} \lambda T_u(Q) \ .$$

From now on, for a normed space X, Δ_X will denote the open unit ball of X.

Lemma 4.5 ([41, Lemma 3.1]). *Let X be a complex normed space, and u an element in X. Then u is a vertex of B_X if and only if $\hat{T}_u(\Delta_X) = 0$.*

Now, speaking about old results applied in the proof of Theorem 4.4, let us enumerate the following:

1. Kaup's algebraic characterization of bounded symmetric domains ([42] and [43]), namely *a complex Banach space X is a JB^*-triple (for some triple product) if and only if Δ_X is a bounded symmetric domain.*

2. The Chu-Iochum-Loupias result [17] that, *if X is a JB^*-triple, and if $T : X \to X'$ is a bounded linear mapping, then T is weakly compact.* In fact we are applying the reformulation of this result (via [55]) that *every product on a JB^*-triple is Arens regular.*

3. The non-associative version of the Bohnenblust-Karlin theorem (see for instance [60, Theorem 1.5]), namely, *if A is a norm-unital complete normed complex algebra, then the unit of A is a vertex of B_A.*

4. The celebrated Dineen's result [20] that *the bidual of every JB^*-triple is a JB^*-triple.*

5. The Braun-Kaup-Upmeier holomorphic characterization of Banach spaces underlying unital JB^*-algebras [13] (reformulated via Lemma 4.5): *a complex Banach space X underlies a JB^*-algebra with unit u if and only if u is a vertex of B_X and Δ_X is a bounded symmetric domain.*

6. The non-associative version of Vidav-Palmer theorem (Theorem 1.3).

Turning back to new results applied in the proof of Theorem 4.4, the main one is Theorem 4.6 which follows.

Theorem 4.6 ([41, Theorem 2.4]). *Let A be a complete normed complex algebra such that A'', endowed with the Arens product and a suitable involution $*$, is a non-commutative JB^*-algebra. Then A is a $*$-invariant subset of A'', and hence a non-commutative JB^*-algebra.*

In the case that A has a unit, Theorem 4.6 follows easily from a dual version of Theorem 1.3, proved in [44], asserting that *a norm-unital complete normed complex algebra A is a non-commutative JB^*-algebra if and only if $S \cap iS = 0$, where S denotes the real linear hull of $D(A, \mathbf{1})$.* The non unital case of Theorem 4.6 is reduced to the unital one after a lot of work (see [41, Section 2] for details).

Now, let us provide the reader with the following

Sketch of proof of the "if" part of Theorem 4.4.- By the second assumption and Result 1 above, there exists a JB^*-triple X and a surjective linear isometry $\Phi : A \to X$. By the first assumption and Result 2, A'' (endowed with the Arens product) has a unit $\mathbf{1}$ with $\| \mathbf{1} \| = 1$. By Result 3, $\mathbf{1}$ is a vertex of $B_{A''}$, and hence $\Phi''(\mathbf{1})$ is a vertex of $B_{X''}$. By Results 4, 1, and 5, X'' underlies a JB^*-algebra with unit $\Phi''(\mathbf{1})$, and therefore (by an easy observation of M. A. Youngson in [72]) we have

$$X'' = H(X'', \Phi''(\mathbf{1})) + iH(X'', \Phi''(\mathbf{1})),$$

and hence $A'' = H(A'', \mathbf{1}) + iH(A'', \mathbf{1})$. By Result 6, A'' is a non-commutative JB^*-algebra. Finally, by Theorem 4.6, A is a non-commutative JB^*-algebra. ∎

The following corollaries follow straightforwardly from Theorem 4.4.

Corollary 4.7 ([41, Corollary 3.4]). *An associative complete normed complex algebra is a C^*-algebra if and only if it has an approximate unit bounded by one and its open unit ball is a bounded symmetric domain.*

Corollary 4.8. *An alternative complete normed complex algebra is an alternative C^*-algebra if and only if it has an approximate unit bounded by one and its open unit ball is a bounded symmetric domain.*

Corollary 4.9 ([41, Corollary 3.5]). *A normed complex algebra is a non-commutative JB^*-algebra if and only if it is linearly isometric to a non-commutative JB^*-algebra and has an approximate unit bounded by one.*

Corollary 4.10. *An alternative normed complex algebra is an alternative C^*-algebra if and only if it is linearly isometric to a non-commutative JB^*-algebra and has an approximate unit bounded by one.*

Corollary 4.11. *An alternative normed complex algebra is an alternative C^*-algebra if and only if it is linearly isometric to an alternative C^*-algebra and has an approximate unit bounded by one.*

Corollary 4.12. *An associative normed complex algebra is a C^*-algebra if and only if it is linearly isometric to a non-commutative JB^*-algebra and has an approximate unit bounded by one.*

Corollary 4.13 ([62, Corollary 1.3]). *An associative normed complex algebra is a C^*-algebra if and only if it is linearly isometric to a C^*-algebra and has an approximate unit bounded by one.*

For surjective linear isometries between non-commutative JB^*-algebras the reader is referred to Section 6 of the present paper.

5. Multipliers of non-commutative JB^*-algebras

Every semiprime associative algebra A has a natural enlargement, namely the so-called multiplier algebra $M(A)$ of A, which can be characterized as the largest semiprime associative algebra containing A as an essential ideal. In the case that A is an (associative) C^*-algebra, $M(A)$ becomes naturally a C^*-algebra which contains A as a closed (essential) ideal. More precisely, in this case $M(A)$ can be rediscovered as the closed $*$-invariant subalgebra of A'' given by $\{x \in A'' : xA + Ax \subseteq A\}$ (see for instance [50, Propositions 3.12.3 and 3.7.8]). Now let A be a non-commutative JB^*-algebra. The fact just commented suggests to define the set of multipliers, $M(A)$, of A by the equality $M(A) := \{x \in A'' : xA + Ax \subseteq A\}$. It is clear that $M(A)$ is a closed $*$-invariant subspace of A'' containing A and the unit of A''. In this way, the equality $M(A) = A$ holds if and only if A has a unit. It is also clear that, if $M(A)$ were a subalgebra of A'', then A would be an ideal of $M(A)$. We are showing in the present section that $M(A)$ is in fact a subalgebra of A'' (and hence a non-commutative JB^*-algebra) which contains A as an essential ideal. We also will show that, in a categorical sense, $M(A)$ is the largest non-commutative JB^*-algebra containing A as a closed essential ideal.

Our argument begins by invoking the next result, which is taken from C. M. Edwards' paper [23].

Lemma 5.1. *Let B be a JB-algebra. Then $M(B) := \{x \in B'' : x \circ B \subseteq B\}$ is a subalgebra of B''.*

It will be also useful the following lemma, whose verification can be made following the lines of the proof of [34, Lemma 4.2]. Let X be a complex Banach space. By a **conjugation** on X we mean an involutive conjugate-linear isometry on X. Conjugations τ on a complex Banach space X give rise by natural transposition to conjugations τ' on X'. Given a conjugation τ on X, X^τ will stand for the closed real subspace of X given by $X^\tau := \{x \in X \ : \ \tau(x) = x\}$.

Lemma 5.2. *Let X be a complex Banach space, and τ a conjugation on X. Then, up to a natural identification, we have $(X^\tau)'' = (X'')^{\tau''}$.*

Taking in the above lemma X equal to a non-commutative JB^*-algebra A, and τ equal to the JB^*-involution of A, we obtain the following corollary.

