SURJECTIONS FROM BIG SPACES TO SMALL SPACES

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We discuss the following very general question:

Suppose that E is a "big" space and that F is a "small" space. Suppose also that $f : E \to F$ is a surjective function. Is there a "smaller" space $G \subset E$ such that the restriction of f to G, $f|_G$, is still surjective?

The spaces E and F will typically be an infinite dimensional Banach spaces. For example, let $T : E \to F$ be a continuous linear surjection. By the Bartle-Graves selection theorem, one can find a subspace $G \subset E$ such that $T|_G : G \to F$ is surjective, where G is the same "size" (i.e., has the same density character) as F. On the other hand, the analogous question for polynomial maps $P : E \to F$ seems to be open.

We show that in many cases, there is a negative answer to this question, even if the mapping f is quite nice. For instance, for every infinite dimensional Banach space E, there is a surjective $\mathcal{C}^\infty$ function $f : E \to \mathbb{R}$ such that for any finite dimensional subspace $G \subset E, f|_G$ is not surjective. Even more striking, for very many (and possibly for all) non-separable Banach spaces E, there is a surjective, smooth mapping $f : E \to \mathbb{R}^2$ whose restriction to any separable subspace is no longer surjective. We also discuss some related open questions, even when both E and F are finite dimensional.