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TEACHING REAL NUMBERS IN THE HIGH SCHOOL: AN ONTO-SEMIOTIC APPROACH TO THE INVESTIGATION AND EVALUATION OF THE TEACHERS' DECLARED CHOICES

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Εις απειρον γαρ διαιρετον το συνεχεσ
Aristotle, *Φυσικής Ακροάσεως*, Book I, 185b, IV century B.C.

If an infinite line were constituted by an infinite number of one-foot sections and if another infinite line were constituted by an infinite number of two-foot sections, these lines would nevertheless have to be equal, since the infinite is not greater than the infinite.
Nicolas Cusanus, De docta ignorantia*, 1440*

[…] The statement is so frequently made that the differential calculus deals with continuous magnitude, and yet an explanation of this continuity is nowhere given. […] It then only remained to discover its true origin in the elements of arithmetic and thus at the same time to secure a real definition of the essence of continuity. I succeeded Nov. 24, 1858
Richard Dedekind, Stetigkeit und irrationale Zahlen *,1872*

"No one will drive us from the paradise which Cantor created for us"
David Hilbert, Über das Unendliche,*1926*

The constructive transition to the continuum of real numbers is a serious affair and I am bold enough to say that not even to this day are the logical issues involved in that constructive concept completely clarified and settled.
Hermann Weyl, Axiomatic versus Constructive Procedures in Mathematics, *1985 (posthumous publication)*

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Introduction

The thesis addresses the topic of evaluating the didactical suitability of teachers' choices concerning real numbers and the continuum in the high school in Italy. Our first research hypothesis was that teachers' choices of teaching sequences concerning real numbers and of representations of real numbers could be very relevant in order to interpret the students' difficulties. After a pilot study in form of a teaching experiment and a literature review concerning students' and teachers' difficulties with real numbers and the continuum, we observed that some causes of potential difficulties could be situated yet in the very beginning of the teaching-learning process: the choice of practices and objects by means of whom introducing and work with real numbers and the continuum. In particular the choice of the mathematical objects involved in the practice and the strategies of connections between different objects seemed to be relevant in the pilot study. Thus we decided to explore first of all the teachers' choices concerning sequences of practices and representation of the mathematical objects used in order to talk about real number and the continuum in the high school. Then we aimed at evaluating the choices suitability in relation to the literature review concerning students' difficulties with real numbers and to the complexity of the mathematical object as it emerge from an historical analysis.

After having analyzed the theoretical frameworks in mathematics education that could permit us to carry out the analysis of practices and to differentiate the meaning of objects involved in the different didactical practices, we decided to use the OSA, onto-semiotic approach (Godino, Batanero & Font, 2007). In this framework is also taken in account the possibility of different interpretation of the signs produced and used by teachers and students at school. We analyzed also other frameworks, in particular the ATD (Chevallard, 2014), but we found the OSA better for the analysis that we would like to carry out. In particular the attention to the personal meaning of mathematical objects and the construct of didactical suitability were more effective for our purposes. Also we choose the Schoenfeld's goal-oriented decision-making approach to the study of teachers' choices (2010) since our research is based merely on questionnaires and interviews that we consider relevant in order to forecast the teachers' concrete choices in their classrooms.

We formulated three general research questions:

GQ - 1 How can we describe the complexity of the teaching-learning processes involving real numbers and continuous sets from an epistemic, cognitive and ecologically point of view?

GQ - 2 Does the didactical suitability depend on the categories we used to describe the teachers' DMK? Which is the relation between them?

GQ – 3 Could our methodology of research be useful in the teachers' training concerning real numbers and the continuum?

We carried out parallel researches: we explored the practices concerning the continuum and real numbers in the history of Mathematics in order to delineate the complex meaning of the continuum and the real numbers; we analyzed some textbooks and manuals; we analyzed the literature in mathematics education concerning the students' and teachers' conceptions and difficulties with the continuum and the real numbers; finally we interviewed 116 in-service teachers by means of a questionnaire, interviews in focus groups and individual interviews. The product of the first analysis is the epistemic meaning of real numbers; the results of the second analysis is a list of "standard" practices and traditional teaching sequences in the high school; the third analysis allowed us both to anticipate some possible teachers' wrong conceptions and to construct a picture of possible difficulties to put in correspondence, if it is possible, to some of the teachers' declared choices; the last analysis allowed us to make emerge the way the teachers approach the teaching-learning processes in this case and to identify categories of teachers. Then we evaluated the epistemic and cognitive suitability of teachers' choices and put it in relation with the teachers' profiles constructed in the previous analysis. In the end we followed in particular 11 high school teacher. In our first pilot study analysis we found out that one of the most relevant object used in the high school practices leading to real numbers was the interval, both of rational and real numbers. Wondering how it is possible to have a wrong conception of the infinity of the points included in a limited segment and also of its continuity (or better completeness, in the topological sense) and, at the same time, work correctly with intervals represented by segments, that are expected to be continuous sets, we faced the complexity of the definition of continuum itself: what is continuum? Has it a unique characterization or are there many conception of continuum that can't be resumed in a only, more abstract, object that include all the others? This research lead us to explore the history of the continuum in the history of mathematics and its relation with real numbers, too often identified at school with the more complex and articulated meaning of the continuum. We ended up in the collection of many different conceptions and images of continuum in the history of mathematics, philosophy and science and also in the didactical practices at school. Are teachers aware of this complexity and the lack of equivalence between different conceptions of the continuum? Are teachers aware that "to pass through" an interval without stops and to say an interval is a subset of the complete field of real numbers is not exactly the same? Do they in general take it in account while they are planning teaching sequences? Are they aware that objects that aroused from previous practices may not have the meaning they are expected to have? In the teachers' interviews we found a very articulated panorama: some teachers' personal meaning are not correct or not complex enough because of a lack of adequate formation in the Master courses or in the teachers training courses; some teachers are partially aware of the complexity but a lack of knowledge in mathematics education leads them to choose sequences without taking care of students' potential difficulties highlighted by researchers in Mathematics education; some teachers are aware of the complexity but, trying to help students, give too much attention to cognitive suitability and compromise the epistemic suitability, generating new causes of potential difficulties, like for instance using concrete examples from real life to represent irrational numbers (what is a rope with a length of $\sqrt{2}$ km?). In fact as we observed in our pilot study and Bagni highlighted in his researches concerning continuous and dense sets, a lack of complexity may cause other problems to students while learning real numbers. In our analysis we found out some profiles of teachers concerning the choice of teaching sequences concerning real numbers. Also studying the epistemic meaning of real numbers we discovered a complexity even greater than it was already stressed in the other researches about the teaching-learning processes concerning real numbers. The evaluation of the didactical suitability from the point of view of epistemic and cognitive suitability of teachers' choices made emerge a heterogeneous panorama that lead us to hypothesize some possible implications of our research in teachers' training: some teachers should be made aware of the possible meanings of the continuum and real numbers, that are some of the most quoted objects in the didactical practices in the high school but are underestimated in their complexity; some teachers that have a good knowledge concerning the continuum and the real numbers may choose potentially wrong teaching sequences because of lack of knowledge in Mathematics education or because of a lack of reflection at an epistemological level; some teachers that have a very good knowledge about the continuum and the real numbers, also from an historical-epistemological point of view may choose to avoid to take in account the complexity of real numbers at school and make it too simple because of a greater to attention to the cognitive suitability than to the epistemic one, creating this way other potential sources of difficulties to their students.

1. Research problem and literature review

1.1 **The complex relation between *continuum* and real numbers**

<< [...] lo studio dell'infinito in quanto tale è tutto sommato recente, e non si afferma che alla fine del secolo scorso. Nei duemila anni che separano la nascita della geometria greca dalle profonde intuizioni di Cantor, la trattazione matematica dell'infinitamente grande non fa registrare che progressi modesti, quasi che l'immensità dell'oggetto valga a precludere ogni sua analisi approfondita. Così chi voglia studiare la storia dell'infinito matematico dovrà rivolgersi piuttosto alla sua immagine speculare, ed indagare l'evoluzione dei temi e delle teorie legate all'infinitamente piccolo.

Tra esse, un posto particolare spetta alle dottrine del continuo, soprattutto a causa del ruolo centrale di quest'ultimo, quasi un ponte gettato tra la geometria, scienza del continuo per definizione, e la filosofia naturale, che nella composizione del continuo trova uno dei temi più dibattuti. [...]

Il rapporto tra teorie geometriche e struttura del continuo va in senso contrario alla successione logica: la discussione delle proprietà del continuo non precede, come sarebbe logico e lecito attendersi, la formulazione e lo sviluppo delle teorie geometriche delle quali esso costituisce per così dire la materia.

Al contrario, il continuo è piuttosto un risultato finale, un sottoprodotto, della geometria; un risultato peraltro che non è quasi mai esplicito, e che è piuttosto suggerito che enunciato, meno che mai dimostrato.

In altre parole, quella del continuo non è una scienza, una teoria, sulla quale si possa fondare la geometria; ma piuttosto un'immagine che si forma nella mente del geometra alla fine delle sue elucubrazioni; immagine costruita pezzo a pezzo mediante le proprietà che al continuo si sono attribuite nel corso delle dimostrazioni, e che vengono via via a modificare immagini precedenti.

La geometria genera immagini del continuo; e così ai cambiamenti di punti di vista in geometria corrisponderanno analoghe revisioni della nozione di continuità, in modo che i periodi di grande attività creatrice come il XVII secolo, sono anche caratterizzati da una forte instabilità fondazionale; periodi in cui nuove immagini del continuo sono create, modificate, e infine rimpiazzate da altre immagini, non più fondate queste ultime, o meno arbitrarie, di quelle che le hanno precedute." (Giusti, 1990).

These words of Giusti (1990) resume in a masterly manner some crucial observations that inspired our research.

The particular nature of the *continuum* as a mathematical objects first of all fascinated us as Mathematics scholars, then became the topic of our investigation. "Magnitudes that may assume all the possible intermediate stages", "Set of infinite points without dimensions", "What we can trace without interruptions or breaks" and so on: the most of the characterizations of continuity are far from being rigorous definitions. If we consider the definitions of continuity in the History of mathematics, the most of them have a deep link with intuition. The numerical characterization of continuity came late in the History of mathematics and the construction of connections between the numbers and the physical and geometrical conceptions of continuity have not been trivial and free from consequences. Weyl, for instance, spent a lot of time investigating this topic. He changed many times his perspective and came to the conclusion that we can't transform completely the intuition of continuity as a product of our consciousness in a formal definition without losing the global sense of continuity. The attempts to formalize the intuition of continuity - motivated by the search for higher precision, more effective representations, manipulation techniques and by the claims for rigor in Calculus' methods – obliged the mathematicians and philosophers of the XIX century to transform the nature of continuous objects and processes. Continuous magnitudes (segments, figures, varieties) and variations (motion, time, increasing or decreasing values of variables) wasn't considered as wholes but as a system of parts related each other (points, numbers, instants of time etc), whose totality could form the whole. John L. Bell, one of the contemporary most experts of infinitesimals and continuity in the history of mathematics, named this process the reduction of *continuum* to discrete. This change cause the abandon of the space-temporal based concept of variation and the consideration of continuous mathematical objects in favor of static and local definitions. It was not the first period in which the composition and decomposition of continuum was investigated, since this topic has been present quite in all the centuries from Anaxagoras to our days. But this was the first time that the intuitive dimension of continuity gave way for the numerical, discrete attempts to determine the continuum, being usually numbers considered useful for calculation and applications but not suitable enough to be the base of mathematical theories (Giusti, 2000).

This operation was nor neutral nor immune from a philosophical point of view. On the contrary this reduction implied a revolution in the way the *continuum* was considered and managed. For instance standing on the famous Aristotle's definition of *continuum* magnitudes the continuum ends to be continuous if it's composed by discrete indivisible parts.

This issue opens the path to other themes that are crucial for the issue of formalization of intuitive *continua*, that we can synthesize in three questions:

1. Is a continuum a set of points, or small increments, or infinitesimal variations, or indivisible things? Is it on the other side the result of motion, supposed continuum? What does it mean that a function is continuous in one point?
2. What do mathematicians and teachers consider "formal enough" and why do they need to be formal talking about the *continua*? Are there practical issues? Do all mathematicians agree in considering rigorous a definition of the continuum rather than another?
3. If formalizing we change in a certain sense the nature of the object, as Weyl and Brouwer stated, can we consider the formal and the intuitive objects the same mathematical object? Can we thus re-frame and re-interpret the difference between concept image and concept definition used in Tall & Vinner (1981) and the cognitive conflicts observing that, in the end, the formalization changes at all the mathematical object? May the incommensurability of discourses concerning the continuity and the continuous functions identified by Nunez (2000) be an alert in this sense?

The first question addresses the ontological issue of continuity in mathematics, both in an absolute sense and in relation to the characterization of discrete magnitudes.

The second question addresses the following anthropological and sociocultural matter: what were and what are the mathematical *continua* used in the concrete life of mathematicians in different epochs, with different sensations, problems, aims, environments? How the mathematical *continua* was transformed by different philosophers, mathematicians, physicians and scientists who were facing new problems, animated by different "spirits"? Giusti proposes an engaging challenge: the *continuum* is << immagine costruita pezzo a pezzo mediante le proprietà che al continuo si sono attribuite nel corso delle dimostrazioni, e che vengono via via a modificare immagini precedenti >>, Giusti, 2000)? A paradigmatic example in this sense is that of Leibniz, as Giusti (1990) stressed using the plural *continua* in the paragraph dedicated to the German mathematician and philosopher's conceptions of continuum. The attention to the personal transformations of the meaning of what it was considered *continuum* by the mathematicians should not mislead the readers: we are not stating that a particular choice could solve the ontological problem of continuum and that the different approaches are not somehow necessary. What is a bit surprising and significant in the historical researches is the existence of "global inconsistencies" in the works of philosophers and mathematicians related to different domains, goals and practices: a mathematician could use different levels of the meaning depending on the system of practices and also could change his perspective when it was necessary, as Leibniz did three times.

The processes of using, defining and re-defining the *continuum* are in a dialectic relation and owe particular importance since, for instance, some mathematicians used the word *continuum* with very different meanings, sometimes opposite.

Even if each mathematician addressed his specific goals and modeled its definition on his necessities, the issue of consistency was faced by the most of them: every mathematician at least was used to declare his position in relation to the definitions used by the Ancients or by their contemporaries.

The third question is crucial in a didactical sense, since it reframes the previous questions in the problem of didactical transposition. May a teacher plan a "vertical" didactical sequence that ends with one of the formalizations of the concept of continuity as the top element without coming back to less formal partial meanings?

To answers the three questions we had to explore deeply the literature review concerning the evolution of the ideas concerning the mathematical *continua* in the history of mathematics, from an epistemological - and, more generally, philosophical - point of view, but also from a semiotic point of view. The analyses that we carried out to answer these questions drove us to choose a theoretical framework for mathematical objects that includes the personal meaning of mathematical objects, the deep relation between objects and practices, and allows to take care of the evolution and generation of objects in the practices of mathematicians community.

1.1.1 Different continua in the History of Mathematics

a. Aristotle's continuum

Aristotle's continuum have been framed in a wider philosophical dissertation.

<< Aristotle distinguishes discrete quantity (*poson*) from continuous (*suneches)* quantity. He includes lines, surfaces, bodies, time, and place in the latter category. Aristotle also distinguishes quantity that is continuous from that which is non-continuous. A magnitude, he says, is quantity that is measurable (as opposed to numerable or countable), and a magnitude is divisible into parts that are continuous. Among magnitudes, "that which is continuous in one [dimension] is length, that in two breadth, and that in three depth. I think that is fair to say that Aristotle's conception of continuous quantity or magnitude is a geometrical conception. I shall postpone discussion of the continuous "quantities" of time and place, concentrating for the moment on magnitudes of the three spatial dimensions. There are both what might be called "structural" (geometrical or topological) and metaphysical features to be found in Aristotle's analysis of *megethos* (magnitudes).

Some of the former, structural features correspond to properties central to the developing geometry of the fourth century BCE. Aristotle's basic structural property, however, is continuity (*sunecheia*); and it does not have an explicit role in Euclidean geometry (where it appears principally in the notion of a "continuous proportion" of three or more terms). A principle of continuity of geometrical magnitude is assumed, however, in many Euclidean constructions: it guarantees the existence of points at the intersection of two lines, the existence of lines at the intersection of two surfaces or planes, etc. (Heath 1956). of. Having stipulated that "something is contiguous (*echomenon*) [to something] that is successive to and touches it" (227a6), he proceed as follows:

<I say that something is continuous (suneches), which is a kind of being contiguous, whenever the limit of both things at which they touch becomes one and the same and , as the word implies , they are "stuck together" (sunechêtai). But this is not possible if the extremities are two. It is clear from this definition that continuity pertains to those things from which there naturally results a sort of unity in virtue of their contact.>

In Phys VI.1 Aristotle argues that his conception of continuity implies that "it is impossible that what is continuous be composed of indivisibles, e.g., a line from points". >> (White, 2009)

In fact if we can distinguish the parts that compose the continuum and they are indivisible, the limits of the these parts don't exist and it doesn't make sense to think they are in contact. Aristotle defines the *continuum* as a determination of *contiguous,* i.e what is consecutive and in contact (Giusti, 2000) so it's impossible for a continuous to be composed of parts. The Aristotle's complete distinction between magnitude is presented by Giusti (2000):

"Aristotele distingue tre tipi di grandezze, a seconda dell'accoppiamento tra le loro parti. In primo luogo la grandezza discreta, le cui parti si susseguono consecutivamente senza che tra di esse vi sia alcunché di simile, pur non escludendo la possibilità che tra esse siano intercalati altri oggetti eterogenei.
Così ad esempio tra due linee consecutive potremo trovare uno spazio, ma non una linea; e tra due case consecutive un prato, ma non una casa. Il consecutivo… è ciò che non presenta alcun intermedio dello stesso suo genere tra sé stesso e quello di cui è consecutivo (dico ad esempio, che non vi siano una linea o più linee dopo la linea, una unità o più unità dopo l'unità, ovvero una casa dopo una casa; nulla però impedisce che vi sia in mezzo qualcosa di altro genere).

Contiguo è ciò che, oltre ad essere consecutivo, è anche in contatto.

Il continuo [suneches] è una determinazione del contiguo, ed io dico che c'è continuità quando i limiti di due cose, mediante i quali l'una e l'altra si toccano, diventano uno solo e il medesimo e, come dice la parola stessa, si tengono insieme. Questo però non può verificarsi quando gli estremi sono due. Tenendo conto di questa precisazione, risulta chiaro che il continuo è in quelle cose da cui per natura vien fuori qualcosa di unico in virtù del contatto."
" Magnitudes of null dimension (points), of one dimension (lines), and of two dimensions (surfaces) are ontologically dependent on physical (changeable and sensible) body. Although this account of mathematical objects is difficult, Aristotle appears to hold that they are not, in reality, "separable" (*chôrista*) from sensible, physical reality. Rather, the geometer considers physical bodies *qua* geometrical, abstracting (*aphairein*) their spatial/geometrical characteristics and considering such properties separately from other kinds of property of such bodies" (White, 2009)

While Aristotle's conception of magnitude (*megethos*) is strongly geometrical, his concept of place (*topos*) finds more direct employment in his physics: motion is indeed motion with respect to a place. Place is classified as a kind of continuous quantity ("how much" – poson). (White, 2009)

Between infinite, space, and time in Aristotle's thought, the last has surely received the most attention. In his "physical" analysis of time: "this, then, is [a] time: the number of motion with respect to the earlier [or 'prior' – *proteron*] and the later [or 'posterior' – *husteron*]". Another important feature of time, << which borders on the ineffable, is the transitory, evanescent, or "flowing" character that attaches to our experience of time>>. Just as a line is not composed of points according to Aristotle, so time is not composed of nows or instants. Any interval or stretch of time is continuous and "infinitely divisible" (in Aristotle's "potential infinity" sense) into smaller sub-intervals. Temporal "points" or instants can be demarcated or "constructed" (as, for example, the boundaries of processes and, perhaps, as instantaneous events).

Resuming the previous nuances of Aristotle's continuum, we can find these main features:

1. infinitely divisible but not composed of infinite elementary parts
2. a particular case of contiguous, whose parts are kept together in order to form a whole
3. allows a potential infinite process of division
4. is coherent with the Eudosso's theory of proportions
5. is strictly connected to physical continuum, since mathematical objects are inseparable from reality
6. the numeric dimension of continuum is allowed just for comparisons (earlier and later, previous or sequent)

Aristotle's vision is partially influential still now, in particular to the mathematicians who deny the use of actual infinite, considered too "artificial", a sort of catch, a joke and - paradoxically - a play with the unreal.

“Whatever the relation between Aristotle’s doctrine of the potential infinite and contemporary ancient mathematical practice, the Aristotelian conception became, in the long run, the orthodox view, particularly in physical and mathematical contexts. Despite the difficulty in working out the foundations of the calculus in terms of “Aristotelian” potential infinity (a problem that was finally solved by B. Bolzano, A. Cauchy, and K. Weierstrass in the nineteenth century), the basic Aristotelian view persisted in technical contexts until the work of Georg Cantor in the late nineteenth century. In contemporary mathematics, the Aristotelian influence is detectable in the so-called “intuitionist” and constructivist traditions. In words that could have been written by Aristotle himself, M. A. E. Dummett writes that “in intuitionistic mathematics, all infinity is potential infinity; there is no completed infinity. This means, simply, that to grasp an infinite structure is to grasp the process which generates it; to recognize it as infinite is to recognize that the process is such that it will not terminate” (Dummett, 1974)

c. Archimedean continuum

Archimedes (287 BC - 212 BC) have been one the most influential thinker of Ellenics. His work was underestimated until the XVII century both because of historical happenings concerning the discovery and interpretation of his writings and because some of the significant and revolutionary ideas that underlie to them was not completely grasped by posterity, at least until '600 (Volterra, 1920)

His conception of continuum was more complex than we can think if we only focus our attention on the most famous of its assumption, that is universally known with Archimedean postulate.

In fact from the Archimedes’ correspondence with Eratostene (discovered by Heiberg in 1906) we can understand that there were at least "two Archimedes": the one who was working to find results and the one who was presenting his results. The "second" Archimedes appears to be coherent with the ban of actual infinity, only seems to work with exhaustion methods and to deal with potential infinity. Archimedes avoid thus to use infinitesimal quantities, and only says that the parts of continuum he consider can be small of your choosing.

As we can see reading this excerpt, his method is very familiar for us, since this is indeed the method that inspired Weierstrass approach to limits and the one we that influenced more actual didactics of Analysis in the secondary school and university:

“*Supponiamo che l’area del cerchio non sia metà del prodotto del raggio per la sua circonferenza; sia allora d la differenza fra la maggiore e la minore delle due quantità . Se circoscriviamo alla circonferenza un poligono di n lati, l’area di tale poligono sarà la somma delle aree degli n triangoli che lo compongono, tutti di altezza 1, e quindi l’area complessiva sarà p , essendo p il semiperimetro del poligono . Preso n sufficientemente grande, possiamo far sì che l’area del poligono differisca dall’area del cerchio meno della metà di d. Poiché il perimetro del poligono differirà dalla circonferenza per meno della metà di d, l’area del cerchio e il semiprodotto del raggio per la circonferenza differiranno meno di d, contro l’ipotesi di partenza. Quindi d deve essere zero".* (Maffini, 1999).

This approach need to include the Archimedean axiom for numbers corresponding to measures of magnitudes, i.e.

For any two segments a and b there is a positive integer $n \in N$ such that $a < n \cdot b$

The Archimedean postulate is very important also for the modern theories of real numbers: fixed by Hilbert as a postulate in its axiomatizations of real numbers, the relation with this postulate is a key element for comparing Cantor's and Dedekind's postulate of continuity.

As highlighted by Maffini, the rigid approach to infinite and infinitesimal quantities adopted by the Greeks is one of the most substantiated reasons why the infinitesimal analysis did not develop in the Ellenics, even if its seeds had already been sown at that time. An interesting suggestion to prove how deep this sentence is came us from Volterra (1920), who reported an anomalous behavior of the Syracusan. It seems that when Archimedes was working in practice to find results he was using not so orthodoxical methods, but rather infinitesimal methods, exactly as Eudosso did but more and more frequently, or series:

"Dalla lettera di Archimede ad Eratostene si rileva come egli facesse uso per le sue scoperte del metodo delle quantità infinitamente piccole , e, solo per esporre i risultati al pubblico, ricorresse al metodo dell'esaustione e a quello delle serie . Basta ricordare le differenti soluzioni ch 'egli ha date , allo scopo di trovare l 'area della parabola , per riconoscere i principî fondamentali , mediante i quali il calcolo infinitesimale si è sviluppato da quell 'epoca remota fino ai nostri giorni. [...] basta infatti classificarli [i metodi, nba], come ora abbiamo fatto, in tre gruppi; quello degli infinitesimi, quello di esaustione ed infine quello delle serie, per veder delinearsi tutte le concezioni fondamentali del calcolo infinitesimale." (Volterra, 1920)

Standing on this consideration it appears evident that the epistemological preciseness concerning the use of infinite infinitesimal quantities is fascinating in his firmness, but is somehow unreasonable if we observe two evidences: 1) the infinitesimal method works and leads to useful results; 2) the Hellenics somehow accepted and used results coming from its unbelievable assumptions, but accepted them just if they were expressed in "acceptable terms", i.e. respecting a strict relation with sensible reality - whatever it means - and with a sort of intuitiveness.

This became more evident in Cusanus and Cavalieri, Barrow and Torricelli methods and, at the same time, in all the harsh critiques moved toward them in the sequent centuries.

Standing on these observation we can hypothesize that there is at least a distance, not to say a trench, between practice with continuum and its parts, connected to infinitesimal methods, both derivative and integral, and the theorization of continuity in its various nuances (arithmetization, algebraization, axiomatization, ...) and that this cut is not easy to mend.

To sum up, in Archimedes we find two conceptions of continuum:

1. practical continuum: actually infinite, composed by infinite infinitesimal parts (infinitesimal segments and rectangle)
2. theoretical continuum: potentially infinite, consistent with Eudosso's exhaustion method and with the ban of actual infinite and infinitesimal quantities

d. Ockham's continuum

William of Ockham (c. 1280–1349) the principal difficulty presented by the continuous is the infinite divisibility of space, and in general, that of any continuum. The treatment of continuity rests on the idea that between any two points on a line there is a third—perhaps the first explicit formulation of the property of density—and on the distinction between a continuum “whose parts form a unity” from a contiguum of juxtaposed things. Ockham recognizes that it follows from the property of density that on arbitrarily small stretches of a line infinitely many points must lie, but resists the conclusion that lines, or indeed any continuum, consists of points. Concerned, rather, to determine “the sense in which the line may be said to consist or to be made up of anything.”, Ockham claims that “no part of the line is indivisible, nor is any part of a continuum indivisible.” While Ockham does not assert that a line is actually “composed” of points, he had the insight, startling in its prescience, that a punctate and yet continuous line becomes a possibility when conceived as a dense array of points, rather than as an assemblage of points in contiguous succession. (Bell, 2014)

Ockham's continuum is so:

1. punctate and dense, but not actually composed of points nor of indivisibles;
2. whose parts form a unity;
3. not set of points in contiguous succession

e. **Cusanus' continua**

Nicolaus Cusanus (1401 – 1464) pointed out an interesting distinction. He asserts that any continuum, be it geometric, perceptual, or physical, is divisible in two senses, the one ideal, the other actual. Ideal division "progresses to infinity"; actual division terminates in atoms after finitely many steps.