Corollary 5.3. *Let A be a non-commutative JB^*-algebra. Then the Banach space identification $(A_{sa})'' = (A'')_{sa}$ is also a JB-algebra identification.*

Proof. The Banach space identification $(A_{sa})'' = (A'')_{sa}$ is the identity on A_{sa}, A_{sa} is w^*-dense in both $(A_{sa})''$ and $(A'')_{sa}$, and the products of $(A_{sa})''$ and $(A'')_{sa}$ are separately w^*-continuous. ∎

Now, putting together Lemma 5.1 and Corollary 5.3, the next result follows.

Corollary 5.4. *Let A be a (commutative) JB^*-algebra. Then $M(A)$ is a subalgebra of A''.*

Now, to obtain the non-commutative generalization of the above corollary we only need a single new fact, which is proved in the next lemma. We recall that every derivation of a JB^*-algebra is automatically continuous [73].

Lemma 5.5. *Let A be a JB^*-algebra, and D a derivation of A. Then $M(A)$ is D''-invariant.*

Proof. Let a, x be in A and $M(A)$, respectively. We have

$$D''(x) \circ a = D(x \circ a) - x \circ D(a) \in A \ .$$

Since a is arbitrary in A, it follows that $D''(x)$ lies in $M(A)$. ∎

Theorem 5.6. *Let A be a non-commutative JB^*-algebra. Then $M(A)$ is a closed $*$-invariant subalgebra of A'' containing A as an essential ideal. Moreover, if B is another non-commutative JB^*-algebra containing A as a closed essential ideal, then B can be seen as a closed $*$-subalgebra of $M(A)$ containing A. In addition we have $M(A) = M(A^+)$.*

Proof. Keeping in mind the equality $(A'')^+ = (A^+)''$, the inclusion $M(A) \subseteq M(A^+)$ is clear. Let b be in A. Then the mapping $D : a \to [b, a]$ from A to A is a derivation of A^+, and we have $D''(x) = [b, x]$ for every x in A''. It follows from Lemma 5.5 that

$D''(M(A^+)) \subseteq M(A^+)$, or equivalently $[b,x] \in M(A^+)$ for every x in $M(A^+)$. Therefore, for x in $M(A^+)$ we have

$$[b^2, x] = 2b \circ [b,x] \in A \circ M(A^+) \subseteq A\ .$$

Since b is arbitrary in A, and A is the linear hull of the set of squares of its elements, we deduce $[A,x] \subseteq A$ for every x in $M(A^+)$. It follows

$$M(A^+)A + AM(A^+) \subseteq A\ ,$$

and hence $M(A^+) \subseteq M(A)$. Then the equality $M(A^+) = M(A)$ is proved.

Now, for x, y in $M(A)$ and A in A we have

$$[x,y] \circ a = [x, y \circ a] - [x,a] \circ y \in [M(A), A] + A \circ M(A) \subseteq A\ ,$$

and hence $[M(A), M(A)] \subseteq M(A^+)$. On the other hand, Corollary 5.4 applies to A^+ giving $M(A) \circ M(A) \subseteq M(A^+) \circ M(A^+) \subseteq M(A^+)$. It follows from the first paragraph of the proof that $M(A)$ is a subalgebra of A''.

Assume that P is an ideal of $M(A)$ with $P \cap A = 0$. Then, since A is an ideal of $M(A)$, we actually have $AP = 0$, so $A''P = 0$, and so $P = 0$. Therefore A is an essential ideal of $M(A)$.

Let B be a non-commutative JB^*-algebra, and $\varphi : A \to B$ a one-to-one (automatically isometric) *-homomorphism such that $\varphi(A)$ is an essential ideal of B. Then φ'' is a one-to-one $*$-homomorphism from A'' to B'' whose range is a w^*-closed ideal of B'' (apply that $*$-homomorphisms between non-commutative JB^*-algebras have norm-closed range [48, Corollary 1.11 and Proposition 2.1], and that w^*-continuous linear operators with norm-closed range have in fact w^*-closed range [49, Lemma 1.3]). By [48, Theorem 3.9] we have $\varphi''(A'') = B''e$ for a suitable central projection e in B''. Now $(B''(1-e)) \cap B$ is an ideal of B whose intersection with $\varphi(A)$ is zero, and hence $(B''(1-e)) \cap B = 0$ because $\varphi(A)$ is an essential ideal of B. Therefore the mapping $\psi : b \to be$ from B to $\varphi''(A'')$ is a one-to-one $*$-homomorphism. Then $\eta := (\varphi'')^{-1}\psi$ is a one-to-one $*$-homomorphism from B to A'' satisfying $\eta(\varphi(a)) = a$ for every a in A. In this way we can see B as a closed $*$-invariant subalgebra of A'' containing A as an ideal. In this regarding we have clearly $B \subseteq M(A)$. ■

Let A be a non-commutative JB^*-algebra. The above theorem allows us to say that $M(A)$ is the **multiplier non-commutative JB^*-algebra** of A. The equality $M(A) = M(A^+)$ in the theorem can be understood in the sense that, if we consider the JB-algebra A_{sa}, then the multiplier JB-algebra of A_{sa} (in the sense of [23]) is nothing but the self-adjoint part of the multiplier JB^*-algebra of A. It is worth mentioning that the concluding paragraph of the proof of Theorem 5.6 is quite standard (see [26, Proposition 1.3], [14, Lemma 2.3], and [18, Proposition 1.2] for forerunners).

Let X be a JB^*-triple. We recall that the bidual X'' of X is a JB^*-triple under a triple product extending that of X [20], and that the set

$$\mathcal{M}(X) := \{x \in X'' \ :\ \{xXX\} \subseteq X\}$$

is a JB^*-subtriple of X'' containing X as a triple ideal [14]. The JB^*-triple $\mathcal{M}(X)$ just defined is called the **multiplier JB^*-triple** of X. Therefore, for a given non-commutative JB^*-algebra A, we can consider the multiplier non-commutative JB^*-algebra $M(A)$ of A, and the multiplier JB^*-triple $\mathcal{M}(A)$ of the JB^*-triple underlying A. Actually, the following result holds.

Proposition 5.7. *Let A be a non-commutative JB^*-algebra. Then we have $M(A) = \mathcal{M}(A)$.*

Proof. The inclusion $M(A) \subseteq \mathcal{M}(A)$ is clear. To prove the converse inclusion, we start by noticing that, clearly, the JB^*-triples underlying A and A^+ coincide, and that, by Theorem 5.6, the equality $M(A) = M(A^+)$ holds, so that we may assume that A is commutative. We note also that, since the equality $\{xyz\}^* = \{x^*y^*z^*\}$ is true for all x, y, z in A'', and A is a $*$-invariant subset of A'', $\mathcal{M}(A)$ is $*$-invariant too, and therefore it is enough to show that $a \circ x$ lies in A whenever a and x are self-adjoint elements of A and $\mathcal{M}(A)$, respectively. But, for such a and x, we can find a self-adjoint element b in A satisfying $b^3 = a$ (see for instance [45, Proposition 1.2]), and apply Shirshov's theorem [76, p.71] to obtain that

$$x \circ a = x \circ b^3 = b \circ (2U_{x,b}(b) - U_{b,b}(x)) = b \circ (2\{xbb\} - \{bxb\})$$

belongs to A. ■

6. Isometries of non-commutative JB^*-algebras

This section is devoted to the non-associative discussion of the following Paterson-Sinclair refinement of Kadison's classical theorem [37] on isometries of C^*-algebras.

