Absolute-valued algebras are fully surveyed. Some attention is also payed to Banach spaces underlying complete absolute-valued algebras.

Introduction

Absolute-valued algebras are defined as those real or complex algebras $A$ satisfying $\|xy\| = \|x\| \|y\|$ for a given norm $\| \cdot \|$ on $A$, and all $x, y \in A$. Despite the nice simplicity of the above definition, absolute-valued algebras have not attracted the attention of too many people. A reason could be that, in the presence of associativity, the axiom $\|xy\| = \|x\| \|y\|$ is extremely obstructive. Indeed, according to an old theorem of S. Mazur, there are only three absolute-valued associative real algebras. Nevertheless, when associativity is removed, absolute-valued algebras do exist in abundance. Some facts corroborating the above assertion are that every complete normed algebra is isometrically algebra-isomorphic to a quotient of a complete absolute-valued algebra (Corollary 3.2), and that every Banach space is linearly isometric to a subspace of a complete absolute-valued algebra (Theorem 5.1). Anyway, the quantity and quality of works on absolute-valued algebras seemed to us enough to deserve a detailed survey paper like the one we are just beginning.

Our paper collects the results on absolute-valued algebras since the pioneering works of Ostrowski, Mazur, Albert, and Wright (see Subsection 1.3) to the more recent developments. The inflexion point in the

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theory, namely the Urbanik-Wright paper \cite{106}, is fully reviewed (see Subsections 2.1, 2.2, 2.3, and 3.1). Among the recent developments, we emphasize the solution \cite{57} to Albert’s old problem \cite{3} if every absolute-valued algebraic algebra is finite-dimensional (see Subsection 2.7), and the study of Banach spaces underlying complete absolute-valued algebras, done in \cite{7} and \cite{69} (see Section 5). A special attention is also payed to the intermediate works of K. Urbanik (\cite{101} to \cite{104}) and M. L. El-Mallah (\cite{35} to \cite{46}). This is done in Subsections 2.4, 2.5, 2.7, 3.1, 3.2, and 3.4. Contributions of other authors (including the one of this paper) are also reviewed (see mainly Subsections 1.4, 2.6, 3.5, and 3.6, and the whole Section 4). The clarifications of the theory at some precise points, done by Gleichgewicht \cite{49} and Elduque-Pérez \cite{33}, are inserted in the appropriate places (see Subsection 3.4, and Subsections 1.3, 2.1, and 3.5, respectively). Our paper contains also some new results, and several new proofs of known results. Known proofs have been included only when they seemed to us specially illuminating.

As far as we know, absolute-valued algebras have been surveyed in exclusive several times (see \cite{86}, \cite{91}, and \cite{110}), but in references not easily available, and never in English. Moreover, references \cite{91} and \cite{110} are relatively short, and references \cite{86} and \cite{110} become today rather obsolete. On the other hand, there are also survey papers on more general topics, devoting to absolute-valued algebras some attention (see \cite{87} and \cite{88}). Finally, let us note that the Ph. Theses \cite{35}, \cite{63}, and \cite{78} are devoted to absolute-valued algebras, and contain both reviews of other people’s results and proofs of results of their respective authors.

1. Finite-dimensional absolute-valued algebras

1.1. Some basic definitions and facts

By an algebra over a field \(F\) we mean a vector space \(A\) over \(F\) endowed with a bilinear mapping \((x, y) \rightarrow xy\) from \(A \times A\) to \(A\) called the product of the algebra \(A\). Algebras in this paper are assumed to be nonzero, but are not assumed to be associative, nor to have a unit element. We suppose that the reader is familiarized with the basic terminology in the theory of algebras. Thus, terms as subalgebra, ideal, or algebra homomorphism are not defined here. For an element \(x\) in an algebra \(A\), we denote by \(L_x\) (respectively, \(R_x\)) the operator of left (respectively, right) multiplication by \(x\) on \(A\). The algebra \(A\) is said to be a division algebra if, for every nonzero element \(x\) of \(A\), the operators \(L_x\) and \(R_x\) are bijective. An algebra is said to be alternative if it satisfies the identities \(x_1^2 x_2 = x_1(x_1 x_2)\) and
We note that alternative algebras are “very nearly” associative. Indeed, by Artin’s theorem (see Theorem 2.3.2 of \cite{113}), the subalgebra generated by two arbitrary elements of an alternative algebra is associative. It is also worth mentioning that every alternative division algebra has a unit (see page 226 of \cite{31}). By an algebra involution on an algebra $A$ we mean an involutive linear operator $x \to x^*$ on $A$ satisfying $(xy)^* = y^*x^*$ for all $x, y \in A$.

Now, let $K$ denote the field of real or complex numbers. An algebra norm (respectively, absolute value) on an algebra $A$ over $K$ is a norm $\| \cdot \|$ on the vector space of $A$ satisfying $\|xy\| \leq \|x\|\|y\|$ (respectively, $\|xy\| = \|x\|\|y\|$) for all $x, y \in A$. By a normed (respectively, absolute-valued) algebra we mean an algebra over $K$ endowed with an algebra norm (respectively, absolute value). We note that absolute-valued finite-dimensional algebras are division algebras. We also note that, if there exists an absolute value on a finite-dimensional algebra $A$ over $K$, then we can speak about “the” absolute value of $A$, understanding that such an absolute value is the unique possible one on $A$. This is a straightforward consequence of the easy and well-known result immediately below. The proof we are giving here is taken from \cite{26}.

**Proposition 1.1.** Let $A$ be a normed algebra over $K$, let $B$ be an absolute-valued algebra over $K$, and let $\phi : A \to B$ be a continuous algebra homomorphism. Then $\phi$ is in fact contractive.

**Proof.** Assume to the contrary that $\phi$ is not contractive. Then we can choose a norm-one element $x$ in $A$ such that $\|\phi(x)\| > 1$. Defining inductively $x_1 := x$ and $x_{n+1} := x_n^2$, we have $\|\phi(x_n)\| = \|\phi(x)\|^{2^{n-1}} \to \infty$. Since $\|x_n\| \leq 1$, this contradicts the assumed continuity of $\phi$.

Looking at the above proof, we realize that Proposition 1.1 remains true if $B$ is only assumed to be a normed algebra over $K$ satisfying $\|y^2\| = \|y\|^2$ for every $y \in B$, and $\phi : A \to B$ is only assumed to be a continuous linear mapping preserving squares. As a consequence of Proposition 1.1, every continuous algebra involution on an absolute-valued algebra is isometric.

Let $A$ be a normed algebra. An element $x$ of $A$ is said to be a left (respectively, right) topological divisor of zero in $A$ if there exists a sequence $\{x_n\}$ of norm-one elements of $A$ such that $\{xx_n\} \to 0$ (respectively, $\{x_nx\} \to 0$). Elements of $A$ which are left or right (respectively, both left and right) topological divisors of zero in $A$ are called one-sided.
(respectively, **two-sided**) **topological divisors of zero** in $A$. The element $x \in A$ is said to be a **joint topological divisor of zero** in $A$ if there exists a sequence $\{x_n\}$ of norm-one elements of $A$ such that $\{xx_n\} \to 0$ and $\{x_nx\} \to 0$. We note that both absolute-valued algebras and normed division alternative algebras have no one-sided topological divisor of zero other than zero. (This is clear in the case of absolute-valued algebras, and is easily verified in the case of normed division alternative algebras, by keeping in mind the fact already pointed out that division alternative algebras have a unit element, and applying the properties of “invertible” elements of unital alternative algebras given in page 38 of 94.) We will review in Theorem 1.1 a much deeper fact implying that, conversely, normed alternative algebras without nonzero joint topological divisors of zero are division algebras.

1.2. **Quaternions and Octonions**

Surveying absolute-valued algebras, we should write something about the algebra $H$ of Hamilton’s **Quaternions**, and the algebra $O$ of **Octonions** (also called “Cayley numbers”). These algebras, together with the fields of real and complex numbers (denoted by $\mathbb{R}$ and $\mathbb{C}$, respectively), become the basic examples of absolute-valued algebras. The algebras $\mathbb{C}$, $\mathbb{H}$, and $\mathbb{O}$ can be built from $\mathbb{R}$ by iterating the so-called “Cayley-Dickson doubling process” (see for example pages 256-257 of 31). Thus, if $A$ stands for either $\mathbb{R}$, $\mathbb{C}$, or $\mathbb{H}$, and if $*$ denotes the standard algebra involution of $A$ (which, for $A = \mathbb{R}$, is nothing other than the identity mapping), then we can consider the real vector space $A \times A$ with the product given by

$$(x_1, x_2)(x_3, x_4) := (x_1x_3 - x_4x_2^*, x_1^*x_4 + x_3x_2),$$

obtaining in this way a new real algebra which is a copy of either $\mathbb{C}$, $\mathbb{H}$, or $\mathbb{O}$, respectively. In this doubling process, the standard involution $*$ and the absolute value $\| \cdot \|$ of the new algebra are related to the corresponding ones of the starting algebra by the formulae

$$(x_1, x_2)^* := (x_1^*, -x_2) \quad \text{and} \quad \|(x_1, x_2)\| := \sqrt{\|x_1\|^2 + \|x_2\|^2},$$

respectively. It follows from the last formula that the **absolute values of $\mathbb{R}$, $\mathbb{C}$, $\mathbb{H}$, and $\mathbb{O}$ come from inner products**. It is also of straightforward verification that the algebra $\mathbb{H}$ is associative but not commutative, whereas the algebra $\mathbb{O}$ is alternative but not associative.

The joint introduction of $\mathbb{H}$ and $\mathbb{O}$ done above is surely the quickest possible one. However, concerning $\mathbb{H}$, there is another more natural approach.
Indeed, in the same way as \( \mathbb{C} \) can be rediscovered as the subalgebra of the algebra \( M_2(\mathbb{R}) \) (of all \( 2 \times 2 \) matrices over \( \mathbb{R} \)) given by

\[
\left\{ \begin{pmatrix} a & b \\ -b & a \end{pmatrix} : a, b \in \mathbb{R} \right\},
\]

\( \mathbb{H} \) can be rediscovered as the real subalgebra of \( M_2(\mathbb{C}) \) given by

\[
\left\{ \begin{pmatrix} z & w \\ -w^* & z^* \end{pmatrix} : z, w \in \mathbb{C} \right\}
\]

(see for example page 195 of 31). Regarded \( \mathbb{H} \) in this new way, the standard involution of \( \mathbb{H} \) corresponds with the transposition of matrices, and the absolute value of an element of \( \mathbb{H} \) is nothing other than the nonnegative square root of its (automatically nonnegative) determinant.

The algebras \( \mathbb{H} \) and \( \mathbb{O} \) are very far from being only exotic objects in Mathematics. By the contrary, they solve many natural problems in the field of the Algebra, the Geometry, and the Mathematical Analysis. Thus, as a consequence of the Frobenius-Zorn theorem (see for example pages 229 and 262 of 31), \( \mathbb{R} \), \( \mathbb{C} \), \( \mathbb{H} \), and \( \mathbb{O} \) are the unique finite-dimensional alternative division real algebras. On the other hand, we have the following.

**Theorem 1.1.** Every normed alternative real algebra without nonzero joint topological divisors of zero is algebra-isomorphic to either \( \mathbb{R} \), \( \mathbb{C} \), \( \mathbb{H} \), or \( \mathbb{O} \).

Theorem 1.1 has been proved by M. L. El-Mallah and A. Micali 45 by applying the forerunner of I. Kaplansky 60 (see also 17) for associative algebras. Keeping in mind the uniqueness of the absolute value on a finite-dimensional algebra, pointed out in Subsection 1.1, it follows that \( \mathbb{R} \), \( \mathbb{C} \), \( \mathbb{H} \), and \( \mathbb{O} \) are the unique absolute-valued alternative real algebras. A refinement of the fact just formulated (see Theorem 2.4) and other interesting characterizations of \( \mathbb{R} \), \( \mathbb{C} \), \( \mathbb{H} \), and \( \mathbb{O} \) (see Theorems 2.1, 2.5, 2.6, and 3.4) will be reviewed later. The reader interested in increasing his knowledge on Quaternions and Octonions is referred to the books 22 and 31, and the survey papers 6 and 98. These works and references therein will provide him with a complete panoramic view of the topic. Nevertheless, let us emphasize the abundance of historical notes and mathematical remarks collected in 31, and take some samples of them.

Thus, in a note written with the occasion of the fifteenth birthday of the Quaternions, W. R. Hamilton says: “They [the Quaternions] started into life, or light, full grown, on the 16th of October, 1843, as I was walking with Lady Hamilton to Dublin, and came up to Brougham Bridge.” (see
page 191 of \cite{31}). It turns out also curious to know that Hamilton tried for many years to built a three dimensional division associative real algebra. In fact, shortly before his death in 1865 he wrote to his son: “Every morning, on my coming down to breakfast, you used to ask me: ‘Well, Papa, can you multiply triplets?’ Where to I was always obliged to reply, with a sad shake of the head: ‘No, I can only add and subtract them’.” (see page 189 of \cite{31}). It is not less curious how, in a very elemental way, one can realize that the attempt of Hamilton just quoted could not be successful. Indeed, refining slightly the content of the footnote in page 190 of \cite{31}, we have the following.

**Proposition 1.2.** Let $A$ be a (possibly nonassociative) division real algebra of odd dimension. Then $A$ has dimension 1, and hence it is isomorphic to $\mathbb{R}$.

**Proof.** Fix $y \in A \setminus \{0\}$, and let $x$ be in $A$. Then the characteristic polynomial of the operator $L_y^{-1} \circ L_x$ must have a real root (say $\lambda$), which becomes an eigenvalue of such an operator. Taking a corresponding eigenvector $z \neq 0$, we have $(x - \lambda y)z = 0$, which implies $x = \lambda y$. Since $x$ is arbitrary in $A$, we have $A = \mathbb{R}y$. \hfill \Box

We note that the above proof is nothing other than a natural variant of the usual one for the fact that finite-dimensional division algebras over an algebraically closed field $F$ are isomorphic to $F$.

According to the information collected in page 249 of \cite{31}, the Octonions were discovered by J. T. Graves in December 1843, only two months after the birth of the Quaternions. Graves communicated his discovery to Hamilton in a letter dated 4th January 1844, but did not publish it until 1848. In the meantime, just in 1845, the Octonions were rediscovered by A. Cayley, who published his result immediately. For a more detailed history of the discovery of Octonions the reader is referred to pages 146-147 of \cite{6}.

### 1.3. The pioneering work of Ostrowski, Mazur, Albert, and Wright

We already know that $\mathbb{R}$, $\mathbb{C}$, $\mathbb{H}$, and $\mathbb{O}$ are the unique absolute-valued alternative real algebras. As a consequence, $\mathbb{R}$, $\mathbb{C}$, and $\mathbb{H}$ are the unique absolute-valued associative real algebras (a fact first proved by S. Mazur \cite{66}). More particularly, we have the following.

**Proposition 1.3.** Let $A$ be an absolute-valued, associative, and commutative algebra over $\mathbb{R}$. Then $A$ is isometrically isomorphic to either $\mathbb{R}$ or $\mathbb{C}$. 
Proof. Since $A$ is an integral domain, we can consider the field of fractions of $A$ (say $F$), and extend (in the unique possible way) the absolute value of $A$ to an absolute value on $F$. Now $F$ is an absolute-valued field extension of $\mathbb{R}$, and hence it is isometrically isomorphic to $\mathbb{R}$ or $\mathbb{C}$ (see Lemma 1.1 below). Since $A$ is a subalgebra of $F$, the result follows.

According to the information collected in pages 243 and 245 of [31], the above proposition and proof are due to A. Ostrowski [75], who seems to have been the first mathematician considering absolute-valued algebras as abstract objects which are worth being studied. The following lemma (today a consequence of the famous Gelfand-Mazur theorem) is also due to him.

Lemma 1.1. Every absolute-valued field extension of $\mathbb{R}$ is isometrically isomorphic to either $\mathbb{R}$ or $\mathbb{C}$.

The first paper dealing with absolute-valued algebras in a general nonassociative setting is the one of A. A. Albert [2], who proves as main result the following.

Proposition 1.4. $\mathbb{R}$, $\mathbb{C}$, $\mathbb{H}$, and $\mathbb{O}$ are the unique absolute-valued finite-dimensional real algebras with a unit.

A surprisingly short proof of Proposition 1.4, based on early works of H. Auerbach [5] and A. Hurwitz [53], can be given. However, since such a proof was not noticed by Albert, nor by anybody at his time, we prefer to postpone it until the conclusion of Subsection 2.2, and continue here with the chronological narration of facts. As we will see in Proposition 1.6 below, Proposition 1.4 was refined shortly later by Albert himself. Thus, the actual interest of Albert’s paper [2] relies on both the introduction of the notion of “isotopy” between absolute-valued algebras, and the proof of the following proposition.

Proposition 1.5. Let $A$ be an absolute-valued finite-dimensional real algebra. Then $A$ is isotopic to either $\mathbb{R}$, $\mathbb{C}$, $\mathbb{H}$, or $\mathbb{O}$. Therefore $A$ has dimension 1, 2, 4, or 8, and the absolute-value of $A$ comes from an inner product.

According to Albert’s definition, two absolute-valued algebras $A$ and $B$ over $\mathbb{K}$ are said to be isotopic if there exist linear isometries $\phi_1, \phi_2, \phi_3$ from $A$ onto $B$ satisfying $\phi_1(xy) = \phi_2(x)\phi_3(y)$ for all $x, y$ in $A$. Albert derives Proposition 1.5 from Proposition 1.4 in a clever but quite simple way. Indeed, choosing a norm-one element $a \in A$, and defining a new product $\odot$ on the normed space of $A$ by $x \odot y := R^{-1}(x) L^{-1}(y)$, we obtain
a finite-dimensional absolute-valued algebra, which is isotopic to $A$ and has a unit (namely, $a^2$). The argument just reviewed has been recently refined in the paper of A. Elduque and J. M. Pérez, yielding Lemma 1.2 immediately below. As we will see later, such a lemma has turned out to be very useful in the theory.