So Cusanus' possible features of continuum are two:

1. Ideal continuum: infinitely divisible (actual infinity)
2. Actual continuum: atomic and only finitely divisible

It's not surprising that precisely in Nicolaus Cusanus we find two somewhat opposite characterizations of the continuum, since it's theological approach to the science attributed to Mathematics the role and the potentiality to realize in the infinite the *coincidence of the opposites (coincidentia oppositorum).* Finite human mind can only investigate finite things, not the infinite (the famous *docta ignorantia*, 1440*).* Knowing is an approximation to the truth, so as an approximation to infinite. Truth is to knowledge what a circle is to a polygonal. In Cusano's infinite - an actual infinite, explicitly in contradiction with all the Greek mathematics' assumptions - the coincidence of opposite reveals itself and finite and infinite are not antithetical but, on the contrary, finite things have a sort of symbolical relation with infinite. The potential infinite is necessary to human mind, that only can think finitely, but this is not a proof of the inexistence of actual infinite, that is included in everything as God is "compacted" in every creature.

<< "....Più cose non sono, dunque, in una qualsiasi cosa in atto, ma tutte sono senza pluralità questa cosa stessa. L'universo è nelle cose in modo contratto , e ogni cosa che esiste in atto contrae i suoi universi , affinché essi siano in atto ciò che essa è. Tutto ciò che esiste in atto è in Dio, perché Dio è l'atto di tutto. Ma l'atto è la perfezione e il fine della pot enza. Ed essendo l'universo contratto in qualsiasi cosa esistente in atto, è chiaro che Dio, che è nell'universo, è in qualsiasi cosa e che qualsiasi cosa che esiste in atto è, come universo, immediatamente in Dio."

Ciò che emerge quindi è che l'uomo pensa all'infinito in termini potenziali, non potendo fare diversamente la finitezza della sua mente, ma ciò non toglie che esista un infinito in atto che Cusano identifica con Dio . >> (Maffini, 1999)

His actual infinite conception of continuum revelead to be not only philosophical speculation, but also very useful tool: for instance lead him to consider *infinilateral* regular polygon - a regular polygon with an infinite number of (infinitesimally short) sides. By dividing it up into a correspondingly infinite number of triangles, its area, as for any regular polygon, can be computed as half the product of the apothem (in this case identical with the radius of the circle), and the perimeter. The idea of considering a curve as an infinilateral polygon was employed by a number of later thinkers, for instance, Kepler, Galileo and Leibniz.

g. Galileo Galilei's continuum

Galileo Galilei faced the issue of the composition of the continuum when he was working on physical problems, in particular motion and dynamics. His idea of decomposing motion in infinitesimal parts was a revolutionary step for physics, since it allowed to consider locally constant the velocity and to use for general situations laws that had been enunciated in simpler cases. This way he brought infinitesimal methods into physics, opening the path to fruitful exchanges between geometry and physics that produced very important developments in both the sciences (Volterra, 1920). But Galileo's fame linked to the interpretation of the continuum is not due to this intuition, although to the re-discussion of the Aristotle's dogma, ancient but still influential at that time, and to the creation of the correspondence between naturals and their squares. Aristotle had stressed that the continuum can't be composed by discrete elements without loosing its continuous characterization, while Galileo's continuum is fragmented, atomic, composed by infinite elements that are results of the infinitive process of division. An important feature of the Galileian continuum is strictly linked to its infinite cardinality. Let's imagine to compare two segments using the number of its elements. Since they are both composed by infinite indivisible elements there is no way to establish an order relation between them. An excerpt from the *Dialogue* clarify Galileo's position, represented as usually by Salviati:

***Simplicio**: Here a difficulty presents itself which appears to me insoluble. Since it is clear that we may have one line segment longer than another, each containing an infinite number of points, we are forced to admit that, within one and the same class, we may have something greater than infinity, because the infinity of points in the long line segment is greater than the infinity of points in the short line segment. This assigning to an infinite quantity a value greater than infinity is quite beyond my comprehension.*

***Salviati**: This is one of the difficulties which arise when we attempt, with our finite minds, to discuss the infinite, assigning to it those properties which we give to the finite and limited; but this I think is wrong, for we cannot speak of infinite quantities as being the one greater or less than or equal to another. To prove this I have in mind an argument, which, for the sake of clearness, I shall put in the form of questions to Simplicio who raised this difficulty. [...]*

***Salviati**: If I should ask further how many squares there are, one might reply truly that there are as many as the corresponding number of roots, since every square has its own root and every root its own square, while no square has more than one root and no root more than one square.*

***Simplicio**: Precisely so.*

***Salviati**: But if I inquire how many roots there are, it cannot be denied that there are as many as there are numbers because every number is a root of some square. This being granted we must say that there are as many squares as there are numbers because they are just as numerous as their roots, and all the numbers are roots. Yet at the outset we said there are many more numbers than squares, since the larger portion of them are not squares. Not only so, but the proportionate number of squares diminishes as we pass to larger numbers. Thus up to 100 we have 10 squares, that is, the squares constitute 1/10 part of all the numbers; up to 10,000 wefind only 1/100th part to be squares; and up to a million only l/1000th part; on the other hand in an infinite number, if one could conceive of such a thing, he would be forced to admit that there are as many squares as there are numbers all taken together.*

***Sagredo**: What then must one conclude under these circumstances?*

***Salviati**: So far as I see we can only infer that the totality of all numbers is infinite, that the number of squares is infinite, and that the number of their roots is infinite; neither is the number of squares less than the totality of all numbers, nor the latter greater than the former; and finally the attributes "equal," "greater," and "less" are not applicable to infinite, but only to finite, quantities. When, therefore, Simplicio introduces several lines of different lengths and asks me how it is possible that the longer ones do not contain more points than the shorter, I answer him that one line does not contain more or less or just as many points as another, but that each line contains an infinite number. Or if I had replied to him that the points in one line segment were equal in number to the squares; in another, greater than the totality of numbers; and in the little one, as many as the number of cubes, might I not, indeed, have satisfied him by thus placing more points in one line than in another and yet maintaining an infinite number in each? So much for the first difficulty.*

Sagredo*: Pray stop a moment and let me add to what has already been said an idea which just occurs to me. If the preceding be true, it seems to me impossible to say that one infinite number is greater than another. ...*

In summary Galileo's continuum had the following features:

1. composed of infinite elementary and invisible parts
2. different continuous segments couldn't be compared since they are composed of infinite elements and all the infinite quantities have the same cardinality

h. Bonaventura Cavalieri's continuum

Bonaventura Cavalieri was a pupil of Galileo Galilei and, to a great extent, he continued the works of his master. This obliged Cavalieri to follow a very curvy path when he had to deal with the composition of the continuum. In fact the assumptions he needed to adopt in order to go on with his quantitative methods for calculating areas and volumes clashed with a fundamental ban introduced by his master: infinitive quantities can’t be compared, it’s denied to define ratios between infinitive quantities since it’s senseless. Getting out of thorny ancient and modern philosophical implications Cavalieri succeeded in “tailoring” a continuum suitable for his goals and not inconsistent with his master’s view, but he had to accept an important restriction: renounce to a global vision of continuum in mathematics and philosophy, in particular natural philosophy, avoiding to consider its *indivisibles* as *atomic elementary parts of the matter.* This renounce can be considered as a crucial point in the history of continua since it sanctioned the separation between the mathematical (Cavalieri's one) and the physical material continuum (Galilei's one).

The first acquired the freedom of being composed of elementary constituents that were pure abstract objects created by the mathematicians imagination, with no needs to represent atoms or monads, pure fruitful inventions useful time after time in different practices involving the infinitesimal quantities.

The second is more complex to analyse, because of its many facets. For a clarification about the issues of continuity in Physics we refer to a specific chapter of this work, but we anticipate here just some excerpts in order to clarify the meaning of the distinction between mathematical andd physical continua operated by Cavalieri. Two big categories of continuity can be individuated in physics: the continuous evolution of processes (in space and time, which in turn can be continuous) and the continuity of matter (opposite to the discretization of matter in atoms).

The freedom acquired by mathematical continua involved the first kind of continuity, when processes, space and time have been modeled through mathematical tools (laws, functions, systems composed by infinite parts). The analysis of the composition of matter (and of the space, if considered full of matter like *ether*) followed instead its own path towards the current quantistic theory of matter.

Cavalieri was interested in Eudosso's exhaustion method in order to simplify it and make it more effective. He needed to conjugate the theory of ratios, on whom Eudosso's methods relied on, and his *Geometry of indivisibles*, that uses n-1 dimensional elements, i.e. segments for calculating areas, rectanguls for calculating volumes. Eudosso's theory of ratios was coherent with the ban of actual infinite and took care of the problem of incommensurability already studied by the Pythagoreans school. The problem of incommensurability was strictly linked to hypothesis that underlies the atomic composition of the Pythagoreans continuum, based on the concept of monad and number, so it was denied to consider a segment composed of indivisible points-monads. In order to be consistent with this constraint, Cavalieri could not consider segments as composed by points-monads. This was fortunately not necessary since in his method Cavalieri was considering both the segments and the figures themselves as wholes. This "stratagem" also preserves partially Cavalieri from the inconsistency with Galileian ban concerning quantifications and comparisons of infinite quantities of atomic elements, that in his approach were really the elementar parts of the continuum and had the same cardinality in two segments of different lenght. Cavalieri had to do more than considering *indivisibles* as wholes, but also he had to justify the core operation of his methods: the comparison between indivisibles. This comparison could not be realized standing on the number of their elements, because, as his master has proved, the number of elements is infinite and so is the same. Cavalieri thus opted for a continuum:

1) composed by *indivisibles* considered as wholes that can be compared through methods not involving infinite quantities;

2) if the figure was considered in his totality, it was considered as a whole and not as a sum of indivisible elements;

3) the *indivisibles* were not elementary parts of the continuum

4) obliquus segment were not comparable with horizontals (in order to avoid the incommensurability diagonal-edge)

i. **Viete's and Descartes' continua: a break-point with tradition**

Until now we have presented merely geometrical approaches to continuum, to which numbers were connected only by a hierarchical relation, in particular after the premature collapse of the numerical theory of continuum, dreamt by Pythagoras and his followers, due to the discovery of incommensurable quantities. This didn't implicate, of course, the abandon of a numerical mathematics, as for instances all the Arabic tradition and the fascinating history of equations testify, but scarred the numerical continuum as a low quality product, afflicted by dangerous bugs. What happened in France by means of Viéte and Descartes is really more than a crucial turning point in the history of continuum, but rather a milestone, a revolution in mathematics, since it reopenend by force the doors between two rooms that hadn't anything to do each other for many centuries. In Classical geometry the numbers had assumed a subordinate position in respect of magnitudes and proportions. Viéte and Déscartes mainteined a strong interest in Geometry, but two main innovations were suggesting a possible improvement of mathematics substained by numbers: the overflow of numerical methods in practical problems of real life - encouraged by decimal system's spread that accelerated calculations and improve approximations; the impressive development in the algebraic field - let's think for instance at Bombelli's *Algebra* - lead them to prefer the generality of Algebra to the rigor of Classical geometry. The Algebraic geometry that they were building up was founded on numbers, in spite of all the ancient recommendations, and were upsetting the hierarchy between numbers and magnitudes (Giusti, 2012). To the ancient limitations of numbers, the new Algebra had already replied with the creative power of signs in at least two ways: the symbol Rx to indicate not rational solutions to rational equations (called *numeri surdi*, absurd numbers) and the imaginary unity *i*.

The sentences by Leibniz concerning *i* let half-view the hybrid and mysterious atmosphere wrapping the Algebra at that time, useful but somewhat incomprehensible in its deep meaning:

"Ex irrationalibus oriuntur quantitates impossibiles seu imaginariae, quarum mira est natura, et tamen non contemnenda utilitas."

"The imaginary number is a fine and wonderful recourse of the divine spirit, almost an amphibian between being and not being."

The terms "real number" and "imaginary number" have been invented precisely by Descartes, who was addressing the topic to manage visually two curves' intersections but trying to merge algebraic and geometrical methods realized that some algebraic solution corresponded to invisible points.

As Giusti (1990) stresses, algebraic questions demanded not only practical solutions, as it was frequent before, but also theoretical answers, usually reserved to geometry. Viéte did more than inventing the syncopated language that made him famous all over the world, but operated a real change of perspective transforming even geometrical problems in algebraic problems through the appointment of a numerical value to magnitudes. This way he enlarged the concept of number itself: a number is solution to a geometrical problem, even if the solution is unknown. Viéte replaced the numeric algebra (*algebra numerosa*) with a new symbolic algebra (*algebra speciosa*) in which objects were not labeled with numbers but with letters: consonants for known numbers, vocal for the unknown ones. The new algebra dealt with forms, not only of with letters. This is the crucial point: to substitute numbers with letters wasn't revolutionary in itself, since it would have been sufficient to provide suitable example in order to express a rule, no need to use the verbal register. The true innovation *algebra speciosa* was the new unity established between algebra and geometry through the algebraic manipulation of forms, whose value was even unknown, in which the introduction of symbols was not the cause but the consequence (ibid.). Nevertheless it's undeniable that the semiotic choice to represent and manipulate letters instead of numbers was determinant for the development of the algebraic geometry: the semantic neutrality of letters, sometimes numbers, sometimes magnitudes made possible the union between algebraic procedures and geometrical methods, towards a new geometry that was founded on the ambiguity of what the letters actually represent. Geometrical magnitudes could thus be represented by a letter avoiding the mediation of numbers. This way a new, very complex relation is established between magnitudes and numbers, so powerful to seem to wipe away the past fears linked to the use of numbers in geometry and the problem of incommensurability. This acquired flexibility allowed, for instance, to express geometrical constructions through algebraic procedures, like compass-and-straightened procedures that were containing square roots - for which a symbol already existed - but also to create new numbers that were arising from new geometrical constructions - like cubic roots. Descartes accepted completely Viéte innovations, indeed accentuated more the numeric aspects of magnitude introducing the measurement unity, towards an algebraic theory of continuum, more and more oriented to abandon geometry and its strict axiomatic constraints towards an algebraic structure able to take the place of the axiomatic structure. This is also revolutionary, and characterize strongly both the Viéte's and Descartes' conception of continuum and the future mathematical approaches to this topic.

Summing up, the "amphibious" Viéte's and Descartes' continuum (ibid.) was characterized by these features:

1. accepted the classical continuum of magnitudes represented by the Euclidean straight and approximation methods
2. substituted to the axiomatic structure of Eudosso's theory of proportions the algebraic structure of the numerical continuum, whose rules were established by operations and descriptions of procedures
3. had no contradictions with the use infinitesimal quantities since it renounced to the precision and the rigour of Eudosso's axiomatization

For this last reason in particular the algebraic continuum is a truly new continuum, on which growed up the new mathematical analysis. It is thus contradictory to frame mathematical analysis in the classical continuum, ignoring this crucial transistion. It's not surprising to recognize similarities with Archimedes infinitesimal procedures, that would have find room in this new continuum, and the fact that, after less of one century, actual infinitesimal quantities and infinitesimal procedures, directly descending from Archimedes', entered mathematics, leading to the impressive steps forward of infinitesimal analysis. Given the reach of this revolutionary change of perspective on continuum it's also not surprising that infinitesimal analysis had to face urgent and frequent critiques of vagueness, lack of exactness and rigour, lack of strong bases, and so on. The abandon of the safe port of classical continuum to the new not-axiomatic conceptions leaved a trace in the newborn analysis. This Achilles' heel took after around two centuries to the born of theories of real numbers.

l. Leibniz's continua

Leibniz is one of the most influential mathematicians in the history of Calculus. Also his philosophical speculations concerning the world (The monadology) had been intertwined with mathematics, enriching his theories with frequent comparisons with the real world, with whom mathematical theories had to be harmonized. Well-known is the Leibniz's principle *Natura non facit saltus*, become a discussion topic of the modern Quantistic Physics thanks to Niels Bohr.

On the topic of continuum Leibniz's position was very articulated (Giusti, 2000). Leibniz never reached a unique theory continuum but rather different images of different continua, changing time after time depending on the case he was analyzing. In the end we will individuate at least three Leibnizian continua, each of whom coexists with the other without hierarchy and fulfills a particular task. In Giusti (1990) the Leibnizian continua are labeled with the following names:

1. Continuum with infinitesimals
2. Formalized classical continuum
3. Hyperdense continuum

Only the third one is suitable for the Calculus, since the others have no sufficient complexity to accomodate "*i valori della variabile, le differenze, le differenze delle differenze, e via sminuzzando, [che] si muovono sempre all'interno del punto strutturato ma inesteso del continuo iperdenso leibniziano. [...] Non sono necessarie infatti solo delle grandezze infinitamente piccole; occorre anche e soprattutto che esse siano paragonabili tra loro in modo che, come dice Leibniz, uno zero sia più grande di un altro. Inoltre si deve poter operare su di esse con le regole formali delle operazioni aritmetiche: sommarle, sottrarle, moltiplicarle tra loro e con altre grandezze finite."* (Giusti, 2000)

The first time in mathematics the term "function" appeared it was written by Leibniz, even if the concept was already "in the air" since many centuries (Volterra, 1920). The conception of Leibnizian function is founded on the third continuum.

I. Continuum with infinitesimals for motion and qualitative analysis of processes

The continuum with infinitesimals is analyzed in order to make room to natural discontinuities and describe qualitative changes in a suitable image of continuous process. His bumps were the discontinuity between life and death and between near and far. The natural reference for this kind of dissertation were the Aristotelic characterizations of continuous and contiguos, that Leibniz take as starting point to abandon them in the end of its analysis. Qualititative differences can not coexist, so one can be alive or died but not the both of them. So life and death are contiguous but their extremes can not coincide since it would implicate that a moment exists in which one is alive and dead at the same time.

1. Contiguous entities are produced by cuts of continuous entities (reversing Aristotle's determination of continuous as a particular case of contiguous)
2. The mental act of separation produces two points, distinct but not distant. The instant-point is splitted in two.
3. The process of split can not be repeated generating consecutive, infinitely near points (that would lead to the ancient contradictions), since after the first cut there are no segments to divide since the two points, let them be, B and C, are not distant and AB=AC. So a straight is not a set of points, nor of indivisibles, but infinitesimal are created by every single cut.
4. The motion doesn't happen in an instant or in a point, but is a whole process; points are results of a cut.

II. Formalized classical continuum for geometrical problems

The formalized classical continuum has a strong relation with the Euclidean Book I since its construction deals with intersections between geometrical entities. As a reader can imagine the problem to face will be partially similar to the one presented before, since the local composition of continuum in proximity of the intersection points concerns cuts and near points. Nevertheless this continuum has a peculiarity that makes it different from the first.

Let's start from Euclid. Lebniz’s objects to the statements concerning intersections between for instance lines and circumferences, in which it's not explicit that the existence of the point of intersection was not obvious but postulated. Lebniz fills the gap making explicit that there is always at least a point in common between curves that intersect each other. In this context the continuum assumes nuances that make it more similar the Aristotle’s one.

To compare Leibniz's and Aristotle's continuum it's necessary to point out that the concept of part was different: Aristotle's parts were intervals resulting from a division, while Leibniz's concept of part was more similar to set (not necessarily ordered). This explains some apparent incongruences but merely justify the - although implicit - use of superpositions between co-integrant parts of a continuum by Leibniz; superpositions that would have been impossible in Aristotle's continuum.

A section is thus an intersection between not superposed parts. To avoid empty intersections Lebniz implies that subsets are closed, as all the parts of the continuum and all the figures.

Leibniz's geometry is geometry of closed sets.

The characterization of the formalized classical continuum is thus this one:

1. closed and connected set
2. every intersection is not empty, so the point now belongs to every part (contrary to the case of motion explored before)
3. no need infinitely near points since the split is not only not necessary, but instead not desiderable.
4. every point belong both to the internal and the external part of an intersection

III. Hyperdense continuum

As we anticipated the continuum with infinitesimals couldn't satisfy all the necessities of the Calculus, since the infinitesimals E had not only to be manipulated algebraically so to keep equal to a segment x the segment x + E, with E the distance between two distinct and not distant points, but every point needed to have a cloud of infinite infinitely close points. Robinson identified with the concept of monad this differential structure of point without extension but composed of infinite infinitely close points; so the point isn't what is not divisible in parts. Distances had to be at the same time zeros but a zero had to be larger than another zero. The operations had to be confined into the infinitely populated cloud around the point, in which we find the arbitrary but assigned sequence of infinitely close values that the Leibnizian variables assumed (Bos, 1974).

Differentials become themselves variables, may be differentiated and may assume values inside the cloud. Everything happens *intra-extensionem*, the micro-world of the cloud have the same structure of the macro-world of the line.

To sum up the hyperdense continuum has the following features:

1. each point has around a cloud of infinite infinitely close points, whose distances are comparable each other
2. variables assume assigned but arbitrary sequences of values *intra-extensionem*

3. differences of any order are all inside the cloud, so as results of every kind of operation involving differentials
4. the whole continuum is not sum of its points

This may be considered a critical point of the hyperdense continuum if we consider integration, as it was for Cavalieri, since if the operation cannot exit the cloud, how can a sum of infinitesimals form a figure? Leibniz never explicit this point but coherently with his model never sums effectively quantities but always traits integration as anti-derivation, being derivation the local operator for whom the model had been tailored. As Cavalieri did indeed in practice, Leibniz considered always the figures as wholes to study locally and to re-obtained inverting the local operation of derivation, but never as sums of elements.

On the contrary further approaches reversed the hierarchy between integration and derivation (f.i. Volterra, 1920). So it's proved that there are no right approaches but only goal-oriented practical choices that delimit a space of action in which mathematicians stay until they do not need new freedoms.

m. Newtonian continuum

Newton had come to treat continuity facing both the issue of abstract description of Space and Time as absolute entities, pursued through mathematical tools, and the analysis of magnitudes variations and motion. The long-term debate concerning the born of the Calculus that involved Leibniz's and Newton's scholars, that has been fascinating students for all the following centuries, is very well-known but it's usually narrated merely from a social point of view, i.e. that of the attribution of the paternity of Calculus to Leibniz or Newton. The intriguing happenings obscured partially the presence a very deep difference between the two approaches to the Calculus, that determined the affirmation of the Leibnizian formalism and, in general, of the Leibnizian method for differential analysis. A key element to grasp the difference between the two methods is precisely the conception of continuity. The dynamic approach to variations proposed and carried out by Newton and its scholars implied a dynamic conception of continuity, framed in a more general mechanical conception of Geometry. In Newton we can read:

< *Quantitates Mathematicas, non ut ex partibus quam minimis costantes, sed ut motu continuo descriptas hie considero. Lineae describuntur ... per motum continuum Punctorum; Superficies per motum Linearum; Solida per motum Superficierum; Anguli per rotationem Laterum; Tempora per fluxum continuum & sic in ceteris. Hae Geneses in rerum naturae locum vere habent, & in Motu Corporum quotidie cernuntur* >
(Giusti, 1988)

The term themselves used by Newton, like fluxions and fluents, lead to think at the intuitive, perceptual dimension of continuity as a flow, something that pass through a points from an endpoint to another, a process of becoming. The deny of space-temporal intuitions of continuity that will characterize the arithmetization of Analysis indeed after a century in favor of punctual, static characterization of continuity (also for the infinitesimal variations) has a strong, maybe causal, relation with the affirmation of the Leibnizian methods on those created by Newton. The Newtonian method obliged him to borrow and then transform deeply and make more effective the Mercator's series, since the approximation is the key of the method. The famous mathematicians MacLaurin, belonging to the Newtonian school, readfirms the preference for a dynamic approach to continuity of his school, in particular talking about limits:

< *Quando la certezza di una parte qualunque della geometria è messa in discussione, la maniera più efficace per ristabilire la verità sulla sua piena luce e prevenire dispute, è di dedurla da assiomi o principi primi di evidenza indiscutibile, con dimostrazioni del tipo più rigoroso alla maniera degli antichi geometri. Questo è il nostro intento nel trattato che segue; nel quale non proponiamo di cambiare la nozione di flussione di Sir Isaac Newton, ma di spiegare e dimostrare il suo metodo deducendolo per disteso da poche*

verità autoevidenti, in maniera rigorosa: e, nel trattare ciò, di fare astrazione da tutti i principi e i postulati che possono richiedere di immaginare altre quantità che non possano essere facilmente concepite come realmente esistenti. Non considereremo una parte qualunque dello spazio o del tempo come indivisibile o infinitesima; ma considereremo un punto come il termine o il limite di una linea, e un momento come un termine o un limite del tempo.> (Cinti, 2013)

The conception of continuum in Netwon was thus:

1. dynamic;
2. associated to functions and variations;
3. related to physical *phenomena*, like motion, magnitudes variation, time lapsing.

n. Euler's continuum

R may be regarded as the space of ratios of microquantities. This was essentially the view of Euler, who regarded (real) numbers as representing the possible results of calculating the ratio 0/0. For this reason Lawvere suggested that R be called the space of Euler reals. In Euler the continuity law was traduced in the fact that a function is expressible through an analytical expression in his whole domain.

o. **D'Alembert's continuum**

D'Alembert studies about continuity was merely connected to the mathematical description of physical laws and so the central concepts are functions and variations. The French scholar in particular was interested in functions that describe waves in a chord, so not only continuous but at least derivable two times (for calculating local speed and acceleration).

In D'Alembert, similarly to Euler, the continuity law was traduced in the expressibility of a function through an analytical expression in his whole domain.

p. Arbogast's continuity

In 1787 in St. Petersburg a curious contest was announced: find a way to determinate the nature of continuous functions. In 1791 the competition was won by Luis-Francois Arbogast, that propose the following definition that broaden the D'Alembert's one, but also the definition proposed by Euler:

"The law of continuity consists in that a quantity cannot pass from one state [value] to another [value] without passing through all the intermediate states [values] ...."

This insight was made rigorous in an 1817 pamphlet by Bernhard Bolzano (1781--1848) and is now know as the Intermediate Value Theorem.

Two cases of not continuous functions were taken in account: *discontinuous*, i.e. the law, so the function, change "completely", or *discontiguous* i.e. different parts of a curve are unconnected. From this characterization also emerges the conception of function as a curve.

The idea of "passing through" is anchored to the intuitive conception of physical motion, assumed to be continuous. To assume all the values is here used equivalently to pass through without interruptions or "saltus". Discontinguity contradict the notion of continuum as something with *no breaks*.

q. Lagrange's continuum

Lagrange faced first the problem of foundation of Calculus with the "rigor of the ancient proofs", trying to provide an algebraic solution (Cinti, 2013). He tries to refer all the Calculus procedures and definitions to Algebra, getting it away from the geometrical evidences that had made room to metaphysical arguments to criticize the differential Analysis. This way Lagrange inaugurated the season of "arithmetization of Analysis" that seemed to sort out all the problems, giving the desiderated rigor to the Calculus.

r. Bolzano's continuum

Bernhard Bolzano's contribution to the debate on continuity is merely to place on the side of the question of methods and rigour in Analysis. It's very important for our research to trace the paths towards contemporary images and conceptions of continuum passing through the role of rigor and precision in the choice of representations of continuity in the mathematicians' practices. In particular we are interested in the forces that transformed the actual infinite arithmetic continuum, forbidden in the Ancient Hellad, in the solution to the issues of credibility, methodological strength, acceptable foundation of the Analysis.

The "[...] platonist orthodoxy [of founding mathematics on set-theory, nba] is a comparatively recent phenomenon, dating back only a century or so. Before then, it was more common to view infinity as potential infinity. It is illuminating to look at how and why the change-over to actual infinity occurred. The transition arose out of the needs of nineteenth-century mathematics, particularly the arithmetization of analysis. Four reasons can be traced" (Fletcher, 2007).