Theorem 6.1 ([47, Theorem 1]). *Let A and B be C^*-algebras, and F a mapping from B to A. Then F is a surjective linear isometry (if and) only if there exists a Jordan-$*$-isomorphism $G : B \to A$, and a unitary element u in the multiplier C^*-algebra of A satisfying $F(b) = uG(b)$ for every b in B.*

We recall that, given algebras A and B, **Jordan homomorphisms** from B to A are defined as homomorphisms from B^+ to A^+. By an **isomorphism** between algebras we mean a one-to-one surjective homomorphism. If follows from Theorem 6.1 that unit-preserving surjective linear isometries between unital C^*-algebras are in fact Jordan-$*$-isomorphisms. This particular case of Theorem 6.1 remains true in the setting of non-commutative JB^*-algebras thanks to the following Wright-Youngson theorem.

Theorem 6.2 ([71, Theorem 6]). *Let A and B be unital non-commutative JB^*-algebras, and $F : B \to A$ a unit-preserving surjective linear isometry. Then F is a Jordan-$*$-isomorphism.*

The above theorem can be also derived from the fact that non-commutative JB^*-algebras are JB^*-triples in a natural way, and Kaup's theorem [42] that surjective linear isometries between JB^*-triples preserve triple products. In any case, the easiest known proof of Theorem 6.2 seems to be the one provided by the implication $(i) \Rightarrow (ii)$ in [38, Lemma 6].

Concerning concepts involved in the statement, the general formulation of Theorem 6.1 could have a sense in the more general setting of non-commutative JB^*-algebras. Indeed, in the previous section we introduced (automatically unital) multiplier non-commutative JB^*-algebras of arbitrary non-commutative JB^*-algebras. On the other hand, a reasonable notion of unitary element in a unital non-commutative JB^*-algebra A can be given, by invoking McCrimmon's definition of invertible elements in unital non-commutative Jordan algebras [46], and saying that an element a in A is **unitary** whenever it is invertible and satisfies $a^* = a^{-1}$. Let A be a unital non-commutative Jordan algebra, and a an element of A. We recall that a is said to be **invertible** in A if there exists b in A such that the equalities $ab = ba = \mathbf{1}$ and $a^2b = ba^2 = a$ hold. If a is invertible in A, then the element b above is unique, is called the inverse of a , and is denoted by a^{-1}. Moreover a is invertible in A if and only if it is invertible in the Jordan algebra A^+. This reduces most questions and results on inverses in non-commutative Jordan algebras to the commutative case. For this particular case, the reader is referred to [36, Section I.11].

Despite the above comments, even the "if" part of Theorem 6.1 does not remain true in the setting of non-commutative JB^*-algebras. Indeed, Jordan-$*$-isomorphisms between non-commutative JB^*-algebras are isometries [69] but, unfortunately, left multiplications by unitary elements of a unital non-commutative JB^*-algebra need not be isometries. This handicap becomes more than an anecdote in view of the following result. Given an element x in the multiplier non-commutative JB^*-algebra of a non-commutative JB^*-algebra A, we denote by T_x the operator on A defined by $T_x(a) := xa$ for every a in A. By a **Jordan-derivation** of an algebra A we mean a derivation of A^+.

Proposition 6.3. *Let A be a non-commutative JB^*-algebra. Then A is an alternative C^*-algebra if and only if, for every unitary element u of $M(A)$, T_u is an isometry.*

Proof. The "only if" part is very easy. Assume that A is an alternative C^*-algebra. Then it follows from Theorem 5.6 and Corollary 2.3 that $M(A)$ is a unital alternative C^*-algebra. Therefore, as we have seen in Section 2, left multiplications on $M(A)$ by unitary elements of $M(A)$ are surjective linear isometries. Since A is invariant under such isometries, it follows that $T_u : A \to A$ is an isometry whenever u is a unitary element in $M(A)$.

Now assume that A is a non-commutative JB^*-algebra such that T_u is an isometry whenever u is a unitary element in $M(A)$. For x in A'', denote by $L_x^{A''}$ the operator of left multiplication by x on A''. We remark that $L_x^{A''} = (T_x)''$ whenever x belongs to $M(A)$ (indeed, both sides of the equality are w^*-continuous operators on A'' coinciding on A). Let h be in $(M(A))_{sa}$ and r be a real number. Then $\exp(irh)$ is a unitary element of $M(A)$, and therefore, by the assumption on A and the equality $L_{\exp(irh)}^{A''} = (T_{\exp(irh)})''$ just established, $L_{\exp(irh)}^{A''}$ is an isometry on A''. Now put $G_r := L_{\exp(irh)}^{A''} L_{\exp(-irh)}^{A''}$. Then G_r is an isometry on A'' preserving the unit of A''. Since $G_0 = I_{A''}$ and the mapping $r \to G_r$ is continuous, there exists a positive number k such that G_r is surjective whenever $|r| < k$. It follows from Theorem 6.2 that, for $|r| < k$, G_r is a Jordan-$*$-automorphism of A''. If $\sum (1/n!) r^n F_n$ is the power series development of G_r, then we easily obtain $F_0 = I_{A''}$, $F_1 = 0$, and $F_2 = 2((L_h^{A''})^2 - L_{h^2}^{A''})$. By [53, Lemma 13], $(L_h^{A''})^2 - L_{h^2}^{A''}$ is a Jordan-derivation of A'' commuting with the JB^*-involution of A''. Now, arguing as in the conclusion of the proof of [53, Theorem 14], we realize that actually the equality $(L_h^{A''})^2 - L_{h^2}^{A''} = 0$ holds. In particular, for x in $M(A)$ we have $h(hx) = h^2x$. Since h is an arbitrary element of $(M(A))_{sa}$, an easy linearization argument gives $y(yx) = y^2x$ for all x, y in $M(A)$. By applying the JB^*-involution of $M(A)$ to both sides of the above equality, it follows that $M(A)$ (and hence A) is alternative. ■

In relation to the above proposition, we note that, if A is an alternative C^*-algebra, then, for every unitary element u of $M(A), T_u$ is in fact a SURJECTIVE linear isometry on A (with inverse mapping equal to T_{u*}). Now that we know that alternative C^*-algebras are the unique non-commutative JB^*-algebras which can play the role of A in a reasonable non-associative generalization of the "if" part of Theorem 6.1, we proceed to prove that they are also "good" for the non-associative generalization of the "only if" part of that theorem.

Lemma 6.4. *Let A be a unital alternative C^*-algebra. Then vertices of B_A and unitary elements of A coincide.*

Proof. Let u be a unitary element of A. Then u is a vertex of B_A because $\mathbf{1}$ is a vertex of B_A [60, Theorem 1.5] and the mapping $a \to ua$ from A to A is a surjective linear isometry sending $\mathbf{1}$ into u.

Now, let u be a vertex of B_A. Then the closed subalgebra B of A generated by $\{\mathbf{1}, u, u^*\}$ is a unital (associative) C^*-algebra. Since the vertex property is hereditary, it follows from [7, Example 4.1] that u is a unitary element of B, and hence also of A. ■

Remark 6.5. Actually the assertion in the above lemma remains true if A is only assumed to be a unital non-commutative JB^*-algebra. This follows straightforwardly from Lemma 4.5 and the equivalence $(i) \Leftrightarrow (iii)$ in [13, Proposition 4.3]. The proof we have given of this fact in the particular case of alternative C^*-algebras has however its methodological own interest.