Lemma 1.2. Let $A$ be an absolute-valued algebra over $K$ such that there exist $a, b \in A$ satisfying $aA = Ab = A$. Then $A$ is isotopic to an absolute-valued algebra over $K$ having a unit element.

Proof. We may assume that $\|a\| = \|b\| = 1$. Then, defining a new product $\odot$ on the normed space of $A$ by $x \odot y := R_b^{-1}(x)L_a^{-1}(y)$, we obtain an absolute-valued algebra over $K$, which is isotopic to $A$, and has a unit (namely, $ab$).

Concerning the assertion in Proposition 1.5 about the dimension of absolute-valued finite-dimensional real algebras, it is worth mentioning that, some years after Albert’s paper (just in 1958), it was proved the following.

Theorem 1.2. Every finite-dimensional division real algebra has dimension $1, 2, 4,$ or $8$.

The paternity of Theorem 1.2 seems to be rather questioned. Indeed, according to $31, 48, 6$, such a theorem was first proved by Kervaire and Milnor, Adams, and Kervaire and Bott-Milnor, respectively. Anyway, in contrast with the case of Proposition 1.2, all known proofs of Theorem 1.2 are extremely deep.

A second paper of Albert contains as main result the following refinement of Proposition 1.4.

Proposition 1.6. Let $A$ be an absolute-valued algebraic real algebra with a unit. Then $A$ is equal to either $\mathbb{R}, \mathbb{C}, \mathbb{H},$ or $\mathbb{O}$.

We recall that an algebra $A$ is called algebraic if all single-generated subalgebras of $A$ are finite-dimensional. As we will see later, Proposition 1.6 has been also refined, in two different directions, and at two very distant dates (see Theorems 2.1 and 2.11). Therefore, Proposition 1.6 has today the unique interest of having been, some years later, one of the key tools in the original proofs of more relevant results in the theory of absolute-valued algebras. Among these results, we limit ourselves for the moment to review the one of F. B. Wright which follows.
**Theorem 1.3.** An absolute-valued algebra over \( K \) is finite-dimensional if (and only if) it is a division algebra.

Albert’s paper also contains the particular case of Theorem 1.3 that absolute-valued algebraic division algebras are finite-dimensional. However, the proof given in for this result seems to us not to be correct. To conclude the present section, let us note that Propositions 1.3 and 1.6, and Theorem 1.3 above become “apéritifs” for Section 2 below.

**1.4. Classification**

For \( A \) equal to either \( C \), \( H \), or \( O \), let us denote by \( \hat{A} \), \( *A \), and \( A^* \) the absolute-valued real algebras obtained by endowing the normed space of \( A \) with the products \( x \odot y := x^*y^* \), \( x \odot y := xy \), and \( x \odot y := xy^* \), respectively, where \( * \) means the standard involution. It follows easily from Proposition 1.5 that \( C, \hat{C}, *C, \) and \( C^* \) are the unique absolute-valued two-dimensional real algebras. Therefore, to be provided with a classification (up to algebra isomorphisms) of all finite-dimensional absolute-valued real algebras, it would be enough to obtain such a classification in dimension 4 and 8. Whereas for dimension 8 the problem seems to remain open, the case of dimension 4 has been solved in the paper of M. I. Ramírez, by applying Proposition 1.5 and the description of all linear isometries on \( H \) (see page 215 of). To this end, the so-called \textbf{principal isotopes} of \( H \) are considered. These are the absolute-valued real algebras \( H_1(a,b), H_2(a,b), H_3(a,b), \) and \( H_4(a,b) \) obtained from fixed norm-one elements \( a, b \) in \( H \) by endowing the normed space of \( H \) with the products \( x \odot y := axyb, x \odot y := ax^*y^*b, x \odot y := x^*ayb, \) and \( x \odot y := axby^* \), respectively. Then it is proved the following.

**Proposition 1.7.** Every four-dimensional absolute-valued real algebra is isomorphic to a principal isotope of \( H \). Moreover two principal isotopes \( H_i(a,b) \) and \( H_j(a',b') \) of \( H \) are isomorphic if and only if \( i = j \) and the equalities \( a'p = \varepsilon pa \) and \( b'p = \delta pb \) hold for some norm-one element \( p \in H \) and some \( \varepsilon, \delta \in \{1, -1\} \).

Proposition 1.7 can be also derived from. A refinement of it can be found in. The paper also contains a precise description of all four-dimensional absolute-valued real algebras with a left unit, as well as many examples of four-dimensional absolute-valued algebras containing no two-dimensional subalgebra.
Eight-dimensional absolute-valued real algebras with a left unit have been systematically studied in the recent paper of A. Rochdi\cite{79}. As a first basic result, Rochdi proves the following.

**Proposition 1.8.** The finite-dimensional absolute-valued real algebras with a left unit are precisely those of the form $A_\varphi$, where $A$ stands for either $\mathbb{R}$, $\mathbb{C}$, $\mathbb{H}$, or $\mathbb{O}$, $\varphi : A \to A$ is a linear isometry fixing 1, and $A_\varphi$ denotes the absolute-valued real algebra obtained by endowing the normed space of $A$ with the product $x \odot y := \varphi(x)y$. Moreover, given linear isometries $\varphi, \phi : A \to A$ fixing 1, the algebras $A_\varphi$ and $A_\phi$ are isomorphic if and only if there exists an algebra automorphism $\psi$ of $A$ satisfying $\phi = \psi \circ \varphi \circ \psi^{-1}$.

It is proved also in\cite{79} that, for $A$ and $\varphi$ as in Proposition 1.8, subalgebras of $A_\varphi$ and $\varphi$-invariant subalgebras of $A$ coincide. Moreover, a linear isometry $\varphi : \mathbb{O} \to \mathbb{O}$ fixing 1 can be built in such a way that $\mathbb{O}$ has no four-dimensional $\varphi$-invariant subalgebra. It follows that there exist eight-dimensional absolute-valued real algebras with a left unit, containing no four-dimensional subalgebra. Such algebras are characterized, among all eight-dimensional absolute-valued real algebras with a left unit, by the triviality of their groups of automorphisms. Such algebras seem to become the first examples of eight-dimensional division real algebras containing no four-dimensional subalgebra.

In Subsection 3.4 we will review in detail the results concerning those absolute-valued real algebras $A$ endowed with an isometric algebra involution $\ast$ which is different from the identity operator and satisfies $xx' = x'x$ for every $x \in A$. In the finite-dimensional case, such algebras have been classified in\cite{80}. The classification theorem has a flavour similar to that of Proposition 1.8.

**Right Moufang algebras** are defined as those algebras satisfying the identity $x_2((x_1x_3)x_1) = ((x_2x_1)x_3)x_1$. Absolute-valued right Moufang algebras are considered by J. A. Cuenca, M. I. Ramírez, and E. Sánchez\cite{24}, who show that such algebras are finite-dimensional. More precisely, they prove Theorem 1.4 immediately below. The formulation of such a theorem involves the notation introduced in Proposition 1.8 above, as well as the result of N. Jacobson\cite{55} that both $\mathbb{H}$ and $\mathbb{O}$ have an “essentially” unique involutive automorphism different from the identity operator.

**Theorem 1.4.** The absolute-valued right Moufang real algebras are $\mathbb{R}$, $\mathbb{C}$, $\mathbb{H}$, $\mathbb{O}$, $\ast\mathbb{C}$, and the algebras $A_\varphi$, where $A$ stands for either $\mathbb{H}$ or $\mathbb{O}$, and $\varphi$ denotes the essentially unique involutive automorphism of $A$ different from
the identity operator.

2. Conditions on absolute-valued algebras leading to the finite dimension

2.1. The noncommutative Urbanik-Wright theorem

Despite the constant scarcity of works on absolute-valued algebras along the history, a relatively short paper of K. Urbanik and F. B. Wright appeared in 1960 and announced the same year in 105, attracted the attention of many people because of the nice simplicity of its powerful results. In fact, Urbanik-Wright theorems have become the key tools in the later development of the theory of absolute-valued algebras. The first surprising result in the Urbanik-Wright paper is the following.

**Theorem 2.1.** For an absolute-valued real algebra $A$, the following conditions are equivalent:

1. There exists $a \in A \setminus \{0\}$ satisfying $ax = xa$, $a(ax) = a^2x$, and $(xa)a = xa^2$ for every $x \in A$.
2. $A$ has a unit element.
3. $A$ is equal to either $\mathbb{R}$, $\mathbb{C}$, $\mathbb{H}$, or $\mathbb{O}$.

We shall call the crucial implication (2) $\Rightarrow$ (3) in Theorem 2.1 above the noncommutative Urbanik-Wright theorem. Such a theorem immediately “works havoc” in the theory. For instance, it follows from it, and Albert’s ideas about isotopes, that an absolute-valued algebra $A$ over $K$ is finite-dimensional if (and only if) there exists $a \in A$ satisfying $aA = Aa = A$. This refinement of Wright’s Theorem 1.3 attains a better form whenever Lemma 1.2 replaces Albert’s ideas. Thus we have the following.

**Theorem 2.2.** An absolute-valued algebra $A$ over $K$ is finite-dimensional if (and only if) there exist $a, b \in A$ satisfying $aA = Ab = A$.

Even, applying an easy argument of completion (see 26 for details), we derive from Theorem 2.2 a still better form of Theorem 1.3. Indeed, an absolute-valued algebra $A$ is finite-dimensional if (and only if) there exist $a, b \in A$ such that $aA$ and $Ab$ are dense in $A$. Theorem 2.2 was first proved by the author 85 with other techniques. The proof given here is taken from the Elduque-Pérez paper 33.
2.2. Kaplansky’s prophetic proof of the noncommutative
Urbanik-Wright theorem

Concerning the proof of the noncommutative Urbanik-Wright theorem, the interested reader could go into the original paper \textsuperscript{106} to see how Urbanik and Wright apply, to commutative subspaces, Schoenberg’s characterization \textsuperscript{95} of pre-Hilbert spaces as those normed spaces $X$ satisfying

$$\|x + y\|^2 + \|x - y\|^2 \geq 4$$

for all norm-one elements $x, y \in X$ (see Remark 2.1 later), and how then, after some technical arguments, they show that the algebra satisfies the requirements in Albert’s Proposition 1.6. However, it seems to us more instructive to sketch how a proof of the noncommutative Urbanik-Wright theorem can be tackled by the light of the present knowledge.

Actually, the proof of the noncommutative Urbanik-Wright theorem can be divided into two parts. The first one, of a purely analytic type, consists in realizing that absolute-values on unital algebras come from inner products. This question was completely clarified twenty years ago. Indeed, it is easy to show that unital absolute-valued algebras become particular cases of the so-called smooth-normed algebras (see the proof of $(b) \Rightarrow (a)$ in Corollary 29 of \textsuperscript{82}), and it follows from Theorem 27 of \textsuperscript{82} that the norm of every smooth-normed algebra derives from an inner product (see also Section 2 of \textsuperscript{84} for a considerable simplification of the arguments in \textsuperscript{82}). We recall that a normed space $X$ over $\mathbb{K}$ is said to be \textbf{smooth} at a norm-one element $x \in X$ if the closed unit ball of $X$ has a unique tangent real hyperplane at $x$, and that \textbf{smooth-normed algebras} are defined as those normed algebras $A$ over $\mathbb{K}$ having a norm-one unit $1$ such that the normed space of $A$ is smooth at $1$. Incidentally, we note that $\mathbb{C}$ is the unique smooth-normed complex algebra, and that $\mathbb{R}, \mathbb{C}, \mathbb{H},$ and $\mathbb{O}$ are the unique smooth-normed alternative real algebras (see \textsuperscript{82} and \textsuperscript{84}, and references therein). We also remark that other arguments of more autonomous nature, showing as well that unital absolute-valued algebras are pre-Hilbert spaces, have been found later by El-Mallah \textsuperscript{40} and the author \textsuperscript{85} (see Theorems 3.2 and 3.5, respectively).

Now that we know that absolute values on unital algebras over $\mathbb{K}$ derive from inner products, the second (and last) part of the suggested proof of the noncommutative Urbanik-Wright theorem (now of a purely algebraic type) begins with an easy observation. Indeed, if an absolute value on a (possibly nonunital) real algebra $A$ comes from an inner product, then we are provided with a nondegenerate quadratic form $q$ on $A$ (namely, the mapping $x \rightarrow \|x\|^2$) satisfying $q(xy) = q(x)q(y)$ for all $x, y \in A$. In this way, we nat-
urally meet the so-called **composition algebras**, and the problem of classifying them. This problem was already considered and solved by Hurwitz under the additional requirements of finite dimension and existence of a unit. Later Kaplansky proved that the assumption of finite dimension in Hurwitz’s theorem is superfluous (see also Chapter 2 of [113]). Applying the Hurwitz-Kaplansky theorem, we obtain that the unique unital composition real algebras are \( \mathbb{R}, \mathbb{C}, \mathbb{R}^2 \) (with coordinate-wise multiplication), \( \mathbb{H}, M_2(\mathbb{R}), \mathcal{O} \), and a certain eight-dimensional alternative nonassociative algebra \( \mathcal{O}' \) which (as for the case of \( \mathbb{R}^2 \) and \( M_2(\mathbb{R}) \)) has nonzero divisors of zero. Since this last pathology is prevented in the case of absolute-valued algebras, the proof of the noncommutative Urbanik-Wright theorem is then concluded.

In the paper just quoted, which was published seven years before the one of Urbanik and Wright, Kaplansky prophesies both the noncommutative Urbanik-Wright theorem and a proof similar to that we have sketched above. Even, it seems that he thinks that the noncommutative Urbanik-Wright theorem was already proved at that time. Thus, he says that “Wright succeeded in removing the assumption [in Albert’s Proposition 1.6] that the algebra is algebraic”. Since we know that the above assertion is not right, we continue reproducing Kaplansky’s words with the appropriate corrections and explanations: “Wright proceeds by proving that the norm [of a unital absolute-valued DIVISION algebra] springs from an inner product [see Lemma 3.2 later], and then that the algebra is algebraic. ... Thus Albert’s finite-dimensional theorem [i.e., Proposition 1.4] can be proved by combining Wright’s result with Hurwitz’s classical theorem on quadratic forms admitting composition [see also Proof of Proposition 1.4 below]”. Immediately, Kaplansky motivates his work by saying that “The main purpose of this paper is to make a similar method possible in the infinite-dimensional case by providing a suitable generalization of Hurwitz’s theorem.”

Concerning the proof of the noncommutative Urbanik-Wright theorem just sketched, let us also comment that, really, the two parts in which we have divided it overlap somewhat. This is so because the proofs of the results in [82, 40, and 85], implying that unital absolute-valued algebras are pre-Hilbert spaces, give simultaneously a rich algebraic information, which is also provided by a part of the proof of the Hurwitz-Kaplansky theorem. In fact, with such an additional information in mind, the proof of the noncommutative Urbanik-Wright theorem can be concluded by applying the Frobenius-Zorn theorem instead of the one of Hurwitz-Kaplansky (see
Remark 31 of \textsuperscript{82} and Remark 4 of \textsuperscript{85} for details).

To conclude the present subsection, let us show how actually Albert could have derived his Proposition 1.4 from Hurwitz’s theorem, if he were aware of a result of Auerbach \textsuperscript{5} (see also Theorem 9.5.1 of \textsuperscript{92}) implying that \textit{finite-dimensional transitive normed spaces are Hilbert spaces}. We recall that a normed space \(X\) is called \textbf{transitive} if, given arbitrary norm-one elements \(x, y \in X\), there exists a surjective linear isometry \(T : X \to X\) such that \(T(x) = y\). The notion of transitivity just introduced will be revisited more quietly in Subsection 5.1.

\textbf{Proof of Proposition 1.4.} Let \(A\) be an absolute-valued algebra over \(\mathbb{K}\). If \(A\) is a division algebra, then the normed space of \(A\) is transitive, since for all norm-one elements \(x, y \in A\) we have \(T(x) = y\), where \(T := L_{R}^{-1}(y)\) is a surjective linear isometry on \(A\). Therefore, when \(A\) is finite-dimensional, Auerbach’s result applies, giving that the norm of \(A\) comes from an inner product. Finally, if \(\mathbb{K} = \mathbb{R}\), if \(A\) is finite-dimensional, and if \(A\) has a unit, then \(A\) is equal to either \(\mathbb{R}\), \(\mathbb{C}\), \(\mathbb{H}\), or \(\mathbb{O}\) (by Hurwitz’s theorem). \(\Box\)

The argument in the above proof is taken from page 156 of \textsuperscript{6}, where no reference to the works of Albert and Auerbach is done. In fact, Proposition 1.4 appears as Theorem 1 of \textsuperscript{6}, and is directly attributed there to Hurwitz \textsuperscript{53}, including shortly later the above argument as a part of the complete proof of such Hurwitz’s theorem. We do not agree with this attribution. Indeed, as far as we know, the observation that absolute-valued division algebras have transitive normed spaces appears first in the proof of Lemma 4 of Wright’s paper \textsuperscript{109} (fifty five years after Hurwitz’s paper). On the other hand, Auerbach’s result, published thirty six years after Hurwitz’s paper, seems to us non obvious.

2.3. \textit{The commutative Urbanik-Wright theorem}

The second surprising result in the Urbanik-Wright paper \textsuperscript{106} is the following.

\textbf{Theorem 2.3.} \(\mathbb{R}, \mathbb{C}, \text{ and } \mathbb{O}\) are the unique absolute-valued commutative real algebras.