One of these reasons is related to Bolzano's methodological issues: the rejection of spatial and temporal intuition.

"Newton based his ideas of limits and differentiation on intuitions of motion; other mathematicians based their ideas of continuity on spatial intuition. These kinematic and geometric conceptions fell into disfavour in the nineteenth century, as they had failed to provide satisfactory theories of negative numbers, irrational numbers, imaginary numbers, power series, and differential and integral calculus (Bolzano, 1810, preface). Dedekind pointed out that simple irrational equations such as $\sqrt{2} \cdot \sqrt{3} = \sqrt{6}$ lacked rigorous proofs. Even the legitimacy of the negative numbers was a matter of controversy in the eighteenth and nineteenth centuries (Ewald, 1996, 314 - 336). Moreover, Bolzano, Dedekind, Cantor, Frege and Russell all believed that spatial and temporal considerations were extraneous to arithmetic, which ought to be built on its own intrinsic foundations" (Fletcher, 2007).

Bolzano didn't spare of course the intuitive ideas of continuity, keeping continuity at a safe distance from motion and time:

"He criticized demonstrations of Kaestner, Clairaut, Lacroix, Metternich, Rösling, Klügel, and Lagrange for the involvement of geometrical and physical images (time and movement, transfer) and the lack of analyticity in their reasoning, i.e. lack of understanding of the continuity as a mathematical notion" (Sinkevic, 2015). Bolzano wrote:

"*As a matter of fact, if we take into account that a proof in science must not at all be just words but argumentation, i.e. be the exposition of objective cause for the true being proved, then it goes without saying, that if an affirmation is correct only for the values in the space, it may not be correct for all variables, whether or not they are in the space. The most common kind of proof depends on a truth borrowed from geometry, namely, that every continuous line of simple curvature of which the ordinates are first positive and then negative (or conversely) must necessarily intersect the x-axis somewhere at a point that lies in between those ordinates. There is certainly no question concerning the correctness, nor indeed the obviousness, of this geometrical proposition. But it is clear that it is an intolerable offense against correct method to derive truths of pure (or general) mathematics (i.e., arithmetic, algebra, analysis) from considerations which belong to a merely applied (or special) part, namely, geometry. No one will deny that the concepts of time and motion are just as foreign to general mathematics as the concept of space. We strictly require only this: that examples never be put forward instead of proofs and that the essence of a deduction never be based on the merely metaphorical use of phrases or on their related ideas, so that the deduction itself would become void as soon as these were changed*» (Bell, 2014, translation of Bolzano, 1817).

"These considerations led them to reject the potential-infinity notion of a quantity capable of augmentation without end, with its connotations of time and change, and to replace it with the static notion of the infinite set of all possible values of the quantity." (Fletcher, 2007)

The paper published by Bolzano in 1817 represents an important stage in the rigorous foundation of analysis and is one of the earliest occasions when the continuity of a function and the convergence of an infinite series are both defined and used correctly (Russ, 1980).

The change of images of continuum from Geometry to Algebra and Arithmetics that we can observe in the following century may be directly connected to Bolzano's statements. Let's finally see in depth which was the conception of the continuum emerging from Bolzano's works.

Bolzano, like Galileo, considered that an actual infinity could arise from aggregation of infinte points (Bell, 2014):

"If we try to form a clear idea of what we call a 'continuous extension' or 'continuum', we are forced to declare that a continuum is present when, and only when, we have an aggregate of simple entities (instants or points or substances) so arranged that each individual member of the aggregate has, at each individual and sufficiently small distance from itself, at least one other member of the aggregate for a neighbour".

Bolzano's continuum is thus:

1. dense (convinced that it was sufficient to make a set continuum)
2. actual infinite and composed, through aggregation, by infinite points
3. admitted the existence of infinitesimal quantities as the reciprocals of infinitely great quantities (algebraic)
4. infinitesimal quantities cannot be considered as zeros (contrary of Euler's position), but also cannot be considered as "infinitely small geometrical entities"; this was true only for isolated values.

s. Cauchy's continuum

Le Cours D'Analyse, published by Cauchy in 1821, can be considered the first didactical manual for Analysis published ever, and published "for a major utility for students". Cauchy's work was determinant in development of the Analysis, also thanks to this book in which he reorganized all the previous knowledge re-founding it in order to give it rigorous foundations, as Lagrange and Bolzano had already tried to do. It may surprise - and it should deal us to deep didactical reflections - that the concept of rigor that guided Cauchy in his mission was exactly the opposite of the two eminent mathematicians that came before him. Cauchy always thought indeed that the maximum rigor was given by Geometry instead of the more generic algebra, whose formulas may be valid only under particular conditions (Cinti,), but was anyway convinced the intuitive and evident concepts were not enough yet. Their success in geometry and physics couldn't satisfy the compelling foundational necessities. So Cauchy aimed to a rigor "like that of Geometry" - but not geometrical - avoiding to base his methods on the algebraic general procedures, that had been supposed and pretended to be more general than it was really. The semiotic dimension of this careful procedure of refoundation of Calculus deserves a special attention. As Leibniz did, Cauchy needed new symbols to concile the rigour with the simplicity - desirable for students' manuals - that came from intuitive infinitesimals. The new concept, with its relative coherent and goal-oriented representations, on which Cauchy re-founded the Analysis was the limit (Cinti, 2013). His symbols are not the most used today - invented by Weierstrass - but it was anyway already clear to Cauchy what they had to permit: to suggest the process of getting closer and closer without using intuitive and "customizable" intuitive ideas, but, on the contrary, delimiting precisely the kind of approach the variables' values had to follow. This is the crucial concept of Cauchy's sequence and leads to define precisely the infinitesimals, leaving aside the old intuitive conceptions: infinitesimals were the differences between two terms – closer and closer - of a sequence of elements that defined a limit.

The first example of limits is precisely the real numbers, that, if are not rational, coincided with the sequence of rational numbers that approached it, following the rules Cauchy had established for getting closer to a fixed point.

Cauchy was aware of the importance of a suitable concept of continuity in the complex structure that the Calculus need to come out on top.

t. Weyl and Brouwer's characterization of the continuum

The formal constructions of the sets of real numbers - the plural is necessary - constitute refined objectives, peaks of the mathematical activity and beautiful products of work, debates, fantasy and compromises. Nevertheless the critiques, the alternative proposals and the open problems were maybe more than the appreciations and the successes. Particularly remarkable are the objections of Luitzen Egbertus Jan Brouwer, Dutch mathematician and philosopher, founder of the mathematical philosophy known as Intuitionism.

A metaphor for the issue emerging in the beginning of the XXI century due to Brouwer – precisely in 1905, one of the most famous year in the history of contemporary science - can be borrowed from Bohr's complementary principle for couples of magnitudes in Quantum physics:

"Within the scope of classical physics, all characteristic properties of a given object can in principle be ascertained by a single experimental arrangement, although in practice various arrangements are often convenient for the study of different aspects of the phenomena. In fact, data obtained in such a way simply supplement each other and can be combined into a consistent picture of the behavior of the object under investigation. In quantum mechanics, however, evidence about atomic objects obtained by different experimental arrangements exhibits a novel kind of complementary relationship. Indeed, it must be recognized that such evidence which appears contradictory when combination into a single picture is attempted, exhaust all conceivable knowledge about the object. Far from restricting our efforts to put questions to nature in the form of experiments, the notion of complementarity simply characterizes the answers we can receive by such inquiry, whenever the interaction between the measuring instruments and the objects form an integral part of the phenomena" (Bohr, 1962, in Favrholdt, (1999), p.4).

We may reformulate an analog principle for the continuum, considering the punctual, or infinitesimal, and the intuitive whole continuum as two aspects hard to see together. Every time we want to see one aspect or the other we will see it through adequate assumption and limitations, but trying to consider the two aspects together we can get lost. In words of Weyl (1987): *"[...] the conceptual world of mathematics is so foreign to what the intuitive continuum presents to us that the demand for coincidence between the two must be dismissed as absurd".*

This argument was supported by eloquent mathematicians and modern and contemporary philosophers like René Thom and Hermann Weyl and is highlighted also in Tall & Vinner (1981) - even if this duality is presented as a tricky problem to overcome in order to create a "concept definition image", supposed to be one and all-embracing – but overall it was grasped and transform in a powerful tool by Lakoff & Nunez (2000).

We need to synthesize here the long-term reflections that characterized the non-linear, troubled path towards *Das kontinuum* and the further developments of Hermann Weyl's thought concerning the continuum. It's however deserved to remind that those years of enthusiastic research couldn't be resumed in a few words.

We report some excerpts from an article by Bell (2000, p. 4):

“In Consistency in Mathematics (1929), *Weyl characterized the mathematical method as the a priori construction of the possible in opposition to the a posteriori description of what is actually given. The problem of identifying the limits on constructing “the possible” in this sense occupied Weyl a great deal. He was particularly concerned with the concept of the mathematical infinite, which he believed to elude “construction” in the idealized sense of set theory. Again to quote a passage from Das Kontinuum (Weyl, 1987):*

<No one can describe an infinite set other than by indicating properties characteristic of the elements of the set. [...] The notion that a set is a “gathering” brought together by infinitely many individual arbitrary acts of selection, assembled and then surveyed as a whole by consciousness, is nonsensical; “inexhaustibility” is essential to the infinite.>

< The continuity given to us immediately by intuition (in the flow of time and of motion) has yet to be grasped mathematically as a totality of discrete “stages” in accordance with that part of its content which can be conceptualized in an exact way. […] Exact time- or space-points are not the ultimate, underlying atomic elements of the duration or extension given to us in experience. On the contrary, only reason, which thoroughly penetrates what is experientially given, is able to grasp these exact ideas. And only in the arithmetic-analytic concept of the real number belonging to the purely formal sphere do these ideas crystallize into full definiteness. When our experience has turned into a real process in a real world and our phenomenal time has spread itself out over this world and assumed a cosmic dimension, we are not satisfied with replacing the continuum by the exact concept of the real number, in spite of the essential and undeniable inexactness arising from what is given>.

Above and beyond the claims of logic, Weyl welcomed Brouwer’s construction of the continuum by means of sequences generated by free acts of choice, thus identifying it as a “medium of free Becoming” which “does not dissolve into a set of real numbers as finished entities”. Weyl felt that Brouwer, through his doctrine of Intuitionism, had come closer than anyone else to bridging that “unbridgeable chasm” between the intuitive and mathematical continua.”

We can synthesize the Weyl’s conceptions of the continuum in three main categories:

1) **Intuitive global continuum**: given to the conscience from the experience of the world, perception of the flow of time and motion, given immediately by the intuition;

2) **Formal punctual continuum:** *”atomistic or discrete conception of continuity, the continuum is composed of individual real numbers which are well-defined and can be sharply distinguished; concept of real number in this extensionally determining and delimiting manner […] an ensemble of individual points, so to speak, picked out from the fluid paste of the continuum.”* (Weyl, 1987).

   The relation between the two is presented by the same Weyl (1987). *“The continuum is broken up into isolated elements, and the flowing-into-each other of its parts is replaced by certain conceptual relations between these elements, based on the “larger-smaller” relationship”.*

3) **Brouwer’s construction of the continuum by means of sequences generated by free acts of choice**: a “medium of free Becoming” which “does not dissolve into a set of real numbers as finished entities”.

u. Surreal, super-real and hyper-real numbers

After the critic to real numbers some alternative proposals appear in the field of the Theories of numbers.
“The surreal number system is an arithmetic continuum containing the real numbers as well
as infinite and infinitesimal numbers, respectively larger or smaller in absolute value than any positive real

number. The surreals share many properties with the reals, including a total order $\leq$ and the usual arithmetic operations (addition, subtraction, multiplication, and division); as such, they form an ordered field. (Strictly speaking, the surreals are not a set, but a proper class.) If formulated in Von Neumann–Bernays–Gödel set theory, the surreal numbers are the largest possible ordered field; all other ordered fields, such as the rationals, the reals, the rational functions, the Levi-Civita field, the super-real numbers, and the hyperreal numbers, can be realized as subfields of the surreals. […] The surreals also contain all transfinite ordinal numbers; the arithmetic on them is given by the natural operations."

To sum up very briefly the intention underlying these new constructions, they emerged from the necessity of formalizing the application of the arithmetical procedures to quantities that were not numbers in the classical sense.

1.1.2 Implications on our research design

This awareness may suggest to use the complementarity of the two classes of images of continuum (whole/intuitive-systemic/formal) as a potential resource for didactics instead of trying to construct a vertical way to modern formalizations.

A first hypothesis that should be evaluated is to avoid both the strategies that aim at forcing the didactical actions to construct one unique path and a global image and the mixed and confusing didactical transposition that use the two classes of images as equivalent. In such a complementarity the field of real numbers - considered as an object of pure mathematics - would find his own position on the side of numerical, decomposed (systemic) continuum. The description of the trace of the whole continuous processes in the intuitive sense in terms of discrete entities like points, "epsilons" and numbers would rightly be presented at school as a Icarus' flight, so fascinating as somehow intrinsically impossible with our wings.

In this sentence we may seem to forgot, or deny, a very important period of mathematics and its wonderful products, like Cantor's continuous or Hilbert's axiomatization or Cauchy's, Dedekind's, Weierstrass' efforts to transform the vague conceptions of continuous magnitudes always expressed by processes in mathematical "static" objects that could make rigorous the infinitesimal analysis.

This is not true, since the constructions or axiomatizations themselves are intrinsically characterized both from the tension to transform the intuition of continuity into a set of numbers algebraically structured and the awareness of the limits that frustrates the complete mission (necessity of infinite existence axioms, need of a postulate of continuity, lack of constructive procedures for the $\aleph_1$ irrational numbers, ...).

The eminent mathematicians that dealt with the construction or the axiomatization of the field of real numbers had clear in their mind the complexity of this issue and pursued their aims aware of the loss of the intuitive dimension of continuity that will have been caused through formalization.

The fields of real numbers (the plural is necessary, since there are at least two logically not equivalent constructions) are endpoints of social processes that involved the mathematicians' community, asked to clarify and make rigorous the foundations of Analysis even if the criteria to determine what is rigorous in Mathematics were not unique and clear at that time as now.

The posterity didn't accept completely this kind of "formal solution" and this is the reason why the critics to the "Cantorian paradise" (Hilbert, 1926) were not lacking. This reply by Wittengstein to Hilbert resumes perfectly the spirit of this reaction:

" If one person can see it as a paradise of mathematics, why should not another see it as a joke"

Proofs of the controversy that Cantor's set theory provoked in mathematicians' debates are easy to find: see for instance Poincaré's rejection of actual infinity (Heinzmann and Stump, 2014); the sharp opposition to Cantor's ideas that gave rise to the Brouwerian intuitionism and in general to all the constructivist thread in mathematics research (Iemhoff, 2015); Hermann Weyl's dissertations about the continuum, in which he criticizes harshly the attempts to transform *tout-court* the continuous space and time variations in assumptions by a variable of values in an interval of real numbers (Bell, 2014). Also the Cantorian axiom of infinity was judged scarsely self-evident at all (Mayberry, 1983) and not adequately justified by analogy to finite sets (Bell, 2014).

Standing on the historical review, that we will present in the following, we can state that there is not a "solution" to the incommensurability between intuitive and numerical continuum, but rather there have been choices oriented to specific goals, both practical and theoretical, not often well explicit, representations and problem-situations have been very important in the development of different nuances and potentialities of the mathematical continuum. Understanding how the traces of these different paths are intertwined and how they have wandered in didactical practices time after time until nowadays is the main interest for the investigation of didactical transposition and thus is one of the aims of our work. In particular we will look for the relations between the search for intuitive examples and representations of continuity, to wish in the high school.The core of our a-priori historical and epistemological review is thus to look for different *continua* in the history of Italian secondary school. This configurations of objects will be compared first of all with the objects emerging from didactical practices declared by the teachers involved in our investigation, in order to find out the potential cognitive conflicts (Tall & Vinner, 1981; Godino et a., 2007) that can be forecasted through the epistemological analysis. Then we will evaluate the didactical suitability of the teachers' choices from the epistemic, cognitive and ecological points of view.

We will avoid to look for a unique general mathematical object including the field of real numbers and the continuity, as is usually done in OSA, but we will rather take care of preserving a certain degree of necessary and fruitful complexity.

This would not have been impossible at all, but we don't consider this attempt useful, nor for the historical analysis (as Giusti explained very well), nor for the analyses of declared practices.

1.2 Difficulties in the teaching-learning processes involving the real numbers and the continuum in the high school

In this Paragraph we resume the empirical an theoretical results concerning the difficulties that may arise in the teaching-learning processes involving real numbers; even though the major attention have been posed to the students, some of these results can be extended to the teachers in this case, taking care of unavoidable differences due to the major experience with Mathematics of the teachers. The reasons why we consider relevant the observation concerning the student also for the teacher are:

a. the complexity of the topic, that make them hard to grasp completely even in case of good mathematical background (it's an epistemological obstacle; Brousseau, 1983; Arrigo & D'Amore, 2002);

b. the scarce attention posed to the epistemological issues, typical of real numbers, in the high school but also at the University and in the training courses;

c. the "reputation" of the real numbers as objects of didactical activities – indispensable but prohibitive - that leads the teachers to try to make sense of the different aspects of real numbers relying on textbooks and manuals just in order to teach them, even if they are not aware of the true reasons why real numbers have been defined in the History of Mathematics.

In this Paragraph we will thus distinguish time after time the kind of subjects involved in the investigations that have leaded to the results we propose without dividing completely the two groups of results.

We relied on this literature review in order to investigate both the teachers' mathematical and didactical knowledge of real numbers.

1.2.1 Students' and teachers' difficulties with real numbers

The literature concerning the students' difficulties with real numbers is wide and many results were confirmed by different researchers since the 70's. In fact the analysis of the difficulties of students dealing with some aspects of real numbers characterized many of the most dated papers reporting researches in Mathematics education. Some of these papers are still very important references both for the specific studies about learning processes involving real numbers and for more general perspectives. We resume here some of the most relevant results that we took in account in all the phases of our research. Some of these results oriented us in the design of the pilot teaching experiment; other results were used as references in order to evaluate the cognitive suitability of the teachers choices. We decided to focus our attention on the teachers' choices suitability, also from a cognitive point of view, thus we took as reference the most frequent students' difficulties to evaluate the cognitive suitability. Many of these results are resumed in Voskoglou & Kosyvas (2012). In a paper extracted from a PhD thesis concerning the primary school teachers' belief concerning the infinite (Sbaragli, 2006), a very detailed literature review about the misconceptions of the infinity is presented, both observed in primary school teachers and in high school students and teachers. We will distinguish time after time between the subjects in which they were spotted.

In this review in particular the author discuss some aspects that are relevant also for our investigations: some typical phenomena that have been observed when students' were asked to describe the composition of the line as an ensemble of points and when they were asked to compare different infinite cardinalities. We report here some of the remarkable references we extracted from her literature review.

The main themes related to real numbers in the literature are:

1. Irrational numbers
2. Infinity
3. Points of a line
4. Density and continuity
5. Number line

We present here a brief analysis of the difficulties grouping them in these categories.

1. Irrational numbers

There are different kind of researches concerning the students difficulties with irrational numbers.

Some of these researches stress the difficulty to make sense of the representations of irrational numbers, while in other researches the difficulties with operation are highlighted.

Voskoglou & Kosyvas (2012, p. 303) first of all highlighted that "an essential pre-assumption for the comprehension of irrational numbers is that students have already consolidated their knowledge about rational numbers and, if this has not been achieved, as it usually happens, many problems are created. It has been observed that pupils, but also university students at all levels, are not able to define correctly the concepts of rational and irrational numbers, neither are in position to distinguish between integers and these numbers (Hart, 1 988; Fischbein, et al., 1 995). It seems that the concept of rational numbers in general remains isolated from the wider class of real numbers (Moseley, 2005; Toepliz, 2007)". In the end of their own research they confirmed that students' difficulties in understanding the real numbers was related with the incomplete understanding of rational numbers, the incommensurability and nondenumerability of irrational numbers.

Being the adjective " irrational" defined as a negation of "rational", this observation is certainly relevant for interpreting the students' difficulties when they have to grasp the meaning of irrational number, complex in itself.

In the field of investigation concerning the difficulties with rational number, one is particularly relevant for us: the students' ideas concerning the relation between 0.999… and 1 (Tall & Schwarzenberger, 1978; Hewitt, 1984; Sierpinska, 1987; Margolinas, 1988; Hirst 1990; Edwards & Ward, 2004; Weller, Arnon, & Dubinsky, 2011). The students consider the two numbers "infinitely close" but not equal, because of the problems in the interpretation of the representation. Indeed the students perceives the irrational numbers as non-whole numbers when they are represented in the decimal register (Fischbein, Jehiam & Cohen, 1995), numbers that are not complete. Thus it's hard for them to accept they are equal to 1 and the usual proof involving their representation as fractions don't convince the students anyway.

A very interesting phenomenon have been reported by Margolinas (1988): some students, asked to represent the difference between 0.999… and 1 invented a symbol, that was defined by the author "infinitesimal answer". The symbol had infinite zeros, in the actual sense of infinity, and then in the end a 1. This confirms that some students, in front of an anomaly in the decimal representation of numbers, in a more or less explicit way try to represent their perception of incompleteness. In that case it was made concrete by means of a symbol: 0, 000.. 1.

Apart from this "there are also other obstacles (cognitive and epistemological) that make the comprehension of irrational numbers even more difficult (Herscovics, 1989; Sierpinska, 1994; Sirotic & Zazkis, 2007a)" (Voskoglou & Kosyvas , 2012) ; in contrast of the amount of researches concerning difficulties with rational numbers, research on irrational numbers is rather slim (Sirotic & Zazkis, 2004). Definitions of rational and irrational numbers rely on number representations, but only a few studies investigated the problem of learning irrational numbers from the perspective of representations (ibid. , 2004).

The most relevant research concerning the students difficulties with real numbers are Schwarzenberger & Tall (1978), Fischbein, Jehiam & Cohen (1995, trad. it. 1996), and Sirotic e Zazkis (2004, 2007).

Fischbein et al. (1995) carried out a research based on the assumption that the irrational numbers could be the intuitive because of the high complexity of the issue, that we already showed in depth in Par. 1.1.1. Analyzing students and pre-service teachers they observed a general lack attention to the hierarchy of numerical sets and a consequent low awareness of the differences between rational and irrational numbers.

Contrary to what they had hypothesized, the difficulties with irrational numbers were not due to a lack of intuitiveness, because many of the students they interviewed overcame the barrier quite easily in suitable contexts. They concluded that paying attention to the potential difficulties and presenting them to the students the known difficulties concerning irrational could be reduced significantly and that ignoring them is a further error. Research focused on the comprehension and proper didactical approach of irrational numbers is rather slim (Voskoglou & Kosyvas , 2012).

As Sirotic & Zazkis (2004) stressed another remarkable – even if underestimated - category of investigation concerning irrational numbers regards the difficulties in managing and make sense of different representations.

Peled & Hershkovitz (1999) observed that pre-service mathematics teachers often were not at ease facing tasks involving irrational numbers in different representation even if they knew the definitions; Sirotic & Zazkis (2007b) reported some pre-service teachers' confusion about the relation between points on the number line and approximation of irrational.

Sirotic and Zazkis (2004) faced the issue of representations of irrational numbers using a theoretical tool: the distinction between opaque and transparent representations. They "examined how the availability of certain representations influenced participants' decisions with respect to irrationality (p. 499)" concluding that a minor but significant part of the pre-service teachers they interviewed showed wrong conception of real numbers; in particular 40% of them (46) didn't identify nonrepeating decimal representations as irrational numbers. Also some trends have been observed: " a tendency to rely on a calculator […] preference towards decimal representation over the common fraction representation; […] confusion between irrationality and infinite decimal representation, regardless of the structure of this representation; the idea of "repeating pattern" in decimal representation of numbers was at times overgeneralized to mean any pattern." (Sirotic & Zazkis, 2004). They attribute this wrong conceptions to a missing link between the procedure that leads to decimal number starting from fractions and viceversa. They recommend a grower attention to "the connections between decimal (binary, etc.) and other representations (geometric, symbolic, common fraction, and even continued fractions) of a number can be an asset. Simply put, we suggest that by directing explicit attention of students to representations and to mathematical connections that render the two representations equivalent" (ibid., 2004).

Voskoglou & Kosyvas (2012), investigating the students' difficulties daling with their multiple representations - 13-14 years old and 18-19 years old students – concluded that:

1) the technologist students (18-19 years) failed almost completely to deal with processes connected to geometric constructions of incommensurable magnitudes;

2) the richness of semiotic representations for the understanding of real numbers didn't prevent the students in answering satisfactorily the other questions; nevertheless in these interviews the students' difficulty dealt with the multiple representations of real numbers;

3) with the exception of the geometric representations , the students' performance was connected to their ability of flexible transformations among the multiple representations of real numbers.

The most of the researches concerning difficulties with irrational numbers had as participants students and preservice teachers and aimed at showing how far they had difficulties in understanding particular aspects of real numbers. Voskoglou & Kosyvas (2012) came to consider the age a determining factor and to state that a major age was correlated with better understanding of real numbers.

Also it was stressed that the teachers' difficulties influence the students' learning and that similar wrong conceptions compare in the students' answers (Gonzales-Martin, 2014).

Another interesting results concerning somehow irrational numbers were presented by Gonzales-Martin (2013) in CERME 8. The investigation concerned students who were attending Mathematics course at the university dealing with infinite series of real numbers: "the notion of convergence, especially in other settings which are not symbolic (e.g. geometric setting), is not clear for students regarding series. The identification and the manipulation of series in settings other than the symbolic setting are not evident for students." (ibid, 2013, p. 9). It seems that the notion of convergence is the geometrical settings, maybe the most useful in order to connect the representation of numbers as points of the line to the real numbers in the decimal representation, is never studied in depth not even at the University.

According to Gonzalez-Martin (2014) upper secondary and college students give more importance to the different writings of these numbers than to their specific properties (Robinet, 1986).

To sum up the favorite representation of irrational numbers seemed to be the decimal one, even if it's the hardest to grasp, especially if it's proposed suddenly without an understanding of convergence, infinite procedures and in lack of connections between the different representations.

2. Infinity

The research concerning the teaching and learning of infinity is wide and multifaceted; we can state this is a sort of research program in Mathematics education. For a very detailed, even if not exhaustive, dissertation about this topic see Fischbein, Tirosh & Hess (1979); Moreno-Armella & Waldegg (1991), Moreno-Armella & Waldegg (1995), Tsamir & Tirosh (1994), Waldegg, G. (1993); Waldegg (1996).

In this Paragraph we resume just a few of these researches that are relevant to our investigation.

Fischbein, Tirosh and Hess (1979), Fischbein (1992a; 2001) , Tall (1980) observed a recurrent phenomenon, that was defined *dependence, "* according to which there are more points in a longer segment than in a shorter one." (Sbaragli 2006, p. 52). Arrigo and D'Amore (1999; 2002) carried out two investigations about the Cantor's diagonal theorem and the real numbers with students attending the last years of the high secondary school and concluded again that this misconception was relevant also in that case. "This phenomenon can be observed not only in geometrical field but it is also evident when referring to dependence of the cardinality on the "size" of numerical sets. For example, since the set of even numbers represents a sub-set of the natural numbers set, the former seems to be by implication formed of a smaller number of elements." (Sbaragli, 2006). Other researches regarding the intuitive approaches to such a complex and counter-intuitive topic were also reported in Sbaragli (2006): the students observed trying to put into biunivocal correspondence infinite sets attributed a greater cardinality to the set expected to contain the other; interesting in this sense are Tsamir and Tirosh (1994, 1997) focusing on the metacognitive aspect and consistency and Tall (2001). "This dependence phenomenon is based on the generalisation to infinite cases of what has been learnt of the biunivocal correspondence of finite cases (Shama and Movshovitz Hadar, 1994)."(Sbaragli, 2006; p. 53).