Let X and Y be JB^*-triples, and $F : X \to Y$ a surjective linear isometry. It follows easily from the already quoted Kaup's Kadison type theorem that $F''(\mathcal{M}(X)) = \mathcal{M}(Y)$. In the particular case of non-commutative JB^*-algebras, we can apply Proposition 5.7 to arrive in the following non-associative generalization of [47, Theorem 2].

Lemma 6.6. *Let A and B be non-commutative JB^*-algebras, and $F : B \to A$ a surjective linear isometry. Then we have $F''(M(B)) = M(A)$. In particular, Jordan-$*$-isomorphisms from B to A extend uniquely to Jordan-$*$-isomorphisms from $M(B)$ to $M(A)$.*

Proof. In view of the previous comments, we only must prove the uniqueness of Jordan-$*$-isomorphisms from $M(B)$ to $M(A)$ extending a given Jordan-$*$-isomorphism (say G) from B to A. But, if R and S are Jordan-$*$-isomorphisms from $M(B)$ to $M(A)$ extending G, then for b in B and x in $M(B)$ we have

$$(R(x) - S(x)) \circ G(b) = R(x) \circ R(b) - S(x) \circ S(b)$$

$$= R(x \circ b) - S(x \circ b) = G(x \circ b) - G(x \circ b) = 0.$$

■

Theorem 6.7. *Let A be an alternative C^*-algebra, B a non-commutative JB^*-algebra, and F a mapping from B to A. Then F is a surjective linear isometry (if and) only if there exists a Jordan-$*$-isomorphism $G : B \to A$, and a unitary element u in $M(A)$ satisfying $F = T_uG$.*

Proof. Assume that F is a surjective linear isometry. Put $u := F''(\mathbf{1})$. By Lemma 6.4, u is a unitary element of A'', and, by Lemma 6.6, u lies in $M(A)$. Write $G := T_{u^*}T$. Then G is a Jordan-$*$-isomorphism from B to A because it is a surjective linear isometry satisfying $G''(\mathbf{1}) = \mathbf{1}$, and therefore Theorem 6.2 successfully applies. Finally, the equality $F = T_uG$ is clear. ■

To conclude the discussion about verbatim non-associative versions of Theorem 6.1, we show that alternative C^*-algebras are also the unique non-commutative JB^*-algebras which can play the role of A in the "only if" part of such versions. Let A be a non-commutative JB^*-algebra, and u a unitary element of $M(A)$. It is easily deducible from [36, Section I.12] and [70, Corollary 2.5] that the Banach space of A with product $\circ_u$ and involution $*_u$ defined by $a \circ_u b := U_{a,b}(u^*)$ and $a^{*_u} := U_u(a^*)$, respectively, becomes a (commutative) JB^*-algebra. Such a JB^*-algebra will be denoted by $A(u)$.

Proposition 6.8. *Let A be a non-commutative JB^*-algebra which is not an alternative C^*-algebra. Then there exists a non-commutative JB^*-algebra B, and a surjective linear isometry $F : B \to A$ which cannot be written in the form T_uG with u a unitary element in $M(A)$ and G a Jordan-$*$-isomorphism from B to A.*

Proof. By Proposition 6.3, there is a unitary element v in $M(A)$ such that T_v is not an isometry on A. Take B equal to $A(v)$, and $F : B \to A$ equal to the identity mapping. Assume that $F = T_uG$ for some unitary u in $M(A)$ and some Jordan-$*$-isomorphism G from B to A. Noticing that the JB^*-algebra B'' is nothing but $A''(v)$ (by the w^*-continuity of the JB^*-involutions and the separate w^*-continuity of the products on JBW^*-algebras), we have $G''(v) = \mathbf{1}$ (because v is the unit of $A''(v)$ and G'' is a Jordan-$*$-isomorphism), and hence $F''(v) = u$. Therefore $v = u$ (because F is the identity mapping). Finally, the equality $F = T_vG$ implies that T_v is an isometry, contrarily to the choice of v. ■

Now that the verbatim non-associative variant of Theorem 6.1 has been altogether discussed, we pass to consider the consequence of that theorem that *linearly isometric C^*-algebras are Jordan-$*$-isomorphic.* In the unital case, the non-associative variant of such an assertion has been fully discussed in [13, Section 5]. In fact, as the next example shows, *linearly isometric non-commutative JB^*-algebras need not be Jordan-$*$-isomorphic.*

Example 6.9 ([13, Example 5.7]). JC^***-algebras** are defined as those JB^*-algebras which can be seen as closed *-invariant subalgebras of A^+ for some C^*-algebra A. Let C be the unital simple JC^*-algebra of all symmetric 2×2-matrices over $\mathbb{C}$, put $S := \{z \in \mathbb{C} : |z| = 1\}$, let A stand for the unital JC^*-algebra of all continuous complex-valued functions from S to C, consider the unitary element u of A defined by $u(s) := diag\{s, 1\}$ for every s in S, and put $B := A(u)$. Then A and B are linearly isometric JB^*-algebras, but they are not Jordan-$*$-isomorphic.

For a non-commutative JB^*-algebra A, consider the property *(P)* which follows.

(P) Non-commutative JB^*-algebras which are linearly isometric to A are in fact Jordan-$*$-isomorphic to A.

Despite the above example, the class of those non-commutative JB^*-algebras A satisfying Property *(P)* is reasonably wide, and in fact much larger than that of alternative C^*-algebras. The verification of this fact relies on the next theorem. We remark that, if u is a unitary element in the multiplier non-commutative JB^*-algebra of a non-commutative JB^*-algebra A, then the operator U_u (acting on A) is a surjective linear isometry on A.

Theorem 6.10. *Let A be a non-commutative JB^*-algebra. The following assertions are equivalent:*

1. *For every non-commutative JB^*-algebra B, and every surjective linear isometry $F : B \to A$, there exists a Jordan-$*$-isomorphism $G : B \to A$, and a unitary element u in $M(A)$ satisfying $F = U_u G$.*

2. *For each unitary element v of $M(A)$ there is a unitary element u in $M(A)$ such that $u^2 = v$.*

Proof. $1 \Rightarrow 2$. Let v be a unitary element of $M(A)$. Take B equal to $A(v)$, and $F : B \to A$ equal to the identity mapping. By the assumption 1, we have $F = U_u G$ for some unitary u in $M(A)$ and some Jordan-$*$-isomorphism G from B to A. Arguing as in the proof of Proposition 6.8, we find $G''(v) = \mathbf{1}$, and hence $F''(v) = u^2$. Therefore $v = u^2$ (because F is the identity mapping).

$2 \Rightarrow 1$. Let B be a non-commutative JB^*-algebra, and $F : B \to A$ a surjective linear isometry. Put $v := F''(\mathbf{1})$. By Remark 6.5, v is a unitary element of A'', and, by Lemma 6.6, v belongs to $M(A)$. By the assumption 2, there is a unitary element u in $M(A)$ with $u^2 = v$. Write $G := U_{u^*} F$. Then G is a Jordan-$*$-isomorphism from B to A because it is a surjective linear isometry satisfying $G''(\mathbf{1}) = \mathbf{1}$, and Theorem 6.2 applies. On the other hand, the equality $F = U_u G$ is clear. ∎

Remark 6.11. An argument similar to the one in the above proof allows us to obtain the following variant of Theorem 6.10. Indeed, *given a non-commutative JB^*-algebra A, the following assertions on A are equivalent:*

1. *For every non-commutative JB^*-algebra B, and every surjective linear isometry $F : B \to A$, there exists a Jordan-$*$-isomorphism $G : B \to A$, together with unitary elements $u_1, ..., u_n$ in $M(A)$, satisfying $F = U_{u_1} U_{u_2} ... U_{u_n} G$.*

2. *For each unitary element v of $M(A)$ there are unitary elements $u_1, ..., u_n$ in $M(A)$ such that $U_{u_1} U_{u_2} ... U_{u_n}(\mathbf{1}) = v$.*

The next corollary extends [13, Lemma 5.2] in several directions.