We shall call Theorem 2.3 above the \textbf{commutative Urbanik-Wright theorem}. We know no proof of Theorem 2.3 other than the original one in \textsuperscript{106}. Starting with a new application of Schoenberg’s theorem \textsuperscript{95}, such a proof is really clever and easy. Therefore we do not resist the temptation of reproducing it here. Some unnecessary complications are of course avoided.
Proof. Let $A$ be an absolute-valued commutative real algebra. Since for all norm-one elements $x, y \in A$ we have

$$4 = 4\|xy\| = \|(x + y)^2 - (x - y)^2\| \leq \|x + y\|^2 + \|x - y\|^2,$$

Schoenberg’s theorem applies giving that $A$ is a pre-Hilbert space. On the other hand, since $\mathbb{R}, \mathbb{C},$ and $\mathcal{C}$ are the unique absolute-valued commutative real algebras of dimension $\leq 2$ (see Subsection 1.4), it is enough to show that the dimension of $A$ is $\leq 2$. Assume to the contrary that we can find pair-wise orthogonal norm-one elements $u, v, w$ in $A$. Then we have $\|u^2 - v^2\| = \|u + v\|\|u - v\| = 2$. Since $\|u^2\| = \|v^2\| = 1$, the parallelogram law implies that $u^2 + v^2 = 0$. Analogously, we obtain $u^2 + w^2 = v^2 + w^2 = 0$. It follows $u^2 = 0$, and hence also $u = 0$, a contradiction.

With the help of Lemma 2.4 below, the commutative Urbanik-Wright theorem can be refined as follows. There is a universal constant $K > 0$ such that every absolute-valued real algebra $A$ satisfying $\|xy - yx\| \leq K\|x\|\|y\|$ for all $x, y \in A$ is in fact equal to either $\mathbb{R}, \mathbb{C},$ or $\mathcal{C}$ (see Corollary 1.4 of 59).

Remark 2.1. For a normed space $X$ over $\mathbb{K}$, consider the property $P$ which follows:

($P$) There exists a normed space $Y$ over $\mathbb{K}$, together with a bilinear mapping $(a, b) \to ab$ from $X \times X$ to $Y$ satisfying $ab = ba$ and $\|ab\| = \|a\|\|b\|$ for all $a, b \in X$.

Arguing as in the beginning of the proof of Theorem 2.3, we see that, if the normed space $X$ satisfies Property $P$, then $X$ is a pre-Hilbert space. The converse is also true (see Theorem 4.4 of 8).

2.4. Power-associativity

Let $A$ be an algebra over a field $\mathbb{F}$. We say that $A$ is of bounded degree if there exists a natural number $n$ such that all single-generated subalgebras of $A$ have dimension $\leq n$, and power-associative if all single-generated subalgebras of $A$ are associative. In the case that the characteristic of $\mathbb{F}$ is different from 2, we will consider the algebra $A^*$ whose vector space is the same as that of $A$, and whose product is defined by $x \odot y := \frac{1}{2}(xy + yx)$. We remark that both the bounded degree and the power-associativity pass from $A$ to $A^*$.
Lemma 2.1. Let $A$ be a normed algebra over $\mathbb{K}$ satisfying $\|x^2\| = \|x\|^2$ for every $x \in A$, and such that $A^\ast$ is power-associative and of bounded degree. Then $A$ has a norm-one unit.

**Proof.** Since $A^\ast$ is a commutative power-associative algebra of bounded degree, and has no nonzero element $x$ such that $x^2 = 0$, it follows from Proposition 2 of 21 that $A^\ast$ has a unit element (say $1$). Moreover, since $\|1\| = \|1^2\| = \|1\|^2$, we have $\|1\| = 1$. Then, since $A$ is a normed algebra, both $L_1$ and $R_1$ lie in the closed unit ball of the normed algebra $\mathcal{L}(A)$ of all continuous linear operators on $A$. Since $\frac{1}{2}(L_1 + R_1) = I_A$ (the identity operator on $A$), and $I_A$ is an extreme point of the closed unit ball of $\mathcal{L}(A)$ (by Proposition 1.6.6 of 93), it follows that $L_1 = R_1 = I_A$, i.e., $1$ is a unit element for $A$. \hfill \Box

Now we can prove the main result in this subsection. It is due to El-Mallah and Micalli 45, and reads as follows.

**Theorem 2.4.** $\mathbb{R}$, $\mathbb{C}$, $\mathbb{H}$, and $\mathbb{O}$ are the unique absolute-valued power-associative real algebras.

**Proof.** Let $A$ be an absolute-valued power-associative real algebra. By Proposition 1.3, $A$ is of bounded degree. Then, by Lemma 2.1, $A$ has a unit. Finally, by the noncommutative Urbanik-Wright theorem, $A$ is equal to either $\mathbb{R}$, $\mathbb{C}$, $\mathbb{H}$, or $\mathbb{O}$. \hfill \Box

The original proof of El-Mallah and Micalli differs not too much of the above one. Of course, they did not know Lemma 2.1, which has been proved here by the first time. Thus, in the El-Mallah-Micalli proof, Lemma 2.1 was replaced with a simpler purely algebraic result (see Lemma 1.1 of 45). Anyway, both Lemma 1.1 of 45 and Proposition 2 of 17 (which has been one of the tools in the proof of Lemma 2.1, and is also of a purely algebraic nature) have a common root, namely the proof of Lemma 5.3 of 94. Before the appearance of the Urbanik-Wright paper, Wright knew that $\mathbb{R}$, $\mathbb{C}$, $\mathbb{H}$, and $\mathbb{O}$ are the unique unital absolute-valued power-associative real algebras (see the introduction of 109). This (today doubly unsubstantial) result was rediscovered by L. Ingelstam 54 (four years after the appearance of the Urbanik-Wright paper!) with a proof essentially identical to the one suggested by Wright in 109. Anyway, the Wright-Ingelstam argument has some methodological interest. Indeed, it shows that, in an autonomous
proof of Theorem 2.4, the noncommutative Urbanik-Wright theorem can be replaced with Albert’s forerunner given by Proposition 1.6.

As we commented in Subsection 2.2, smooth normed algebras are pre-Hilbert spaces. A converse to this fact is proved in Proposition 2.1 immediately below. The key tools are Lemma 2.1 and the result of B. Zalar (see also Theorem 3 of [112]) that $\mathbb{R}$ and $\mathbb{C}$ are the unique pre-Hilbert associative commutative real algebras $A$ satisfying $\|x^2\| = \|x\|^2$ for every $x \in A$.

**Proposition 2.1.** Let $A$ be a normed real algebra. Then the following conditions are equivalent:

1. $A$ is a smooth-normed algebra.
2. $A$ is power-associative, the norm of $A$ derives from an inner product, and the equality $\|x^2\| = \|x\|^2$ holds for every $x \in A$.
3. $A^*$ is power-associative, the norm of $A$ derives from an inner product, and the equality $\|x^2\| = \|x\|^2$ holds for every $x \in A$.

**Proof.** The implication (1) $\Rightarrow$ (2) is a consequence of Theorem 27 of [82], whereas the one (2) $\Rightarrow$ (3) is clear. Assume that Condition (3) is fulfilled. Then, by Zalar’s result quoted above, the algebra $A^*$ is of bounded degree. Therefore, by Lemma 2.1, $A$ has a norm-one unit. Since pre-Hilbert spaces are smooth at all their norm-one elements, it follows that $A$ is a smooth-normed algebra.

The following result of Zalar [111] follows straightforwardly from Proposition 2.1 above and Hurwitz’s theorem (see Subsection 2.2).

**Theorem 2.5.** Let $A$ be an absolute-valued real algebra whose norm springs from an inner product, and such that $A^*$ is power associative. Then $A$ is equal to either $\mathbb{R}$, $\mathbb{C}$, $\mathbb{H}$, or $\mathbb{O}$.

In relation to Proposition 2.1, it is worth mentioning that smooth normed algebras are precisely those unital normed algebras $A$ satisfying $\|1 - x^2\| = \|1 + x\|\|1 - x\|$ for every $x \in A$, as well as those unital normed algebras $A$ satisfying $\|U_x(y)\| = \|x\|^2\|y\|$ for all $x, y \in A$, where $U_x(y) := x(yx) + (yx)x - yx^2$ (see Corollary 29 of [82]). Another characterization of smooth normed algebras is given in the next proposition.

**Proposition 2.2.** Let $A$ be a normed real algebra. Then the following conditions are equivalent:

1. $A$ is a smooth-normed algebra.
(2) \( A \) is power-associative, and the equality \( \|U_x(y)\| = \|x\|^2\|y\| \) holds for all \( x, y \in A \).

**Proof.** In view of Proposition 2.1 and the comments immediately above, it is enough to show that (2) implies that \( A \) has a unit. Assume that (2) is fulfilled. Then for \( x, y \) in any single-generated subalgebra of \( A \), we have
\[
\|x\|^2\|y\| = \|U_x(y)\| = \|xyx\| \leq \|xy\|\|x\|.
\]
Therefore, all single-generated subalgebras of \( A \) are absolute-valued algebras. By Proposition 1.3, \( A \) is of bounded degree. Finally, by Lemma 2.1, \( A \) has a unit. \( \square \)

Proposition 2.2 was first proved by M. Benslimane and N. Merrachi\(^{10}\) with slightly different techniques. More information about smooth normed algebras can be found in Subsection 3.5.

To conclude the present subsection, let us comment that Theorem 2.4 is "almost" contained in the early paper of Urbanik\(^{102}\). Indeed, it could have been very easy for him to establish such a theorem by selecting, among the many auxiliary results in that paper, the appropriate ones for the goal. However, Urbanik does not do this, since he completely devotes his paper\(^{102}\) to characterize \( \mathbb{R}, \mathbb{C}, \mathbb{H}, \) and \( \mathbb{O} \) in terms conceptually far from the power-associativity. An element \( x \) of an algebra \( A \) is said to be **reversible** if there exists \( y \in A \) satisfying \( x + y - xy = x + y - yx = 0 \). The algebra \( A \) is said to fulfill the **reversibility condition** if all its elements, except those in some countable set, are reversible. Now the main result in\(^{102}\) reads as follows.

**Theorem 2.6.** \( \mathbb{R}, \mathbb{C}, \mathbb{H}, \) and \( \mathbb{O} \) are the unique absolute-valued real algebras satisfying the reversibility condition.

Note that for \( A \) equal to either \( \mathbb{R}, \mathbb{C}, \mathbb{H}, \) or \( \mathbb{O} \), all elements of \( A \), except the unit of \( A \), are reversible.

### 2.5. Flexibility

An algebra is said to be **flexible** whenever it satisfies the identity \((x_1x_2)x_1 = x_1(x_2x_1)\). Since single-generated subalgebras of flexible algebras are commutative, the commutative Urbanik-Wright theorem applies successfully to single-generated subalgebras of absolute-valued flexible algebras. After a lot of work, involving the information obtained from the procedure just pointed out, El-Mallah and Micali\(^{46}\) prove the following.
Lemma 2.2. Absolute-valued flexible algebras are finite-dimensional.

Later, El-Mallah, in a series of papers (see 36, 37, 38, 39, and 40), refines deeply the result just reviewed, by considering absolute-valued algebras satisfying the identity $xx^2 = x^2x$ (which is of course implied by the flexibility), and proving the following.

Theorem 2.7. For an absolute-valued real algebra $A$, the following assertions are equivalent:

1. $A$ is flexible.
2. $A$ is a pre-Hilbert space and satisfies the identity $x^2x = xx^2$.
3. $A$ is finite-dimensional and satisfies the identity $x^2x = xx^2$.
4. $A$ is equal to either $\mathbb{R}$, $\mathbb{C}$, $\mathbb{H}$, $\mathbb{O}$, $\mathbb{\hat{O}}$, or the algebra $\mathbb{P}$ of pseudo-octonions.

According to Theorem 2.7 just formulated, the algebra $\mathbb{P}$ of pseudo-octonions is the unique absolute-valued flexible real algebra which has not been still introduced in our development. Such an algebra was discovered by S. Okubo 72 (see also pages 65-71 of 70). The vector space of $\mathbb{P}$ is the eight-dimensional real subspace of $M_3(\mathbb{C})$ consisting of those trace-zero elements which remain fixed after taking conjugates of their entries and passing to the transpose matrix. The product $\odot$ of $\mathbb{P}$ is defined by choosing a complex number $\mu$ satisfying $3\mu(1 - \mu) = 1$, and then by putting

$$x \odot y := \mu xy + (1 - \mu)yx - \frac{1}{3}T(xy)1.$$  

Here $T$ denotes the trace function on $M_3(\mathbb{C})$, 1 stands for the unit of the associative algebra $M_3(\mathbb{C})$, and, for $x, y$ in $\mathbb{P}$, $xy$ means the product of $x$ and $y$ as elements of such an algebra. If for $x, y \in \mathbb{P}$ we define $(x|y) := \frac{1}{2}T(xy)$, then $(\cdot|\cdot)$ becomes an inner product on $\mathbb{P}$ whose associated norm is an absolute value.

In relation to Theorem 2.7, it seems to be an open problem (see the abstract of 41) if every absolute-valued real algebra satisfying the identity $x^2x = xx^2$ is finite-dimensional. According to Theorem 2.7 itself, the answer is affirmative if $A$ is a pre-Hilbert space. The answer is also affirmative if $A$ is algebraic 41, but, as we will see in Subsection 2.7, this result is today unsubstantial. As a more ambitious problem, we can wonder whether every absolute-valued algebra satisfying some identity is finite-dimensional.

The classification of absolute-valued flexible real algebras contained in Theorem 2.7 was tried in 63, with a partial success. Actually, such a clas-
sification can be derived from Lemma 5.3, Proposition 1.5, and Theorem 2.7 has inspired the result in that finite-dimensional composition algebras satisfying the identity \( x^2 x = xx^2 \) are in fact flexible.

2.6. \( H^* \)-theory

The following theorem has been proved by J. A. Cuenca and the author.

**Theorem 2.8.** Let \( A \) be an absolute-valued algebra over \( K \). Assume that there exists a complete inner product \((\cdot|\cdot)\) on \( A \), together with an involutive conjugate-linear operator \(*\) on \( A \), satisfying \((xy|z) = (x|zy^*) = (y|x^*z)\) for all \( x, y, z \in A \). Then we have:

1. \( A \) is finite-dimensional.
2. The Hilbertian norm \( x \rightarrow \sqrt{(x|x)} \) is a positive multiple of the absolute-value of \( A \).
3. The operator \(*\) is an algebra involution on \( A \).
4. The equality \( x^* (xy) = (yx^*) x = \|x\|^2 y \) holds for all \( x, y, z \in A \).

With the terminology of, the assumptions on \((\cdot|\cdot)\) and \(*\) in Theorem 2.8 mean that, forgetting the absolute value of \( A \), \((A,(\cdot|\cdot),*)\) is a semi-\( H^* \)-algebra over \( K \). The conclusion, that \(*\) is in fact an algebra involution, then reads as that \((A,(\cdot|\cdot),*)\) is an \( H^* \)-algebra over \( K \). Besides a little \( H^* \)-theory, the proof of Theorem 2.8 involves some results on absolute-valued algebras previously reviewed (as Wright’s Theorem 1.3), and others to be reviewed later (as for example Theorem 3.8). Such a proof, as well as that of Theorem 2.9 below, also includes some easy facts first pointed out in. Among these, we emphasize the following one for later reference.

**Lemma 2.3.** Let \( A \) be an absolute-valued algebra over \( K \). Assume that the absolute-value of \( A \) comes from an inner product \((\cdot|\cdot)\), and that, for every \( x \in A \), there exists \( x^* \in A \) satisfying \((xy|z) = (y|x^*z)\) for all \( x, y, z \in A \). Then we have \( x^* (xy) = \|x\|^2 y \) for all \( x, y, z \in A \).

**Proof.** For \( x, y \in A \), we have \((xy|xy) = \|x\|^2 (y|y)\). Linearizing in the variable \( y \), we obtain that the equality \((xz|xy) = \|x\|^2 (z|y)\) holds for all \( x, y, z \in A \). Since \((xz|xy) = (z|x^*(xy))\), we deduce \((z|x^*(xy)) = \|x\|^2 (z|y)\), which, in view of the arbitrariness of \( z \), yields \( x^* (xy) = \|x\|^2 y \).

The Cuenca-Rodríguez paper also contains a precise determination of the algebras \( A \) in Theorem 2.8. Since the case that \( K = \mathbb{C} \) is unsub-
stantial (see Subsection 2.8 later), only the case that \( \mathbb{K} = \mathbb{R} \) merits to be considered. Thus, in view of Theorem 2.8, we are dealing in fact with an absolute-valued finite-dimensional real algebra \( A \) endowed with an algebra involution \(*\), and whose norm derives from an inner product \((\cdot|\cdot)\) satisfying \( x^*(xy) = (yx)x^* \) and \((xy|z) = (x|yz^*) = (y|x^*z)\) for all \( x, y, z \in A \). Since \(*\) is isometric, we can consider the isotope of \( A \) (say \( B \)) consisting of the normed space of \( A \) and the product \( x \odot y := x^*y^* \). Now, we trivially realize that the absolute-valued real algebra \( B \) is flexible and satisfies \((x \odot y|z) = (x|y \odot z)\) for all \( x, y, z \in B \). Then, we deduce from El-Mallah’s Theorem 2.7 that \( B \) is equal to either \( \mathbb{R}, \mathbb{C}, \mathbb{H}, \mathbb{O}, \mathbb{P} \). Moreover, \(*\) becomes an algebra involution on \( B \), and the correspondence \((A, *) \rightarrow (B, *)\) is categorical and bijective. After the laborious classification of algebra involutions on \( \mathbb{C}, \mathbb{H}, \mathbb{O}, \mathbb{P} \) made in \( \text{26} \), the determination of the algebras in Theorem 2.8 concludes. In this way, three new distinguished examples of absolute-valued finite-dimensional real algebras appear. These are the natural isotopes of \( \mathbb{H}, \mathbb{O}, \mathbb{P} \) (denoted respectively by \( \hat{\mathbb{H}}, \hat{\mathbb{O}}, \hat{\mathbb{P}} \)) built as follows. For every absolute-valued algebra \( A \), and every linear isometry \( \psi \) on \( A \), the \( \psi\)-twist of \( A \) is defined as the absolute-valued algebra consisting of the normed space of \( A \) and the product \( x \odot y := \psi(x)\psi(y) \). For \( A \) equal to either \( \mathbb{H} \) or \( \mathbb{O} \), we define \( \hat{A} \) as the \( \phi\)-twist of \( A \), where \( \phi \) stands for the essentially unique involutive automorphism of \( A \) different from the identity operator (see Subsection 1.4). On the other hand, there exists an “essentially” unique algebra involution \( \sigma \) on \( \mathbb{P} \), which allows us to define \( \hat{\mathbb{P}} \) as the \( \sigma\)-twist of \( \mathbb{P} \). Now we have the following.