Waldegg (1993) and Tsamir and Tirosh (1994) explored another aspect of the learning of infinity: the way the students combine intuition and representation dealing with the correspondence between infinites sets. Infinite sets are considered to have the same cardinality just because they are infinite, over-extending the correspondence between N and Z, Z and Q. This is known to the literature as "flattening" and it was also treated in the works of Arrigo and D'Amore (1999, 2002), who showed it is not only due to epistemological obstacles, but also to didactical obstacles. Garbin (2005) added to this misconceptions another negative factor: "the incidence [...] of representations and the different language codes of mathematics in the perception of the concept of infinity performed on students aged 16 – 20 […] and the study of D'Amore et al. (2004) conducted in Colombia, Italy and Switzerland on the "sense of infinity", demonstrating that this sense does exist, but it can be reached only in some specific cases" (Sbaragli, 2006; p.53).

Talking about infinity we can always intend it in two nuances: potential and actual infinity. One of the most important aspect of the Cantor's dissertation about infinity was indeed the "consecration" of the actual infinity, transformed after centuries of denies and uncertainity in a number in all respects thanks to the theory of transfinite numbers. Nevertheless the potential conception of infinity still pervade rightly the image and the language of contemporary Mathematics because of two main reasons: 1) the potential acception of infinity can't be assimilated to the actual one - think f.i. to the limits; 2) the actual infinity is hard to conceive and it's a cultural conquest, ever challenging, to grasp and accept this sort of "quantum jump" from a finite characterization of infinity, still near to our intuition, to a purely infinite one. Tirosh (2000), in a research work concerning the teachers' belief about infinity, observed that the teachers confused potential and actual conceptions of infinity, trying - more or less explicitly – to bring the actual back to potential infinity.

3. Points of a line

The relation between the line and the set of real numbers is one of the most critical issues in the field of the Foundations of Mathematics. Nevertheless to represent the real numbers in a synthetic, not formal and somehow intuitive manner many teachers could be tempted to use it, especially if they explore the other representations of the set of real numbers, much more complex from a semiotic point of view and apparently more difficult to grasp. As many researchers highlighted clearly unfortunately this shortcut is scattered of a lot of dangers.

Furthermore while there are many ways to show that a line can be divided with the imagination as long as we wish, there is no agreement about the fact that we can re-construct a line putting together this small resulting elements.

What may seem to be only a philosophical speculation resulted to be borne out by concrete facts in many researches involving both very young students and high school teachers with a lot of experience. It's so hard to perceive the line as a set of infinite and no-dimensional points that both in the examples proposed by the teachers since the primary school and in the students' representations of points the necessity to find strategies to make this representation more understandable emerge clearly. The results of didactical and personal representations of the line as a set points that the researchers reported were all quite far from the original meaning and from the set of real numbers.

Tall's (1980) found out that an intuitive model of the line was very widespread among the students: they had an image of the line as a "necklace of beads", thus every segment is considered "as a thread formed of beads-points put one next to the other. This model derives from the misconceptions of the mathematical point considered as an entity provided with dimension and from the wrong ideas related to the topology of the straight line" (Sbaragli, 2006) and for sure this is one of the most frequent reasons why the density of the line is not grasped by the students and the teachers as well (Sbaragli, 2006; Bagni; 2000). In particular Bagni (2000), standing on very important statements by Gagatsis et al. (2003) had already alerted from the false illusions concerning the introduction of properties of real numbers in the graphical domain; indeed the visualization of such a property is thoughtless if considered a priori after a detailed historical analysis, but have also been confirmed by the an analysis of teaching experiments carried out by Bagni (2000). In particular the author stress the more general aspect of visualization in the didactics of Calculus relying on Fischbein's considerations: the use of graphical representation is not only not so intuitive as it may seem to be, but it's also a potential source of unavoidable conflicts if it's not use with particular attention and carefulness. Sbaragli (2006) also quoted the research work of Gimenez (1990) identifying primary school students’ difficulty to conceive the density concept. Arrigo and D’Amore (1999, 2002) confirmed the existence of this model in two investigations concerning high school students.

4. **Density and continuity: the critical role of visualization**

A sentence we read in D’Amore e Fandiño Pinilla (2009) resumes in a very effective way the results of the research concerning the students difficulties with real numbers: «Le incredibili convinzioni che hanno anche studenti maturi (fine secondaria superiore, anche dopo corsi di analisi) su densità e continuità, proposti in aula come puri oggetti matematici da apprendere, senza alcuna attenzione epistemologica cautelativa, sono evidenziate da infiniti autori in ricerche didattiche su campo.» (p. 12).

In Gonzales-Martin (2014) is stressed that some teachers’ conceptions about the structure of the real line and the notion of density coincide with those present in their students (Dias, 2002); the students are oriented towards atomistic models and almost discrete conceptions (Dias, 2002; Robinet, 1986).

David Tall and Giorgio T. Bagni faced from different points of view the problem of understanding how far the visualization of the properties of real numbers could be helpful or, in the contrary, misleading. Of course this topic has many intersections with the students' conception of the numbers line, but with a more specific attention to the possibility of visualizing properties in the case of infinitesimal quantities.

Such properties are really very far from intuition, merely if the representation we use is the graphic one.

The author reported the results of two researches in which the surveyed the high school students' knowledge of these properties of real numbers.

Tall & Schwarzenberger (1978), talking about limits and real numbers, stressed that some conscious and unconscious conflicts between different conception of infinite, between processes and objects, may emerge interviewing also students with a good mathematical background; in the interviews they carried out with first year University students they observed many of these conflicts. This kind of conflicts between different meanings associated to real numbers, infinite, infinitesimals, continuity permeate many other interviews carried out by Tall for his researches, so much that the conflicts became a tool to analyze the students’ knowledge of the real numbers and other crucial notions in the Calculus.

In Tall & Vinner (1981) the topic was explored in depth and a theoretical approach to the analysis of students' and teachers' answers have been proposed. The authors define some terms useful to carry out the analysis: concept image, concept definition and cognitive conflict. The main issue faced by the authors is the relation between the formal and the intuitive dimensions of the mathematical objects involved in the Calculus. This aspect is crucial for our investigation since the teachers' orientations about this point may affect significantly their choices and a lack of awareness of the problem may lead to inconsistent choices (Tall & Vinner, 1981). The subject concerning real numbers in which the conflict between the intuitive and the formal dimensions emerges more clearly is with no doubt continuity. As we will show, f.i. many teachers identify completeness, a considerably formal property of the set of real numbers, and continuity, a very indefinite and many-sided property.

Tall & Vinner (1981), studied limits and continuous functions from the perspective of the relation between concept image and concept definition (Par. 2.1), regarding expecially the possibility to develop formal theories consistently with the informal images that form the whole concept images. Tall & Vinner, as examples, refer to limits and continuous functions; to the last ones they refer as “*bête noire* of analysis” (ibid., 1981; p. 13), since they reveal suddenly the presence of conflicts hard to overcome, that can seriously impede the students to access the formal dimension of the Calculus. This inevitably has a strong impact on the conceptualization of real numbers.

According to Tall (1991) students generally have very weak visualization skills in the calculus, that cause lack of meaning in the formalities of mathematical analysis. Even if the author highlight the relevance of visualization in mathematics, we will focus here only on the aspects that makes it problematic in the teaching-learning of real numbers, according to the aim of this Paragraph.

Tall (1991) highlighted that what is intuitive for the teachers should not be intuitive for the students, because it often becomes intuitive in the years of study also what is formal, by means of adequate practices. Trying to avoid to make intuitive the formal dimension the teacher can only make the situation worse. In words of Tall (1991, p. 2): "My analysis of the difficulty is that we will certainly not do this by making the concepts simpler. The alternative is to make them more complicated!"

If in a formal proof one try to use intuitive conceptions of continuity - for instance, a function whose graph has "no gaps"; its graph can be drawn "without taking the pencil off the paper" (Tall & Vinner, 1981) - she uses "ideas are not the intuitive beginnings of continuity but of "connectedness" which is mathematically linked, but technically quite different […] By introducing *suitably complicated* visualizations […], examples which work and examples which fail, it is possible for the students to gain the visual intuitions necessary to provide powerful formal insights. Thus intuition and rigour need not be at odds with each other. By providing a suitably powerful context, intuition naturally leads into the rigour of mathematical proof." (ibid., 1991; p. 19).

The researches carried out by Bagni were inspired by Tall, but Bagni focused his attention more on the high school students and on the reality of the teaching-learning processes in the classrooms. In his research published in 2000 he compared the students answers to the same test before and after the study of infinity and real numbers, concluding that a lack of specific attention to the didactics of infinity sets is a direct cause of the students' difficulties reveled by many studies. In particular "some ingenuous attempts to visualise dense sets and connected sets are inefficacious without a strong and careful introduction of the concepts of dense set and of connected set, and they can cause dangerous misconceptions […] it is impossible to rely their use on spontaneous interpretation of pictures and of images" (ibid., 2000; p. 23).

Indeed he observed anomalous interpretations of the properties of density and completeness in the visual representation:

1. dense and connected sets are confused;
2. the difference between dense sets and connected sets aren't correctly visualized.

5. Number line

In occasion of the *Ideas for the Research* project, funded by the Italian National Institute for the Educational Evaluation of Instruction (INVALSI), we had the opportunity to take part in a research concerning the students' difficulties dealing with the number line (Lemmo et al., 2015). The project aimed at analyse the outcomes of the Italian mathematics standardized tests collected by INVALSI from 2010 to 2013 in grades 6 and 8, thus the students were not high school students. Nevertheless the analysis of the literature review concerning the students' difficulties when they manipulate the line as a representation of sets of numbers are useful for our investigation. In particular, since to succeed in this task with fractions is considered highly predictive of future good results in mathematics (Jordan et al., 2013), Lemmo et al. (2015) analysed some grade 6 students' solutions of a task about rational numbers' different representations and the number line. We will resume here briefly the literature review analysis concerning the number line that we setted up in that occasion, in order to show that the relation between the students and the number line is anything but intuitive and innate. On the contrary many researches report students' difficulties until the end of the 8^{th} level, i.e. just one before the highs school. This is especially relevant for our research since many teachers are convinced of the opposite thesis and base on this orientation relevant choices concerning real numbers.

In Skoumpourdi (2010) the number line is presented as a didactical tool with high potential, especially since it provides a simple way to picture mathematical concepts: as a matter of fact, the number line is used for counting, for estimations and for representing time, but also for the representation of different number sets.

Moreover, number line can be used for providing geometric models of the arithmetical operations, for measuring, and comparing quantities. In the same work it is also pointed out that many studies report difficulties and limitations in the use of the number line and propose educational activities to overcome these difficulties.

> The potential of the number line for organizing thinking about numbers and operations on the one hand, and the difficulties that arise from its use on the other hand, lead many researchers to propose learning strands and teaching sequences for its use in the teaching/learning process of mathematics. (Skoumpourdi, 2010, p. 2)

Skoumpourdi stresses that the number line can be presented in different versions: structured or semi structured, with or without numbers and other symbols, but also empty. For each of these representations, students may use different approaches in finding solutions.

In this paper we will focus on the difficulties that students could have facing tasks in which they have to place rational numbers on the number line.

In the following, we briefly present some results from the educational research about this specific topic.

Concerning the placing of rational numbers on the number line, we initially have to distinguish the difficulties about the management of decimals and fractions. According to Iuculano and Butterworth (2011) both adults and children are more accurate when performing this task with decimals rather than fractions because "decimals afford direct mapping onto a mental number line and, therefore, allow for easier magnitude assessment than do fractions" (De Wolf et al., 2014, p.2136).

Students may also have difficulties in conversion between representations in different semiotic registers (Duval, 1993): they could have troubles in finding strategies to pinpoint numbers on the number line because the number line is a hybrid representation (a line with a scale on it). Every geometric operation can be translated into an arithmetic operation and carried out algorithmically and vice versa (Gagatsis, et al., 2003). Some studies on the number line and fractions highlight the distinction between making partitions and reading pre-marked partitions (Mitchell & Horne, 2008). In fact, the identification of the unit in number lines seems to be problematic; in particular students' strategies may change whether the line is partitioned or not, since marks may act as perceptual distractors (Lesh et al, 1982). Students may have difficulties if one unit of length is divided into parts (Behr & Bright 1984): for instance some students, in order to determine the fraction denominator, ignore the endpoint and count only internal hash marks. If intervals between points already drawn have unequal lengths, students can count the number of points ignoring the distance. Other difficulties may stem from an over-generalization of part-whole partitioning strategies in measurement contexts. For example, Saxe and colleagues (2007) show that a student can progressively divide one unit by 2 and then an half by 2 and find ¾, but this strategy does not work in general: e.g. 2/7 . Therefore many mistakes can be generated by the interlacement of misconception about rational numbers and number line management. Other studies (e.g. Hartnett & Gelman,1998) show the conflict between ordering natural numbers on the number line (when numbers get bigger as values increase) and fractions (when the denominator gets bigger the fraction is smaller). A common error is to put the fraction close to the value of numerator or of the denominator: this can be explained in terms of the "whole number bias", that means considering fractions as two separated whole numbers (Ni & Zhou, 2005) and comparing them separately (Stafylidou & Vosniadou, 2004).

Lemmo et al. (2015) reported the most of the known difficulties, but also observed something more.

They observed that many difficulties depended on the specific num1ber set that was considered (Iuculano & Butterworth, 2011; Saxe et al., 2007), but the representation of the numbers played a further role. Indeed some tasks involving particular categories of rational numbers resulted more hard to face in the decimal register, while other were more difficult if the numbers were represented as fractions (Lemmo et al., 2015).

Furthermore some students’ answers let see a kind of *implicit model* (Fischbein et al., 1985): hash marks are only “reserved” to integers.

1.3 Teachers' choices and students' learning: two *case - studies*

1.3.1 Discrete, dense and continuous sets after Dedekind's cuts and the Calculus: a teaching experiment results analysed by Bagni (2000)

Bagni (2000) carried out a research in order to analyse the importance of discrete, dense and continuous sets in the secondary school (16-19 years-old students). In particular in the first part the idea of connected set referred to the set of real numbers have been studied. Two classes of high school (Liceo Scientifico) have been tested with the same questionnaire and the results have been compared. Then the students' knowledge have been explored by means of interviews.
The research was carried out with:

a. a class of 26 students, 16-17 years-old (11-degree), with a previous knowledge of sets and cardinality, who had studied real numbers as Dedekind's cuts and had never studied infinite sets (definition, transfinite numbers).
b. a class of 22 students, 18-19 years-old (13-degree), with a previous knowledge of limits and continuous functions and who had never studied infinite sets (definition, transfinite numbers).

Comparing the test 1 and 2 (before and after the introduction of the main concept of the Calculus in the tradition curriculum of Liceo Scientifico) the authors aim at analysing the role of these concepts in the learning process of the notions of dense and continuous set.

We will resume the results that were presented by Bagni (2000):

Test 1
"All the students of the first group (11-degree) learnt in a satisfying manner the use of the set theory’s symbols: the first item was indeed answered correctly by all the students.
The density of the set Q was apprehended by the most of the students as the property of existence of an element of Q that lies of a couple of given elements, whatever the elements are chosen.
The infinite cardinality of Q (third item, 73 %) and R (fifth item, 85 %) were correctly learnt by the most of the students, even if it's necessary to analyse with attention the answers by means of interviews. Difficulties emerged concerning the concept of irrationality (fourth item): only the 58 % of students state that $\sqrt{26} \notin Q$. This will be explored through interviews.
The answers to the sixth item were even more uncertain, so interviews were carried out also to interpret them.

Interviews concerning test 1
The author invited the students to reflect about the answers and to justify them by means of interviews; in particular he focused on the sixth item.
7 on 11 students who answered that the cardinality of R is bigger than that of Q:
$N \subset Q \subset R \Rightarrow$ the cardinality of R is bigger than Q and N
"The cardinality is that of R: the other sets are included in their subsets [proper]" (Dara)
The students who answered that the bigger cardinality is that of R and Q justified their answers stated that two infinite sets have the same cardinality. For instance an interesting statement is the following:

"I_N only occupies the whole points and I_Q occupies all the fractions; I_R occupies all the spaces between the fractions.

But since I_Q and I_R are infinite and their cardinality is the same. With the rationals between 3 and 10 I can cover the whole segment because I can draw points as close as I want." (Marco,)

Analysis of interviews and conclusions (test 1)

For what concerns the misconceptions linked to the concepts of density and continuity and infinite sets, there is an interesting partition of the group in two parts:

- some students found their reasonings on the consideration : $N \subset Q \subset R \Rightarrow$ the cardinality of R is bigger than Q and N; i.e. reason in the same way with finite and infinite sets and don't consider the fact that an infinite set can be put in correspondence with one of its subsets (the definition itself);
- other students think that to compare two infinite quantity is impossible and that a segment of a real line contains the same number of rational and real numbers.

The first result was predictable since the notion of cardinality had not been introduced before and the aim of the research was to highlight the lack of a right introduction of infinite.

Marco's answer is very interesting since highlights the role of the graphic representations (Duval, 1993). Some students may thus be tempted to approach the concepts of density and continuity in terms of graphic representations, probably because of the habit of using graphic representations in the high school in Italy (Bagni, 2007). While the difference between a discrete and continuous set is visualizable (as the student did representing N), but it's not true for the difference between dense and continuous sets (the student indeed talked about the subsets of Q and R but was not able to represent them). The graphic approach was thus misleading.

Test 2 (13th degree)

Confirming the results of the previous test the students showed good knowledge about the symbology of set theory and the density of Q and the infinite cardinality of the sets Q and R. By means of interviews the answers have been then checked.

Some difficulty emerged concerning the irrationality (72%) and in the sixth item as in the previous case.

Interviews (test 2)

As the author observed in the previous interviews the most of the students who stated that the cardinality of R is bigger than the others (9 on 11) use the argument: $N \subset Q \subset R \Rightarrow$ the cardinality of R is bigger than Q and N.

The following justification is interesting: "For sure the cardinality of I_N is lower than the other since it's finite and the other infinite. But in the other case can an infinite be bigger than another?"

The students who answered that R and Q have the biggest cardinality state that two infinite have the same cardinality (5 students, 23 %) and the same happened with the students that stated that all the sets have the same cardinality.

The infinite sets are considered to have the same cardinality.

The issue of irrationality was faced in this ways by the students who answered correctly (16):

- $\sqrt{26} \notin Q$ since $\sqrt{26} = \sqrt{2} \cdot \sqrt{13} \wedge \sqrt{2} \notin Q \wedge \sqrt{26} \notin Q \Rightarrow \sqrt{26} \notin Q$ (8 on 16)

√26 is not rational but I can't explain the reason why. It's hard to imagine to calculate the power 2 of a decimal number with an infinity of digits. Maybe the digits ends at any point but I can know it.
(operational difficulties didn't end up in argumentations; for similar results see Romero i Chesa & Azcarte Gimenes, 1994; Fischbein, Jehiam & Cohen, 1995).

Interviews analysis and conclusions (test 2)
Apart from small quantitative differences the 13th degree students show analogies with the 11th degree students we analysed before. In particular the argument concerning the inclusion $N \subset Q \subset R$ is significant and no differences have been detected between the arguments concerning finite and infinite sets.

The case of the student who justifies the irrationality of $\sqrt{26}$ with the irrationality of $\sqrt{2}$ and $\sqrt{13}$ is interesting since the students could present the argumentation concerning the irrationality of $\sqrt{13}$ for $\sqrt{26}$ itself. The reference to $\sqrt{2}$ may be interpreted as a matter of assurance linked to the didactical contract (Brousseau, 1987)

In the high school the presentation of the continuity of the number line appeared to be weak and incomplete. The introduction had been realized in the traditional way it's presented in the Scientific high school in Italy. In particular the difference between density and continuity didn't seem to be understood by the students.
In particular after the presentation of the Dedekind's cuts in the 10th degree the students of the both of the classroom never referred to them to evaluate the irrationality of the numbers and the sets' cardinality. Also even after studying the Calculus the situation was not so much better.
Some students tried to use graphic representations to distinguish dense and continuous sets but this attempt of course failed: as Duval highlighted "learning by means of graphic representation makes necessary a particular work and it's not possible to use spontaneous interpretations of figures and images" (Duval, 1994, translated by us).
Indeed the "double nature" of mathematical objects, being both abstract and real (figural concepts, Fischbein, 1993) didn't seem to be very useful in this case since to advantage of this double nature with continuous sets is hard requires to make choices carefully and opens the way to misleading ambiguities (Bagni, 1997); the coordination of different registers of representation in general is useful (Duval, 1993; Kaldrimidou, 1987; Vinner, 1992) but in the case should be analysed in depth the way to carry it out.
A specific teaching sequence concerning infinite sets and transfinite numbers is thus necessary to carry out a theoretical approach to the study of real numbers.

1.3.2 A teaching experiment concerning real numbers as Dedekind's cuts: emergent relations between students' difficulties and teachers' choices

In the Master thesis discussed in 2011(Branchetti, 2011) a teaching experiment concerning real numbers in the high school was described and partially analysed within an heterogeneous framework composed merely by the Theory of obstacles (Brousseau, 1986) and the first conception of Didactical transposition (Chevallard, 1985). The aim of the teaching experiment was trying to implement a didactical proposal for introducing real numbers in the high school based on some lessons presented in an academic course of Foundations of Geometry (Coen, 2010). The core element of the lessons concerning real numbers was the analogy (and the possibility of defining an isomorphism) between real numbers as Dedekind's cuts and "rational segments" introduced by Euclid in the
, based on Eudosso's theory of ratios of homogeneous magnitudes. This analogy had been presented as a possible didactical resource since the Euclid's construction was somehow more intuitive and elementary. We decided to investigate wether and how a didactical sequence in which the Euclid's construction was introduced before the Dedekind's cuts could have been useful in order to introduce Dedekind's cuts as a

modern version of the same theory. The modern theory would have been presented as just more formal and expressed through more effective mathematical tools that simplify the rigorous but complicated Eudosso's definition (functions, minimum and maximum, algebraic operations, set theory).
The complexity of real numbers from an historical point of view at that time had not been considered very much since it was quite unknown at all to the teacher and to the researcher to. The only aspects of the complexity of real numbers taken in account had been:

1. the lack of equivalence between different theories of real numbers and, in particular, the exemplar case of the asymmetry between the postulate fixed by Cantor and Dedekind for the continuity of the line;
2. the analogies and the differences between the definition of ratios of homogeneous magnitudes proposed by Eudosso, reported in Euclid, *Book V* and the Dedekind's field of real numbers as "rational cuts";
3. the relation between space and time (supposed continuous and corresponding through a bijection to real numbers) and numbers as presented in many textbooks' problems is mediated by the practice of approximation, that changes the properties of the set of numbers really used and the manipulation rules.

The teaching experiment has been planned in order to:

1. respect some constraints decided after a literature review concerning students' difficulties in the learning process;
2. pursue some intermediate disciplinary objectives necessary to reach the construction of real numbers as Dedekind's cuts, spending also time in verifying the students' previous knowledge concerning the objects involved in the teaching experiment;
3. use the history of mathematics in order to motivate students on one hand, and to present the cultural problem that characterize continuous and discrete sets since the very origins of Western mathematics;
4. avoid to use the concept of limit and infinitesimal and the language of the Calculus, since the teaching experiment was addressed to students that hadn't studied them yet.

The choice of Dedekind's cuts as a way to construct the field of real numbers was due to some features of this construction that permit to address the most of the previous objectives:

1. discussing with the teacher before the teaching experiment, the objects involved in the construction should have been known quite well by the students;
2. the time the teacher considered available for the teaching experiment was 12 hours, including the final questionnaire and the discussion. To respect this constraint the Dedekind's construction was considered good because it was planned to be implemented in 4 hours and this feature allowed to re-analyse with the students in classroom discussions the previous knowledge (rational numbers, representation of numbers on the line, approximation, ...);
3. from an historical point of view the Dedekind's cuts are inspired by Eudosso's theory of ratios
4. the two construction have a similar structure (Coen,);
5. in the whole process of construction of the field of real numbers as Dedekind's cuts the concept of limit or infinitesimal are never involved explicitly.

In order to make students as much as possible aware of the epistemological status of the construction of a set of real numbers and to avoid to make them confused about the necessity of constructing the set of real

numbers, the teaching experiment was planned to start with tasks involving computers and discrete sets of finite numbers, showing what it was possible to do with finite numbers. Then the line was used to represent the "set of all the numbers" and the set of finite numbers that a computer can use (fixed the limitations for the mantissa and the exponents of their representations) have been represented on the number line, showing its distribution on the line. This decision was inspired by the book
(Villani, 2000) in which different sets of numbers that may be involved in high schools' didactical practices, included finite, rational and real numbers, were presented. The introduction of the set of finite numbers was planned in order to highlight some of its properties starting from its decimal and graphic representation:

1. the set of finite numbers doesn't complete the whole line;
2. there is a finite quantity of finite numbers;
3. the distribution on the line is not homogeneous but there is a different density near the number 0 and near the upper and lower endpoints;
4. the set is not closed for any operation, i.e. for any arithmetic operation between finite numbers there is always at least couple of numbers whose addition or multiplication (and inverse operations, when possible) is not a representable finite number;
5. there is an effect of numerical elimination of terms of the mantissa that transform a result in a truncated or approximated number.

Introducing these properties in relation to the graphic representations we aimed at creating the opportunity to involve the classroom in a discourse concerning the properties of this set, in order to compare it with other number sets. In particular we aimed at driving the students to reflect about the differences and the relation between concrete problems concerning numbers and their approximations and theoretical problems concerning the definitions, the local and the global structure and the properties that can characterize a set of numbers as an algebraic and topological structure with operations.

Secondly we decided to plan a phase in which the students could recall to their memory the previous knowledge concerning rational numbers in different representations (verbal definitions, fractions, graphic representation onto the number line, ...). Then the already known elements of the set would have been characterized by properties and organized in a structure in relation to the set of the finite numbers.

In a further phase some problems concerning the limitations of rational numbers when trying to describe physical phenomena and geometrical constructions with numbers would have been presented in order to justify the introduction of new numbers that could complete the line. First of all we planned to present the problem of the incommensurability of $\sqrt{2}$ in a modern form, more familiar to the students, using the fractions to represent ratios and the decomposition in prime factors of natural numbers to prove the irrationality of $\sqrt{2}$ by means of the Theorem of Arithmetic.

Then we planned to propose two other problems to solve in group works: the problem of the bottle and the definition of Cantorian set (in the Appendices). These two problems had been chosen because they permit to stress the difference between an object or a process supposed to be continuous and the numerical description of it.
In the problem of the bottle the instant of time
in which the bottle is empty isn't finite nor rational. The instant exists because it's identified by a process - the bottle's quantity of water decrease - but the rational numbers are not sufficient to capture that instant of time.
The Cantorian set is possible to define through a process of division but its description requires more than the rational numbers.