Corollary 6.12. *A non-commutative JB^*-algebra A satisfies Property (P) whenever one of the following conditions is fulfilled:*

1. *A is of the form $B^{(\lambda)}$ for some alternative C^*-algebra B and some $0 \leq \lambda \leq 1$.*

2. *For each unitary element v of $M(A)$ there is a unitary element u in $M(A)$ such that $u^2 = v$.*

3. *A is a non-commutative JBW^*-algebra.*

Proof. Both Conditions 1 and 2 are sufficient for Property *(P)* in view of Theorems 6.7 and 6.10, respectively. To conclude the proof, we realize that Condition 3 implies Condition 2. Indeed, if A is a non-commutative JBW^*-algebra, and if v is a unitary element in A, then the w^*-closure of the subalgebra of A generated by $\{v, v^*\}$ is an (associative) W^*-algebra, and it is well-known that W^*-algebras fulfill Condition ii). ∎

We conclude this section by determining the hermitian operators on a non-commutative JB^*-algebra. Our determination generalizes and unifies both that of Paterson-Sinclair [47] for the associative case and that of M. A. Youngson [73] for the unital non-associative case.

Theorem 6.13. *Let A be a non-commutative JB^*-algebra, and R a bounded linear operator on A. Then R is hermitian if and only if it can be expressed in the form $T_x + D$ for some self-adjoint element x of $M(A)$ and some Jordan-derivation D of A anticommuting with the JB^*-involution of A.*

Proof. Let x be in $(M(A))_{sa}$. Since the mapping $y \to T_y$ from $M(A)$ to $BL(A)$ is a linear isometry sending $\mathbf{1}$ to I_A, and the equality $(M(A))_{sa} = H(M(A), \mathbf{1})$ holds, we obtain that T_x belongs to $H(BL(A), I_A)$, i.e., T_x is an hermitian operator on A. Now let D be a Jordan-derivation of A anticommuting with the JB^*-involution of A. Then, for every λ in $\mathbb{R}$, $\exp(i\lambda D)$ is a Jordan-$*$-automorphism of A, and hence we have $\| \exp(i\lambda D) \| = 1$, i.e., D is a hermitian operator on A.

Conversely, let R be an hermitian operator on A. Then, for λ in $\mathbb{R}$, $\exp(i\lambda R)$ is a surjective linear isometry on A, so, by Lemma 6.6, we have

$$\exp(i\lambda R'')(M(A)) = (\exp(i\lambda R))''(M(A)) = M(A) \ ,$$

and so

$$x := R''(\mathbf{1}) = \lim_{\lambda \to 0} \frac{\exp(i\lambda R'')(\mathbf{1}) - \mathbf{1}}{i\lambda}$$

lies in $M(A)$. On the other hand, since R'' is a hermitian operator on A'', and the mapping $S \to S(\mathbf{1})$ from $BL(A'')$ to A'' is a linear contraction sending $I_{A''}$ to $\mathbf{1}$, we deduce that x belongs to $H(A'', \mathbf{1})$. It follows that x lies in $(M(A))_{sa}$. Put $D := R - T_x$. By the first paragraph of the proof, D is a hermitian operator on A. Now D'' is a hermitian operator on A'' with $D''(\mathbf{1}) = 0$, so that, by [73, Theorem 11], D'' is a Jordan-derivation of A'' anticommuting with the JB^*-involution of A''. Therefore D is a Jordan-derivation of A anticommuting with the JB^*-involution of A. Since the equality $R = T_x + D$ is obvious, the proof is concluded. ∎

7. Notes and remarks

7.1.- The following refinement of Theorem 1.4 is proved in [16]. *If A is a unital complete normed complex algebra, and if there exists a conjugate-linear vector space involution* □

on A satisfying $\mathbf{1}^\square = \mathbf{1}$ and

$$\| a^\square a \| = \| a^\square \| \| a \|$$

for every a in A, then A is an alternative C^*-algebra for some C^*-involution $*$. If in addition the dimension of A is different from 2, then we have $\square = *$.

7.2.- As noticed in [58, Corollary 1.2], the proof of Theorem 2.2 given in [48] allows us to realize that, *if a normed complex algebra B is isometrically Jordan-isomorphic to a non-commutative JB^*-algebra, then B is a non-commutative JB^*-algebra.* On the other hand, it is known that *the norm of every non-commutative JB^*-algebra A is minimal*, i.e., if $|||.|||$ is an algebra-norm on A satisfying $|||.||| \leq \| \cdot \|$, then we have in fact $|||.||| = \| \cdot \|$ [51, Proposition 11]. Now, keeping in mind the above results, we can show that, *if a normed complex algebra B is the range of a contractive Jordan-homomorphism from a non-commutative JB^*-algebra, then B is a non-commutative JB^*-algebra.* The proof goes as follows. Let A be a non-commutative JB^*-algebra and φ a contractive Jordan-homomorphism from A onto the normed complex algebra B. Since closed ideals of A^+ are ideals of A (a consequence of [48, Theorem 4.3]), and quotients of non-commutative JB^*-algebras by closed ideals are non-commutative JB^*-algebras [48, Corollary 1.11], we may assume that φ is injective. Then we can define an algebra norm $|||.|||$ on A^+ by $|||a||| := \| \varphi(a) \|$. Since φ is contractive, and the norm of A^+ is minimal we obtain that $|||.||| = \| \cdot \|$ on A. Now B is isometrically Jordan-isomorphic to A, and hence B is a non-commutative JB^*-algebra.

The result just proved implies that, *if a normed complex alternative algebra B is the range of a contractive Jordan-homomorphism from a non-commutative JB^*-algebra, then B is a an alternative C^*-algebra* (compare [51, Corollary 12]).

7.3.- *Every non-commutative JB^*-algebra A has minimum norm topology*, i.e., the topology of an arbitrary algebra norm on A is always stronger than that of the natural norm (see [5], [51], and [24]). As pointed out in [51, Remark 14], this fact can be applied, together with [49, Theorem 3.5], to derive the theorem, originally due to J. E. Galé [28], asserting that, *if a normed complex algebra B is the range of a weakly compact homomorphism from a non-commutative JB^*-algebra, then B is bicontinuously isomorphic to a finite direct sum of simple non-commutative JB^*-algebras which are either quadratic or finite-dimensional.*

Now, *let B be a normed complex alternative algebra, and assume that B is the range of a weakly compact Jordan-homomorphism from a non-commutative JB^*-algebra.* By the above, B^+ is a finite direct sum of simple ideals which are either quadratic or finite-dimensional. Moreover, such ideals of B^+ are in fact ideals of B (use that, for b in B, the mapping $x \to [b, x]$ is a derivation of B^+, and the folklore fact that direct summands of semiprime algebras are invariant under derivations [58, Lemma 7.5]). Therefore those simple direct summands of B^+ which are quadratic actually are simple quadratic alternative algebras, and hence finite-dimensional [76, Theorems 2.3.4 and 2.2.1]. *Then B is finite-dimensional.*

We note also that *the range of any weakly compact Jordan-homomorphism from an alternative C^*-algebra into a complex normed algebra is finite-dimensional* (compare [51, Corollary 13]). The proof of this assertion involves no new idea, and hence is left to the reader.