**Theorem 2.9.** Let \( A \) be an absolute-valued real algebra fulfilling the requirements in Theorem 2.8. Then \( A \) is equal to either \( \mathbb{R}, \mathbb{C}, \mathbb{H}, \hat{\mathbb{H}}, \hat{\mathbb{O}}, \hat{\mathbb{P}} \).

A slight variant of the proof of Theorem 2.9 sketched above, involving Corollary 7 of \( \text{74} \) instead of Theorem 2.7, can be seen in Remark 2.9 of \( \text{26} \).

### 2.7. Algebraicity

Albert’s Proposition 1.6, although obsolete after the noncommutative Urbanik-Wright theorem, has had the merit of encouraging the work on the question if every absolute-valued algebraic algebra is finite-dimensional. Since for complex algebras such a question has an almost trivial affirmative answer (see the concluding paragraph of Subsection 2.8 below), the interest centers in the case of real algebras. Some partial affirmative answers have
been provided by El-Mallah. Thus, an absolute-valued algebraic real algebra is finite-dimensional whenever there exists a nonzero idempotent in $A$ commuting with every element of $A$ \textsuperscript{36}, or there exists a continuous algebra involution $\ast$ on $A$ satisfying $xx^\ast = x^\ast x$ for every $x \in A$ \textsuperscript{39}, or $A$ satisfies the identity $xx^2 = x^2x$ \textsuperscript{41}. We note that the result in \textsuperscript{36} would become later a consequence of the one in \textsuperscript{39} (see El-Mallah’s Theorem 3.2), and that the result in \textsuperscript{41} was already commented at the conclusion of Subsection 2.5.

To specify that an algebra $A$ is of bounded degree, let us say that $A$ is of degree $n \in \mathbb{N}$ if $n$ is the minimum natural number such that all single-generated subalgebras of $A$ have dimension $\leq n$. It follows from Proposition 1.4, that absolute-valued algebraic real algebras are of bounded degree, and, more precisely, of degree 1, 2, 4, or 8. Then, since $\mathbb{R}$ is the unique absolute-valued algebraic algebra of degree 1 (see again the concluding paragraph of Subsection 2.8), the strategy of studying separately the cases of degree 2, 4, and 8 could seem tempting in order to answer affirmatively the question we are considering. Unfortunately, such a strategy has turned out to be unsuccessful for the moment, unless for the case of degree 2, for which we have the following result of the author \textsuperscript{89}.

\textbf{Theorem 2.10.} The absolute-valued real algebras of degree two are $\mathbb{C}$, $\mathbb{C}^*$, $*\mathbb{C}$, $\mathbb{H}$, $\mathbb{H}^*$, $*\mathbb{H}$, $\mathbb{O}$, $*\mathbb{O}$, $\mathbb{O}^*$, and $\mathbb{P}$.

Via the commutative Urbanik-Wright theorem, Theorem 2.10 above contains both Theorem 2.4 and the classification of absolute-valued flexible real algebras included in Theorem 2.7. However, this is quite deceptive because, in fact, the proof of Theorem 2.10 involves Theorem 2.4 and the whole Theorem 2.7. In any case, by keeping in mind again the commutative Urbanik-Wright theorem, Theorem 2.10 shows for the first time that, for absolute-valued algebras, power-commutativity and flexibility are equivalent notions. We recall that an algebra is said to be \textbf{power-commutative} if all its single-generated subalgebras are commutative, and that flexible algebras are power-commutative \textsuperscript{76}. Theorem 2.10 has inspired the classification of composition algebras of degree two, done in \textsuperscript{34}.

Returning to the general problem if absolute-valued algebraic algebras are finite-dimensional, we must say that, six years ago, A. Kaidi, M. I. Ramírez, and the author \textsuperscript{57} succeeded in solving it. Thus we have the following.

\textbf{Theorem 2.11.} An absolute-valued real algebra is finite-dimensional if (and only if) it is algebraic.
We know no proofs of Theorem 2.11 above others than the original one in $^{57}$, and the slight variant of it given in $^{58}$. We do not enter here the details of such proofs, nor even give a sketch of them. Referring the reader to $^{58}$ for such a sketch, we limit ourselves here to say that both arguments are long and complicated, and involve in an essential way the techniques of normed ultraproducts $^{52}$. Thus, by the first time in the theory of absolute-valued algebras, the following folklore result shows useful.

**Lemma 2.4.** The normed ultraproduct of every ultrafiltered family of absolute-valued algebras over $\mathbb{K}$ becomes naturally an absolute-valued algebra over $\mathbb{K}$.

Concerning the proof of Theorem 2.11, let us also revisit a minor auxiliary result (namely, Lemma 4.2 of $^{57}$). Such a result can be refined as follows.

**Lemma 2.5.** Let $X$ be a normed space over $\mathbb{K}$, let $F : X \rightarrow X$ be a linear contraction, and let $M$ be a finite-dimensional subspace of $X$. Assume that $F$ is the identity on $M$, and that $X$ is smooth at every norm-one element of $M$. Then there exists a continuous linear projection $\pi$ from $X$ onto $M$ such that $\ker(\pi)$ is invariant under $F$.

**Proof.** Let $M^*$ denote the dual space of $M$. By a theorem of Auerbach $^5$ (see also Lemmas 7.1.6 and 7.1.7 of $^{92}$), there are bases $\{m_1, \ldots, m_k\}$ and $\{g_1, \ldots, g_k\}$ of $M$ and $M^*$, respectively, consisting of norm-one elements and satisfying $g_i(m_j) = \delta_{ij}$. Extending each $g_i$ to a norm-one linear functional $\phi_i$ on $X$ (via the Hahn-Banach theorem), and considering the mapping $x \rightarrow \sum_{i=1}^{k} \phi_i(x)m_i$ from $X$ to $M$, it is easily seen that such a mapping satisfies the properties asserted for $\pi$ in the statement of the lemma (see the proof of Lemma 4.2 of $^{57}$ for details).

Lemma 2.5 above was proved in $^{57}$ under the additional assumption that the restriction to $M$ of the norm of $X$ springs from an inner product. The refinement we have just made does not matter there because, when the lemma applies, $X$ is an absolute-valued algebra, and $M$ is a subspace of a finite-dimensional subalgebra of $X$, so that the superfluous requirement in the original formulation of the lemma is automatically fulfilled (by Proposition 1.5).
2.8. A remark on complex algebras

All conditions we have considered above, leading absolute-valued real algebras to the finite-dimension, in the case of absolute-valued complex algebras yield that the algebra is \( \mathbb{C} \). Indeed, if an absolute-valued complex algebra fulfills some of those conditions, then, by restriction of scalars, we obtain an absolute-valued real algebra satisfying the same condition, and hence the corresponding result applies. But we know that absolute-valued finite-dimensional algebras are division algebras, and that \( \mathbb{C} \) is the unique finite-dimensional division complex algebra.

In some cases, the result obtained in this way can be refined still more. For example, the complex version of Theorem 2.2 is that, if \( A \) is an absolute-valued complex algebra, and if there exists \( a \in A \) such that \( aA \) is dense in \( A \), then \( A = \mathbb{C} \) (see Lemma 1.1 of \(^{26}\)). On the other hand, the joint complex version of Theorems 2.8 and 2.9 is that, if \( A \) is an absolute-valued complex algebra, and if there exists a complete inner product \( (\cdot|\cdot) \) on \( A \) making the product continuous, and an involutive conjugate-linear operator * on \( A \) satisfying \( (xy|z) = (x|zy^*) \) for all \( x,y,z \in A \), then \( A = \mathbb{C} \) (see Theorem 1.2 of \(^{26}\)). None of the two results just quoted remains true (with the finite-dimensionality of \( A \) instead of \( A = \mathbb{C} \) in the conclusion) whenever real algebras replace complex ones. Concerning the second result, in the real case nor even can be expected the Hilbertian norm \( x \mapsto \sqrt{(x|x)} \) to be equivalent to the absolute value of \( A \) (see Example 1.7 of \(^{26}\)). These pathologies give rise to an interesting development of the theory of absolute-valued algebras, which will be reviewed in Subsection 3.5.

As a consequence of Theorem 2.11 and the comments at the beginning of the present subsection, \( \mathbb{C} \) is the unique absolute-valued algebraic complex algebra. However, this can be proved elementarily. Indeed, notice that, by the same comments, absolute-valued algebraic complex algebras are of degree one, and that, if \( \mathbb{F} \) is a field containing more than two elements, if \( A \) is an algebra over \( \mathbb{F} \) of degree one, and if there is no nonzero element \( x \in A \) with \( x^2 = 0 \), then \( A = \mathbb{F} \) (see for example page 297 of \(^{57}\)).

3. Infinite-dimensional absolute-valued algebras

3.1. The basic examples

The first example of an absolute-valued infinite-dimensional algebra appears in the celebrated paper of Urbanik and Wright \(^{106}\). Indeed, they show that the classical real Hilbert space \( \ell_2 \) becomes an absolute-valued algebra under a suitable product. Looking at their argument, many other
similar examples can be built. To get them, let us start by fixing an arbitrary nonempty set \( U \), and a mapping \( \vartheta : U \times U \to X \), where \( X = X(U, \mathbb{K}) \) stands for the free vector space over \( \mathbb{K} \) on \( U \). We denote by \( \mathcal{A} = \mathcal{A}(U, \vartheta, \mathbb{K}) \) the algebra over \( \mathbb{K} \) whose vector space is \( X \), and whose product is defined as the unique bilinear mapping from \( X \times X \) to \( X \) which extends \( \vartheta \). From now on, we assume that \( U \) is infinite, and, accordingly to such an assumption, we choose \( \vartheta \) among the injective mappings from \( U \times U \) to \( U \) or, more generally, of the form \( fg \), where \( g : U \times U \to U \) is injective and \( f : U \times U \to \mathbb{K} \) satisfies \( |f(u, v)| = 1 \) for every \( (u, v) \in U \times U \). With these restrictions in mind, we are going to realize that there are “many” absolute values on \( \mathcal{A} \). To this end, let us involve a new ingredient, namely an extended real number \( p \) with \( 1 \leq p \leq \infty \). Then, for \( x \) in \( \mathcal{A} \), we can think about the family \( \{x_u\}_{u \in U} \) of coordinates of \( x \) relative to \( U \), and define
\[
\|x\|_p := \left( \sum_{u \in U} |x_u|^p \right)^{\frac{1}{p}} \text{ if } p < \infty \quad \text{and} \quad \|x\|_\infty := \max\{|x_u| : u \in U\}.
\]
Invoking the properties of \( \vartheta \), we straightforwardly verify that \( \| \cdot \|_p \) is an absolute value on \( \mathcal{A} \). We denote by \( \mathcal{A}_p = \mathcal{A}_p(U, \vartheta, \mathbb{K}) \) the absolute-valued algebra over \( \mathbb{K} \) obtained by endowing \( \mathcal{A} \) with the norm \( \| \cdot \|_p \). By considering the completion of \( \mathcal{A}_p \), we obtain a complete absolute-valued algebra over \( \mathbb{K} \), denoted by \( \mathcal{C}_p = \mathcal{C}_p(U, \vartheta, \mathbb{K}) \), whose Banach space is nothing other than the familiar space \( \ell_p(U, \mathbb{K}) \) if \( p \neq \infty \), or \( c_0(U, \mathbb{K}) \) otherwise. Now, the Urbanik-Wright example is just the algebra \( \mathcal{C}_2(N, \vartheta, \mathbb{R}) \), with \( \vartheta : N \times N \to N \) equal to any bijection.

Returning to our general setting, let us remark that, since \( \mathcal{C}_p \) is a Hilbert space if and only if \( p = 2 \), it follows from the above construction that

\textit{composition algebras need not be finite-dimensional}, and that, contrarily to what is conjectured in \textit{40}, \textit{absolute-values need not come from inner products}. Another consequence of our construction is that \textit{there exist complete absolute-valued algebras without uniqueness of the (noncomplete) absolute value}. Indeed, for \( 1 \leq p < q \leq \infty \), the complete absolute-valued algebra \( \mathcal{C}_p \) can be algebraically regarded as a subalgebra of \( \mathcal{C}_q \), but the topology of the restriction of the absolute value of \( \mathcal{C}_q \) to \( \mathcal{C}_p \) does not coincide with the natural one of \( \mathcal{C}_p \). The straightforward fact, that \( \| \cdot \|_p \geq \| \cdot \|_q \) on \( \mathcal{C}_p \), is not anecdotic. Indeed, as a consequence of Theorem 3.8 below, \textit{every complete algebra norm on an absolute-valued algebra is greater than the absolute value}. In particular, \textit{two complete absolute values on the same algebra must coincide}.

The refinement of the Urbanik-Wright example, done above, is implicitly known in some works on Banach spaces (see for instance the proof of
Theorem 3.1. There exists a complete absolute-valued infinite-dimensional ordered real algebra.

A simplification of Urbanik’s argument is the following.

Proof. Let \( \vartheta : \mathbb{N} \times \mathbb{N} \to \mathbb{N} \) be defined by \( \vartheta(n,m) := 2^n 3^m \), and let us fix \( 1 \leq p \leq \infty \). Since \( \vartheta \) is injective, we can consider the complete absolute-valued infinite-dimensional real algebra \( C_p = C_p(\mathbb{N}, \vartheta, \mathbb{R}) \). The natural inclusion \( \mathbb{N} \to C_p \) converts \( \mathbb{N} \) into a Schauder basis of \( C_p \). For \( x \in C_p \), let \( \{x_n\}_{n \in \mathbb{N}} \) stand for the family of coordinates of \( x \) relative to such a basis, and, when \( x \neq 0 \), define \( n(x) := \min\{n \in \mathbb{N} : x_n \neq 0\} \). Finally, put \( C_p^+ := \{x \in C_p \setminus \{0\} : x_{n(x)} > 0\} \). Keeping in mind that \( \vartheta \) is increasing in each one of its variables, it is easily seen that \( C_p^+ \) fulfills the properties required above for the sets of positive elements of ordered real algebras. \( \square \)

3.2. Free normed nonassociative algebras

Let us fix a nonempty set \( V \). Nonassociative words with characters in \( V \) are defined inductively (according to their “degree”) as follows. The nonassociative words of degree 1 are just the elements of \( V \). If \( n \geq 2 \), and if we know all nonassociative words of degree \( < n \), then the nonassociative words of degree \( n \) are defined as those of the form \( (w_1)(w_2) \), where \( w_1 \) and \( w_2 \) are nonassociative words with \( \deg(w_1) + \deg(w_2) = n \). Although the use of brackets is essential in the above definition, some natural simplifications in the writing are usually accepted. For example, brackets covering a word of degree 1 are omitted, and words of the form \( (w)(w) \), for some other word \( w \), are written as \( (w)^2 \). Two nonassociative words are taken to be equal only if they have exactly the same writing. Thus for example, for \( v \in V \), the nonassociative words \( vv^2 \) and \( v^2v \) are different. Now, denoting by \( U \) the set of all nonassociative words with characters in \( V \), and by \( \vartheta \) the mapping \( (w_1, w_2) \to (w_1)(w_2) \) from \( U \times U \) to \( U \), we can think about the
algebra \(A(U, \vartheta, K)\) constructed in the preceding subsection. Since such an algebra depends only on \(V\) and \(K\), we denote it by \(\mathcal{F}(V, K)\). The algebra \(\mathcal{F}(V, K)\), called the \textbf{free nonassociative algebra} on \(V\) over \(K\), contains \(V\) in a natural manner, and is characterized up to algebra isomorphisms by the following \textit{universal property}: If \(A\) is any algebra over \(K\), and if \(\varphi : V \to A\) is any mapping, then \(\varphi\) extends uniquely to an algebra homomorphism from \(\mathcal{F}(V, K)\) to \(A\) (see Theorem 1.1.1 of [113]). Now, since the mapping \(\vartheta\) above is injective (by Proposition 1.1.2 of [113]), we invoke again the preceding subsection to realize that there are \textit{“many” absolute-values on} \(\mathcal{F}(V, K)\). In the original proof\(^{104}\) of Theorem 3.1, Urbanik already knows that, when \(V\) reduces to a singleton, \(\mathcal{F}(V, \mathbb{R})\) becomes an absolute-valued algebra under the norm \(\|\cdot\|_2\). The general case of such an observation is due to M. Cabrera and the author (who announced it in \(^{16}\)), and appears formulated with the appropriate precisions first in \(^{85}\). For \(1 \leq p \leq \infty\), we denote by \(\mathcal{F}_p(V, K)\) the absolute-valued algebra over \(K\) obtained by endowing \(\mathcal{F}(V, K)\) (\(= A(U, \vartheta, K)\) for \(U\) and \(\vartheta\) as above) with the absolute value \(\|\cdot\|_p\). As we are seeing in the proof of Proposition 3.1 immediately below, the absolute-valued algebra \(\mathcal{F}_1(V, K)\) has a special relevance in the general theory of normed algebras.