Once introduced three problems to show cases that rational numbers don't permit to describe, we planned to propose a definition of all the possible numbers, rational and not, as all the points of the line and to propose also a formal definition. In order to define formally the real numbers we opted for a sequencial and gradual introduction of collateral definitions, images, procedures, starting from Euclid and reaching in the end the Dedekind's cuts. In the end we planned to reframe the rational numbers as Dedekind's cuts and to promote the use of the formal definition and algebraic considerations rather than graphical procedures in order to establish if a couple of rational sections was a Dedekind's cut or not and to identify eventually the cut with a rational or irrational number.
In this passage we didn't stress (since we were not completely aware of it) that the new numbers are created and that the correspondence between points, obtained by means of rational cuts, and real numbers is not provable but it's rather postulated. Also we used the line as an image both for representing rational and real numbers, distinguishing the two only as figural concepts, whose properties had been introduced in practices involving other representations of the line.
In the following Paragraph we report the designs' description that we wrote before the experience.

1. Computer
A support to face this issue is the computer, that students use in the everyday life and at school. While usually the computer is used in order to motivate students or to take advantage of its potentialities during the lessons, in this case the computer is the very object of the analysis. Its way of representing and using number is so explored showing its rules and its limitations in comparison with the pure mathematics.
Numbers in the Greek ancient mathematics
One of the objectives of this sequence is to make students' models of numbers emerge, keeping in our mind that the images correlated to this models origin in the first years of school. As we showed in the literature review the first use of numbers is related to quantification and is deeply linked to concrete objects (fingers, little stones, ...). The conception of number as tool for indicate and compare quantities may be used in order to associate numbers and segments by means of the conception of a segment as an aggregation of points, or better, atomic elements. This is precisely the Pythagoreans idea of aggregating a given quantity of monads in order to form a segment. We can associate the measure of a segment and the practice of computation of elements in order to introduce the crisis of this model, that may be an intuitive model also formed by the didactical practice in the primary school (Sbaragli, 2006), and to present the concept of commensurable magnitudes and the problem of incommensurability from an historical point of view.

3. The set of finite numbers
The set of finite numbers is not one of the sets of numbers usually defined at school. Usually teachers talks about natural, whole, rational, real numbers, even if they very often use numbers that arise from operations with computers or approximations. The set of finite numbers that we can generate with a computer has not a unique definition because of the arbitrary number of elements of its decimal representation, depending time after time on the particular computer. Also if we don't take in account the limitation the set of finite numbers with any possible lengths we can't define on it an algebraic structure and it's poor of properties. For instance it's not closed for operations and it has more than one neutral element for addition (Villani,). Nevertheless its use in the didactical practice is so widespread that these numbers are very often used by the students and they are used indeed to represent and manipulate the most of the numbers by means of approximation of every rational or irrational number.
There are important consequences of this fact on students' conception of numbers:

1. in this set, once fixed the maximum length, it's possible to identify the consecutive number of every element;

2. all the numbers, rational or irrational, are reduced to finite numbers with the same methods of approximation, implying implicitly the lack of importance of the so stressed difference between rational or irrational.

We could use an a-didactical situation including computers in order to make emerge the students' conceptions of numbers, with high probability linked in the definitions to rational or irrational numbers but in the practice to finite numbers.
A possible way to make the limitations of the set of finite numbers emerge is to modify a non-vectorial image coded, for instance, in jpg. The image reconstruction after a modification is often different from the initial one and this is because This way the students may be motivated by graphical operations and, by means of suitable transformations of graphical operations into arithmetical ones, students may see "in action" the limitations of a set of numbers that doesn't own the property of continuity, but instead is:

1. discrete
2. not equally distributed.

The set of finite numbers may be introduced through attempts to define it and to construct its elements. Since the computers' codes are binary, we could present first numbers in the binary representation and then make explicit the similar structure of decimal numbers. The scientific notation adapted to computers may be introduced in order to define a particular set of finite numbers, i.e. that of the computer the students are using, that can be determined once the computer's characteristics are known. After writing the definitions, the numbers are placed on a line in order to show their distribution.

4. Decimal numbers
The set of finite numbers is a subset of the set of decimal numbers, that should already have been introduced following the planning we are presenting. Now, in order to permit the students to recall their conceptions about decimal numbers and so make them aware of some properties of the numbers they are used to manipulate, the set of decimal numbers may be explored better and compared with the already known sets: natural, integers, rational numbers. Also, since the students should already know some irrational numbers, the properties and definitions of this decimal numbers are explored in order to distinguish them from the rational ones. Now the challenge is the creation of a bridge between the set of all the decimal numbers and all the points of a line, that will be created after an exploration of the representation of rational numbers on the number line.

5. Rational numbers as ratios
The practice of placing rational numbers onto the line is linked to the conception of rational numbers as ratios between two segments, one of whom is the unit. This is the original definition of ratios, proposed by Eudosso when numbers where still considered segments, and in modern terms, it deals to the representation of rational numbers as fractions. The adjective 'rational' comes itself from 'ratio'. Thus we could introduce the Eudosso's definition of ratios of homogeneous magnitudes for segments in order to connect rational numbers and segments of the line.
As we have already pointed out, the relation between points and line (i.e., at an abstract level, between an object and its elementary parts) expressed through graphical representations of the both of them, may lead students with high probability to the "chain model" (Sbaragli, 2006).
Starting from the Pythagoreans conception of segment as result of an aggregation of monads, we could propose an analysis of what is the meaning of a number that represent a "rational segment". How can the Pythagoreans conception of number/segment generate a wider conception as that of rational number? Why

do we indicate points with rational numbers? The strategies of placement of rational numbers onto the line could be put in correspondence with the Eudosso's definition.

This approach may allow the teacher to rethink with the students in a classroom discussion a critical point concerning the structure of the line: are the points small pearls? How far can we divide them? How many "pearls-points" there should be in the unit? Reflecting on the possible fractions we can identify in the interval [0,1] it will emerge a property of rational numbers that also is useful to change the image of the line as an aggregate of small pearls: there is an infinity of numbers in the interval [0,1] and also in the segment we used as a unit. In other words: the line is dense as long as the set of rational numbers. The property of the set of all the possible ratios included in the interval [0,1] may this way be assumed by the line itself. This could be a useful step to introduce the density of real numbers using the line as a model of them, since in the graphic register it's hard to represent the property of density (Bagni, 2000)

6. Different representations of rational numbers

The representations of rational numbers as decimals has a particular feature: there is not a bijective mapping between representations and numbers. In fact there are special cases in which two different decimal representations identify indeed the same number. This is the case of any decimal with 9 as period and the decimal whose antiperiod has the last digit equal to the consecutive of the digit of the first at the same position and the other digits equal to 0, like f.i. 0.(9) and 1.(0), 1,2(9) and 1.3(0). To show this anomaly is interesting in order to highlight the difference between a number and its representation. Also this could be used in order to discuss the possibility of introducing or not a limitation, like avoiding the representations which 9 as a period, to construct the map. This meta-discourses may help to shift the discussion from practical to theoretical issues.

7. Irrationality of √2

Once analyzed in depth the set Q, it's possible to go forward and construct R starting from Q. This process is coherent with a famous sentence pronounced by Dedekind himself:

Here comes the issue of the previous knowledge and the way the constrution can be realized. Being R an algebraic structure we have to operate an enlargement taking in account the comparison between the two structure from the algebraic point of view.
Students are expected to know:

1. the standard order of Q;
2. some irrational numbers

Since the students already know some irrational numbers, both algebraic and not, like √2, e and π, the enlargement can't ignore them. A problem that may justify the enlargement coherently with the whole teaching experiment is the prove of the existence of numbers that are not rational using a geometrical example. The classical construction of sqrt 2 as diagonal of a square with edge's lenght equal to a unit may be a good example in this sense since the number is represented by a segment. The segment can be projected onto the line by means of a compass showing graphically that the segment does belong to the line but proving arithmetically, at the same time, that there are no rational numbers that can describe it. So rational numbers are not enough to describe all the segments that we can individuate onto a line. This example is still framed in a geometrical construction in which numbers and lenghts of segment are identified but we have to stress we are providing an example that only may be used for positive numbers. Once proved the existence of one segment that can't be described in terms of rational numbers we may go on looking for other irrational numbers as points of a line. Other example can be presented that the students already know and then formulate the crucial questions concerning irrational numbers and segments: how can I describe all the "irrational points"? To answer this question we will introduce the Dedekind's cuts.

8. Dedekind's cuts

In order to present the set of rational and irrational numbers and to describe the line in terms of properties, like we did before with the density of rational numbers, we need to define a complete set and to do this also we need the notion of extreme, minimum and maximum of a set.
An example of not complete set is Q and the previous introduction of √2 may be used to explain that Q is not complete using the representation of the segment √2 ad a set of points that are indicated with numbers whose square is less than 2. This transformation may be proposed both in the algebraic and graphic register coordinating the two semiotic registers. After this example other examples, algebraic and trascendental, may be presented, like √5, ln 2, e, and so on. Since we need a procedure to enlarge Q to create R numbers we use to define the segments are rational. The endpoint of the segment may be rational or not. We can define the point using the Dedekind's rational cuts , i.e. two subsets of Q whose intersection is empty, whose union is Q and with the lower set without the maximum. To do this we can work in the graphical register. After defining the Dedekind's rational cuts we can define all the numbers, rational or irrational, through a couple of sections.
The implementation of the planned teaching experiment have been realized in a high school classroom of twenty-five 17-18 years-old students (Liceo scientifico, grade 12). The planning has been discussed with the teacher. The students had not studied the Calculus yet and they had never studied limits and the properties of infinite sets. The teacher had declared that in his in mind the most important practices in which students should use a not elementary knowledge about real numbers is the extension of the exponential function from Q to R, since the lack of a procedure to go closer and closer to a point with rational numbers it's impossible to make sense of expressions like e^√2. The teacher is expert since he have been teaching for 20 years more or less and has got a Master degree in Physics. The teacher accepted our request to experiment our sequence in one of his classroom for 6 lessons, including the assessment and a final discussion. The teacher has always been present during the teaching experiment but it has been conducted by us.
The first lesson took place in the Informatics Lab. Students didn't know the content of the teaching experiment. The first assigned task was to open a folder containing images and to modify the image using a sofware, with particular attention to make it bigger or smaller and to observe how it changed. Their observation have been collected and shared with the classmates. The categories of answers were:

1. the images are blurry;
2. if we try to change the colour filling a part of the image, we only colour one pixel;
3. if we enlarge and then we try to return the image to its original status we fail since the quality of the image is worse than it was in the beginning
4. every is similar to the whole.

The last observation was referred to the particular image the student were modifying, since it contained a fractal. Even if the observation didn't concern the activity as it was planned it have been written down on the blackboard since we decided not to filter the observations since the very beginning.

Then the students were asked to formulate hypothesis about the way a computer colour an image in order to go a step towards the numerical intepretation of the manual procedures of enlargement and restriction and to start a discourse concerning the quantity of numbers a computer can use to carry out its tasks.

The proposals have been again collected onto the blackboard. The proposals have been synthesized in a unique proposal after a brief discussion, reaching in a few minutes an agreement in the classroom about the method but not about the quantity of colours a computer can use.

The method they decided to propose was the costruction of a scale from black to white, deciding a quantity of colours to use and assigning 0 to black and the higher number to white.
The quantity of numbers to use divided the classroom in groups:

- 30, as in the interface of the software Paint for Windows.
- 50, since it can be enough
- infinite
- a finite number since in the reality they are infinite but in the computer it's impossible
- a lot.

As we were wishing before starting the lesson, the keywords came from the students in a classroom discussion. After that we decided to institutionalize the quantity of number as finite.

In the first step we aroused curiosity about the way a computer colour, construct and reconstruct an image. In the second phase we presented the RGB (red, green, blue) methods used by many softwares: to every pixel a colour is assigned through a number, but the number is not a casual number on a linear scale but a 3D vector with a number assigned to any component. Students had studied and used vectors in the Physics lessons so we could stand on their previous knowledge about vectors. The third phase of the first lesson was the transformation of the vector into a number using the sum of powers of 16 with the three components as coefficients.
In the last phase of the first lesson we came up to highlight that the computer modify the image by means of numerical operations: the color of every pixel must be computed. Is this the reason why we can't use with the "inverse operation" of enlargement of reduction coming back to the starting point?
To fix this point ax example concerning the reduction by 1/4 was presented, showing the difference between the "pure operation", the expected invertibility of the procedure and the observed image that contradict the invertibility. This issue has been used for introducing the set of finite numbers that a computer can represent, using the notions of mantissa and exponent and the usual limitations imposed to the quantity of digits (Basic single - 32 bit, 1 for the sign, 8 for the exponent, 23 for the mantissa, Basic Double - 64 bit, 1 for the sign, 11 for the exponent, 52 for the mantissa)

The lessons end with a question: how are the finite numbers distributed onto the number line? Which is the relation between them and the numbers the students' already know and use in the didactical practices? How many numbers we can represent with a computer? Are there holes in the line if we only use finite numbers?
The students' answers to the questions posed in the end of the first lesson have been collected in the beginning of the second one. These are the proposed answers:

- All the numbers can be represented by means of a computer
- Numbers with too many digits can't be represented

Some further questions were posed in order to make the classroom discuss about the two different positions. How many numbers can we represent with these computers? Does any number that we can't represent exist? A group of students convinced all the others that the second thesis was correct, proposing argumentations like this: the numbers with a quantity of significant digits bigger than the quantity of bits available for the mantissa can't be represented. An interesting fact is that the examples presented were all about irrational numbers, to whom all the students associated the infinity of digits, but no examples concerning rational numbers were analysed. In particular they analyzed the case of $\sqrt{3}$. Also no students faced the problem of underflow or overflow for the exponent but only for the mantissa. Other examples were proposed by the teacher and by us and in the end we came to a definition of finite number, before referring to computable finite numbers and then a more general one:

A number n is a finite number if exists a number k so that a k-truncation of n is equal to n. For the distribution of computable finite numbers an image was proposed by us, showing how many numbers we can't represent.

The question concerning the cardinality of the general set of finite numbers was answered by the students without institutionalization. A temporary agreement was found in the classroom, fixing that the numbers were infinite but less than all the possible numbers. The discussion have been interrupted since we didn't feel it was the right moment to deepen this aspect. Instead we decided to focus their attention on the distinction between rational and irrational numbers starting from a student's previous sentence.

"One of you said that we can't represent $\sqrt{3}$ with a computer since it has an infinite quantity of digits, like all the irrational numbers. What about rational numbers? Do all of them have a finite quantity of digits?". Once again there was a lack of agreement in the classroom and this made emerge a new debate: "Considering all the finite numbers do we take in account all the rational numbers?". The teacher proposed a reformulation of the problem: "All the finite numbers are rational. Is the reverse statement true?". All the students that were discussing answered "Yes". A not so convinced student posed the issue of the definition of rational numbers and the students tried to formulate definitions comparing them in a classroom discussion. The main question was "What is a rational number? How can we define it?". A student while discussing posed a crucial question: "Are the period numbers rational or irrational?", driving the group to focus the very problem. The two possibilities have been taken in account seriously by the classroom and two groups aroused that weren't finding any agreement. Reasoning in the decimal register the students' were discussing and proposing arbitrary argumentations based on the memory of previous teaching-learning happenings. The choice of the decimal register was a natural implication of the previous activities. To restart the discussion we proposed a question. "Is 5 a rational number?". All the students answered "Yes.". When asking for explanations we collected some argumentations and a student proposed to answer using the algorithm of transformation af a fraction in a decimal number and the inverse one in order to decide if a decimal number was rational or not. By means of examples proposed by the teacher the group lead to the statement that not only the finite numbers, but also the periodic ones are rational. This statement was institutionalized as a definition of rational number.

N is a rational number if N is a finite decimal number or a periodic decimal number.

The ambiguity of the representation of decimal periodic numbers with 9 as period is presented in the case of 0.(9) and 1. We proposed the following argumentation:

1/3 = 0,(3)

So:

0.(3) · 3 = 0.(9)

$1/3 \cdot 3 = 1$

Thus:

$0.(9) = 1$

One student tried to confute the argumentation using his calculator and we took advantage of this happening to recall the main concept of the previous lesson: a calculator, like every computer, substitute the most of the numbers with its approximations, in particular the number with an infinity of digits. In this moment we observed that the most of the students were reminding successfully the previous lesson. The issue of periodic numbers with period 9 was proposed as a matter of choice, giving to the students the freedom of deciding if:

1. exclude the periodic numbers by means of an explicit deny in the definition;
2. accept that some rational numbers may have two different decimal representations.

This open question obliged the students to reconsider the definition of rational numbers.

N is a rational number if N is a finite decimal number or a periodic decimal number with period different from 9.

N is a rational number if N is a finite decimal number or a periodic decimal number. If a number has 9 as period it's equal to a number that has the same digits of the ante-period but the last, that's incremented by 1.

In the end we proposed a further definition:

N is a rational number if we can express it as ratio between two whole numbers, i.e. if exist in Z so that q=m/n.

The second lesson ended with a question for the next day: "Does any not rational numbers exist"
The third lesson was planned to be frontal and dedicated to the History of Mathematics. The theme was the conception of numbers in the Ancient Greece, from the Pythagoreans to Archimedes and Euclid, with a special attention to the
The lesson was supported by a Power Point presentation. The main contents have been the concept of numbers for the Pythagoreans, the notion of commensurability between homogeneous magnitudes by Eudoxus, the issue of irrationality of $\sqrt{2}$ as it was presented by Plato in the
and the solution proposed by Euclid in the Elements
The conceptions of numbers emerging from this historical path were: number as quantity (natural number), number as ratio between homogeneous magnitudes (rational numbers). Furthermore the Hellenics already were aware of the necessity of introducing a new kind of numbers. Using the definition of rational number formulated during the previous lesson the irrationality of $\sqrt{2}$ have been proved by means of an arithmetic prove, i.e. the decomposition in prime numbers of 2 and the impossibility of finding a rational number whose square was equal to 2. So an irrational number was defined as a number that couldn't be expressed as a ratio of whole numbers.

The fourth lesson was the last in terms of introduction of new concepts. The 4 hours left had already been planned to be dedicated to assessment and final discussion. The aim of the last lesson was the introduction of real numbers as Dedekind's cuts in order to finish the teaching experiment with a construction of a field of real numbers. The prevailing lesson methodology was frontal as the previous one but the lesson started with a group problem solving. The problems concerned irrational numbers and the numerical solution of problems contextualized once in the geometry and once in the real-life. The problems were the two we presented before in the planning description: the problem of the bottle and the description of a generic element of the Cantorian set. The class was split in two groups and to one group only one problem was assigned.
Both the problems concerns the logarithmic and exponential functions since the teacher had to work on them the next week. The problems were proposed in order to pose the problem of the relation between real numbers and continuity. In fact the solution of the problem of the bottle is irrational but we measure time using a discrete system (seconds, minutes, hours, ...). The numerical solution represent an instant in temporal continuity, while we usually represent the instants by means of a discrete set. A student had an interesting reaction: once understood the problem she asked "Does that instant exist anyway?". We answered that the instant exists even if our usual measures don't permit to represent it. After this problem solving session we introduced some preparatory concepts, like maximum, minimum and endpoint, and the definition of rational section in the sense of Dedekind. We expressed the definition of Dedekind's sections in the language of set theory but also representing them as half lines in the graphical register. Some slides are reported. Also we used the definition of maximum to justify the third part of the definition, showing examples of couples of subsets of Q without maximum in the first section. In particular the example of $\sqrt{2}$ defined as a couple (A,B), A,B, subsects of Q, so that: $A=\{x \text{ in } Q \mid x < 0 \vee x^2 <2\}$, $B= \{x \text{ in } Q \mid x > 0 \ \& \ x^2 >2\}$. Also we presented a transcendental example defining ln 3. In the end we showed how the same procedure may be used for rational numbers. We defined R as the set of all the possible sections of Q, precising that if the number is rational we say it a separator element for (A,B), if it's not rational we create the irrational number. It was stressed the analogy between this definition and that presented by Euclid in
, highlighting differences and analogies.
The fifth day of the teaching experiment was used for a formative assessment in written form (whose complete version is reported in the Appendices). In the test the notion of consecutive number, never explicitly introduced before, have been briefly introduced and then used in order to explore the students' knowledge avoiding to use the same language we used during the lessons. This choice aimed at investigating their deep understanding of the concept of density.
The sixth and last meeting with the classroom have been divided in two phases. In the first part the students have been asked to read a resume of the last lesson concerning real numbers and to answer two questions, similar to one of the most critical questions of the test (Exercise 6). In the last part a discussion concerning the test results have been carried out in order to close the teaching experiment with a final collective reflection. The 6th question had been re-proposed to the students since the answers were significantly worse than the others and, merely, they were difficult to interpret.
A mass phenomenon had emerged from the tests analysis and we tried to better investigate it by means of a further brief test similar to the first but with more precise questions. Then we guided a discussion concerning all the questions so to make order in the different conception emerged from the answers and to institutionalize the concepts that any question was planned to investigate.

- all periodics decimals (1)
- all irrationals (1)
- all but irrationals (1)
- all whole numbers (1)
- all natural numbers (1)
- all but natural numbers (1)

- naturals and rationals (1)
- irrational and rational, ut not whole numbers (2)
- all the kind of numbers (2)
- natural and irrational numbers (2)
- all kind of numbers but all positive (3)

The first question concerns the enlargement of the set Q. We wished to verify if the students had correctly conceptualized the enlargement or the considered real numbers as complementary set of rationals. The answers showed many different students' profiles but none but one listed only irrational numbers.

- Every animal is a figure, so quantity of figures = quantity of animals (1)
- infinite, but in a finite square the infinite can only be imagined (1)
- A very big quantity, but not infinite, since in the reality the infinite doesn't exist because of spatial limitations (1)
- Infinite, since I can divide infinite times the figures (9)
- Theoretically infinite, but finite in the practice (4)
- Potentially infinite, but finite in the practice (1)
- A very big quantity, but not infinite, since the infinite isn't realizable (1)
- I can't determine it since they are so much and so small (1)
- Infinite (2)
- Theoretically infinite (1)
- Infinite, but there is an optical effect (1)
- Infinite, because they are so small that to count them is impossible (1)
- They have the order 10^2, since there should be about 50 figures in every circle (1).

18 students answered that the figures are infinite (72%), 4 answered that they couldn't be infinite (16%), 1 answered that we can't determine it, 1 provided a finite estimation and 1 didn't answer (the last three are together 12 %).

- it's not periodic but it's unlimited and not periodic (1)
- we can express it as a ratio between whole numbers (3)
- it has a finite quantity of decimal digits or we know all its decimal digits (1)
- we can express it completely by means of digits, I don't need other symbols (1)
- a finite quantity of digits (5)
- it's possible to transform its decimal representation into a fraction representation (1)
- we can represent it as a fraction (5)
- it's represented in base 10 and it has a finite quantity of digits or it's periodic (1)
- we can represent it as a ratio between rational numbers, in particular whole numbers (2)
- decimal finite and not periodic
- we can express it as a ratio between natural numbers (1)
- no answer (2)

The mathematically correct answers are 15 (60%), one of which is referred only to naturals (Eudosso's conception of rational number); 8 answers identify rational numbers with decimals with a finite quantity of digits; 2 students didn't answer.

In 8 definitions the correct conception of consecutive number has been defined (32%), even if the language was still natural. Some definitions were good for N and Z (48%) even if there was a lack of generality. 4 students (16%) identified consecutive with bigger. 1 students wrote down a tautology "consecutive the consecutive". All the students answered correctly the 2nd and the 3rd questions, but only 10 students (40%) answered that it's impossible to find the consecutive number of a rational number because of the density. Other 4 students didn't write down the consecutive number but didn't comment; maybe they understood or not, so the percentage of correct answers is between 40% and 56%. 4 (16%) students showed a partial understanding: the first stated that the consecutive exists but we couldn't write it because of the density; the second stated that there are infinite numbers between -1 and 0 but then wrote down the consecutive numbers adding one to the last digit to the decimal representation; the third stated that there are consecutives in the et of rational numbers because of the density, but answered that there was always the consecutive in the set of real numbers and it was the minimum of the set of the numbers bigger than a given number; the last stated that the consecutive element didn't exist for algebraic irrational numbers expressed in form of radicals and rational numbers expressed in form of fraction but wrote down the consecutive number of rational numbers expressed in form of finite decimals. Furthermore there were 3 (12%) answers completely wrong and 2 (8%) missing answers.

All the students said that the first couple of subsets of Q was a Dedekind's cut. 24 students (96%) answered correctly that the number was rational, but 4 of them didn't identify the correct number. In the second case just 3 student answered correctly that the couple was a Dedekind's cut. There have been other interesting answers, even if not correct, in which the errors didn't concern the new concepts. The most of the students made mistakes solving the inequalities that were defining the subsets; many students also didn't interpret correctly the connectives. The representation we chose for the couple of sets revealed to be very unsuitable for the students. In particular the use of polynomial 2-degree inequalities affected in a very negative way their strategies since they recognized the representation as a 2-degree inequality they were used to solve and used the usual procedures, ignoring that the numbers were rational. The students who transformed the inequality in the graphical register in particular came to contrasting results and in the end they got lost in the procedures because of a loss of sense. At a closer view this undesired happening let us reason about a crucial point: students are used to solve inequalities without taking in account the domain and use the number line in ways that veil the properties of R as a complete and the structure of the set Q as it's represented onto the line. Also an important remark concerns the algebraic numbers. The students are not used to distinguish between rational numbers and rational numbers representation onto the number line since they act in the same way when they have to put the two kind of numbers on to number line, i.e. transforming any other possible representation into the decimal one and then approximate by truncation, at least arroundment. This may cause serious problems when trying to teach them real numbers since properties of the sets of numbers N, Z, Q , R vanish using the line, realm of finite numbers instead of being a privilidged representation of the set R. The link between the line and the set R is its continuity, properties that we can only partially associate the completeness of the numerical set R but rather belongs to intuitive, space-time representation of motion and trajectory.
The answers to the fifth item lead to an interesting phenomenon: all the students but one didn't recognize in the couple:

$C = \{x \in Q \mid x < 0 \lor x^2 < 5\}$, $D=\{x \in Q \mid x > 0 \lor x^2 > 5\}$

the irrational number √5. This was the only item to cause so many problems to the students and the quantity of right answers isn't comparable at all with all the others. In particular it has exactly an opposite profile of answers in respect of 5.a, that was formulated in the same way but changing the inequalities that represented the set and the nature of the number, once rational, once irrational.
There are significant differences between the strategies and the argumentations that lead to wrong answers, but a common point is the attempt to solve the inequality instead of interpreting the expression ' $x \in \mathbb{Q} : x^2 < 5$ ' as ' all the rational numbers whose square is less than 5 '. The strategies used by the students to solve the inequality in the domain Q showed habits in her previous didactical practices that we hypothesize to be in a causal relation with their errors.

First of all the students used the same procedure without considering the domain: if the domain had been Q or R or another they should have used the same procedure. This emerge clearly from the answers.

Also the loss of connection between the teaching experiment and the strategies is an interesting phenomenon to investigate. In fact we just modified the way to present the question and the students' behaviors changed in the respect of what we observed during the classroom discussion. In particular the changes we made affected the semiotic dimension of the presentation.
The slides we used to introduce the Dedekind's cuts always presented at the same time two representations of subsets of Q, one in the graphic register, the other in the algebraic register. In the test we omitted the graphic representation but rather we stressed more the algebraic dimension, also using connectives and other logical symbols.

The answers to the two item were the following:
11 students stated that the second couple of sets identifies an irrational numbers (44%) but only 3 of them (15%) answer that this is a Dedekind's cut. This implies that 8 students on 11 were accepting that an irrational number can be not representable by Dedekind's cuts. We couldn't infer certain conclusions since the students didn't state it explicitly but we deduced it by their brief answers. Since the phenomenon was very interesting we categorized the answers trying to figure out possible students' profiles to investigate better.