7.4.- Most criteria of associativity and commutativity for non-commutative JB^*-algebras reviewed in Section 2 rely in the fact that *a non-commutative JB^*-algebra is associative and commutative if (and only if) it has no non-zero nilpotent element* [33]. A recent related result is the one in [6] that *a non-commutative JB^*-algebra A is commutative if (and only if) there exits a positive constant k satisfying* $\| ab \| \leq k \| ba \|$ for all a, b in A.

7.5.- The structure theorem for prime non-commutative JB^*-algebras (Theorem 3.2) becomes a natural analytical variant of the classification theorem for prime nondegenerate non-commutative Jordan algebras, proved by W. G. Skosyrskii [66]. We recall that a non-commutative Jordan algebra A is said to be **nondegenerate** if the conditions $a \in A$ and $U_a = 0$ imply $a = 0$. As it always happens whenever people work with quite general assumptions, the conclusion in Skosyrskii's theorem becomes lightly rough, and involves some complicated notions, like that of a "central order in an algebra", or that of a "quasi-associative algebra over its extended centroid". However, in a "tour de force", Theorem 3.2 actually can be derived from Skosyrskii's classification and some early known results on non-commutative JB^*-algebras. This is explained in what follows.

We begin by establishing a purely algebraic corollary to Skosyrskii's theorem, whose formulation avoids the "complications" quoted above. According to [25] (see also [15]) a prime algebra A over a field $\mathbb{F}$ is called **centrally closed** over $\mathbb{F}$ if, for every non zero ideal M of A and for every linear mapping $f : M \to A$ satisfying $f(ax) = af(x)$ and $f(xa) = f(x)a$ for all x in M and a in A, there exists λ in $\mathbb{F}$ such that $f(x) = \lambda x$ for every x in M. Now it follows from the main result in [66] that, *if A is a centrally closed prime nondegenerate non-commutative Jordan algebra over an algebraically closed field $\mathbb{F}$, then at least one of the following assertions hold:*

1. *A is commutative.*
2. *A is quadratic (over $\mathbb{F}$).*
3. *A^+ is associative.*
4. *A is* **split quasi-associative** *over $\mathbb{F}$, i.e., there exists an associative algebra B over $\mathbb{F}$, and some λ in $\mathbb{F} \setminus \{1/2\}$ such that A and B coincide as vector spaces, but the product ab of A is related to the one $a\Box b$ of B by means of the equality $ab = \lambda a\Box b + (1-\lambda) b\Box a$.*

We do not know whether the result just formulated is or not explicitly stated in Skosyrskii's paper (since it is written in Russian, and we only know about its main result thanks to the appropriate note in Mathematical Reviews). In any case, the steps to derive the corollary above from the main result of [66] are not difficult, and therefore are left to the reader.

Now let A be a prime non-commutative JB^*-algebra which is neither commutative nor quadratic. Since, clearly, non-commutative JB^*-algebras are nondegenerate, and prime non-commutative JB^*-algebras are centrally closed [63], the above paragraph applies, so that either A^+ is associative or A is split quasi-associative over $\mathbb{C}$. In the last case, it follows easily from [57, Theorem 2] and [56, Lemma] that A is of the form $C^{(\lambda)}$ for some prime C^*-algebra C and some $\frac{1}{2} < \lambda \leq 1$. Assume that A^+ is associative. Then, since A^+ is a JB^*-algebra, A^+ actually is a commutative C^*-algebra. Since commutative C^*-algebras have no non zero derivations [64, Lemma 4.1.2], and for a in A the mapping

$b \to ab - ba$ from A to A is a derivation of A^+ [65, p. 146], we have $A = A^+$, and therefore A is commutative. Now Theorem 3.2 has been re-proved.

7.6.- Prime non-commutative JB^*-algebras with non zero socle were precisely classified in [52, Theorem 1.4]. As a consequence of such a classification, *prime non-commutative JB^*-algebras with non zero socle are either commutative, quadratic, or of the form $C^{(\lambda)}$ for some prime C^*-algebra C with non zero socle, and some $\frac{1}{2} < \lambda \leq 1$.* We note that all quadratic non-commutative JB^*-algebras have non zero socle.

7.7.- It is shown in [62, Theorem A] that, *given a C^*-algebra A, a (possibly non-associative) normed complex algebra B having an approximate unit bounded by one, and a surjective linear isometry $F : B \to A$, there exists an isometric Jordan-isomorphism $G : B \to A$, and a unitary element u in $M(A)$ satisfying $F = T_u G$.* It is worth mentioning that the above result follows straightforwardly from Corollary 4.9 and Theorem 6.7. Even Corollary 4.9 and Theorem 6.7 give rise to the result in [62] just quoted with "alternative C^*-algebra" instead of "C^*-algebra". This "alternative" generalization of [62, Theorem A] motivated most results collected in Sections 4, 5, and 6 of the present paper. In a first attempt we tried to obtain such a generalization by replacing associativity with alternativity in the original arguments of [62]. All things worked without relevant problems until the application made in [62] of a result of C. A. Akemann and G. K. Pedersen [1] asserting that, *if A is a C^*-algebra, and if u is a unitary element in A'' such that au^*a lies in A for every a in A, then u belongs to $M(A)$.* In that time we were unable to avoid associativity in the proof of the result of [1] just mentioned, and were obliged to get round the handicap and design a new strategy, which apparently avoided the "alternative" version of the result of [1], and whose main ingredient was Theorem 4.6. Now we claim that Theorem 4.6 germinally contains the "alternative" generalization of the Akemann-Pedersen result, and even its general non-associative extension. The proof of the claim goes as follows.

Let A be a non-commutative JB^*-algebra, and u a unitary element in A'' such that $U_a(u^*)$ lies in A for every a in A. Regarding A (and hence A'') as a JB^*-triple in its canonical way, we have that $\{aub\} = U_{a,b}(u^*)$ belongs to A whenever a and b are in A. Therefore we can consider the complete normed complex algebra B consisting of the Banach space of A and the product $a \circ b := \{aub\}$ (apply either [70, Corollary 2.5] or [27, Corollary 3] for the submultiplicativity of the norm). Let us also consider the JB^*-algebra $A''(u)$ in the meaning explained immediately before Proposition 6.8. Then the product of $A''(u)$ is a product on the Banach space of B'', which extends the product of B and is w^*-continuous in its first variable. Therefore B'' (with the Arens product relative to that of B) coincides with $A''(u)$, and hence is a JB^*-algebra. Now Theorem 4.6 applies, so that B is invariant under the JB^*-involution of $B'' = A''(u)$. Therefore $\{uau\} = U_u(a*)$ lies in A whenever a is in A. By the main identity of JB^*-triples, for a, b in A we have

$$\{uab\} = 2\{uab\} - \{u\{uba\}u\} = 2\{ua\{buu\}\} - \{u\{uba\}u\} = \{bu\{uau\}\} \ ,$$

and hence $\{uab\}$ lies in A. This proves that u lies in $\mathcal{M}(A)$. *Then*, by Proposition 5.7, *u belongs to $M(A)$*. As a consequence of the fact just proved, *if A is an alternative C^*-algebra, and if u is a unitary element in A'' such that au^*a lies in A for every a in A, then u belongs to $M(A)$.*

7.8.- In the proof of Theorem 4.4 we applied a characterization of Banach spaces underlying unital JB^*-algebras proved in [13]. An independent characterization of such Banach spaces is the one given in [61] that *a non zero complex Banach space X underlies a JB^*-algebra with unit u if and only if $\|u\| = 1$, $X = H(X, u) + iH(X, u)$, and*

$$\inf\{\|f\| : f \in \Pi(X) \text{ and } f(x, u) = f(u, x) = x \ \forall x \in X\} = 1.$$

Here, as in the case that X is a norm-unital complete normed complex algebra and u is the unit of X, $H(X, u)$ stands for the set of those elements h in X such that $V(X, u, h) \subset \mathbb{R}$. A refinement of the result of [61] just quoted can be found in [60, Theorem 4.4].