**Proposition 3.1.** Let \(V\) be a nonempty set. Then, up to isometric algebra isomorphisms, there exists a unique normed algebra \(N = N(V, K)\) over \(K\) satisfying the following properties:

1. \(V\) is a subset of the closed unit ball of \(N\).
2. If \(A\) is any normed algebra over \(K\), and if \(\varphi\) is any mapping from \(V\) into the closed unit ball of \(A\), then \(\varphi\) extends uniquely to a contractive algebra homomorphism from \(N\) to \(A\).

Moreover, we have:

3. The normed algebra \(N\) is in fact an absolute-valued algebra.
4. The set \(V\) consists only of norm-one elements of \(N\).

**Proof.** Take \(N = \mathcal{F}_1(V, K)\). Clearly \(N\) satisfies Properties (1), (3), and (4) in the statement. Let \(A\) be a normed algebra over \(K\), and let \(\varphi\) be a mapping from \(V\) into the closed unit ball of \(A\). Since, forgetting the norm, \(N\) is nothing other than \(\mathcal{F}(V, K)\), the universal property of this last algebra provided us with a unique algebra homomorphism \(\psi : \mathcal{F}_1(V, K) \to A\) which extends \(\varphi\). Let \(x\) be in \(N\). We have \(x = \sum_{w \in U} x_w w\), where \(U\) denotes the set of all nonassociative words with characters in \(V\), and \(\{x_w\}_{w \in U}\) stands
for the family of coordinates of $x$ relative to $U$. Therefore

$$\|\psi(x)\| = \left\| \sum_{w \in U} x_w \psi(w) \right\| \leq \sum_{w \in U} |x_w| \|\psi(w)\| \leq \sum_{w \in U} |x_w| = \|x\|.$$  

(Starting from the fact that $\psi(V)$ is contained in the closed unit ball of $A$, the inequality $\|\psi(w)\| \leq 1$ just applied is proved by induction on the degree of $w$.) Now that we know that $N$ also satisfies Property (2), let us conclude the proof by showing that $N$ is the “unique” normed algebra over $K$ satisfying (1) and (2). Let $N'$ be a normed algebra over $K$ satisfying (1) and (2) with $N'$ instead of $N$. Then we are provided with contractive algebra homomorphisms $\phi : N \to N'$ and $\phi' : N' \to N$ fixing the elements of $V$. Therefore $\phi' \circ \phi$ and $\phi \circ \phi'$ are contractive algebra endomorphisms of $N$ and $N'$, respectively, extending the corresponding inclusions $V \to N$ and $V \to N'$. By the uniqueness of such extensions, we must have $\phi' \circ \phi = I_N$ and $\phi \circ \phi' = I_{N'}$. It follows that $\phi$ is an isometric algebra isomorphism from $N$ onto $N'$ respecting the corresponding inclusions of $V$ in each of the algebras.  

Now, if $A$ is a normed algebra over $K$, if $V$ denotes the closed unit ball of $A$, and if $\Phi : N(V, K) \to A$ is the unique contractive algebra homomorphism which is the identity on $V$, then we easily realize that the induced algebra homomorphism $N(V, K)/\ker(\Phi) \to A$ is a surjective isometry. Therefore, we have the following.

**Corollary 3.1.** Every normed algebra over $K$ is isometrically algebra-isomorphic to a quotient of an absolute-valued algebra over $K$.

The absolute-valued algebra $N(V, K)$ in Proposition 3.1 has its own right to be called the **free normed nonassociative algebra** on the set $V$ over $K$. The variant of Proposition 3.1, with “complete normed” instead of “normed” everywhere, is also true, giving rise to the **free complete normed nonassociative algebra** on the set $V$ over $K$. This algebra is implicitly involved in the proof of the following result.

**Corollary 3.2.** Every complete normed algebra over $K$ is isometrically algebra-isomorphic to a quotient of a complete absolute-valued algebra over $K$.

**Proof.** Let $A$ be a complete normed algebra over $K$. Choose any subset $V$ of $A$ whose closed absolutely convex hull is the closed unit ball of $A$. By Proposition 3.1, $N(V, K)$ is an absolute-valued algebra over $K$ whose
closed unit ball contains \( V \), and there exists a contractive algebra homomorphism from \( \mathcal{N}(V, \mathbb{K}) \) to \( A \) fixing the elements of \( V \). By passing to the completion of \( \mathcal{N}(V, \mathbb{K}) \), and invoking the completeness of \( A \), we are in fact provided with a complete absolute-valued algebra \( B \) over \( \mathbb{K} \) whose closed unit ball contains \( V \), and a contractive algebra homomorphism \( \Phi : B \to A \) fixing the elements of \( V \). Let \( A_1 \) and \( B_1 \) denote the closed unit balls of \( A \) and \( B \), respectively. Since \( \Phi(B_1) \) is an absolutely convex subset of \( A \) containing \( V \), and \( A_1 \) is the closed absolutely convex hull of \( V \), the closure of \( \Phi(B_1) \) in \( A \) contains \( A_1 \). Now, from the main tool in the proof of Banach’s open mapping theorem (see for example Lemma 48.3 of \(^{11}\)) we deduce that \( \Phi(B_1) \) contains the open unit ball of \( A \). Since \( \Phi : B \to A \) is a contractive algebra homomorphism, it follows from the above that the induced algebra homomorphism \( B/\ker(\Phi) \to A \) is a surjective isometry.

Of course, the most confortable choice of \( V \) in the above proof is the one \( V = A_1 \). However, finer selections of \( V \) allow us to realize that the absolute-valued algebra \( B \) can be chosen with the same density character as that of \( A \). We recall that the density character of a topological space \( E \) is the smallest cardinal among those of dense subsets of \( E \).

Gelfand-Naimark algebras are defined as those complete normed complex algebras \( A \) endowed with a conjugate-linear algebra involution \( * \) satisfying \( ||x^*x|| = ||x||^2 \) for every \( x \in A \). Their name is due to the celebrated Gelfand-Naimark theorem \(^{30}\) that there are no Gelfand-Naimark associative algebras others than the closed \( * \)-invariant subalgebras of the Banach algebra \( \mathcal{L}(H) \) of all continuous linear operators on some complex Hilbert space \( H \), when this last algebra is endowed with the involution \( * \) determined by \( (x(\eta)|\zeta) = (\eta|x^*(\zeta)) \) for every \( x \in \mathcal{L}(H) \) and all \( \eta, \zeta \in H \). The nonassociative Gelfand-Naimark theorem \(^{81}\) asserts that unital Gelfand-Naimark algebras are alternative. Moreover, every alternative Gelfand-Naimark algebra can be seen as a closed \( * \)-invariant subalgebra of a unital Gelfand-Naimark algebra, and the study of alternative Gelfand-Naimark algebras can be reasonably reduced to that of associative ones and to that of the complexification of \( \mathbb{O} \) with suitable norm and involution. For these and other interesting results in the theory of Gelfand-Naimark alternative algebras the reader is referred to \(^{56}\) and references therein. Now, absolute-valued algebras provide us with examples of Gelfand-Naimark algebras which are not alternative. Indeed, it follows easily from Proposition 3.1 that, for any nonempty set \( V \), the absolute-valued algebra \( \mathcal{N}(V, \mathbb{C}) \) has an isometric conjugate-linear algebra involution fixing the elements of \( V \). By
passing to the completion, we obtain an absolute-valued Gelfand-Naimark algebra which is not alternative (nor even satisfies any identity when \( V \) is infinite). As pointed out in \( 87 \), the same remains true if we start from \( \mathcal{F}_p(V, \mathbb{C}) \) (\( 1 \leq p \leq \infty \)) instead of \( \mathcal{N}(V, \mathbb{C}) \) (\( = \mathcal{F}_1(V, \mathbb{C}) \)).

### 3.3. Center, centroid, and extended centroid

Let \( A \) be an algebra over a field \( F \). For \( x, y, z \in A \), we write \( [x, y] := xy - yx \) and \( [x, y, z] := (xy)z - x(yz) \). The **center** of \( A \) (denoted by \( Z(A) \)) is defined as the set of those elements \( x \in A \) such that \( [x, A] = [A, x, A] = [A, A, x] = 0 \), and is indeed an associative and commutative subalgebra of \( A \). The **centroid** of \( A \) (denoted by \( \Gamma(A) \)) is defined as the set of those linear operators \( f \) on \( A \) satisfying \( f(xy) = f(x)y = xf(y) \) for all \( x, y \in A \), and becomes naturally an associative algebra over \( F \) with a unit. Under the quite weak assumption that there is no nonzero element \( x \in A \) with \( xA = Ax = 0 \), the associative algebra \( \Gamma(A) \) is also commutative, and, by identifying each element \( z \in Z(A) \) with the operator of left multiplication by \( z \) on \( A \), \( Z(A) \) imbeds naturally into \( \Gamma(A) \). From now on, assume that \( A \) is prime (i.e., \( PQ \neq 0 \) whenever \( P \) and \( Q \) are nonzero (two-sided) ideals of \( A \)). Then \( \Gamma(A) \) becomes an integral domain, and hence it can be enlarged to its field of fractions. However, such an enlargement does not provide any additional information on the structure of \( A \). By the contrary, a larger field extension of \( \Gamma(A) \), called the **extended centroid** of \( A \) (denoted by \( C(A) \)), has turned out to be very useful to determine the behaviour of \( A \). The elements of \( C(A) \) are those linear mappings \( f : P_f \to A \), where \( P_f \) is some nonzero ideal of \( A \), satisfying \( f(xp) = xf(p) \) and \( f(px) = f(p)x \) for every \( (x, p) \in A \times P_f \). Two elements \( f, g \in C(A) \) are considered to be “equal” if they coincide on \( P_f \cap P_g \). Summing and composing elements of \( C(A) \) as is usually done for partially defined operators, such sum and composition are compatible with the notion of “equality” settled above, and convert \( C(A) \) into a field extension of \( F \). Moreover, \( \Gamma(A) \) imbeds naturally into \( C(A) \).

Now, if \( A \) is an absolute-valued real algebra, then, by Theorem 2.1, we have \( Z(A) = 0 \) unless \( A \) is equal to either \( \mathbb{R} \), \( \mathbb{C} \), \( \mathbb{H} \), or \( \mathbb{O} \). As a consequence, \( Z(A) = 0 \) for every absolute-valued complex algebra \( A \) different from \( \mathbb{C} \). Noticing that every absolute-valued algebra \( A \) is a prime algebra, the determination of \( \Gamma(A) \) follows from the inclusion \( \Gamma(A) \subseteq C(A) \), and Proposition 3.2 immediately below.
Proposition 3.2. Let $A$ be an absolute-valued algebra over $K$. Then $C(A) = C$ if $K = \mathbb{C}$, and $C(A)$ is equal to either $\mathbb{R}$ or $\mathbb{C}$ if $K = \mathbb{R}$.

Proof. In view of Lemma 1.1, it is enough to show that $C(A)$ can be endowed with an absolute value. To this end, we claim that, if $f, g$ are in $C(A)$, if $f$ is “equal” to $g$, and if $p$ and $q$ are norm-one elements of $P_f$ and $P_g$, respectively, then $\|f(p)\| = \|g(q)\|$. Indeed, $pq$ lies in $P_f \cap P_g$, so we have $f(p)q = f(pq) = g(pq) = g(q)p$, and hence $\|f(p)\| = \|f(p)q\| = \|g(q)p\| = \|g(q)\|$, as desired. Now $f \to \|f(p)\|$, with $f$ and $p$ as above, becomes a (well-defined) real valued mapping on $C(A)$, and it is easily seen that such a mapping is an absolute value. \qed

Proposition 3.2 above can be derived either from Theorem 3 and Remark 2 of 16 (by keeping in mind Lemma 2.4), or straightforwardly from Corollary 1 of 17. The autonomous proof given here is taken from 86.

As a consequence of Proposition 3.2, if $A$ is an absolute-valued algebra over $K$, then $\Gamma(A) = C$ if $K = \mathbb{C}$, and $\Gamma(A)$ is equal to either $\mathbb{R}$ or $\mathbb{C}$ if $K = \mathbb{R}$. Let $A$ be an absolute-valued real algebra. We can have either $C(A) = \Gamma(A) = \mathbb{R}$ (as happens in the case $A = \mathbb{R}, \mathbb{H}, \text{or } \mathbb{O}$), $C(A) = \Gamma(A) = \mathbb{C}$ (which happens if and only if $A$ is the absolute-valued real algebra underlying a complex one), or $C(A) = \mathbb{C}$ and $\Gamma(A) = \mathbb{R}$. To exemplify the last possibility, note that it is easily deduced from Proposition 3.1 the existence of a complete absolute-valued complex algebra $B$, together with a continuous nonzero algebra homomorphism $\phi$ from $B$ to $C$. Taking $v \in B$ with $\phi(v) = 1$, and putting $A := \mathbb{R}v \oplus \ker(\phi)$, $A$ becomes a closed real subalgebra of $B$ (and hence, a complete absolute-valued real algebra) such that $C(A) = \mathbb{C}$ and $\Gamma(A) = \mathbb{R}$. Although Proposition 3.1 was not explicitly known in 86, the example just reviewed appears there with an argument essentially equal to that we have given here.

3.4. Algebras with involution

The following result is due to Urbanik 101.

Proposition 3.3. Let $A$ be an absolute-valued real algebra endowed with an isometric algebra involution $*$ which is different from the identity operator and satisfies $xx^* = x^*x$ for every $x \in A$. Then self-adjoint elements commute with skew elements, and there exists an idempotent $e \in A$ such
that the equality $x^*x = \|x\|^2 e$ holds for every $x \in A$. As a consequence, the absolute value of $A$ comes from an inner product.

Looking at B. Gleichgewicht’s paper 49, we discovered that the first assertion in the conclusion of Proposition 3.3 is nothing other than a joint reformulation of Lemmas 1, 2, and 3 of 101. Keeping in mind such a reformulation, the consequence that $A$ is a pre-Hilbert space, proved in Lemma 4 of 101, seems to us obvious.

Seventeen years after the appearance of Urbanik’s paper 101, El-Mallah 39 shows that, if $A$ is an absolute-valued real algebra fulfilling the requirements in Proposition 3.3, then the commutant of $e$ in $A$ (say $B$) is in fact a self-adjoint subalgebra of $A$, and we have $B = \Re \oplus A_s$, where $A_s$ stands for the space of all skew elements of $A$. Shorty later, he proves the remarkable converse which follows.

**Theorem 3.2.** 40 Let $A$ be an absolute-valued real algebra containing a nonzero idempotent $e$ which commutes with all elements of $A$. Then the absolute-value of $A$ derives from an inner product $(\cdot | \cdot)$. Moreover, the isometric mapping $x \rightarrow x^* := 2(x|e)e - x$ becomes an algebra involution on $A$ satisfying $x^*x = xx^*$ for every $x \in A$.

The conclusion in Theorem 3.2, that $A$ is a pre-Hilbert space, remains true if the requirement of the existence of a nonzero idempotent which commutes with all elements of $A$ is relaxed to that of the existence of a nonzero element $a$ which commutes with all elements of $A$ and satisfies $aa^2 = (a^2)^2$ (see 42). El-Mallah’s paper 39, already quoted, contains results non previously reviewed, some of which merits a methodological comment. For instance, the proof of Theorem 5.6 of 39 (asserting that an absolute-valued algebra $A$ is finite-dimensional whenever so is the subspace of $A$ spanned by squares and there exists $a \in A \setminus \{0\}$ satisfying $ax = xa$ for every $x \in A$) can be concluded after its two first lines. Indeed, we have the following.

**Lemma 3.1.** Let $A$ be an absolute-valued algebra over $\mathbb{K}$ such that there exists $a \in A \setminus \{0\}$ satisfying $ax = xa$ for every $x \in A$. Then $A$ imbeds linearly and isometrically into the subspace $S(A)$ of $A$ spanned by squares.

**Proof.** We may assume $\|a\| = 1$. Since for $x \in A$ we have $(a + x)^2 = a^2 + 2ax + x^2$, we deduce $L_a(A) \subseteq S(A)$. But $L_a$ is a linear isometry.
Now, let us return to Urbanik’s paper \(^{101}\) to review its main results. These are a construction method producing in abundance absolute-valued real algebras \(A\) fulfilling the requirements of Proposition 3.3, and a theorem characterizing the algebras obtained from such a construction. The ingredients of the construction are an infinite set \(U\), a nonempty subset \(T\) of \(U\) such that \(#(U \setminus T) = \#U\) (where \# means cardinal number), an element \(u_0 \in T\), an injective function \(\phi\) from the family of all binary subsets of \(U\) to \(U\) whose range does not intersect \(T\), and a function \(\psi : U \times U \to \{1, -1\}\) satisfying \(\psi(u, v) + \psi(v, u) = 0\) whenever \((u, v) \in (T \times T) \cup ((U \setminus T) \times (U \setminus T))\), and \(\psi(u, v) = 1\) otherwise. Now, putting \(\varepsilon(u) := \pm 1\) depending on whether or not \(u\) belongs to \(T\), and defining \(\vartheta : U \times U \to X(U, \mathbb{R})\) by

\[
\vartheta(u, v) := \psi(u, v)\varphi(\{u, v\}) \quad \text{if } u \neq v \quad \text{and} \quad \vartheta(u, u) := \varepsilon(u)u_0,
\]

we consider the associated real algebra \(A = \mathcal{A}(U, \vartheta, \mathbb{R})\) in the meaning of Subsection 3.1. After a careful calculation, we realize that \(\mathcal{A}\) becomes an absolute-valued algebra under the norm \(\|x\| := (\sum_{u \in U} |x_u|^2)^{\frac{1}{2}}\), where \(\{x_u\}_{u \in U}\) is the family of coordinates of \(x\) relative to \(U\). Moreover, the unique linear operator \(\ast\) on \(\mathcal{A}\) which extends the mapping \(u \mapsto \varepsilon(u)u\) from \(U\) to \(\mathcal{A}\) becomes an isometric algebra involution satisfying \(x^\ast x = xx^\ast\) for every \(x \in \mathcal{A}\). If in addition we put \(((xy)|zt)) = ((xz^\ast|ty^\ast))\) for all \(x, y, z, t \in \mathcal{A}\). Passing to the completion of \(\mathcal{A}\), we obtain a complete absolute-valued real algebra, denoted by \(\mathcal{R} = \mathcal{R}(U, T, u_0, \vartheta, \psi)\), which is endowed with an isometric algebra involution \(\ast\) satisfying \(x^\ast x = xx^\ast\) and \(((xy)|zt)) = ((xz^\ast|ty^\ast))\) for all \(x, y, z, t \in \mathcal{R}\), where \(((xy)|)) := \frac{1}{2}(xy^* + yx^*)\). Following \(^{101}\), we codify the information on \(\mathcal{R}\) just collected by saying that \(\mathcal{R}\) is a \textbf{regular absolute-valued \(\ast\)-algebra}.