11 students on the 22 that answered solved the inequality on the margin of the text using the methodology that defines the set R4. We didn't know which procedures or argumentations they used the other students to answer, but we hypothesized that in the most of the cases also the students that didn't explain their procedures could have found problems in the conversion between one representation and the other of the sets C and D, that lead them to change the practices and indeed to face another problem. For instance the students who transformed the inequality used to represent the set of the rational numbers whose square is less of 5 in a task that we can resume in "Solve the inequality" changed the nature of the problem ignoring the domain Q and applying a usual procedure. In the end some of them abandoned the attempt and didn't go on, the others tried to match the solution with the problem, but the core of the problem was indeed in the difference between the two procedures.

The previous practices carried out to solve inequalities ignoring the domain and, even if the set of real numbers had not been studied, the solution had always been represented with segments and expressed in the verbal register as "all the numbers between *a* and *b*".

The procedure can be summarized in two steps:

1. solve the equation obtained by changing the sign < or ≤ with =
2. represent the solutions, if they exists in the real numbers, on the number line and study the intervals that make the function positive or negative

The intervals endpoints may be rational or irrational, algebraic or transcendental. What numbers lie in the middle of the endpoints doesn't seem to matter. In this procedure the algorithm is more important than the concept and following the steps of solving the equation before solving the inequality it makes sense to use respectively < or ≤ to exclude or include the endpoints. In fact, if we can solve the equation, the numbers to use as endpoints are known and we can include or exclude them. In fact there are no doubts about the existence of a number to put in the place of the variable to make the expression equal to 0 since it is exactly the solution of the equation solved at the first step.
This way the line is already manipulated as if it was complete, even if the numbers associated by teachers and students to the points are only a subset of real numbers (rational and algebraic numbers).
This makes very hard to use this procedure to introduce new numbers that make it possible to find numbers in any cut, since students may already believe it's possible with the numbers the know.
(D'Amore, Fandino Pinilla, Santi, Sbaragli, 2011)

The categories, not incompatible, are:
In order to investigate better the phenomenon we decided to use a part of the last lesson to re-administer a similar version of the question after giving the students a written note in which the two representations, graphic and algebraic, were used at the same time like we had done during the lessons. The relation between the two representations had been explained again. Only 19 on 25 students answered the new questionnaire.

The new questionnaire was composed by 2 questions.

For every couple of sets say if it is a Dedekind's cut and, if so, say if it's rational or irrational.

1) $A = \{x \in Q \mid x < 0 \vee x^2 \leq 5\}$, $B=\{x \in Q \mid x > 0 \vee x^2 > 5\}$

2) $C = \{x \in Q \mid x < 0 \vee x^2 \leq 5\}$, $D=\{x \in Q \mid x > 0 \vee x^2 > 5\}$

The first couple is very similar to the same with presented before, but the sign used in the first case was ≤ rather than <.

The second couple was defined by polynomial inequalities but of a different grade. The first subset was defined by means of the sign < , while for the second we used ≤, i.e. the contrary respect of the previous one.

We made two clear variation: once we differentiated between < and ≤, in the other case we changed the degree of the inequality in order to analyse the case of inequalities the students were not used to solve at school. We tried to figure out which criteria the students were using to decide if a couple was or not a Dedekind's cut.
The results were the following:
1)
a. Right answer (8)
b. No, because $\pm\sqrt{5}$ are not in Q (2)
c. No, because $A \cap B = \sqrt{5}$ (4)
d. No (3)

e. No answers (2)

5 students reported the solution of the procedure to solve the inequality on a margin of the text. We will use the letters to indicate a category we attributed to a student who answered the first question in order to see if the student remained in the same category in the second or not.

2) To answer the second question some students changed their position.

One student belonging to 'a' changed strategy in the second case. In the first case he didn't solve the inequality but answered immediately. In the second case he solved the inequality and interpreted the sign $\leq$ as inclusion of the element and turned to the wrong strategy.
Two other students belonging to 'a' wrote only moved to 'd', without explanations.

Two students belonging to 'c' and one student belonging to 'd' changed their position in the second case stating that it's a section because $C \cap D = \emptyset$. In this case probably the sign $<$ instead of $\leq$ may have lead the students to think that the intersection was empty.

This changes confirmed that the algebraic representation is deeply linked to equations and to the practice we named R4 and that this representation changes indeed the nature of the problem from which the real numbers as Dedekind cuts emerge as a solution: to complete the set of numbers we put in correspondence with the line.

2. Research framework

2.1 The formal and the intuitive dimensions of real numbers: remarkable research tools and results

The fascinating challenges that infinity, continuum and real numbers pose to our minds have lead mathematicians and mathematics education researchers to spent a significant amount of years investigating and trying to deepen more and more in our relation with such a topic. We can state that precisely the eluding nature of this human experience drove them to carry out researches from which some research tools emerged that became very important in the community of mathematics education researchers.

The works by Fischbein (1979;) on intuition have been inspired also by the problem of learning infinity, infinitesimals and real numbers, with a special attention to the conceptualization of the relation between representations, meanings and students' interpretations. In words of Tirosh and Dreyfus: "Fischbein's desire to understand the nature of intuitive thinking and the relationship between intuitive and other forms of thinking is evidenced in his further work on infinity, on implicit models of multiplication and division, on irrational numbers, on the relationship between intuitions and proofs, on the interaction between the formal, the algorithmic and the intuitive components in mathematical activities, and other topics." (1998, p. 1).

In the fundamental work published in 1981 Tall & Vinner defined two notions, concept image and concept definition, indeed to describe the cognitive processes that could lead the students to decouple intuitive and formal meanings of the objects involved in the Calculus, with special attention to continuity, limits, sequences and decimal representation of numbers.

The book "Where mathematics come from?" by Lakoff and Núñez (2000) had a strong impact on the community of research in mathematics education. Since the first examples to the last Chapters real numbers, continuum and infinitesimals are the recurrent topics they use to show the relevance and the effectiveness of their approach (*embodied cognition*). In particular Núñez went in depth in many papers.

Even if the book published by Lakoff & Núñez (2000) have also raised doubts and some critics problematized their approach (for an overview see Schiralli & Sinclar, 2003), we still consider it very powerful in the specific case of real numbers, in particular for analyzing the partial meanings of this mathematical object that are considered intuitive. Indeed in spite of the logical and technical articulation of the mathematical object "Set of real numbers" in Advanced mathematics, in the high school the topic if often transformed in order to make it intuitive, whatever it means. Is what the teacher considered intuitive referred to basic concepts of everyday life? To what other meanings they try to refer when they choose a representation wishing their students to feel it intuitive? To do this we needed a tool to describe adequately the mathematical from this quite particular point of view. The authors themselves analyzed the case of real numbers, providing us of a reliable starting point.

The tools we used in our research in the thread traced by these milestones, in a suitable relation with the other component of our research framework, were mainly three:

1) Primary and secondary intuition (Fischbein, 1987)
2) Concept image and concept definition (Tall & Vinner, 1981)
3) Grounding and linking conceptual metaphors (Lakoff & Nunez, 2000)

The authors of the three research works played in cascade a significant role in the further works, so, from the perspective of the cognitive dynamics that underlie the learning processes involving infinity, continuum and real numbers, we consider them linked since their births and consistent in the principles (Radford, 2008;). As Tall wrote in a tribute to Efraim Fischbein in 1999: "His work on primary and secondary intuitions, on children's probabilistic thinking, on the complex meaning of infinite concepts and on intuition in both mathematics and science have been seminal. They provide us with fundamental notions on which we can continue to build into the future." (p.3)

Tall is one of the most influential experts in the field of the investigation about teaching-learning of infinity and infinitesimals; he was deeply inspired by the work of Fischbein, so much that he (p. 2): "I remember vividly the talk he gave at PME in 1978, for it was to change my whole professional life. He presented his empirical and theoretical ideas on individual conceptions of infinity […] I remember explaining to him that I could "see" an infinitesimal as a graph that tended to zero. He challenged me forcefully, saying: "Show me an infinitesimal". I was taken aback. I could not do it. Though I could formulate the formal mathematical framework, I had never analyzed what it was that made the ideas work cognitively. […]"(ibid., 1999).

Tall (2008) refer to mathematical thinking as a blend of differing knowledge structures and takes precisely as the real numbers as an example, since they are "a blend of *embodiment* as a number line, *symbolism* as (infinite) decimals and *formalism* as a complete ordered field" (ibid., 2008; p. 9; italics by the author).

Lakoff & Núñez, inside of the paradigm called *embodied cognition,* use precisely the example of infinity continuity to introduce their studies about "human conceptual system that makes mathematical ideas possible and in which mathematics makes sense" (Lakoff e Núñez, 2000, p.8). In the book one of the most challenges concerning the Philosophy of Mathematics is treated: how a finite mind can grasp the infinity. We report here an excerpt in order to provide an example of the analysis they carried out; we present in depth the conceptual metaphors relevant for our topic in the Research framework (Par. 2.1.3). "The real numbers do not "fill" the number line. There is a mathematical subject matter, the hyperreal numbers, in which the real numbers are rather sparse on the line [..]. The modern definition of continuity for functions, as well as the so-called *continuum*, do not use the idea of continuity as it is normally understood. […] There is no absolute yes-or-no answer to whether 0.99999. . . . = 1. It will depend on the conceptual system one chooses. There is a mathematical subject matter in which 0.99999. . . . = 1, and another in which 0.99999. . . . ≠ 1. [...] The original notion of continuity for a function was conceptualized in terms of a continuous process of motion—one without intermediate ending points. " (ibid., 2000, p.8)

The Lakoff & Núñez's works came after years of intense researches in the field of mathematics education and have been influenced by these. The authors usually are not focused on empirical researches but rather theorization based on the previous researches. Nevertheless we use them as an introduction to the Paragraph concerning students and teachers' difficulties with real numbers in order to introduce the reader to the complexity of the topic; indeed the so-called difficulties may lie on so different but intertwined levels in this case that it's not easy to organize the literature review without a general overview on crucial questions concerning the didactics of real numbers and continuity.

In Lakoff & Núñez (1998) in particular the relation between every-day continuity and the formal continuity heir tradition *à la* Weierstrass – known as ε/δ definition – is discussed from the point of view of cognitive science and embodied cognition. The authors state that the difference between this two kind of continuity is not mathematical, neither philosophical but they are two cognitively different concepts, connected by means of a conceptual metaphor (Lakoff, 1993; Lakoff & Núñez, 2000); Weierstrass himself to conceptualize the first in terms of the second used a conceptual metaphor (ibid., 1998).

To treat here widely the definition and the enormous potentiality of conceptual metaphor in Mathematics education is not appropriate and really useful for our pourposes; we only report an attempt of definition by

Torgny (1997) and some excerpts from Lakoff (1992), a quite dated but essential review about the teories of metaphor.

“So, what is a metaphor? The most general description is that metaphor is an expression with two conceptual domains (knowledge fields) - where one is understood in terms of the other (Gibbs 1979). The two concepts are referred to as ”source” and ”target”, where the source is the domain where the actual statement is generated, and target is the domain that will be used to explain the statement. Often, the two domains even help to explain each other. (Fauconnier, 1996) There have been several different theories about metaphors and their use, where metaphors have been described as something to be used for rhetorical purposes, as a decoration of language, as a way to create mental model (Black 1979) or even as the basis of all human thinking (Lakoff and Johnson 1980).” (Torgny, 1997, p. 5)

“Metaphor is the main mechanism through which we comprehend abstract concepts and perform abstract reasoning. […] Metaphor is fundamentally conceptual, not linguistic, in nature. Metaphorical language is a surface manifestation of conceptual metaphor. [...] Metaphor allows us to understand a relatively abstract or inherently unstructured subject matter in terms of a more concrete, or at least a more highly structured subject matter. [...] Metaphors are mappings across conceptual domains, [...] asymmetric and partial. Each mapping is a fixed set of ontological correspondences between entities in a source domain and entities in a target domain. [...] A conceptual system contains thousands of conventional metaphorical mappings, which form a highly structured subsystem of the conceptual system.” (Lakoff, 1992; p. 39-40).

This can lead us to grasp what Lakoff & Núñez (1998) meant when they stated that the every-day continuum and the formal one are two cognitively different concepts, connected by means of a conceptual metaphor: there are two different conceptual domains to which they belong; the first plays as a source and the second as a target.

Núñez, Edwards e Matos (1999) proposed a detailed analysis of the distinction between formal and natural definitions of continuity from the perspective of the embodied cognition. The authors, first of all, stress that the both of the concepts are referred - of course – to cognitive primitives, but to different cognitive primitives: in the first case (natural continuity) the inferential structure of the everyday understanding of motion, flow, and wholeness, have been applied to a specific domain of human understanding, functions and variations; in the second case the part-whole schemata and container schemata. Cauchy-Weierstrass definition denies motion, flow and wholeness, dealing exclusively with static, discrete, and atomistic elements.

“Although it is true that the so-called ‘rigorous’ definition deals better with complex and ‘pathological’ cases (such as $f(x) = x \sin 1/x$) for certain purposes, it is not because it captures better the ‘essence of continuity’. Within an embodied, non-objectivist cognitive science, there is no transcendental ‘essence’ of a concept, even in mathematics” (Edwards and Núñez, 1995) (ibid., 1999; p. 55).

The authors conjecture that the pedagogical problem can be summarized as follows: “students are introduced to natural continuity using concepts, ideas, and examples which draw on inferential patterns sustained by the natural human conceptual system. Then, they are introduced to another concept – Cauchy-Weierstrass continuity – that rests upon radically different cognitive contents (although not necessarily more complex). These contents draw on different inferential structures and different entailments that conflict with those from the previous idea. The problem is that students are never told that the new definition is actually a completely different human-embodied idea. Worse, they are told that the new definition captures the essence of the old idea, which, by virtue of being ‘intuitive’ and vague, is to be avoided.” (ibid, 1999; p. 55)

The authors found out three different conceptual continuities: one is *Natural continuity* (as in the ‘informal’, ‘intuitive’ definition) and the other two (implicit in the ‘rigorous’ Cauchy-Weierstrass definition) are *Gaplessness* (for lines as sets of points) and *Preservation of closeness* (for functions).

We will resume here the aspect concerning real numbers leaving aside as much as possible what concerns explicitly functions. In the beginning we introduce briefly what kind of issues the authors faced in their work, while afterwards we explain in depths how this metaphors structure the real numbers and the continuum as cognitive objects.

Natural continuity deals with the following features: the continuity arises from the motion; since there is motion, there is some entity moving; the motion results in a static line with no 'jumps'; the static line that results lost directionality. From the perspective of embodied cognition, the authors conceive of the mobile and static aspects of a continuity as founded on an everyday human conceptual process: the fictive motion metaphor (Talmy, 1988), that can be summarized as follows:

• **A line is the motion of a traveler tracing that line.**

In everyday language many times we refer to motion, direction, and so on; this concepts, belonging to the everyday language domain are used metaphorically in mathematics to talks about points, functions, variations, both by the students and the experts. "These embodied natural and everyday human cognitive mechanisms are the ones that make possible the intuitive dynamic and static conceptualization of a continuous function." (ibid., 1999, p. 57).

Euler's continuity is characterized by motion in the Fictive Motion metaphor.

The Cauchy-Weierstrass definition was motivated by complex mathematical objects that mathematicians developed in the 19th century, that all required conceptualizing lines, planes, and n-dimensional spaces as sets of points. A system of three conceptual metaphors constitute its structure.

These metaphors are:

• *A line is a set of points*

• *Natural continuity is gaplessness*

• *Approaching a limit is preservation of closeness near a point*

A line is a set of points

A line can be conceptualized in two completely different ways: a holistic one, according to which the line is not made up of discrete elements, and the one we are analyzing here, i.e. the points are rather entities constituting the line; we observed many times this complementarity in the History of mathematics in Par. 1.1.1.

The first characterization is congruent with natural continuity and the second with Cauchy-Weierstrass' definition.

"Both conceptions are natural, in that both arise from our everyday conceptual system. Neither is 'right' or 'wrong' *per se*; however, they have very different cognitive properties, and provide different inferential structure." (ibid., 1999; p. 58)

The authors consider this as crucial point to take into account in the teaching and learning process.

Natural continuity is gaplessness

According to our everyday intuition, a line constitutes a natural continuum: we move continuously along a line from a location A to a location B, we go through all point-locations on the line between A and B, without skipping over any, that is, without leaving any gaps between the point-locations.

"The metaphor underlying the Cauchy-Weierstrass definition identifies the point-locations on a line as constituting the line itself. Such a metaphorical 'line' is not a natural continuum, but only a set of points: the metaphor 'A line is a set of points' entails the metaphor 'Natural continuity is gaplessness'." (ibid., 1999; p. 58).

Therefore, a line conceptualized as a set of points cannot be – cognitively – naturally continuous but only gapless.

Approaching a limit is preservation of closeness near a point

In Cauchy-Weierstrass' definition of limit there are only static elements, as claimed by Bernard Bolzano, because it had explicitly been chosen to avoid motion and time. The gaplessness of the set of real numbers in the open interval is Cauchy-Weierstrass' metaphorical version replacing the natural continuity of the intuitive line in Newton's geometric idea of a limit. In order to arithmetize the 'approaching' avoiding motion, the new metaphor for continuity uses the same basic idea as their metaphor for a limit: preservation of closeness. Continuity at a real number is conceptualized as preservation of closeness, not just near a real number but also at it.

Given the metaphor that a line is a set of real numbers, then natural continuity can only be conceptualized metaphorically as gaplessness.

To sum up an important remark concerning the line and the numbers, as we observed in the Par. 1.1.1, is that the line is an infinite ensemble of point is anything but obvious or "absolutely true". A point can be seen as a result of a cut, that though generate at least points, lying on the two parts resulting from the cut (Giusti, 2000); this point are infinitely close a, "distinct but not distant". If we try to avoid this approach and to present a point as a small elementary, atomic part, of the line the path is even more risky; the historical debates concerning the nature of the infinitesimal quantities, sometimes 0, sometimes not, depending on the necessity should discourage everyone to present the point in such a way before an articulated discussion about this ancient, maybe eternal problem. As Tall (1980) asserted: "When we speak of 'existence' [referring to infinitesimal quantities, nba], we may do so from several different points of view. Bishop's criticism is directed at the fact that the axioms of non-standard analysis assert that certain concepts exist which cannot be constructed in any genuine sense." (p. 3).

"[…] not only do embodiment and symbolism act as a foundation for ideas that are formalized in the formal-axiomatic world, structure theorems can also lead back from the formal world to the worlds of embodiment and symbolism. This means that those who use mathematics as a tool can use the embodiment and symbolism to imagine problem situations and model them symbolically. In this way, engineers, economists, physicists, biologists and others often use embodiment and symbolism as a foundation for their work. The new embodiments depend not just on experience in the world, but on concept definition and formal deduction, leading to new formal insights." (Tall, 2008; p. 16-17)

1. Primary and secondary intuition

Fischbein (1978) describes intuition as immediate knowledge characterized by self-evidence, coercive effect, extrapolative capacity and globality. (Tall, 1980).

According to Fischbein (1987), intuitive knowledge is a self-explanatory cognition that we accept with certainty as being true. It is a type of immediate, coercive, self-evident cognition, which leads to generalizations going beyond the known data. Fischbein differentiated between primary intuitions and secondary intuitions. Primary intuitions were defined as intuitions that "develop in individuals independently of any systematic instruction as an effect of their personal experience" (Fischbein, 1987, p. 202). Secondary intuitions were defined as "those that are acquired, not through natural experience, but through some educational intervention" (Fischbein 1987, p. 71). Secondary intuitions were defined as evident when formal knowledge becomes immediate, obvious, and accompanied by confidence. Secondary intuitions about a certain concept or process are often inconsistent with the related primary intuitions about the same concepts. (Singer & Voica, 2008).

2.1.2 Concept image and concept definition

Tall & Vinner (1981) is a milestone of the research in Mathematics education. In this paper the authors try to grasp and to define the processes underlying the human thinking in mathematics with particular attention to symbols, images and relation between the formal definition and the inner logic of everyone's mental mechanisms. A lot of tools and theoretical approaches have been developed since 1981 to face this issue, so crucial for Mathematics education; nevertheless it is relevant still now, in particular for the topic we are analyzing, that have been the first example they proposed in order to show how to use their theoretical tools. We quote here a very significant excerpt, that permit the reader to frame the work and to foresee the seeds of pillars of contemporary Mathematics education:

"The human brain is not a purely logical entity. The complex manner in which it functions is often at variance with the logic of mathematics. It is not always pure logic which gives us insight, nor is it chance that causes us to make mistakes. To understand how these processes occur, both successfully and erroneously, we must formulate a distinction between the mathematical concepts as formally defined and the cognitive processes by which they are conceived. Many concepts which we use happily are not formally defined at all, we learn to recognize them by experience and usage in appropriate contexts. Later these concepts may be refined in their meaning and interpreted with increasing subtlety with or without the luxury of a precise definition. Usually in this process the concept is given a symbol or name which enables it to be communicated and aids in its mental manipulation. But the total cognitive structure which colours the meaning of the concept is far greater than the evocation of a single symbol. It is more than any mental picture, be it pictorial, symbolic or otherwise." (ibid. 1981; p. 1).

The main terms introduce by Tall & Vinner are four:

- *concept image* : the total cognitive structure that is associated with the concept, which includes all the mental pictures and associated properties and processes; it includes all the mental attributes of a concept, conscious or unconscious, that also could generate future conflicts;
- *concept definition*: a form of words used to specify that concept;
- *concept definition image*: concept image generated by a concept definition;
- *cognitive conflict factors*: incoherent aspects of the same concept image that, activated simultaneously, may give a sense of conflict or confusion.

As the concept image develops, it need not be coherent at all times. The brain does not work that way. Sensory input excites certain neuronal pathways and inhibits others. In this way different stimuli can activate different parts of the concept image, developing them in a way which need not make a coherent whole. We shall call the portion of the concept image which is activated at a particular time the *evoked concept image.*

Only when conflicting aspects, that can coexist in general without annoying the student, are evoked simultaneously, the students perceive their inconsistency and feel a sense of confusion.

Every students associate to a concept her own "concept image definition", not necessarily coherent with other parts of the concept image. Such factors, evoked at the same time be potential conflict factors.

A serious type of potential conflict factor consists in the coexistence of a concept image that is incoherent with the formal concept definition itself, because the conflict is not perceived as something generative but the formal definition is perceived as unrelated to the concept image. In other words, in order to generate a productive conflict, the formal definition should activate concept image definitions that can simultaneously be compared to other parts of the concept image; until this event doesn't happen the "students […] may be secure in their own interpretations of the notions concerned and simply regard the formal theory as inoperative and superfluous" (ibid., 1981; p. 4)

Limit and continuity are some of the most risky concepts in this sense.

The concept image of continuity is usually built up from informal images and references to everyday experiences; the authors quote two images of continuity that are very important in the teaching-learning of Calculus:

1) *No gaps*: "the rail is continuously welded" (meaning the track has no gaps)

2) *No breaks*: "it rained continuously all day" (meaning there were no breaks in the rainfall).

Tall & Vinner, as examples, refer to limits and continuous functions; to the last ones they refer as "bête noire of analysis" (ibid., 1981; p. 13).

Even if the two concepts are not exactly our main topics they deal very much with real numbers in the didactical practices; indeed, as we will show in the analysis, many teachers declare to introduce real numbers precisely to treat limit and continuous functions and limits of sequences are necessary to define real numbers at a formal level as Cantor's contiguous classes.

According to the author the usual practices involving functions carried out before the Calculus reinforce the intuitive idea that the graph has "no gaps" and may be drawn freely without lifting the pencil from the paper; the situation didn't seem to change so much when the students had studied the Calculus in the high school since in their investigations the students of the first year of University - 41 students with an A or B grade in A level mathematics in the UK - had the same backgrounds.

To investigate their concept image of continuity they were asked to answer the following question in 5 cases:

"Which of the following functions are continuous? If possible, give your reason for your answer."

$$f_1(x) = x^2$$

$$f_2(x) = 1/x \quad (x \neq 0)$$

$$f_3(x) = \begin{cases} 0 & (x \leq 0) \\ x & (x \geq 0) \end{cases}$$

$$f_4(x) = \begin{cases} 0 & (x \leq 0) \\ 1 & (x \geq 0) \end{cases}$$

$$f_5(x) = \begin{cases} 0 & (x \text{ rational}) \\ 1 & (x \text{ irrational}) \end{cases}$$

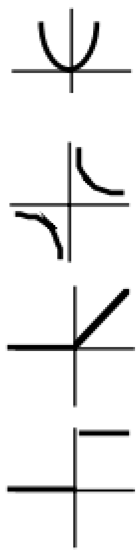

Fig. (Tall & Vinner, 1981; p. 14-15)

Also the ”mathematically right” answers had been given sometimes relying on parts of the concept image that were conflicting with the formal one or, at least, not the concept image definitions that were expected.

Functions can be said to be continuous for the following reasons:

1) “because it was given by only one formula.”
2) “it is all in one piece.”
3) “it has a continuous pattern of definition.”
4) “it has it a smoothly varying graph”

The function (f_5) yet conflict with all three evoked images mentioned above.

The concept image is a global property, the notion of continuity over an interval, not continuity at a point.

Conversely functions was declared not continuous because:

1) “There are gaps in the picture”
2) “the graph is not in one piece”
3) “it is not defined at the origin”
4) “there is a jump at the origin,”
5) “it gets infinite at the origin.”
6) “it is not given by a single formula”
7) “there is a sudden change in gradient.”
8) “it is impossible to draw”

It emerged the difficulty of forming an appropriate concept image, in spite of an inappropriate one having potential conflicts, that can seriously compromise the development of the formal theory in the mind of the individual student (ibid., 1981).

The most of these images can be recovered for continuous sets and segments and thus, at a more formal level, to intervals of real numbers and the set of real numbers itself.

3. Grounding and linking metaphors for real numbers

“What we have found is that there are two types of conceptual metaphor used in projecting from subitizing, counting, and the simplest arithmetic of newborns to an arithmetic of natural numbers. The first are what we call *grounding metaphors*—metaphors that allow you to project from everyday experiences (like putting things into piles) onto abstract concepts (like addition). The second are what we call *linking metaphors*, which link arithmetic to other branches of mathematics—for example, metaphors that allow you to conceptualize arithmetic in spatial terms, linking, say, geometry to arithmetic, as when you conceive of numbers as points on a line.” (Lakoff & Núñez, 2000)

Since conceptual metaphors play a major role in characterizing mathematical ideas, grounding and linking metaphors provide for two types of metaphorical mathematical ideas:

1. *Grounding metaphors* “yield basic, directly grounded ideas. Examples: addition as adding objects to a collection, subtraction as taking objects away from a collection, sets as containers, members of a set as objects in a container. These usually require little instruction.”

2. *Linking metaphors* “yield sophisticated ideas, sometimes called abstract ideas. Examples: numbers as points on a line, geometrical figures as algebraic equations, operations on classes as algebraic operations. These require a significant amount of explicit instruction.”

We propose here the analysis by Lakoff & Núñez (2000) of metaphors underlying the conceptualization of real numbers and continuity.

First of all, since we are investigating numbers, we took in account the four grounding metaphors of arithmetic.

Arithmetic is object collection : mapping from the domain of physical objects to the domain of numbers.