7.9.- *If A and B are non-commutative JB^*-algebras, and if $F : A \to B$ is an isomorphism, then there exists a $*$-isomorphism $G : A \to B$, and a derivation D of A anticommuting with the JB^*-involution of A, such that $F = G\exp(D)$* [48, Theorem 2.9]. It follows from this result and Example 6.9 that *linearly isometric non-commutative JB^*-algebras need not be Jordan-isomorphic.* A similar pathology does not occur for JB-algebras [35].

Acknowledgments.- Part of this work was done while the third author was visiting the University of Almería. He is grateful to the Department of Algebra and Mathematical Analysis of that university for its hospitality and support.

REFERENCES

[1] C. A. AKEMANN and G. K. PEDERSEN, Complications of semicontinuity in C^*-algebra theory. *Duke Math. J.* **40** (1973), 785-795.

[2] E. M. ALFSEN, F. W. SHULTZ, and E. STORMER, A Gelfand-Neumark theorem for Jordan algebras. *Adv. Math.* **28** (1978), 11-56.

[3] K. ALVERMANN and G. JANSSEN, Real and complex non-commutative Jordan Banach algebras. *Math. Z.* **185** (1984), 105-113.

[4] R. ARENS, The adjoint of a bilinear operation. *Proc. Amer. Math. Soc.* **2** (1951), 839-848.

[5] A. BENSEBAH, Weakness of the topology of a JB^*-algebra. *Canad. Math. Bull.* **35** (1992), 449-454.

[6] M. BENSLINANE, L. MESMOUDI, and A. RODRIGUEZ, A characterization of commutativity for non-associative normed algebras. *Extracta Math.* **15** (2000), 121-143.

[7] H. F. BOHNENBLUST and S. KARLIN, Geometrical properties of the unit sphere of a Banach algebra. *Ann. Math.* **62** (1955), 217-229.

[8] F. F. BONSALL and J. DUNCAN, *Numerical ranges of operators on normed spaces and of elements of normed algebras.* London Math. Soc. Lecture Note Series **2**, Cambridge, 1971.

[9] F. F. BONSALL and J. DUNCAN, *Numerical ranges II.* London Math. Soc. Lecture Note Series **10**, Cambridge, 1973.

[10] F. F. BONSALL and J. DUNCAN, *Complete normed algebras.* Springer-Verlag, Berlin, 1973.

[11] R. B. BRAUN, Structure and representations of non-commutative C^*-Jordan algebras. *Manuscripta Math.* **41** (1983), 139-171.

[12] R. B. BRAUN, A Gelfand-Neumark theorem for C^*-alternative algebras. *Math. Z.* **185** (1984), 225-242.

[13] R. B. BRAUN, W. KAUP, and H. UPMEIER, A holomorphic characterization of Jordan C^*-algebras. *Math. Z.* **161** (1978), 277-290.

[14] L. J. BUNCE and C. H. CHU, Compact operations, multipliers and Radon-Nikodym property in JB^*-triples. *Pacific J. Math.* **153** (1992), 249-265.

[15] M. CABRERA and A. RODRIGUEZ, Extended centroid and central closure of semiprime normed algebras: a first approach. *Commun. Algebra* **18** (1990), 2293-2326.

[16] M. CABRERA and A. RODRIGUEZ, New associative and nonassociative Gelfand-Naimark theorems. *Manuscripta Math.* **79** (1993), 197-208.

[17] C. H. CHU, B. IOCUM, and G. LOUPIAS, Grothendieck's theorem and factorization of operators in Jordan triples. *Math. Ann.* **284** (1989), 41-53.

[18] C. H. CHU, A. MORENO, and A. RODRIGUEZ, On prime JB^*-triples. In *Function Spaces; Proceedings of the Third Conference on Function Spaces, May 19-23, 1998, Southern Illinois University at Edwardsville* (Ed. K. Jarosz), *Contemporary Math.* **232** (1999), 105-109.

[19] M. J. CRABB, J. DUNCAN, and C. M. McGREGOR, Characterizations of commutativity for C^*-algebras. *Glasgow Math. J.* **15** (1974), 172-175.

[20] S. DINEEN, The second dual of a JB^*-triple system. In *Complex Analysis, Functional Analysis and Approximation Theory* (ed. by J. Múgica), 67-69, North-Holland Math. Stud. **125**, North-Holland, Amsterdam-New York, 1986.

[21] J. DUNCAN, Review of [44]. *Math. Rev.* **87e**:46067.

[22] C. M. EDWARDS, On Jordan W^*-algebras. *Bull. Sci. Math.* **104** (1980), 393-403.

[23] C. M. EDWARDS, Multipliers of JB-algebras. *Math. Ann.* **249** (1980), 265-272.

[24] S. EJAZIAN and A. NIKNAM, A Kaplansky theorem for JB^*-algebras. *Rocky Mt. J. Math.* **28** (1988), 977-962.

[25] T. S. ERICKSON, W. S. MARTINDALE 3*rd*, and J. M. OSBORN, Prime nonassociative algebras. *Pacific J. Math.* **60** (1975), 49-63.

[26] A. FERNANDEZ, E. GARCIA, and A. RODRIGUEZ, A Zel'manov prime theorem for JB^*-algebras. *J. London Math. Soc.* **46** (1992), 319-335.

[27] Y. FRIEDMAN and B. RUSSO, The Gelfand-Naimark theorem for JB^*-triples. *Duke Math. J.* **53** (1986), 139-148.

[28] J. E. GALE, Weakly compact homomorphisms in non-associative algebras. In *Nonassociative algebraic models* (Ed. S. González and H. Ch. Myung), 167-171, Nova Science Publishers, New York, 1992.

[29] H. HANCHE-OLSEN and E. STORMER, *Jordan operator algebras*. Monograph Stud. Math. **21**, Pitman, 1984.

[30] S. HEINRICH, Ultraproducts in Banach space theory. *J. Reine Angew. Math.* **313** (1980), 72-104.

[31] G. HORN, Coordinatization theorem for JBW^*-triples. *Quart. J. Math. Oxford* **38** (1987), 321-335.

[32] T. HURUYA, The normed space numerical index of C^*-algebras. *Proc. Amer. Math. Soc.* **63** (1977), 289-290.

[33] B. IOCHUM, G. LOUPIAS, and A. RODRIGUEZ, Commutativity of C^*-algebras and

associativity of JB^*-algebras. *Math. Proc. Cambridge Phil. Soc.* **106** (1989), 281-291.

[34] J. M. ISIDRO, W. KAUP, and A. RODRIGUEZ, On real forms of JB^*-triples. *Manuscripta Math.* **86** (1995), 311-335.

[35] J. M. ISIDRO and A. RODRIGUEZ, Isometries of JB-algebras. *Manuscripta Math.* **86** (1995), 337-348.