To classify regular absolute-valued \(\ast\)-algebras, Urbanik introduces a particular appropriate type of isotopy, called similarity. If \(A\) is a regular absolute-valued \(\ast\)-algebra, and if \(F : A \to A\) is a surjective linear isometry commuting with \(\ast\), then the Banach space of \(A\) with the same involution becomes a new regular absolute-valued \(\ast\)-algebra under the product \(x \circ y := F(xy)\). Algebras obtained from \(A\) by the above procedure are called \textbf{similar} to \(A\). By the way, two algebras \(\mathcal{R}(U, T, u_0, \vartheta, \psi)\) and \(\mathcal{R}(U', T', u'_0, \vartheta', \psi')\) are similar if and only if \(#U = \#U'\), \(#T = \#T'\), and \(#S = \#S'\), where \(S\) stands for the set of those elements of \(U\) which are neither in \(T\) nor in the range of \(\vartheta\). Thus, each similarity class of the algebras in Urbanik’s construction depends only on three cardinal numbers \(\wp_1, \wp_2, \wp_3\) with \(\wp_1\) infinite, \(\wp_3 \leq \wp_1\), and \(0 \neq \wp_2 \leq \wp_1\). Denoting by
\[ \chi(\varpi_1, \varpi_2, \varpi_3) \] such a similarity class, Urbanik’s structure theorem for regular absolute-valued \(\ast\)-algebras reads as follows.

**Theorem 3.3.** Every regular absolute-valued \(\ast\)-algebra is similar to either \(\mathbb{R}\), \(\mathbb{C}\) (with \(\ast\) equal to either the identity or the standard involution), or one in the class \(\chi(\varpi_1, \varpi_2, \varpi_3)\) for suitable cardinal numbers \(\varpi_1, \varpi_2, \varpi_3\) as above.

Let \(A\) be an absolute-valued real algebra endowed with an isometric algebra involution \(\ast\) which is different from the identity operator and satisfies \(xx^\ast = x^\ast x\) for every \(x \in A\). By replacing the product of \(A\) with the one \(x \circ y := x^\ast y\), and applying Proposition 3.3, we are provided with an absolute-valued real algebra \(B\) satisfying \(x \circ x = \|x\|^2 e\) for every \(x \in B\) and some fixed idempotent \(e \in B\). This implies \(\|x \circ x + y \circ y\| \geq \|y\|^2\) for all \(x, y \in B\). Since, in view of Urbanik’s construction, the algebra \(A\) (and hence \(B\)) can be chosen infinite-dimensional, we arrive in Gleichgewicht’s counterexample 49 to Urbanik’s problem 103 if every absolute-valued real algebra \(A\) containing a nonzero idempotent and satisfying \(\|x^2 + y^2\| \geq \|y\|^2\) for all \(x, y \in A\) is isomorphic to \(\mathbb{R}\). Finite-dimensional counterexamples are \(*\mathbb{C}\), \(*\mathbb{H}\), and \(*\mathbb{O}\). The converse of Gleichgewicht’s construction is also true. Indeed, as proved by Urbanik 104, if \(B\) is an absolute-valued real algebra such that the linear hull of squares is one-dimensional, then there exists an absolute-valued real algebra \(A\) with an isometric algebra involution \(\ast\) satisfying \(xx^\ast = x^\ast x\) for every \(x \in A\), such that \(B\) consists of the normed space of \(A\) and the product \(x \circ y := x^\ast y\). Gleichgewicht’s absolute-valued infinite-dimensional algebras were rediscovered by Ingelstam in a more direct way (see Proposition 4.4 of 54).

Given an algebra \(A\), let us define inductively \(x^{(1)} := x, x^{(n+1)} := (x^{(n)})^2\) \((x, n) \in A \times \mathbb{N}\), and let us say that \(A\) is semi-algebraic if for every \(x \in A\) there exists \(n \in \mathbb{N}\) such that the subalgebra of \(A\) generated by \(x^{(n)}\) is finite-dimensional. Clearly, the infinite-dimensional absolute-valued real algebra \(B\) in Gleichgewicht’s counterexample is semi-algebraic. This gives some interest to El-Mallah result 44 that If \(A\) is an absolute-valued semi-algebraic real algebra fulfilling the requirements in Proposition 3.3, then \(A\) is finite-dimensional.

We conclude this subsection with another result of El-Mallah.

**Theorem 3.4.** Let \(A\) be an absolute-valued real algebra endowed with an isometric algebra involution \(\ast\) such that the equality \(xx^\ast = x^\ast x\) holds for every \(x \in A\). If \(A\) satisfies the identity \(x(xx^2) = (x^2)^2\), then \(A\) is
isomorphic to either $\mathbb{R}$, $\mathbb{C}$, $\mathbb{H}$, or $\mathbb{O}$.

**Proof.** If $*$ is different from the identity operator, then the original proof in \ref{43} works without problems. Otherwise, $A$ is commutative, and hence equal to either $\mathbb{R}$, $\mathbb{C}$, or $\mathbb{O}$ (by the commutative Urbanik-Wright theorem). But $\hat{\mathbb{O}}$ does not satisfy the identity $x(xx^2) = (x^2)^2$.

### 3.5. One-sided division algebras

An algebra $A$ is said to be a left- (respectively, right-) division algebra if, for every nonzero element $x \in A$, the operator $L_x$ (respectively, $R_x$) is bijective. Since absolute-valued one-sided division complex algebras are equal to $\mathbb{C}$ (see Subsection 2.8), our interest centers in the real case. Then, refining an argument of Wright \ref{109}, we can prove the following.

**Lemma 3.2.** Let $A$ be an absolute-valued left-division real algebra. Then $A$ is a pre-Hilbert space.

**Proof.** First assume that $A$ has a left unit $e$. Then, since $L_e = I_A$ (the identity operator on $A$), for every norm-one element $x \in A$, we have
\[
4 = 4 \|L_x\| = \|(L_x + I_A)^2 - (L_x - I_A)^2\|
\leq \|L_{x+e}\|^2 + \|L_{x-e}\|^2 \leq \|x + e\|^2 + \|x - e\|^2.
\]

Now remove the assumption that $A$ has a left unit, and note that, for each norm-one element $e \in A$, the normed space of $A$ becomes an absolute-valued algebra with left unit $e$ under the product $x \circ y := L_e^{-1}(xy)$. It follows $4 \leq \|x + e\|^2 + \|x - e\|^2$ for all norm-one elements $e, x \in A$. Finally, apply Schoenberg’s theorem.

Now we can prove one of the main results in this subsection.

**Theorem 3.5.** Let $A$ be an absolute-valued real algebra with a left unit $e$. Then the absolute-value of $A$ derives from an inner product $(\cdot | \cdot)$, and, putting $x^* := 2(x|e)e - x$, we have $(xy|z) = (y|x^*z)$ and $x^*(xy) = \|x\|^2y$ for all $x, y, z \in A$.

**Proof.** We may assume that $A$ is complete. Then an argument, involving connectedness and elementary Operator Theory, shows that $A$ is a left
division algebra (see Lemma 2.2 of 59). By Lemma 3.2, the norm of $A$ comes from an inner product $\langle \cdot | \cdot \rangle$. For $y, u \in A$ with $\langle e | u \rangle = 0$, we have
\[
(1 + \|u\|^2)\|y\|^2 = \|e + u\|^2\|y\|^2 = \|(e + u)y\|^2
\]
and hence $\langle uy | y \rangle = 0$. By linearizing in the variable $y$, we deduce $\langle uy | z \rangle = -\langle y | uz \rangle$ for all $u, y, z \in A$ with $\langle e | u \rangle = 0$, or, equivalently, $\langle xy | z \rangle = \langle y | x^*z \rangle$ for all $x, y, z \in A$. Finally, apply Lemma 2.3.

The following corollary follows straightforwardly from Theorem 3.5 above, Lemma 2.3 just applied, and the fact that every absolute-valued left-division algebra is isotopic to an absolute-valued algebra with a left unit (see the proof of Lemma 3.2).

**Corollary 3.3.** An absolute-valued algebra is a left-division algebra if and only if it is isotopic to an absolute-valued algebra $A$ whose norm derives from an inner product $\langle \cdot | \cdot \rangle$ such that, for each $x \in A$, there exists $x^* \in A$ satisfying $\langle xy | z \rangle = \langle y | x^*z \rangle$ for all $y, z \in A$.

Theorem 3.5 and Corollary 3.3 were first proved by the author (see 85 and 86, respectively). The proof of Theorem 3.5 in 85 is different from that we have given here, and can seem more involved, since Theorem 3.5 is derived there from a more general principle (namely, Theorem 1 of 85). Really, if we take from the proof of Theorem 1 of 85 the minimum necessary to get Theorem 3.5, then most complications disappear. From Theorem 3.5 we derive that absolute-valued real algebras with a left unit are left-division algebras. More generally, we have the following.

**Corollary 3.4.** An absolute-valued real algebra $A$ is a left-division algebra if (and only if) there exists $e \in A$ such that $eA = A$.

We do not know if Corollary 3.4 remains true when the requirement $eA = A$ is replaced with the one that $eA$ is dense in $A$.

In view of Lemma 3.2, absolute-valued left-division real algebras are composition algebras. In 61, Kaplansky proved that composition division algebras are finite-dimensional, and commented on his attempts to show that the same is true when “division” is relaxed to “left-division”. We are going to realize that such attempts could not be successful, by constructing absolute-valued infinite-dimensional left-division real algebras. To this end, is it convenient to reformulate Theorem 3.5 in a more sophisticated way.
We recall the facts, already commented in Subsection 2.2, that smooth-normed real algebras are pre-Hilbert spaces, and that their algebraic structure is well-understood. Some precisions, taken from [82], are needed here.

For instance, if $A$ is a smooth-normed real algebra, then the mapping $x \rightarrow x^* := 2(x|1)1 - x$ becomes an algebra involution on $A$, which is uniquely determined by the fact that, for every $x \in A$, both $x + x^*$ and $x^*x$ lie in $\mathbb{R}1$. Here $1$ denotes the unit of $A$, and $(\cdot|\cdot)$ stands for the inner product from which the norm of $A$ derives. If the smooth-normed real algebra $A$ is commutative, then actually the unit and the inner product determine the algebra product by means of the equality

$$xy = (x|1)y + (y|1)x - (x|y)1.$$  \hspace{1cm} (1)

Now note that, conversely, every nonzero real pre-Hilbert space $H$ becomes a smooth-normed commutative real algebra by choosing any norm-one element $1 \in H$ and then by defining the product according to the equality (1). Note also that the choice of the norm-one element $1$ above is structurally irrelevant because pre-Hilbert spaces are transitive normed spaces. It follows that smooth-normed commutative real algebras and nonzero real pre-Hilbert spaces are in a bijective categorical correspondence. Now, Let $A$ be a smooth-normed commutative real algebra, and let $H$ be a nonzero real pre-Hilbert space. By a unital $\ast$-representation of $A$ on $H$ we mean any linear mapping $\phi : A \rightarrow \mathcal{L}(H)$ satisfying $\phi(1) = I_H$, $\phi(x^2) = (\phi(x))^2$, and $(\phi(x)(\eta)|\zeta) = (\eta|\phi(x^*)(\zeta))$ for every $x \in A$ and all $\eta, \zeta \in H$. The first assertion in Theorem 3.6 immediately below is easily verified (see [85] for details), whereas the second one is the desired reformulation of Theorem 3.5.

**Theorem 3.6.** If $A$ is a smooth-normed commutative real algebra, and if $\phi$ is a unital $\ast$-representation of $A$ on its own pre-Hilbert space, then the normed space of $A$ with the new product $\odot$ defined by $x \odot y := \phi(x)(y)$ becomes an absolute-valued real algebra with a left unit. Moreover, there are no absolute-valued real algebras with a left unit others than those given by the above construction.

One of the main results in the mathematical modelling of Quantum Mechanics is the possibility of representing the so-called “Canonical Anticommutation Relations” by means of bounded linear operators on complex Hilbert spaces [14]. Applying such a result, it is proved in [85] that every complete smooth-normed infinite-dimensional commutative real algebra has a unital $\ast$-representation on its own Hilbert space. Therefore we have the following.
Theorem 3.7. Every infinite-dimensional real Hilbert space becomes an absolute-valued algebra with a left unit, under a suitable product.

In the case that the infinite-dimensional real Hilbert space is separable, Theorem 3.7 was proved simultaneously and independently by Cuenca. Cuenca’s proof is of course easier than the one in for the general case. The key idea in consists of a “doubling process” which, after an induction argument, assures that, for every \( n \in \mathbb{N} \), the smooth normed commutative real algebra \( A_n \) of dimension \( n \) has a unital \( * \)-representation \( \phi_n \) on the real pre-Hilbert space \( H_n \) of dimension \( 2^{n-1} \). Moreover, regarding

\[
A_1 \subseteq A_2 \subseteq \ldots \subseteq A_n \subseteq \ldots \text{ and } H_1 \subseteq H_2 \subseteq \ldots \subseteq H_n \subseteq \ldots
\]

in a convenient way, we have \( \phi_{n+1}(x)(\eta) = \phi_n(x)(\eta) \) whenever \( n, x, \) and \( \eta \) are in \( \mathbb{N}, A_n, \) and \( H_n \), respectively. Then \( A := \bigcup_{n \in \mathbb{N}} A_n \) is a smooth normed commutative real algebra having a unital \( * \)-representation on the real pre-Hilbert space \( H := \bigcup_{n \in \mathbb{N}} H_n \). Since \( H \) can be identified with the pre-Hilbert space of \( A \), the separable version of Theorem 3.7 follows from the first assertion in Theorem 3.6 by passing to completion.

The proof of Theorem 3.7 given in shows in addition that the product, converting the arbitrary infinite-dimensional real Hilbert space into an absolute-valued algebra with a left unit, can be chosen in such a way that the corresponding algebra has no nonzero proper closed left ideals.

Recently, Elduque and Pérez have proved that every infinite-dimensional real vector space can be endowed with a pre-Hilbertian norm and a product which convert it into an absolute-valued algebra with a left unit. Since, in the construction of , the pre-Hilbertian norm and the product can be chosen in such a way that an arbitrarily prefixed algebraic basis becomes orthonormal, it follows that Theorem 3.7 can be derived from the Elduque-Pérez result by an easy argument of completion.

Very recently, relevant progresses about the representations of the Canonical Anticommutation Relations on separable real Hilbert spaces have been done in the paper of E. Galina, A. Kaplan, and L. Saal. As pointed out by these authors, their results give rise to a classification, up to an isotopy, of all separable complete absolute-valued left-division real algebras.

Now that the existence of absolute-valued infinite-dimensional left-division real algebras is not in doubt, Propositions 3.4 and 3.5, and Corollary 3.5 below have their own interest.

Proposition 3.4. Let \( A \) be an absolute-valued real algebra with a left unit. Then the following assertions are equivalent:
(1) For all $x, y \in A$, there exists $z \in A$ such that $L_x \circ L_y = L_z \circ L_x$.

(2) The dimension of $A$ is equal to either 1, 2, or 4.

Proof. Keeping in mind that $\mathbb{R}$, $\mathbb{C}$, and $\mathbb{H}$ are associative division algebras, the implication (2) $\Rightarrow$ (1) is an easy consequence of Proposition 1.8. Let $L$ denote the space of all left multiplication operators on $A$. It follows easily from Theorem 3.5 that $F^2$ lies in $L$ whenever $F$ is in $L$. Therefore both

$$F \circ G := \frac{1}{2} (F \circ G + G \circ F) = \frac{1}{8} ((F + G)^2 - (F - G)^2)$$

and

$$F \circ G \circ F = 2F \circ (F \circ G) - F^2 \circ G$$

lie in $L$ whenever $F, G$ are in $L$. Assume that (1) is true. Let $x, y$ be in $A$ with $x \neq 0$. Then, keeping in mind that the operator $L_x$ is bijective (by Theorem 3.5), the assumption (1) reads as $L_x \circ L_y \circ L_x^{-1} \in L$. But, again by Theorem 3.5, the norm of $A$ derives from an inner product $\langle \cdot | \cdot \rangle$ such that, denoting by $e$ the left unit of $A$, and putting $x^* := 2(x|e)e - x$, we have $L_x^{-1} = ||x||^2 L_{x^*}$. Thus $L_x \circ L_y \circ L_x \in L$. Since $L_x \circ L_y \circ L_x \in L$ and $x + x^* = 2(x|e)e$, we deduce $(x|e)L_x \circ L_y \in L$ or, equivalently, $L_x \circ L_y \in L$ whenever $(x|e) \neq 0$. Since the set $\{ t \in A : (t|e) \neq 0 \}$ is dense in $A$, and the mapping $t \to L_t$ from $A$ to the normed algebra $\mathcal{L}(A)$ (of all continuous linear operators on $A$) is a linear isometry, we obtain $L_x \circ L_y \in L$ without any restriction. In this way, $L$ becomes a subalgebra of $\mathcal{L}(A)$ containing the unit of $\mathcal{L}(A)$. Since the algebra $\mathcal{L}(A)$ is associative, and $L$ is a pre-Hilbert space for the operator norm, it follows from Theorem 3.1 of [34] that $L$ is a copy of $\mathbb{R}$, $\mathbb{C}$, or $\mathbb{H}$. Therefore $A$ has dimension equal to 1, 2, or 4.