Source domain **Object collection**	Target domain **Arithmetic**
Collection of objects of the same size	Numbers
Size of the collection	The size of the number
Bigger	Greater
Smaller	Less
The smallest collection	The unity
Putting collections together	Addition
Taking a smaller collection to a bigger one	Subtraction

Evident properties and actions in the sources become laws of Arithmetic: numbers have a size; the lack of objects is anyway a number; the results of the operations are stable; inverse operations exist since we can come back to the a previous collection adding and taking off a collection; unlimited iteration for addition and limited iteration for subtraction; properties of addition like equivalence in case of addition or subtraction of the same collection, commutativity and associativity, order properties.

While addition and subtraction and their correspondent actions “exist” in both the domain independently, for operation like multiplication and division it’s necessary to cross domains and to refer both to the first and the second since numbers are necessary to represent the times we have to take collections of the same size; we need thus what by Lakoff & Núñez (2000) call a *conceptual blending* to extend the metaphor.

The path towards the extension of the metaphor may be two: by union of collections and partition in sub-collections or repeating addition and subtraction a certain number of times.

Also this is a particularly interesting example, even if very elementary from a mathematical point of view, because it allows to explain that we can extend a metaphor in different ways.

Another possible metaphorical mapping for grounding Arithmetic in embodied actions involves the metaphor that has as source domain the Construction of objects.

Arithmetic is objects construction

Source domain **Object construction**	Target domain **Arithmetic**
Objects (consisting of ultimate parts of unit size)	Numbers
The smallest whole object	Unity
The size of the object	The size of the number
Smaller	Less
Bigger	Greater
Acts of object construction	Operations
A constructed object	Result of operation
A whole object	A whole number

Also in this case *metaphorical blending* allows to extend operations to multiplication and division in two ways: fitting together and splitting in parts or iterating actions.

In this metaphor it's possible to conceptualize fractions as results of splitting units and zero as lack of whole objects.

Another important metaphor for Arithmetic, very relevant for our topic is the *Measuring stick* one. As the authors stress in the introduction to the Paragraph the practice of measuring using a unitary segment is very primitive and also the actions in which operations are grounded are very elementary: we can put end-to-end segments in order to obtain another "physical segment".

Measuring sticks

Source domain **The use of a measuring stick**	Target domain **Arithmetic**
Physical segments (consisting of ultimate elementary parts)	Numbers
The basic physical segment	Unity
The length of the physical segment	The size of the number
Shorter	Less

Longer	Greater
Acts of physical segments placement	Operations
A physical segment	Result of operation

In this metaphor the mapping involves addition and subtraction putting end-to-end physical segments and as in the previous metaphors we can extend in two ways to multiplication and division: fitting together and dividing or iterating addition and subtraction; in the same way we did it before we can go on dividing obtaining fractions, defining zero and so on.

There is a very high resemblance between the last two metaphors, so high that it can seem useless to distinguish them at a quick look. Nevertheless physical segments, both in their concrete and imaginary version, "are very special constructed objects […] they are unidimensional and they are continuous […] the blend of source and target domain of this metaphor has a very special status […] (physical) segment with numbers that specify their lengths […] once you form the blend, a fateful entailment arises " by (Lakoff & Núñez, 2000; p. 71).

In the blend *Numbers are physical segments* the mapping from positive rational numbers and segment is one-directional: it's not always true that a segment correspond to a positive rational number.

Thus the blend goes beyond the metaphor itself: for every physical segments there *must* be a number, as Eudoxus had already observed in 370 B.C, using implicitly the metaphor. The blend makes thus arise the irrational numbers.

Referring again to segments, but this time as imaginary traces of motion, we have immediately another metaphor*: Arithmetic is motion along a path.*

Source domain **Motion along a path**	Target domain **Arithmetic**
Physical segments (consisting of ultimate elementary parts)	Numbers
Act of moving along a path	Operations
A point-location along the path	The result of an operation
The origin of the path	Zero
Further from the origin than	Greater
Closer from the origin than	Less
Moving from a point-location A to a point as much as the distance of a point B from the origin	The addition A + B
Moving towards the origin from a point-location A	The subtraction A – B

as much as the distance of a point B from the origin	

The extension to multiplication and division in this case can only be realized by means of iterations of addition and subtraction. Simple fraction 1/n correspond to points such that repeating n times the movement one can reach the one.

The analogies with the metaphor of the *Measuring stick* are a lot, but there is a remarkable difference: this metaphor permit to define negative numbers and constitute a unitary representation of numbers that allowed the mathematicians to conceptualize the numbers “all together”, not without significant resistances in the community of the mathematicians itself. This metaphor makes sense of utterances like “the number lies between other two”.

A very important remark is that what makes effective arithmetic in real life is precisely the existence of these grounding metaphors and merely of the resulting metaphorical blending.

Other fundamental metaphors useful to describe the complex cognitive structure of real numbers are certainly those underlying the set theory.

Classes are containers

Source domain **Container schemata**	Target domain **Classes**
Internal part of container schemata	Classes
Object in the internal part	Elements
To be an internal object	To belong
An internal part of a container schemata inside a bigger one	Subclass of a bigger one
The partial superposition of container schemas	Intersection of two classes
The totality of internal parts of two container schemas	Union of two classes
The external part of a container schemata	The complementary of a class

We can build a mapping between *Classes* and *Arithmetic* - making correspond addition to union and multiplication to intersection; between *Sets and Natural numbers* - making correspond numbers to cardinalities; *Maps* and *Enumeration* – making correspond the possibility to create a biunivocal correspondence between two sets to have the same number of elements , as established by the Cantor’s metaphor.

Fundamental for our analysis are the metaphors involving infinity.

There are two basic ideas of infinity, related to processes that don't end up: *continuative processes* – the continuous ones - and *iterative processes* – that are characterized by intermediate results. In the common language usually iterative verbs are used to refer to continuous actions, like "The ball was rolling continuously". This generates the metaphor *Indefinite continuous processes are iterative processes.* This metaphor is very important in Mathematics since through it we can divide continuous processes in indefinitely iterated processes, being every step discrete – inspired by Galileo's conception and commonly used in Physics but also in the Calculus.

Relying on these grounding infinite processes we can operate a crucial distinction, historically very significant, between potential and actual infinity. Actual infinity is indeed the result of an infinite process in a suitable metaphor; however this obliges us, standing on the mapping rules, to conceptualize it as an indefinite but finite process, since we can consider its result, an 'infinite thing'.

Lakoff & Núñez (2000) hypothesized there is a unique Basic Metaphor of Infinity (BMI), in which both the source and the target domain ha a starting configuration, indefinite numbers of iterations with an intermediate result. BMI is synthesizable this way:

Basic Metaphor of Infinity

Source domain **Iterative completed processes**	Target domain **Iterative processes that ever go on**
Initial state	Initial state
Resulting state from the initial state	Resulting state from the initial state
Process of production of the consecutive state starting from a given one	Process of production of the consecutive state starting from a given one
The intermediate result after a given iteration	The intermediate result after a given iteration
The partial superposition of container schemas	Intersection of two classes
The final resulting state	**"The final resulting state" (actual infinity)**
Consequence E: The final state is unique and precedes every non-final state	**Consequence E: The final state is unique and precedes every non-final state**

The final resulting state has no correspondents in the mapping. The conceptual metaphor imposes the existence of a "final state" that is nothing real but rather a product of the human process of production of knowledge. The "final state" is something peculiar since it's not part of the process but rather something external to the process, and is the actual infinity, while the potential infinity has no end and no resulting final states. We can also think a process as a thing and indicate it with a symbol: . This may generate controversies since the unique result of BMI is the actual infinity should be unique, contrarily to the Cantor's distinction between different infinite cardinalities but multiple applications of BMI can generate different grades of infinity.

The authors state indeed that also real numbers are conceptualized by means of BMI and, with a wordplay, they observe that there's nothing "real" in real numbers as cognitive objects, but rather they are intrinsically metaphoric.

Many significant observations would be necessary to clarify the reasons why this so ambiguous adjective – *real* – is used for an object that is so far from having anything to do with the usual meaning of the term *real*. Descartes used it first in order to distinguish these numbers from the imaginary ones, not to signify these set was anything concrete, as we read in some books. Lévy-Leblond, one of the most influential living experts of Quantum physics, proposed an interesting interpretation of this use talking about science and its limitations: indeed we call *real* what we can't reach, so reality is truly not what we live but the limit of our imagination.

According to the author all the mathematical objects necessary to characterize real numbers can be conceptualized in terms of BMI: decimal numbers, polynomial representations, limits of infinite sequences, upper extremes, infinite intersections of intervals and also Dedekind cuts.

We present here the metaphors that lead to real numbers from a cognitive point of view, all results in the BMI.

Numeral numbers of natural numbers

Source domain **Iterative processes that ever go on**	Particular case **Infinite set of numeral numbers of natural numbers**
Initial state (1)	N(1): empty set
Resulting state (2) from the initial state	N(2): 1, …, 9
Process of production of the consecutive state *n* starting from a given one *(n-1)*	N(n-1): set of numerals whose number of digits is less n -1
The intermediate result after a given iteration	N(n): set of numerals whose number of digits is less n
"The final resulting state" (actual infinity ∞)	**N(∞):** set of numerals of all the natural numbers
Consequence E: The final state is unique and precedes every non-final state	**Consequence E: N(∞) is unique and contains all the numeral of all the natural numbers**

Infinite decimals

Target domain **Iterative processes that ever go on**	Particular case **Infinite set of numeral numbers of natural numbers**
Initial state (0)	R(0): set containing the elements of **N(∞)** followed by a comma
Resulting state (1) from the initial state	R(1): every string of digits in R(0) ; s1, s2, ---, s9

Process of production of the consecutive state *n* starting from a given one *(n-1)*	R(n): set of the strings s1, s2, ---, s9 with s element of R(n-1)
The intermediate result after a given iteration	R(n): set of numerals with *n digits* after the comma
"The final resulting state" (actual infinity ∞)	**R(∞):** infinite set of numerals with ∞ digits after the comma
Consequence E: The final state is unique and precedes every non-final state	**Consequence E: N(∞) is unique and its elements has more digits than every other element of R(n)**

Easily we can create a correspondence between the conceptual domain of *Infinite polynomial representations* and the symbolic domain of *Infinite decimals.*

While it's clear what is a iteration and its respective result talking about natural numbers, the matter is more complicated if we want to conceptualize the convergence of infinite sequences of real numbers, for which we use in Mathematics concepts like "approaching indefinitely", the index of the terms "tends to infinity" and so on . The researchers challenge us proposing to conceptualize precisely this concept of convergence using BMI to describe infinitely close terms of a sequence, using the metaphors *Numbers are points of a line* and the reverse one, i.e. the line is a set of points.

Infinite sequences

Target domain **Iterative processes that ever go on**	Particular case **Infinite sequences with limit L**
Initial state (0)	Frame Sequences and Limit
Resulting state (1) from the initial state	S(1): set containing the first term of the sequence
Process of production of the consecutive state *n* starting from a given one *(n-1)*	S(*n)* is formed starting from S(*n-1*) that contains the first *n-1* terms of the sequence
The intermediate result after a given iteration	S(n); R(n) is the that contains all the real positive numbers that are less |x(n)-L| and include R(n-1)
"The final resulting state" (actual infinity ∞)	**S(∞):** set that contains all the terms of the sequence ∞ digits after the comma, R(∞) is the empty set since no positive real numbers are less than |x(n)-L|
Consequence E: The final state is unique and precedes every non-final state	**Consequence E: L is the unique limit of the sequence**

Some values ε are relevant, while other values are not; we usually using BMI don't care of those that don't allow to go closer to a limit. This particular case of BMI, joined to fictive motion and the metaphor *A line is a set of points* (Par. 1.2) we gain the metaphor *Approaching a limit when n approaches infinity,* useful also to conceptualize *Infinite sums* as target domain in the metaphor involving limits of infinite sequence of partial sums .

Another very important particular case of BMI is that of upper extremes, crucial in the Dedekind definition of real numbers. It's important to notice that this concept only makes sense in the actual infinity metaphor.

Upper extremes

Target domain **Iterative processes that ever go on**	Particular case **Upper extremes**
Initial state (0)	S is a set of real numbers; b is a number ≥ every numbers of S, b(0) is an upper extreme if it's < every b; B is an infinite set of numbers ≥ every numbers of S.
Resulting state (1) from the initial state	Choice of b(1)
Process of production of the consecutive state *n* starting from a given one *(n-1)*	Given b(n-1) a lower b(n) is chosen
The intermediate result after a given iteration	A finite sequences of decreasing elements b(i), i=1,….. n
"The final resulting state" (actual infinity ∞)	b (∞) is an upper extreme
Consequence E: The final state is unique and precedes every non-final state	**Consequence E:** b (∞) is the unique upper extreme and it's lower than any other one

BMI helps in understanding why adding the Axiom of the Upper extreme lead to extend from rational to real numbers; also in this metaphor 0.999.. is equal to 1,0000… but this is not always true – see hyperreal numbers in Par. 1.1.1.

Infinite intersection of encapsulated intervals

Target domain **Iterative processes that ever go on**	Particular case **Infinite sequences of encapsulated intervals**
Initial state (0)	A sequence of closed encapsulated intervals defined by [l(n),u(n)], with l(n) > l(n-1) and u(n) < u(n-1)
Resulting state (1) from the initial state	The first term of the sequence [l(1),u(1)]
Process of production of the consecutive state *n* starting from a given one *(n-1)*	Intersection between [l(n),u(n)] and [l(n-1),u(n-1)],
The intermediate result after a given iteration	The interval [l(n),u(n)],

"The final resulting state" (actual infinity ∞)	$[l(\infty),u(\infty)]$, $l(\infty) > l(n)$, $u(\infty) < u(n)$; the distance between $l(\infty)$ and $u(\infty)$; the interval is reduced to a point $p = l(\infty) = u(\infty)$
Consequence E: The final state is unique and precedes every non-final state	Consequence E: p is the only point > l(n) and <u(n)

In the history of real numbers another important thread is characterized by infinitesimal objects. In this case the grounding metaphors involve as a basic source domain that of grains, i.e. not zero something smaller than everything we can imagine. Mathematically speaking these objects became *infinitesimals,* maybe the most discussed mathematical objects ever. Lakoff & Nunez (2000) trated in depth all the details concerning the original problem of tangents, while we will extract only some metaphors and conceptual blending useful to characterize the real numbers as cognitive objects, as we did until now. Also the infinitesimals are particular cases of BMI:

Target domain **Iterative processes that ever go on**	Particular case **Infinitesimals**
Initial state (0)	Consider the set of positive numbers less than 1/n that satisfy the first 9 axioms of real numbers
Resulting state (1) from the initial state	S(1): The set of positive numbers less than 1/1 that satisfy the first 9 axioms of real numbers
Process of production of the consecutive state *n* starting from a given one *(n-1)*	From S(n-1) you form S(n)
The intermediate result after a given iteration	S(1): The set of all the positive numbers less than 1/n that satisfy the first 9 axioms of real numbers
"The final resulting state" (actual infinity ∞)	S(∞): The set of all the positive numbers less than every real number that satisfy the first 9 axioms of real numbers
Consequence E: The final state is unique and precedes every non-final state	Consequence E: S(∞)

With an analogous metaphor we can create *Granular numbers,* taking as correspondent of the initial state in the mapping numbers expressed in the form 1/k reaching in the end an object δ less than a 1/k, k natural. To develop the infinitesimal Calculus it was necessary yet to Leibniz to go on this metaphor passing from granular numbers to clouds of infinitely close monads, obtained working with granular arithmetic, i.e. the arithmetic applied to granular numbers. Associating to this metaphor the *Line is a set of point* – possible thank to the linear order imposed by the axioms to the set of granular numbers – we obtain a *Line of granular numbers* in which the real numbers are sparse. To accommodate this new metaphor the hyperreal numbers are necessary, i.e. a rather independent infinite world in respect of that of real numbers. We didn't deepen this topic, even taking it in account in our analysis, but rather we went on analyzing the most relevant metaphors concerning the continuum, the points and the real numbers, coming back to the metaphor of motion that until this point we have only sketched.

Taking of line and numbers the authors starts provoking this way:

1) "The real line" is not a line
2) "The continuum hypothesis" doesn't deal with the continuum

The second question is maybe too subtle for our investigation, even if it's very interesting, while the first is really fundamental. The core argumentations they proposed stem from the distinction between natural continuous space and its discretization, topic that we treated in details in Par. 1.1.1, to which we refer for a better understanding of the issue. Here we propose only the basic metaphors underlying this very problematic node. The first metaphor is the emblem of discretization: *Space is a set of points.*

Space is a set of points

Source domain **A set with elements**	Target domain **Space naturally continuous with positions identified by points**
A set	A n-dimensional space
The elements of the set	Points are positions in the space
The elements exists independently on the set they belong to	The positions are intrinsic in the space
The elements of two sets are distinct if they are different entities	The positions of two sets are distinct if they are different positions
The relation between the elements of the set	The space properties

To conceptualize a point using embodied domains the most natural frame is that of discs, the spatial version of intervals; in this frame points are discs with diameter 0. An intermediate step is the creation of a blend between segments and discs:

The blend disc/segment

Frame Disc	**Frame segment**
A disc and its distinct parts (center, internal, diameter, circumference)	A segment and its distinct parts (extremes, center, internal, length)
Diameter	Length
Center	Center
Opposite points on the circumference	Extremes

BMI succeed in transforming a disc not in a 0-diameter disc, but rather in a disc with infinitesimal diameter. The problem that immediately arise in this metaphor is the following: infinitesimal points are touch each other? If the line is – as we take for granted – "full" the points must touch each other, but these should imply they are the same point. This would make impossible the extensionality, as Giusti (1990) stressed in his profound analysis of the conceptions hidden in the works of Leibniz and Cavalieri, troubled by the relation between the whole and it indivisible or infinitely divisible components.

As usually with metaphors, there are no certain answers but rather truths depending on different choices. The BMI is crucial in the discretization of the space in infinitesimal points, that we stress again is something completely different from the natural continuity of everyday life.

The natural continuous space has the following properties: metric properties; neighborhoods properties; limit point properties; accumulation point properties; open set properties.

Numbers are points of a line

Source domain **Points of a line**	**Target domain** **A collection of numbers**
A point P of a line	A number P'
A point O	The number 0
P is on the left in respect of Q	The number P' is greater than Q'
Distance between O and P	Absolute value of P'

We can also create a blend between a space with positions of points and sets, making correspond the blend *Numbers as points of the line* and a set, the points that are positions to elements of the set and so on. This blend discretize indeed the line and this is the reason why can state that no continuous lines are in the "real line" using the natural continuity as a reference.

The resulting discretized version of *Numbers are points of the line* is the complex result of a double mapping between the blend Line-set, the sets, the numbers.

In the blend *Numbers are points of the line* there is a correspondence between elements-points of a line in the source domain and the numbers in the target domain. We can't the state the same for the continuous line: Numbers-Naturally continuous line since only some positions are correspondent of numbers while the whole line, the means, doesn't correspond to anything. Being the positions zero-dimensional they can't occupy space so independently from the numbers we are considering as target domain in the means an infinite quantity of points "remains".

Even if the set of real numbers is "complete" in the sense that they respect the axioms, in the naturally continuous line the real numbers don't exhaust the position of the points on the line, because the line is a naturally continuous mean.

The "completeness", i.e. the closure in respect with the limit points, is just a way to conceptualize what a line was with the specific aim of discretizing. Lakoff & Nunez commented promptly on this "tradition", or better, criticized vehemently the fact that all the choices that underlie this conceptual blending are omitted, traded for the truth of a rigorous Mathematics. We report here some of these critiques:

1. The natural continuous line is not clearly distinguished from the line as set of points, that indeed are not point but elements
2. The real numbers are not clearly distinguished from the points of the line
3. The real line is not a line
4. What is usually called continuum is not distinguished from a naturally continuous line
5. The myth that discretizing mathematic becomes more rigorous is still living, even if indeed the classical ideas are not formalized but rather changed and substituted with new ideas.

The BMI allows to conceptualize also the continuity paying the penalty of accepting its discretization; this became evident and explicit in Dedekind, who used implicitly the BMI clarifying since the very beginning of his work that the real numbers were something different from the points of a naturally continuous line.
" The statement is so frequently made that the differential Calculus deals with continuous magnitude, and yet an explanation of this continuity is nowhere given. […] It then only remained to discover its true origin in the elements of arithmetic and thus at the same time to secure a real definition of the essence of continuity. I succeeded Nov. 24, 1858" (Dedekind, 1872).

Lakoff & Nunez (2000) stress how important it was the Dedekind's choices in respect of the constitution of the blending Numbers-Line: Dedekind specific that the analogy between rational numbers and points of a line becomes actual just when on a line are fixed: an origin and a unity.

The strength of the Dedekind's metaphor is that the "real correspondence" is biunivocal, i.e. that not only for every a correspondent numbers exist, but also the reverse implication, and that if the numbers considered are rational some points have no associated numbers; the line is supposed *absolutely continuous.* The representation of functions by means of curves is thus itself metaphorical- being the' *thus'* a metaphorical inference of course.

In Dedekind the naturally continuous line, that before was not included in the mapping as correspondent o something else is now included and a new – crucial – implication arise: if the line constituted by points is continuous so the set is continuous too.

The Dedekind's metaphor is 'realized' in the measurement tools, even if of course real numbers are not on the physical tools but the results is rather something that have a double nature, conceptual and physical.

Physical position play the role of positions on a line and the measures are numbers. In this frame to say there are no holes generically may be misleading since it's necessary to stress that there are no holes between numbers and not between points.

The idea of the Dedekind's cut consist of an original structure posed on the blend *Numbers-Line,* called *Frame of the Dedekind's cut,* in which the blend regards rational numbers and a POINT divide the line in two DISJOINED SETS, so that every ELEMENT of A is on the LEFT, so is LESS, than every ELEMENT of B.

At a close view the blend is highly complex because in the same sentence are involved terms belonging to geometry, arithmetic and set theory and a Dedekind cut is the result of three conceptual metaphors: *Cut*, *Geometrical cut* and *Arithmetical cut.*

Only this metaphor can make sense of sentences like “R is continuous”: the metaphors *Spaces are sets* & *Numbers are point of a line* are used together to form a suitable blend that permit this.

Following this fascinating argumentations one could get used to consider these remarks unquestionably relevant, but investigating the teachers’ interviews it’s evident how much this program is infiltrated in the teachers’ declared choices in an almost total absence of awareness.

The author much more into the topic while we will only accept this result as a reference to interpret the students’ and teacher’s resistance when they face this proposal to construct real numbers.

A totally different approach to the continuity is proposed by Weierstrass; it’s very important to remember this remark reading the teachers’ interviews.

Dealing with the continuity of functions, the German mathematician have been lead to other metaphors, suitable for his different aim and that went down in history as Analysis arithmetization. In a suggestive expression this metaphors are associated the “birth of monsters”. The metaphor of continuity proposed quite implicitly by Weierstrass is the following:

Continuity of functions is proximity

Source domain **Numbers**	Target domain **Naturally continuous space**
Discrete numbers	Discrete positions of points
Sets of numbers	Curves
Numbers in the sets of numbers	Positions of points on the continuous line
Functions are maps from discrete numbers, belonging to sets, into discrete numbers, belonging to sets.	Functions are maps from points on continuous curve to points onto continuous curves.
Conservation of numerical proximity for functions in discrete domains	Continuity of functions on continuous curves.

The BMI for encapsulated intervals, using the frame ε-discs, can be used to conceptualize these ideas, creating immediately a good background for the introduction of limits of functions.

The heritage of the Weierstrass metaphor can be easily recognized in the following metaphors, that are accepted and never discussed, as Lakoff & Nunez stress:

1. The spaces are sets of point
2. The points on a line are numbers
3. The functions are ordered couples of numbers
4. The continuity of a line is numerical absence of holes
5. The continuity of a function is conservation of proximity.

We conclude with a crucial observation proposed by the authors, inspired to the reflections by Weyl that we reported extensively in Par. 1.1.1.

Weyl was wondering why should mathematicians understand the continuum in terms of discrete entities, or better, why they should feel obliged to assimilate to the new program all the possible perceptions of continuity, like flows of space and time, so far from being discrete.

Every attempt to do characterize the continuum in terms of discrete entities is necessarily metaphoric, since we are trying to grasp something in terms of something completely different, namely its opposite, and we have to be aware that many aspects of the continuum are lost.

If "arithmetizing" we look for absolute foundations of Mathematics, our hopes are destined to be disregarded.

2.2 Onto-semiotic approach (OSA) to Mathematics teaching-learning processes

The onto-semiotic approach to mathematics teaching learning (EOS, using the Spanish acronym; OSA, using the English one) stresses the relevance of the personal dimension of knowledge, that is not necessary identifiable with the institutional one (D'Amore & Godino, 2006). This is very important in our analysis since the Italian context is very heterogeneous from the point of view of teachers' formation because mathematics teacher can have a Master Degree in Mathematics but also in Physics, Statistics and Engineering.

2.2.1 Mathematical "objects" and meaning in OSA: system of practices and configurations of objects

Since this research aims to analyze teachers choices of representation of mathematical objects in the didactical transposition of real numbers, first of all we will clarify what we mean with the term *mathematical object* .

In mathematics education, as in epistemology and philosophy of language, both the terms mathematical concepts and mathematical objects are used frequently. These terms are often used intuitively as synonyms, but the distinction hides deep reasons. The different meanings of these terms depends first of all on different epistemological perspectives, that can be gathered in the main threads of pragmatism and realism (D'Amore and Godino, 2006). Also different nuances of theories, so pragmatic as realist, make sense of these terms in different ways according to his uses (Font, Godino and Gallardo, 2013). Both the interpretation of the terms *concept* and *object* are too complex to be explained fully in this work; for a complete dissertation see (D'Amore, 2001). In this paper we will draw upon terms of the ontosemiotic theory (Godino and Batanero, 2003; Godino, 2006; Godino and Font, 2007) and merely we will use the term *object.* We clarify the meaning of the term *object* and other terms used in this context in order to make it consistent with the onto-semiotic perspective we rely on.

Since the onto-semiotic approach to mathematics education is pragmatic, it is based on the definition of practice (P). Godino and Batanero (1998) consider "mathematical practice any action or manifestation (linguistic or otherwise) carried out by somebody to solve mathematical problems, to communicate the solution to other people, so as to validate and generalize that solution to other contexts and problems". The practices can be idiosyncratic of a person or institutional. We are merely interested in systems of practices (SP) that a teacher associate to the problems-situations concerning the domain of her didactical transposition. In this case we are interested in systems of practice involving single real numbers, subset of real numbers and the set R as a whole.

We opted for the definition of object and meaning by Godino and Batanero (1998), according to which we use the expression:

* *mathematical object O* indicating any entity or thing to which we refer, or talk about it, be it real or imaginary and that intervenes in some way in mathematical activity;

* *meaning of the object O* indicating the system of practices that a person carries on (personal meaning), or are shared within an institution (institutional meaning), related to the object O.

In the following part of this paragraph we resume the elements of the onto-semiotic approach that will be used in this paper. All of these were presented in the synthesis paper by Godino, Batanero and Font (2007), that can be used as reference for the complete description of the onto-semiotic approach. We label the different tools in order to use them in the analysis.

Meaning can be characterized with different nuances.

Institutional meanings of the object O are linked to these types of practices of teachers:

- *Implemented*: system of practices effectively implemented related to O (ImM);

- *Assessed*: system of practices related to O used to assess students' learning (AM);

- *Intended*: system of practices related to O planned (InM);

- *Referential*: system of practices related to O used as reference to elaborate the intended meaning after an historical and epistemological study of the object O (RM).