[36] N. JACOBSON, *Structure and representations of Jordan algebras.* Amer. Math. Soc. Coll. Publ. **39**, Providence, Rhode Island 1968.

[37] R. V. KADISON, Isometries of operator algebras. *Ann. Math.* **54** (1951), 325-338.

[38] A. M. KAIDI, J. MARTINEZ, and A. RODRIGUEZ, On a non-associative Vidav-Palmer theorem. *Quart. J. Math. Oxford* **32** (1981), 435-442.

[39] A. KAIDI, A MORALES, and A. RODRIGUEZ, Geometric properties of the product of a C^*-algebra. *Rocky Mt. J. Math.* (to appear).

[40] A. KAIDI, A MORALES, and A. RODRIGUEZ, Prime non-commutative JB^*-algebras. *Bull. London Math. Soc.* **32** (2000), 703-708.

[41] A. KAIDI, A MORALES, and A. RODRIGUEZ, A holomorphic characterization of C^*- and JB^*-algebras. *Manuscripta Math.* (to appear).

[42] W. KAUP, Algebraic characterization of symmetric complex Banach manifolds. *Math. Ann.* **228** (1977), 39-64.

[43] W. KAUP, A Riemann mapping theorem for bounded symmetric domains in complex Banach spaces. *Math. Z.* **183** (1983), 503-529.

[44] J. MARTINEZ, J. F. MENA, R. PAYA, and A. RODRIGUEZ, An approach to numerical ranges without Banach algebra theory. *Illinois J. Math.* **29** (1985), 609-625.

[45] A. MORENO and A. RODRIGUEZ, On the Zel'manovian classification of prime JB^*-triples. *J. Algebra* **226** (2000), 577-613.

[46] K. McCRIMMON, Noncommutative Jordan rings. *Trans. Amer. Math. Soc.* **158** (1971), 1-33.

[47] A. L. T. PATERSON and A. M. SINCLAIR, Characterisation of isometries between C^*-algebras. *J. London Math. Soc.* **5** (1972), 755-761.

[48] R. PAYA, J. PEREZ, and A. RODRIGUEZ, Non-commutative Jordan C^*-algebras. *Manuscripta Math.* **37** (1982), 87-120.

[49] R. PAYA, J. PEREZ, and A. RODRIGUEZ, Type I factor representations of non-commutative JB^*-algebras. *Proc. London Math. Soc.* **48** (1984), 428-444.

[50] G. K. PEDERSEN, *C^*-algebras and their automorphism groups.* Academic Press, London 1979.

[51] J. PEREZ, L. RICO, and A. RODRIGUEZ, Full subalgebras of Jordan-Banach algebras and algebra norms on JB^*-algebras. *Proc. Amer. Math. Soc.* **121** (1994), 1133-1143.

[52] J. PEREZ, L. RICO, A. RODRIGUEZ, and A. R. VILLENA, Prime Jordan-Banach algebras with non-zero socle. *Commun. Algebra* **20** (1992), 17-53.

[53] A. RODRIGUEZ, A Vidav-Palmer theorem for Jordan C^*-algebras and related topics. *J. London Math. Soc.* **22** (1980), 318-332.

[54] A. RODRIGUEZ, Non-associative normed algebras spanned by hermitian elements. *Proc. London Math. Soc.* **47** (1983), 258-274.

[55] A. RODRIGUEZ, A note on Arens regularity. *Quart. J. Math. (Oxford)* **38** (1987), 91-93.

[56] A. RODRIGUEZ, Mutations of C^*-algebras and quasiassociative JB^*-algebras. *Collectanea Math.* **38** (1987), 131-135.

[57] A. RODRIGUEZ, Jordan axioms for C^*-algebras. *Manuscripta Math.* **61** (1988), 279-314.

[58] A. RODRIGUEZ, An approach to Jordan-Banach algebras from the theory of nonassociative complete normed algebras. In *Algèbres de Jordan and algèbres normées non associatives, Ann. Sci. Univ. "Blaise Pascal", Clermont II, Sér. Math.* **27** (1991), 1-57.

[59] A. RODRIGUEZ, Jordan structures in Analysis. In *Jordan algebras, Proceedings of the Conference held in Oberwolfach, Germany, August 9-15, 1992* (Ed. W. Kaup, K. McCrimmon, and H. P. Petersson), 97-186, Walter de Gruyter, Berlin-New York 1994.

[60] A. RODRIGUEZ, Multiplicative characterization of Hilbert spaces and other interesting classes of Banach spaces. In *Encuentro de Análisis Matemático,* Homenaje al profesor B. Rodríguez-Salinas (Ed. F. Bombal, F. L. Hernández, P. Jiménez y J. L. de María), *Rev. Mat. Univ. Complutense Madrid* **9** (1996), 149-189.

[61] A. RODRIGUEZ, A Banach space characterization of JB^*-algebras. *Math. Japon.* **45** (1997), 217-221.

[62] A. RODRIGUEZ, Isometries and Jordan-isomorphisms onto C^*-algebras. *J. Operator Theory* **40** (1998), 71-85.

[63] A. RODRIGUEZ and A. R. VILLENA, Centroid and extended centroid of JB^*-algebras. In *Nonassociative algebraic models* (Ed. S. González and H. Ch. Myung), 223-232, Nova Science Publishers, New York, 1992.

[64] S. SAKAI, *C^*- and W^*-algebras.* Springer-Verlag, Berlin, 1971.

[65] R. D. SCHAFER, *An introduction to nonassociative algebras.* Academic Press, New York, 1966.

[66] V. G. SKOSYRSKII, Strongly prime noncommutative Jordan algebras (in Russian). *Trudy Inst. Mat. (Novosibirk)* **16** (1989), 131-164, 198-199, *Math. Rev.* **91b**:17001.

[67] H. UPMEIER, *Symmetric Banach Manifolds and Jordan C^*-algebras.* North-Holland, Amsterdam, 1985.

[68] A. R. VILLENA, *Algunas cuestiones sobre las JB^*-álgebras: centroide, centroide extendido y zócalo.* Tesis Doctoral, Universidad de Granada 1990.

[69] J. D. M. WRIGHT, Jordan C^*-algebras. *Michigan Math. J.* **24** (1977), 291-302.

[70] J. D. M. WRIGHT and M. A. YOUNGSON, A Russo Dye theorem for Jordan C^*-algebras. In *Functional analysis: surveys and recent results* (Ed. by K. D. Bierstedt and F. Fuchssteiner), 279-282, North Holland Math. Studies **27**, Amsterdam, 1977.

[71] J. D. M. WRIGHT and M. A. YOUNGSON, On isometries of Jordan algebras. *J. London Math. Soc.* **17** (1978), 339-344.

[72] M. A. YOUNGSON, A Vidav theorem for Banach Jordan algebras. *Math. Proc. Cambridge Philos. Soc.* **84** (1978), 263-272.

[73] M. A. YOUNGSON, Hermitian operators on Banach Jordan algebras, *Proc. Edinburgh Math. Soc.* **22** (1979), 93-104.

[74] M. A. YOUNGSON, Non unital Banach Jordan algebras and C^*-triple systems. *Proc. Edinburgh Math. Soc.* **24** (1981), 19-31.

[75] E. I. ZEL'MANOV, On prime Jordan triple systems III. *Siberian Math. J.* **26** (1985), 71-82.

[76] K. A. ZHEVLAKOV, A. M. SLIN'KO, I. P. SHESTAKOV, and A. I. SHIRSHOV, *Rings that are nearly associative*. Academic Press, New York, 1982.