Proposition 3.5. Let $A$ be an absolute-valued real algebra with a left unit, and let $|| \cdot ||$ be an algebra norm on $A$. Then we have $|| \cdot || \leq || \cdot ||$.

Proof. Let $e$ denote the left unit of $A$, and let $x$ be in $A$. According to Theorem 3.5, we have $L_x \circ L_x = ||x||^2 I_A$, where $x^* := 2(x|e)e - x$. It follows

$$||x||^2 \leq ||L_x \circ L_x|| \leq ||x^*|| ||x|| \leq (2||x|| ||e|| + ||x||) ||x||,$$

so $(1 + ||e||^2) ||x||^2 \leq (||x|| + ||x|| ||e||)^2$, and so $(\sqrt{1 + ||e||^2} - ||e||) ||x|| \leq ||x||$. Now apply Proposition 1.1.

We do not know if Proposition 3.5 remains true whenever the requirement of the existence of a left unit in $A$ is relaxed to the one that $A$ is a left-division algebra. In any case, we have the following.
Corollary 3.5. Let $A$ be a left-division real algebra. Then there exists at most one absolute value on $A$.

Proof. Let $\| \cdot \|$ and $\| \| \|$ be absolute values on $A$. Fix $e \in A$ with $\| e \| = 1$, and consider the absolute-valued real algebra $B$ consisting of the vector space of $A$, the norm $\| \cdot \|$, and the product $x \odot y := L^{-1}_e(xy)$. Since $B$ has a left unit, and $\| e \|^{-1} \| \cdot \|$ is an algebra norm on $B$, Proposition 3.5 applies giving that $\| \cdot \| \leq \| e \|^{-1} \| \cdot \|$. Then, keeping in mind that $\| \cdot \|$ is an algebra norm on $A$, and that $\| \cdot \|$ is an absolute value on $A$, we deduce from Proposition 1.1 that $\| \cdot \| \leq \| \cdot \|$. By symmetry, we have also $\| \cdot \| \leq \| \cdot \|$. □

Corollary 3.5 above was first proved in \cite{86} with craffer techniques.

3.6. Automatic continuity

Minor changes to the proof of Corollary 3.5 could allow us to realize that if $A$ is an absolute-valued left-division algebra, and if $\| \cdot \|$ is a complete algebra norm on $A$, then we have $\| \cdot \| \leq \| \cdot \|$. However, this fact becomes unsubstantial in view of the result which follows.

Theorem 3.8. Let $A$ be a complete normed algebra over $\mathbb{K}$, let $B$ be an absolute-valued algebra over $\mathbb{K}$, and let $\phi : A \rightarrow B$ be an algebra homomorphism. Then $\phi$ is contractive.

Keeping in mind Proposition 1.1, the actual message of Theorem 3.8 above is that algebra homomorphisms from complete normed algebras to absolute-valued algebras are automatically continuous. We do not enter here the original proof of Theorem 3.8 in \cite{85}. Limiting ourselves to mention its main ingredients (namely, Theorem 2.2 and a little Operator Theory, including Lemma 3.1 of \cite{83}), we prefer to review here how such a proof has inspired further developments of the theory of automatic continuity in settings close to that of absolute-valued algebras. To this end, we note that, replacing the absolute-valued algebra $B$ with the completion of the range of $\phi$, the proof of Theorem 3.8 reduces to the case that $B$ is complete and $\phi$ has dense range. Thus, for $\mathbb{K} = \mathbb{C}$, Theorem 3.8 follows straightforwardly from Theorem 3.9 immediately below.

Theorem 3.9. Algebra homomorphisms from complete normed complex algebras to complete normed complex algebras with no nonzero two-sided topological divisor of zero are continuous.
To prove Theorem 3.9, we introduced in quasi-division algebras. These are defined as those algebras $A$ such that $L_x$ or $R_x$ is bijective for every $x \in A \setminus \{0\}$. Then we proved that, if $A$ and $B$ are complete normed algebras over $K$, if $B$ is not a quasi-division (respectively, division) algebra, and if $B$ has no nonzero two-sided (respectively, one-sided) topological divisors of zero, then dense range algebra homomorphisms from $A$ to $B$ are continuous. With the help of, this implies that, if $A$ and $B$ are complete normed algebras over $K$, and if $B$ has no nonzero two-sided topological divisors of zero, then surjective algebra homomorphisms from $A$ to $B$ are continuous. Now, Theorem 3.9 follows from the results just quoted and Proposition 3.6 immediately below.

**Proposition 3.6.** Every complete normed quasi-division complex algebra has dimension $\leq 2$.

For $K = \mathbb{R}$, Theorem 3.8 can be also derived from the results in quoted above, by applying Wright's Theorem 1.3 instead of Proposition 3.6. Some additional information, related to the discussion of the proof of Theorem 3.8 just done, is collected in the next remark.

**Remark 3.1.** i) In view of Corollary 3.4, absolute-valued quasi-division algebras are in fact one-sided division algebras.

ii) We do not know if Theorem 3.9 remains true when real algebras replace complex ones. Even if the range algebra has no nonzero one-sided topological divisors of zero, the question remains open. The point is that the old problem, if every complete normed division real algebra is finite-dimensional, remains still unsolved. In relation to this problem, let us note that, as a consequence of Theorems 3.5 and 3.7, there exist complete normed infinite-dimensional real algebras $A$ such that, for every $x \in A \setminus \{0\}$, the operators $L_x$ and $R_x$ are surjective (see Proposition 8 of for details).

iii) The question of the automatic continuity of homomorphisms into finite-dimensional algebras has been definitively settled in. Indeed, given a normed finite-dimensional algebra $B$ over $K$, all algebra homomorphisms from all complete normed algebras over $K$ to $B$ are continuous if and only if $B$ has no nonzero element $x$ with $x^2 = 0$.

From now on, let $\Omega$ be a locally compact Hausdorff topological space. Given a normed algebra $A$ over $K$, the space $C_0(\Omega, A)$, of all $A$-valued continuous functions on $\Omega$ which vanish at infinity, becomes a normed algebra over $K$ under the product defined point-wise and the sup norm. If follows
from Lemma 2.10 of \cite{71} and Theorem 3.8 that, if $A$ is an absolute-valued algebra over $\mathbb{K}$, then algebra homomorphisms from complete normed algebras over $\mathbb{K}$ to $C_0(\Omega, A)$ are continuous. The paper of M. M. Neumann, M. V. Velasco and the author \cite{71}, a minor result of which has been just applied, contains a deeper variant of the fact reviewed above. By an $\mathcal{F}$-algebra we mean a real or complex algebra endowed with a complete metrizable vector space topology making the product continuous. Now we have the following.

**Theorem 3.10.** \cite{71} Let $F$ be an $\mathcal{F}$-algebra over $\mathbb{K}$, let $A$ be an absolute-valued algebra over $\mathbb{K}$, and let $\phi : F \to C_0(\Omega, A)$ be an algebra homomorphism. Assume that $\Omega$ has no isolated points, and that the range of $\phi$ separates the points of $\Omega$. Then $\phi$ is continuous.

Here, that a subset $S$ of $C_0(\Omega, A)$ separates the points of $\Omega$ means that, whenever $\omega_1$ and $\omega_2$ are different points of $\Omega$, we can find $f \in S$ such that $f(\omega_1) = 0$ and $f(\omega_2) \neq 0$. Other results of a flavour similar to that of Theorem 3.10 are also proved in \cite{71}. For example, if $A$ is an absolute-valued algebra over $\mathbb{K}$, if $\Omega$ has no isolated points, and if $F$ is a subalgebra of $C_0(\Omega, A)$ which separates the points of $\Omega$ and is endowed with an $\mathcal{F}$-algebra topology, then every derivation of $F$ is continuous. As a consequence, if $A$ is a complete absolute-valued algebra over $\mathbb{K}$, and if $\Omega$ has no isolated points, then every derivation of $C_0(\Omega, A)$ is continuous. We recall that a derivation of an algebra $A$ is a linear operator $D$ on $A$ satisfying

$$D(xy) = xD(y) + D(x)y$$

for all $x, y \in A$. The results in \cite{71} we have reviewed are in fact corollaries to more general facts. In particular, all these results remain true when absolute-valued algebras are replaced with normed algebras without nonzero left topological divisors of zero. Appropriate variants of such results also hold when absolute-valued algebras are replaced with normed algebras with a unit. The associative side of \cite{71} has its own interest, and appears as Section 5.6 of \cite{64}. The announcement of \cite{71} done in \cite{107} centres more in the nonassociative aspects, paying special attention to the applications in the theory of absolute-valued algebras.

In contrast with Theorem 3.8, we do not know if derivations of complete absolute-valued algebras are automatically continuous. Anyway, we have the following.

**Proposition 3.7.** Absolute-valued complex algebras have no nonzero continuous derivations.
Proof. Let $A$ be an absolute-valued complex algebra (which can be assumed complete), and let $D$ be a continuous derivation of $A$. Then, for every complex number $\lambda$, $\exp(\lambda D)$ is an algebra automorphism of $A$, and hence we have $\|\exp(\lambda D)\| = 1$ (by Proposition 1.1). Now apply Liouville’s theorem to deduce $D = 0$.

Proposition 3.7 does not remain true when real algebras replace complex ones. Indeed, $\mathbb{H}$ and $\mathbb{O}$ have nonzero derivations in abundance. With the language of “numerical ranges”\(^{12}\), the general version of Proposition 3.7 is that continuous derivations of an absolute-valued algebra over $K$ have numerical ranges equal to zero. Proposition 3.7 then follows since, by the Bohnenblust-Karlin theorem, continuous linear operators on a complex normed space must be zero provided their numerical ranges are zero.

4. Some deviations of the theory

4.1. Nearly absolute-valued algebras

For every normed algebra $A$ over $K$, let us define $\rho(A)$ as the largest non-negative real number $\rho$ satisfying $\rho\|x\|\|y\| \leq \|xy\|$ for all $x, y \in A$. Those normed algebras $A$ such that $\rho(A) > 0$ are called nearly absolute-valued algebras. Let $A$ be a normed finite-dimensional algebra over $K$. Then, by the compactness of spheres, $A$ is nearly absolute-valued if (and only if) it is a division algebra. Moreover, if this is the case, then $A$ is isomorphic to $\mathbb{C}$ when $K = \mathbb{C}$, and the dimension of $A$ is equal to 1, 2, 4, or 8 when $K = \mathbb{R}$ (by Theorem 1.2). On the other hand, by Hopf’s theorem (see page 235 of 31), nearly absolute-valued finite-dimensional commutative real algebras have dimension $\leq 2$. We note that, since every finite-dimensional algebra over $K$ can be endowed with an algebra norm, nearly absolute-valued finite-dimensional real algebras and finite-dimensional division real algebras essentially coincide.

By Theorem 1.1, every nearly absolute-valued alternative real algebra is isomorphic to either $\mathbb{R}$, $\mathbb{C}$, $\mathbb{H}$, or $\mathbb{O}$, so that no much more can be said about such an algebra. The consequence that nearly absolute-valued alternative algebras are algebra-isomorphic to absolute-valued algebras is no longer true if alternativeness is removed (even for finite-dimensional normed algebras $A$ with $\rho(A)$ near one). For instance, for $0 < \varepsilon < \frac{1}{2}$, consider the normed real algebra $A$ consisting of the normed space of $\mathbb{H}$ and the product $x \odot y := (1 - \varepsilon)xy + \varepsilon yx$. Then we have $\rho(A) \geq 1 - 2\varepsilon$, but $A$ cannot be algebra-isomorphic to an absolute-valued algebra. Indeed,
A is not associative and has a unit, whereas every absolute-valued four-dimensional real algebra with a unit is isomorphic to $\mathbb{H}$ (by Theorem 2.1).

The above example shows in addition how a theory of nearly absolute-valued algebras parallel to that of absolute-valued algebras cannot be expected. Another notice in the same line is that, in contrast with Theorem 2.3, there exist nearly absolute-valued infinite-dimensional commutative algebras over $\mathbb{K}$. Indeed, as proved in Example 1.1 of 59, for every infinite set $U$ and every injective mapping $\vartheta : U \times U \to U$, the complete normed real algebra $A$ obtained by replacing the product of $C_2(U, \vartheta, \mathbb{K})$ (see Subsection 3.1) with $x \circ y := \frac{1}{2} (xy + yx)$ satisfies $\rho(A) \geq 2^{-\frac{1}{2}}$.

Despite the above limitations, in the paper of Kaidi, Ramírez, and the author just quoted we wondered whether there could be a theory of nearly absolute-valued algebras “nearly” parallel to that of absolute-valued algebras. More precisely, we raised the following.

**Question 4.1.** Let $P$ be anyone of the purely algebraic properties leading absolute-valued real algebras to the finite dimension. Is there a universal constant $0 \leq K_P < 1$ such that every normed real algebra $A$ satisfying $P$ and $\rho(A) > K_P$ is finite-dimensional?

We were able to answer Question 4.1 for the most relevant choices of Property $P$. Thus we have the following.

**Theorem 4.1.** 59 Question 4.1 has an affirmative answer whenever Property $P$ is equal to the existence of a unit, the commutativity, or the algebraicity. Moreover, for such choices of $P$, the universal constant $K_P$ can be (uniquely) chosen in such a way that there exists a normed infinite-dimensional real algebra satisfying $P$ and $\rho(A) = K_P$.

Now, fundamental Theorems 2.1, 2.3, and 2.11 are “nearly” true when nearly absolute-valued algebras replace absolute-valued algebras. As a consequence, a normed power-commutative real algebra $A$ is finite-dimensional whenever $\rho(A)$ is near one. Many other results of the same flavour can be obtained (see for example the variants of Theorems 2.2, 2.8, and 3.8 proved in Corollaries 3.2 and 3.4, and Theorem 3.3 of 59, respectively). We already know that $K_P \geq 2^{-\frac{1}{2}}$ when $P$ means commutativity. When $P$ means algebraicity or existence of a unit, we do not know whether or not the equality $K_P = 0$ holds. In any case, Nearly absolute-valued complex algebras are isomorphic to $\mathbb{C}$ whenever they have a left unit or are algebraic (see Remark 2.8 of 59 and our Subsection 2.8, respectively). If they are commutative, then a result similar to the one given by Theorem 4.1 (with
\( P \) equal to the commutativity) holds.

A normed space \( X \) is said to be uniformly non-square if there exists \( 0 < \sigma < 1 \) such that the inequality \( \min\{\|x+y\|, \|x-y\|\} < 2\sigma \) holds for all \( x, y \) in the closed unit ball of \( X \). We note that pre-Hilbert spaces are uniformly non-square, and that the completion of every uniformly non-square normed space is a superreflexive Banach space (see Theorem VII.4.4 of 27). We also remark that neither absolute-valued algebras nor nearly absolute-valued finite-dimensional algebras with a unit need be uniformly non-square (by our Subsection 3.1 and Example 2.1 of 59, respectively). These facts give its own interest to the following variant of Theorem 3.5.

**Theorem 4.2.** 59 Let \( A \) be a normed real algebra with \( \rho(A) > 2^{-\frac{1}{4}} \). If there exists \( a \in A \) such that \( aA \) is dense in \( A \), then \( A \) is uniformly non-square.

4.2. **Other deviations**

By a trigonometric algebra we mean a pre-Hilbert real algebra \( A \) satisfying \( \|xy\|^2 + (x|y)^2 = \|x\|^2\|y\|^2 \) (or, equivalently, \( ||xy|| = ||x||||y|| \sin \alpha \), where \( \alpha \) is the angle between \( x \) and \( y \)) for all \( x, y \in A \setminus \{0\} \). By cleverly applying Hurwiz’s theorem (see Subsection 2.2), P. A. Terekhin 100 shows that the dimensions of finite-dimensional trigonometric algebras are precisely 1, 2, 3, 4, 7, and 8. The existence of complete trigonometric algebras of arbitrary infinite Hilbertian dimension is implicitly known in 59. Indeed, if \( U \) is an infinite set, and if \( \vartheta \) is an injective mapping from \( U \times U \) to \( U \), then the real Hilbert algebra obtained from the absolute-valued algebra \( C_2(U, \vartheta, R) \) (see Subsection 3.1) by replacing its product with the one \( x \odot y := \frac{xy - yx}{\sqrt{2}} \) becomes a trigonometric algebra (see Remark 1.6 of 59 for details). Actually, all infinite-dimensional trigonometric algebras can be constructed in a transparent way from the absolute-valued real algebras with involution considered in Subsection 3.4 (see 9).

By a triple system over a field \( F \) we mean a nonzero vector space \( T \) over \( F \) endowed with a trilinear mapping \( (\cdot, \cdot) : T \times T \times T \rightarrow T \). Absolute-valued triple systems over \( \mathbb{K} \) are defined as those triple systems \( T \) over \( \mathbb{K} \) endowed with a norm \( \| \cdot \| \) satisfying \( ||xyz|| = ||x||||y||\|z\| \) for all \( x, y, z \in T \). Each absolute-valued triple system gives rise to “many” absolute-valued algebras. Indeed, if \( T \) is an absolute-valued triple system, and if \( u \) is a norm-one element in \( T \), then the normed space of \( T \) becomes an absolute-valued algebra under the product \( x \odot y := \langle xyu \rangle \). As pointed out in 18, this implies (by Proposition 1.5) that the norm of every absolute-valued finite-dimensional real triple system springs from an inner product.
It follows that, if $T$ is an absolute-valued finite-dimensional real triple system, then the mapping $q : x \rightarrow \|x\|^2$ is a quadratic form on $T$ satisfying $q((xyz)) = q(x)q(y)q(z)$ for all $x, y, z \in T$. Now we can mimic Albert’s definition of isotopy (see Subsection 1.3), and apply the main result in K. McCrimmon’s paper 67, to get that, up to an isotopy, the absolute-valued finite-dimensional real triple systems are $\mathbb{R}$, $\mathbb{C}$ (with $\langle xyz \rangle = xyz$ in both cases), $\mathbb{H}$ (with $\langle xyz \rangle$ equal to either $xyz$, $xzy$, or $yxz$), and $\mathbb{O}$ (with $\langle xyz \rangle$ equal to either $(xyz)z$, $(xz)y$, $(yx)z$, $x(yz)$, or $y(xz)$). This result is a sample of how the study of absolute-valued triple systems can be promising (see 18 and 20).

An algebra $A$ is said to be two-graded if it can be written as $A = A_0 \oplus A_1$, where $A_0$ and $A_1$ are nonzero subspaces of $A$ satisfying $A_i A_j \subseteq A_{i+j}$ for all $i, j \in \mathbb{Z}_2$. Two-graded absolute-valued algebras are defined as those normed two-graded algebras $A = A_0 \oplus A_1$ over $\mathbb{K}$ satisfying $\|x_i x_j\| = \|x_i\|\|x_j\|$ for all $i, j \in \mathbb{Z}_2$ and every $(x_i, x_j) \in A_i \times A_j$. The work of A. J. Calderón and C. Martín 19 deals with these objects, and starts with the observation that, if $A$ is a two-graded absolute-valued algebra, then, in a natural way, $A_0$ is an absolute-valued algebra, and $A_1$ is an absolute-valued triple system. As a consequence, two-graded absolute-valued finite-dimensional real algebras have dimension 2, 4, 8, or 16.

5. Absolute-valuable Banach spaces

In this concluding section we are going to deal with those Banach spaces which underlie complete absolute-valued algebras. Such Banach spaces will be called absolute-valuable. The finite-dimensional side of this topic is definitively solved by Albert’s Proposition 1.5. Indeed, the absolute-valuable finite-dimensional real Banach spaces are precisely the real Hilbert spaces of dimension 1, 2, 4, and 8. On the other hand, it is clear that $\mathbb{C}$ is the unique absolute-valuable finite-dimensional complex Banach space. Therefore, the interest of absolute-valuable Banach spaces centers into the infinite-dimensional case.

5.1. The isometric point of view

We already know that, for every infinite set $U$, the classical Banach spaces $l_p(U, \mathbb{K})$ ($1 \leq p < \infty$) and $c_0(U, \mathbb{K})$ are absolute-valuable (see Subsection 3.1). In fact, the role played there by $\mathbb{K}$ can be also played by any absolute-valued algebra, and hence we have that, given an infinite set $U$ and a Banach space $X$, the Banach spaces $l_p(U, X)$ ($1 \leq p < \infty$) and
$c_0(U, X)$ are absolute-valuable whenever so is $X$. Even, the value $p = \infty$ is allowed above (by Proposition 5.2 below). Other stability properties of the class of absolute-valuable Banach spaces can be also derived from previously reviewed results. For example, it follows from Lemma 2.4 that the normed ultraproduct of every ultrafiltered family of absolute-valuable Banach spaces is an absolute-valuable Banach space. More examples of absolute-valuable Banach spaces are given in Proposition 5.1 immediately below. As usual, given Banach spaces $X$ and $Y$ over $\mathbb{K}$, we denote by $\mathcal{L}(X, Y)$ the Banach space over $\mathbb{K}$ of all bounded linear operators from $X$ to $Y$, and by $\mathcal{K}(X, Y)$ the closed subspace of $\mathcal{L}(X, Y)$ consisting of all compact operators from $X$ to $Y$. Moreover, we write $X^*$, $\mathcal{L}(X)$, and $\mathcal{K}(X)$ instead of $\mathcal{L}(X, \mathbb{K})$, $\mathcal{L}(X, X)$, and $\mathcal{K}(X, X)$, respectively.

**Proposition 5.1.** Let $1 \leq p \leq \infty$, let $U_1$ be an infinite set, and let $X_1$ stand for $\ell_p(U_1, \mathbb{K})$. Then $X_1^*$ is absolute-valuable. Moreover, if $U_2$ is another infinite set, and if $X_2$ stands for $\ell_p(U_2, \mathbb{K})$, then $\mathcal{L}(X_1, X_2)$ and $\mathcal{K}(X_1, X_2)$ are absolute-valuable.

As a consequence, infinite-dimensional Hilbert spaces over $\mathbb{K}$ are absolute-valuable, and moreover, if $H_1$ and $H_2$ are infinite-dimensional Hilbert spaces over $\mathbb{K}$, then $\mathcal{L}(H_1, H_2)$ and $\mathcal{K}(H_1, H_2)$ are absolute-valuable.

The paper of J. Becerra, A. Moreno, and the author, also contains Theorem 5.1 immediately below. Given a topological space $E$, we denote by $\text{dens}(E)$ the density character of $E$ (see Subsection 3.2).

**Theorem 5.1.** Every Banach space $X$ over $\mathbb{K}$ is linearly isometric to a subspace of a Banach space $Y$ over $\mathbb{K}$ with $\text{dens}(Y) = \text{dens}(X)$ and such that $Y$, $Y^*$, $\mathcal{L}(Y)$, and $\mathcal{K}(Y)$ are absolute-valuable.

The ideas in the proof of Theorem 5.1 are not far from those in Proposition 5.2 immediately below. In what follows, $\Omega$ will denote a compact Hausdorff topological space.

**Proposition 5.2.** Assume that there exists a continuous surjection from $\Omega$ to $\Omega \times \Omega$, and let $X$ be an absolute-valuable Banach space. Then $C(\Omega, X)$ is absolute-valuable.

**Proof.** Let us choose a product $(x, y) \to xy$ on $X$ converting $X$ into an absolute-valued algebra, let $\phi : \Omega \to \Omega \times \Omega$ stand for the continuous surjection whose existence is assumed, and, for $i = 1, 2$, let $\pi_i : \Omega \times \Omega \to \Omega$
denote the $i$-th coordinate projection. Then the product $\diamond$ on $C(\Omega, X)$ defined by $(f \diamond g)(\omega) := f(\pi_1(\phi(\omega)))g(\pi_2(\phi(\omega)))$ (for every $\omega \in \Omega$ and all $f, g \in C(\Omega, X)$) converts $C(\Omega, X)$ into an absolute-valued algebra.

We note that the choice $\Omega = [0, 1]$ is allowed in Proposition 5.2 (see 4 and references therein). According to that proposition, the existence of a continuous surjection from $\Omega$ to $\Omega \times \Omega$ is a sufficient condition for $C(\Omega, K)$ to be absolute-valuable. In 69, a partial converse is shown. Indeed, we have the following

**Theorem 5.2.** If $C(\Omega, K)$ is absolute-valuable, then there exists a closed subset $F$ of $\Omega$, and a continuous surjection from $F$ to $\Omega \times \Omega$.

As a consequence, $C(\Omega, K)$ is not absolute-valuable whenever $\Omega$ is the one-point compactification of a discrete infinite space (a fact first proved in 7). In particular, the classical space $c$ of all real or complex convergent sequences is not absolute-valuable. Theorem 5.2 is also applied in 69 to prove that, in the case that $\Omega$ is metrizable, $C(\Omega, K)$ is absolute-valuable if and only if $\Omega$ is uncountable. The arguments for Theorem 5.2 mimic those in the proof of a theorem of W. Holsztynski on nonsurjective isometries between $C(\Omega)$-spaces (see Section 22 of 96).

Given a Banach space $X$, we denote by $\mathcal{G}$ the group of all surjective linear isometries on $X$. We recall that a Banach space $X$ is said to be transitive (respectively, almost transitive) if, for every (equivalently, some) norm-one element $u$ in $X$, $\mathcal{G}(u)$ is equal to (respectively, dense in) the unit sphere of $X$. The reader is referred to the book of S. Rolewicz 92 and the survey papers of F. Cabello 15 and Becerra-Rodríguez 8 for a comprehensive view of known results and fundamental questions in relation to the notions just introduced. Hilbert spaces become the natural motivating examples of transitive Banach spaces, but there are also examples of non-Hilbert almost transitive separable Banach spaces, as well as of non-Hilbert transitive non-separable Banach spaces. However, the Banach-Mazur rotation problem, if every transitive separable Banach space is a Hilbert space, remains unsolved to date. Since almost transitive finite-dimensional Banach spaces are indeed Hilbert spaces, the rotation problem is actually interesting only in the infinite-dimensional setting. Then, since infinite-dimensional Hilbert spaces are absolute-valuable, we feel authorized to raise the following strong form of the Banach-Mazur rotation problem.

**Problem 5.1.** Let $X$ be an absolute-valuable transitive separable Banach space. Is $X$ a Hilbert space?
We hope Problem 5.1 to have an affirmative answer in the next future. In the meantime, we must limit ourselves to review the following.

**Proposition 5.3.** There exists a non-Hilbert absolute-valuable almost transitive separable Banach space $X$ such that $L(X,Y)$ and $K(X,Y)$ are absolute-valuable for every absolute-valuable Banach space $Y$.

Actually, the space $X$ in Proposition 5.3 can be taken equal to $L_1([0,1])$. Proposition 5.3 implies (applying Lemma 2.4 among other tools) that there exists a non-Hilbert absolute-valuable transitive non-separable Banach space. One of the tools in the proof of Proposition 5.3 is the following.

**Lemma 5.1.** Let $X$ and $Y$ be Banach spaces over $\mathbb{K}$. Assume that the complete projective tensor product $X\tilde{\otimes}_\pi X$ is linearly isometric to a quotient of $X$, and that $Y$ is absolute-valuable. Then $L(X,Y)$ and $\mathcal{F}(X,Y)$ are absolute-valuable. Here $\mathcal{F}(X,Y)$ stands for the space of all finite-rank operators from $X$ to $Y$.

We conclude the present subsection by applying Lemma 5.1 to prove the following

**Theorem 5.3.** Every Banach space $X$ over $\mathbb{K}$ is linearly isometric to a quotient of an absolute-valuable Banach space $Y$ over $\mathbb{K}$ satisfying $\text{dens}(Y) = \text{dens}(X)$, and such that $L(Y,Z)$ and $K(Y,Z)$ are absolute-valuable for every absolute-valuable Banach space $Z$ over $\mathbb{K}$.

**Proof.** Let $U$ be a set whose cardinal number equals $\text{dens}(X)$, and let $Y$ stand for the absolute-valuable Banach space $\ell_1(U,\mathbb{K})$. Clearly, we have $\text{dens}(Y) = \text{dens}(X)$. On the other hand, it is well-known that $X$ is linearly isometric to a quotient of $Y$. (In fact, noticing that $X$ becomes a complete normed algebra under the zero product, we had to show a little more when we proved Corollary 3.2.) Finally, noticing that $Y \tilde{\otimes}_\pi Y = \ell_1(U,\mathbb{K}) \tilde{\otimes}_\pi \ell_1(U,\mathbb{K}) = \ell_1(U \times U,\mathbb{K}) = \ell_1(U,\mathbb{K}) = Y$ (see Ex 3.27 of 28) and that $Y^*$ has the approximation property (see 5.2 of 28, the proof is concluded by applying Lemma 5.1 (see 5.3 of 28).

5.2. The isomorphic point of view

Most isomorphic properties on Banach spaces considered in the literature are inherited by quotients and/or subspaces. Therefore, by Theorems 5.1
and 5.3, none of such properties can be implied by the absolute valuableness. Now, recall that a Banach space \( X \) is called **weakly countably determined** if there exists a countable collection \( \{ K_n \}_{n \in \mathbb{N}} \) of \( w^* \)-compact subsets of \( X^{**} \) in such a way that, for every \( x \) in \( X \) and every \( u \) in \( X^{**} \setminus X \), there exists \( n_0 \) such that \( x \in K_{n_0} \) and \( u \notin K_{n_0} \). If \( X \) is either reflexive, separable, or of the form \( c_0(\Gamma) \) for any set \( \Gamma \), then \( X \) is weakly countably determined. In fact, the class of weakly countably determined Banach spaces is hereditary, and contains the non hereditary class of weakly compactly generated Banach spaces (see Example VI.2.2 of \( \text{51} \) for details). Among the results proved in \( \text{7} \) concerning the isomorphic aspects of absolute-valuable Banach spaces, the main one is the following.

**Theorem 5.4.** Every weakly countably determined real Banach space, different from \( \mathbb{R} \), is isomorphic to a real Banach space \( X \) such that both \( X \) and \( X^* \) are not absolute-valuable.

We do not know if the requirement of countable determination can be removed in Theorem 5.4.

A Banach space \( X \) is said to be **hereditarily indecomposable** if, for every closed subspace \( Y \) of \( X \), the unique complemented subspaces of \( Y \) are the finite-dimensional ones and the closed finite-codimensional ones. According to the paper of W. T. Gowers and B. Maurey \( \text{51} \), the existence of infinite-dimensional hereditarily indecomposable separable reflexive Banach spaces over \( \mathbb{K} \) is not in doubt. On the other hand, we have proved in \( \text{7} \) that infinite-dimensional hereditarily indecomposable Banach spaces over \( \mathbb{K} \) fail to be absolute-valuable. Thus, since the hereditary indecomposability is preserved under isomorphisms, we are provided with an infinite-dimensional Banach space over \( \mathbb{K} \) which is not isomorphic to any absolute-valuable Banach space. In other words, the property of absolute valuableness is not isomorphically innocuous. In the case \( \mathbb{K} = \mathbb{C} \), more can be said. Indeed, we have the following.

**Proposition 5.4.** Let \( X \) be an infinite-dimensional hereditarily indecomposable complex Banach space. Then \( X \) cannot underlie any complete normed algebra without nonzero two-sided topological divisors of zero.

**Proof.** By Corollary 19 of \( \text{51} \), \( X \) is not isomorphic to any of its proper subspaces. Assume that, for some product, \( X \) is a complete normed algebra without nonzero two-sided topological divisors of zero. Then, for every \( x \in X \setminus \{0\} \), \( L_x \) or \( R_x \) is an isomorphism onto its range, and hence it is
bijective. Now, $X$ is a quasi-division algebra, and hence finite-dimensional (by Proposition 3.6).

Let us say that a Banach space $X$ over $\mathbb{K}$ satisfies the **Shelah-Steprans property** whenever $X$ is not separable and, for every $F \in \mathcal{L}(X)$, there exist $\lambda = \lambda(F) \in \mathbb{K}$ and $S = S(F) \in \mathcal{L}(X)$ such that $S$ has separable range and the equality $F = S + \lambda I_X$ holds. In our present discussion, Banach spaces enjoying the Shelah-Steprans property play a role similar to that of infinite-dimensional hereditarily indecomposable Banach spaces. Indeed, reflexive Banach spaces satisfying the Shelah-Steprans property do exist (see 97 and 108), the Shelah-Steprans property is preserved under isomorphisms, and Banach spaces over $\mathbb{K}$ fulfilling the Shelah-Steprans property fail to be absolute-valuable. By the way, the proof of the result of 7 just reviewed can be slightly refined to get the following.

**Proposition 5.5.** Let $X$ be a Banach space over $\mathbb{K}$ satisfying the Shelah-Steprans property. Then $X$ cannot underlie any complete normed algebra without nonzero two-sided topological divisors of zero.

**Proof.** First, we note that, for $F$ in $\mathcal{L}(X)$, the couple $(\lambda(F), S(F))$ given by the Shelah-Steprans property is uniquely determined, and that the mappings $\lambda : F \rightarrow \lambda(F)$ and $S : F \rightarrow S(F)$ from $\mathcal{L}(X)$ to $\mathbb{K}$ and $\mathcal{L}(X)$, respectively, are linear. Now, since $X$ is not separable, and $\ker(\lambda)$ consists of those elements of $\mathcal{L}(X)$ which have separable range, we have $\lambda(F) \neq 0$ whenever the operator $F$ on $X$ is an isomorphism onto its range. Assume that, for some product, $X$ is a complete normed algebra without nonzero two-sided topological divisors of zero. Then, since $L_x$ or $R_x$ is an isomorphism onto its range whenever $x$ is in $X \setminus \{0\}$, it follows that the linear mapping $x \rightarrow (\lambda(L_x), \lambda(R_x))$ from $X$ to $\mathbb{K}^2$ is injective. Therefore $X$ is finite-dimensional, a contradiction.

Concerning the topic of the present section, let us say that the study of absolute-valuable Banach spaces is just started, so that there are more problems than results on the field. Since we have mainly emphasized the results, let us conclude the paper with one of the problems non previously collected. Let us say that a Banach space is **nearly absolute-valuable** if it underlies some complete nearly absolute-valued algebra (see Subsection 4.1). It is easy to see that the near absolute valuableness is preserved under isomorphisms. Consequently, **isomorphic copies of absolute-valuable Banach spaces are nearly absolute-valuable.** However, we do not know whether
or not every nearly absolute-valuable Banach space is isomorphic to an absolute-valuable Banach space.

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