Personal meaning of the object O is characterized by these types of practices:

- *Global:* set of personal practices related to a specific mathematical object (GM).

- *Declared*: the personal practices effectively shown in assessment tasks and questionnaires (institutionally correct or wrong) (DM)

- *Achieved*: personal practices that fit the institutional meaning fixed by an institution (AM)

Personal practices are usually referred to students but in this paper we will refer them also to teachers as past students since the personal practices can affect teachers' choices the didactical transposition.

In mathematical practices are always involved ostensive (symbols, graphs etc.) and non-ostensive objects (objects brought to mind, unperceivable).

The six types of primary mathematical objects are defined in Godino, Batanero & Font (2007) this way:

• *Language* (terms, expressions, notations, graphics) (LAN);

• *Situations* (problems, applications, exercises, ...) (SIT);

• *Concepts*, given by their definitions or descriptions (number, point, straight line, mean, function, etc.)(CON);

• *Propositions*, properties or attributes; (PROP)

• *Procedures* (operations, algorithms, techniques, ...)(PROC);

• *Arguments* used to validate and explain the propositions and procedures (deductive, inductive, etc.) (ARG).

New objects can emerge from the system of practices, with an organization and a structure (types of problems, procedures, definitions, properties, arguments). These objects will form configurations (CO), i.e. network of objects characterized by relationships established between them.

Objects and relationships (CO) emerge by mean of sequences of practices, called processes. The primary objects emerges throughout the respective primary mathematical processes of:

- Communication (COMM);
- problem posing (POS)
- definition (DEF);
- enunciation (EN);
- elaboration of procedures (algorithms, routines, ...) (PROC);
- argumentation (ARG).

2.2.2 Dualities

Objects can have different facets, that will be coupled in pairs since they're complementary:

- *Personal – institutional*: personal objects emerge from practices carried on by a person; institutional objects emerge from systems of practices shared within an institution
- *Ostensive – non ostensive*: mathematical objects (both at personal or institutional levels) are non perceptible. However their associated ostensives (notations, symbols, graphs, etc.) allow to use them in practices. The distinction between ostensive and on-ostensive depends on the language game in which they take part. Ostensive objects can also be thought, imagined by a subject or be implicit.
- *Extensive – intensive:* this duality deals with the use of generic elements in mathematics pratices and merely let us distinguishing the particular and the general.
- *Unitary – systemic: "*in some circumstances mathematical objects are used as unitary entities (they are supposed to be previously known), while in other circumstances they are seen as systems that could be decomposed to be studied. For example, in teaching, addition and subtraction, algorithms, the decimal number system (tens, hundreds, …) is considered as something known, or as unitary

entities. These same objects in other grades are to consider as complex objects to be learned." (Godino, Batanero & Font, 2007)

- *Expression – content*: expression and content are the antecedent and consequent of semiotic functions. All kind of objects can play both the roles for a subject that establish the relation through a semiotic function.

2.2.3 Strategies to connect old and new configurations of objects: metaphors, representation and semiotic functions

To explain the nature of the relationship between objects, the definition of semiotic function is crucial, since it allows to address one goal of OSA we refer to: capturing the intrinsically relational nature of mathematics.

Semiotic function is "the correspondences (relations of dependence or function) between an antecedent (expression, signifier) and a consequent (content, signified or meaning), established by a subject (person or institution) according to certain criteria or a corresponding code." (Godino, Batanero and Font, 2007). All the six different types of objects can be expression or content of the semiotic functions. The relations of dependence between antecedent and consequent can be:

- representational (one object put in place of another),
- instrumental (an object uses another as an instrument),
- structural (two or more objects are parts of a system from which new objects emerge).

Font (2007) framed in the OSA four characteristic aspects of mathematical activity and of the emergence of mathematical objects: the duality particular/general, representation, metaphor, and contextualization/decontextualization, that have a family-resemblance, since they all involve the relation 'A is B'. The explanation of the different nuances and relations between these tools resulted to be very useful for us in order to analyze the kind of networking strategies we could use to connect the different parts of our research framework.

The main reference for Font (2007) in the field of metaphors was Lakoff & Núñez (2000), that we described in Par. 2.1.3 and we used to describe the emergent configurations from the cognitive point of view. The mapping between domains, the conceptual metaphor and conceptual blending have been put in relation with the dualities that the configurations may assume when they are used in a classroom discourse interpreted as a linguistic game in the sense of Wittengstein (Font, 2007; Godino, Batanero & Font, 2007).

We combined, as we will show in the Ch.4, the cognitive analysis proposed by Lakoff & Nunez (2000), that we resumed for our specific goal in Par. 2.1.3, and a possible analysis in terms of dualities and semiotic functions inspired to Font (2007) and Rondero & Font (2015).

We observe that while the dualities are referred to the role that a configuration plays in a classroom discourse or in a task, being it intensive or extensive, expression or content in a semiotic function, the conceptual metaphors are explicitly framed in a cognitive approach, only interested to understand how humans connect and carry out continuous double-sided analogies, basing on these quite unconscious exchanges between domains even very complex and highly abstract reasoning.

First of all we chose precisely to use this tool because it permit to us to analyze and interpret what happens in the teachers mind when they decide - maybe not being completely aware of the feelings that drive their choices - to propose to the students one problem or another, a system of practices or another. Indeed the mapping that underlie the teachers inner conceptual metaphors are the basis on which they constructed their meanings and the ways they think to mathematical objects. Thus we expect the teachers who design the teaching-learning sequences concerning real numbers to act accordingly. Since the objects emerge from systems of practices designed by teachers and mathematic is essentially relational (Godino, Batanero & Font, 2007) we expect the teachers to design the teaching-learning sequences so to make the students re-create the same – maybe unaware - conceptual metaphors.

The further aim of this comparison is to transform the precious cognitive analysis in a tool to analyze the teachers' choices in a didactical perspective; in particular to check the didactical suitability of teachers' choices in relation to the students' "proximal meanings", using thus the cognitive analysis results to forecast potential students' difficulties.

The tools of OSA are indeed deeply anchored to didactics, quite far from pure – even interesting – theoretical speculation; Lakoff & Nunez declared explicitly that their work was not empirical nor interested to be mathematical nor pedagogical, but rather in the field of cognitive science, even if sometimes didn't renounced to pedagogical suggestions and interesting conclusions about possible implications of their studies on the ways Mathematics may be taught at school.
The fellowship between the two has the potentiality to let the cognitive analysis infiltrate effectively into the analysis of teachers' practices and maybe, in the course of time. become a resource for improving innovations by means of suitable teachers' training programs. The teachers indeed are not used to reason – at least explicitly – in cognitive terms but rather in terms of "mathematical contents", problems, practices.

The cognitive analysis provides us a measure of the relevance of the metaphors that makes possible to conceptualization of real number and continuity in the learning processes, stressing that they are not certainly relegated to "a matter of curiosity" or "something that only regards smart students". The OSA analysis inflects the cognitive complexity in didactical terms.

Rondero & Font (2015; p. 1) declared in the abstract: "In this paper we present a theoretical reflection, inside the framework of the Onto-Semiotic Approach of the Cognition and Mathematical Instruction, that explores the mechanisms of articulation of the complexity associated to the mathematical object. We present an integrated view of different types of articulation covered in previous research works that have been using this approach as a theoretical framework (layers of generality, metaphoric projection and sequence of semiotic functions)." Rondero & Font (2015) proposed a reflections on the complexity of the mathematical objects base as it is frame in the OSA but also in relation to other theoretical frameworks. The articulation have been already investigated by researchers adopting this approach, but less than other strong suits. The author looked for the development of a theoretical tool framed in the OSA concerning the articulation of the mathematical objects a necessary step that lead to the unitary vision of the object, reflecting on the previous research works. We resume here part of the paper published by Rondero & Font (2015), in order to clarify the relation between different elements of our research framework, and use them to anticipate how we used the framework to analyze the very complex articulation of real numbers and the continuum as epistemic objects.

In the previous research works concerning the complexity and the articulation of the mathematical objects the articulation between the components of a configuration were described in terms of:

a) Semiotic functions (Godino, Font, Wilhelmi & Lurduy, 2011; Rojas; 2012, 2013 & 2015);

b) Generalization (Godino, Font, Wilhelmi & Arriechem, 2009; Wilhelmi, Godino & Lacasta, 2007; Pino, Godino & Font, 2011);

c) Metaphoric projections (Acevedo, 2008).

According to Rondero & Font (2015) the cognitive approach proposed by Lakoff & Nunez (2000) have strong connections to some aspect of the OSA, in particular the duality unitary-systemic.

On one hand the metaphor is unitary, since the metaphor is precisely a relation that lead to 'A is B'; in the other hand the metaphor, connecting a source domain and a target domain, lead to a new system of practices (systemic perspective) (ibid., 2015). In terms of this duality we can say that Lakoff & Núnez (2000) studied the duality unitary-systemic of different metaphors: indeed the domains, in order to be put in a metaphoric correspondence, must necessarily be decomposed in parts by means of an analytical process that isolates concepts, properties, relations in order to transfer them from a source domain into a target domain.

In order to analyze the complexity of the configurations, a first step is thus to search for the six kind of elementary objects (Par. 2.2) that constitute the epistemic configurations. In the OSA the duality systemic-unitary permit to reformulate the ingenuous sentence "a mathematical object has different representations" (Rondero & Font, 2015): we may better say that in the History of mathematics, in order to solve concrete or theoretical problems in extra or intra-mathematical contexts, appeared definitions, propositions, argumentations, representations that, time after time, were recognized to form a unique configuration that permit to carry out a system of practices. The new configuration may play the role of a generalization of the previous ones, or may constituted a *conceptual blending*, to use the terminology of Lakoff and Nunez.

Once recognized some problems and the related systems of practices, the epistemic configurations that permit to carry out the system of practices emerge, keeping a relation with the system of practices but also being recognizable as an entity that can be related with other entities emerged in other systems of practices. The new configurations can still be re-organized in other configurations that correspond to new systems of practices and so on. From all these configurations emerge a global meaning, usually identified by a unique name, like *arithmetic mean*, to quote the example analyzed by Rondero & Font (2015). In the case of real numbers and continuity, as we showed in Par. , not all the configurations of objects and the system of practices that are usually referred to the global meaning *real numbers* in the high school are really included in this global meaning. As Lakoff & Nunez stressed the real numbers are anything but a domain, yet very complex in itself, that must be metaphorically related to another domain, that of natural continuity, that is not included in the discretized blend *Numbers are point of the line* that is the core with BMI of the of real numbers as results of cognitive activities.

Looking at the duality unitary-systemic in the History of mathematics to understand how the different configurations or, on the other side, their different components, were used in the development of mathematical activities, Rondero & Font (2015) observed that the relations of generalization made emerge the versatility of the arithmetic means thanks to its possibility to be disarticulated in partial meanings and systems of practices that find a place in many fields of Mathematics.

The authors presented three articulation mechanisms: levels of generality, metaphors and semiotic articulation.

The metaphorical projection involving real numbers have been already reported in Par. The articulation of the epistemic meaning is presented in Par. , as long as its semiotic articulation.

Here we present only a resume of the theoretical aspects concerning the semiotic articulation.

Font (2007) highlighted the existence of an analogy between the following processes:

1) representation
2) metaphorical
3) particular-general

The analogy is based on the fact that all of them regards the relation “A is B”, but in different grades of subordination: from a total subordination, the is represented by the relation particular-general, to a total “parity” between the two, represented by the metaphor, in which the mapping may at most not be biunivocal. In the middle finds a place the representation, relying on the position expressed by Peirce (1978): a representation may be identified or differentiate from its referential object depending on the goals. The mechanisms underlying the representation process may be of different kinds.

A *semiotic concatenation* is a process through which the same expression is associated to different objects that are related in some ways (Rojas, 2012).

A *meaning articulation* is produced when a semiotic function between two different meanings of the same primary object is established and one of the two becomes an expression of another, that becomes the content, by means of a further semiotic function. The emerging object is thus the referential meaning (content) of the two objects and then the two object get connected each other . The result is a *semiotic function chain,* that is the elementary part of the intertwining we identify with the global meaning. If the teachers don’t take care of this complex structure, as a result, the students won’t connect each other the meanings and will consider them different objects even in the cases in which they are used in the place of the other or to represent the other as a content in the didactical practices.

Once made emerge the structure and the different connections another important question is: how the components are transformed in a unitary object whose role is to constituted somehow the “commonality” of the objects that form the meaning as a net? Since the OSA relies on a pragmatic ontology, i.e. the objects are considered emerging from system of practices, far from a realist perspective, the objects are considered virtual entities “re-created” by people in discourses, thought as *linguistic games* in the sense of Wittengstein (1953).

The virtual object change time after time his nature going on weaving the net of partial configurations: in the beginning it’s the referential object of a first configuration of primary objects – representations, definitions, propositions, procedures, problems and argumentations - then of the growing system of configurations (Font, Godino & Gallardo, 2013). How the play of complex references leads to a unique virtual object can generate a unique referential objects is explained in the OSA by means of combined effects of dualities – intensive-extensive, personal-institutional, ostensive-not ostensive, expression-content, unitary-systemic (Rondero & Font, 2015). The global reference comes into being as an epistemic configuration that can be manipulated in the mathematical activity sometimes as a unique entity, sometimes as an articulated object composed by epistemic configurations, sometimes identified with the partial configuration that is useful for a specific goal. In words of Rondero & Font (p. 44, translated by us): “Somehow we can consider this object as one and many at the same time.”

2.2.4 Didactical suitability

“The didactical suitability of an instructional process is defined as the consistent and systematic articulation of the following six components (Godino, Batanero & Font, 2007):

1. *Epistemic suitability*
2. *Cognitive suitability*

3. *Resource suitability*

4. *Affective suitability*

5. *Interactional suitability*

6. *Ecological suitability*"

The components of Didactical suitability concerning our analysis are some of those "objective criteria that serve to improve the teaching and learning and guide the evaluation of the teaching/learning processes. [...] *Epistemic suitability* measures the extent to which the implemented meaning represents adequately the intended meaning (the curricular guidelines for this course or classroom). *Cognitive suitability* is the degree to which [...] the implemented meaning is included in the students' zone of proximal development, and whether the students' learning (personal meaning achieved) is close to the intended meaning. *Ecological suitability* is the extent to which the teaching process is in agreement with the school and society educational goals, and takes into account other possible social and cultural factors." (Godino, Ortiz, Roa & Wilhemi, 2011).

In particular we investigated epistemic representativeness, declared cognitive proximity and declared strategies of negotiation of meanings in the didactical interaction, that we describe in depth in this Paragraph. In Ordoñez y Wilhelmi (2010) a typical methodology to investigate teachers' reflection on their practices is presented; in particular in relation to the epistemic meaning the teacher should be able to explicit the objects and processes involved in the didactical practice and to propose changes to make it as best as it can be (Pino-Fan, Godino & Font, 2014).

To evaluate the didactical suitability of the teachers' proposals we refer to Breda & Lima (2016), who specified some criteria to analyze the teachers reports of didactical sequences and, in particular, the reasons why the teachers thinks their proposals are good.

We report here the Table they propose in the paper we quoted before.

Epistemic suitability

EPS1: Errors: No practices are wrong from a mathematical point of view

EPS2: Ambiguity: We can observe no ambiguities that can confuse the students: wrong or not clear definitions, not adequate definitions or explications for the students' level

EPS3: Processes richness: The sequence include relevant processes from the mathematical point of view (modeling, argumentation, problems solutions)

EPS4: Representativeness: the partial meanings are representative examples of the complexity of the mathematical objects that the teachers is going to teach and is included in the national curricula; for every partial meaning problems are presented; for every partial meaning different ways to represent objects and to connect them each other are presented.

Cognitive suitability

CS1: Previous knowledge: the students own the necessary previous knowledge to study the new topic; the expected meanings to reach are possible to achieve in all their components.

CS2: Adaptation to the individual differences: reinforce activities are included in the curriculum

CS3: Learning: the different assessment methods show the knowledge and skills appropriation

CS4: High cognitive request: Relevant cognitive and metacognitive processes are activated

Ecological suitability

ECS1: Adaptation to the national curricula: the contents, their implementation and evaluation correspond to the curricular indications

ECS2: Intra and inter-disciplinary connections: the contents are related with other mathematical contents (with advanced mathematics and the other curricular contents) and with contents belonging to other disciplines (extra-mathematical contexts or other educational steps)

ECS3: Socio-working utility: The contents are useful for the students' future job.

EC4: Didactical innovation: Innovation based on researches and reflections is taken in account.

2.2.5 DMK (Didactical-mathematical knowledge): a proposal for the analysis of the teachers' knowledge

In the model DMK, proposed by Godino (2009), the teachers' knowledge's dimensions that may affect the design and realization of teaching sequences are three: 1) mathematical; 2) didactical; 3) meta didactical-mathematical.

The mathematical dimension concerns the teachers' knowledge of the mathematical configurations of objects and processes, categorized in:

- common knowledge (CK)
- advanced knowledge (AK)

The didactical dimension concern the knowledge about the six aspects of the teaching-learning processes: epistemic, cognitive, affective, interactional, mediational, ecologic.

The meta-didactical-mathematical dimension regards the knowledge the teacher need to reflect on and evaluate the suitability of the teaching-learning environments and actions they design and realize.

The notion of configuration of objects and processes is a tool that allows to describe the mathematical practices' complexity and to use that complexity to interpret the cognitive conflicts (Tall & Vinner, 1981) emerging in the teaching-learning processes.

2.2.6 ATD and OSA: a comparison

2.2.6.1 Didactical transpositions in ATD: past and current definitions

The concept of didactical transposition in mathematics education was introduced by Chevallard in 1985. As it happened for many other constructs, it was declined in many forms by the author itself and other researchers, also belonging to other research fields. In this paper we will refer to the review article by Chevallard and Bosch (2014), according to whom didactical transposition is an analytical instrument to avoid the <illusion of "authenticity" of the knowledge taught at school>. Didactical transposition consists in rebuilding <an appropriate environment with activities aimed at making this knowledge "teachable," meaningful, and useful>.

The institutional dimension is the most stressed in this framework, so much that the concept of didactical transposition was generalized to institutional transposition: the knowledge is transposed from one social institution to another . In this change it "is thus important to understand the choices made in the designation of the knowledge to be taught and the construction of the taught knowledge to analyze what is transposed and why and what mechanisms explain its final organization and to understand what aspects are omitted and will therefore not be diffused." Teachers, as long as producers of knowledge and curriculum designers, contribute in this decision process. The different weights of teachers choices depend on the autonomy of teachers in the school system.

The previous conceptions of didactical transposition, proposed by Chevallard in 1985 and then in 1999, stressed more the importance of the transformation of the knowledge in three main phases and three related "identities" of the mathematical knowledge at school: 1) Savoir savant 2) Knowledge to teach; 3) Taught knowledge. The passage from the 1st to the 2nd step was called Didactical transposition, while the second was called Didactical Engineering.

2.2.6.2 Institutional and personal relation with the mathematical objects in ATD and OSA

In order to rebuild the knowledge to make it teachable, meaningful and useful making choices the teachers need to decompose *mathematical objects'* structure and explore their nature, so as to select aspects to omit and to diffuse and to better understand what to change, how to transform the and finally how to rebuild the aspects of the mathematical object we chose to teach.

While in the first formulation the concept of mathematical object was more static and somehow universal, the second version, developed thanks to the new conception of *praxeology* (Chevallard,), took in account more the pragmatic dimension of mathematical objects and its relation with the social human development in communities and institutions. *Praxeologies* – composed by praxis and logos – include <the study, not only of what people do, and how they do it, but also of what they think, and how they do so> (Chevallard, 2005). Chevallard proposed the quatern (task, tecnique, technologies, theories). Mathematics turned to be considered as a product of human communities of practices, more and more shared, widespread and consolidated through reflexive practices that lead progressively to theories. These processes were named institutionalizations of knowledge, whose products were indeed a set of cultural objects, that we call Mathematics. This conception of Mathematics inspired a true revolution in Mathematics education, describing in a complex manner the different natures of the mathematical practices and permitting to rethink the design of didactical activities and curricula.

In the framework called ATD, Anthropology Theory of Didactics (Chevallard,), it was also stressed the role of the personal relationship between the subject and the mathematical objects, but this aspect was explored more by the authors of another framework, inspired to ATD and many other previous contributions and theories of Mathematics education: the onto-semiotic approach to Mathematics teaching and learning investigation (OSA).

2.3 Schoenfeld's theory of goal-oriented teachers' decision-making

As theoretical framework for teachers' choice we chose the goal-oriented decision-making theory by Schoenfeld (2010). Drawing on this theoretical framework we will consider teachers as decision-makers, whose choices are determined by "their *resources* (their knowledge, in the context of available material and other resources), *goals* (the conscious or unconscious aims they are trying to achieve), and *orientations* (their beliefs, values, biases, dispositions, etc.) [...] at both macro and micro levels." (Schoenfeld, 2010, p. 14). The choices are generally embedded in the institutional context, but this framework deals in particular with choices of the teachers in real-time.

This is particularly interesting for us since our research had only a few contacts with the empirical research in the classrooms: we only carried a pilot teaching-experiment, that inspired our research; analyzed a research work proposed by Bagni (2000) concerning the students' understanding of properties of real numbers; observed some lessons carried out by the teachers we involved in our research.

In the Schoenfeld's model the teacher is analyzed as a decision-maker who has to solve continuously didactical problems. Schoenfeld hypothesized that the teachers' behaviors – answers to the students' questions, decision to go on discussing with the classroom or interrupting a discussion even if it was interesting, to foster the students' argumentations skills, and so on – not only could be imagined, but even forecasted in an almost deterministic way once known the teachers' orientations, goals and resources.

Adopting this perspective we hypothesize that even if we don't enter the classroom collecting data in this phase, an a priori analysis of teachers' orientations, goals and resources may provide us significant information and thus we carry out the analysis of the teachers' choices in the design phase hypothesizing that this information could:

- help us in foreseeing potential teachers' behavior in the classroom
- help us in planning the research methodology
- permit to us to design a long-term research based on the comparison of *a priori* hypotheses about the teachers' real-time choices and the observed teachers' reactions in the contexts in which they really implement their teaching-learning sequences

This approach seems to contradict the researches concerning the frequent inconsistencies between the teachers' declarations and the teachers' observed behaviors. What we decided to plan using this approach was a research methodology that could permit to us to interview the teachers in such a way that the data we collected could bypass the teachers' defenses and somehow circumscribe the unavoidable effects of the *contract* that is established by the teachers and the researcher (a sort of variant of the *didactical contract*).

In particular in the questionnaire we tried to avoid too direct questions when we wished to have information about the teachers orientations and to ask the teachers to comment on videos, materials, students' answers, in order to make them react to something and expressing their orientations about the specific situation rather than talking generally.

We are not sure that this approach could really permit to forecast the teachers' behaviors and real-time choices during the implementations, but we prepared the field for further investigations that can play the role of a counter-example or rather can confirm the Schoenfeld's hypothesis.

An important concern the possibility of using this framework in a networked research framework (Radford, 2008; Bikner-Ahsbahs & Prediger, 2014).

We have reasonable motivations to consider this framework compatible with OSA:

- the work of Schoenfeld inspired the researchers who elaborated the OSA since the very beginning, thus we are lead to consider the approaches consistent at least in the basis;
- there are common assumptions and resemblances: in particular in the both of the frameworks the teachers' disciplinary and didactical-pedagogical knowledge is considered very important in order to analyze the teachers' decision-making processes; in the OSA it was elaborated the teachers' DMK as a result of many elaborations of previous works by other authors and the previous OSA researches involving teachers; in the Schoenfeld's model the resources include both mathematical and pedagogical knowledge

but some choices are necessary because there is also a critical point: while the development of the OSA lead the author to detail more and more the analysis distinguishing clearly what the teachers do in the design phase, during the classroom activities and after the classroom activity, the Schoenfeld' model seems to suggest that the second and the third phase should be "deduced" by the first one. We consider very valuable the distinctions operated by the OSA but in this case we will analyze the teachers' answers as if the *a priori* and the *a posteriori* empirical analysis could be definitely not so different at least in the case of the teaching-learning processes involving real numbers and the continuum, since in such a complex case the design and the orientations may affect significantly the development of the whole activity.

We gave thus a great importance to the analysis of the teachers' design, trusting the reliability of the Schoenfeld's hypothesis.

3. Research design

What happens really when the teachers enter a classroom and in the students' mind depends on many factors, as it was highlighted in many researches and it is theorized in the OSA – see the many facets of Didactical suitability in the Paragraph 2.2.4. We investigated how the teachers' cognitive configurations (Godino, Batanero & Font, 2007) and the teachers' choices are intertwined in order to understand better some possible relations between the teachers' personal mathematical-didactical knowledge (DMK) and the didactical practices:

1) how the personal meaning of real numbers affects the teachers' choices;

2) how far the teachers' choices are responsible for potential students' difficulties with real numbers;

3) which factors affect more the eventual unsuitable choices (formation, experience as students, experience as teachers, didactical and pedagogical orientations,)

We conjecture that if in the design phase we can already find the seeds of potential difficulties, the whole teaching-learning processes can't be anything but unsuitable, being the other phases even more complex and exposed to greater distorsions. Thus we decided to investigate merely this aspect.

3.1 Research questions and research hypotheses

1) In order to evaluate the instructional processes didactical suitability we need to analyze deeply the complexity of real numbers as a mathematical object from an epistemic and cognitive point of view (in the sense of OSA).

GQ - 1 How can we describe the complexity of the teaching-learning processes involving real numbers and the continuum from an epistemic, cognitive and ecological point of view?

PQ - 1.1 What is the epistemic meaning of real numbers?

PQ - 1.2 What is the teachers' mathematical knowledge of real numbers?

PQ - 1.3 What relations between teachers' formation and their mathematical knowledge of real numbers emerged?

PQ - 1.4 What are the teachers' declared goals and orientations?

PQ - 1.5 What systems of practices concerning real numbers do the teachers declare to prefer and to choose and to prefer?

PQ - 1.6 What are the relations between teachers' mathematical knowledge, orientations and goals and their declared choices?

PQ - 1.7 Is any categorization of teachers' profiles possible using the answers to the questionnaires? If yes, what categories emerged from the investigation?

2) If the systems of practices associated to a configuration is not rich enough in terms of epistemic representativeness, is not cognitively suitable or isn't well connected with other systems of practices, when the teachers tries to involve students in classroom discourses, the students may not be able to participate in the discourses and to grasp the new configuration. Maybe the choices suitability is related to the teachers profile we identified before.

GQ - 2 Does the didactical suitability depend on the categories we used to describe the teachers' DMK? Which is the relation between them?

PQ - 2.1 Which is the relation between the national curricula and the epistemic meaning?

PQ - 2.2 Are the teachers' choices epistemically, cognitively and ecologically suitable?

PQ - 2.3 May teachers' search for cognitive suitability cause the lack of epistemic suitability in the case of real numbers?

PQ – 2.4 Is any overall categorization of teachers' profiles possible? If yes, what categories emerged from the investigation?

3) We observe interesting dynamic interviewing the teachers.

GQ – 3 Could our methodology of research be useful in the teachers' training concerning real numbers and the continuum?

Our research hypothesis is that often the configurations of objects and processes used to introduce other situations-problems and to construct other configurations are not complex and rich enough to support the new attempt or constitute different meanings in respect of the needed ones (see for instance the conception of R as union of rational and irrational numbers and the definition of R as set of the limit points of Q; it's not the same to use one or the other to introduce limits of real functions in the real domain).

The relation between the institutional and the teachers' cognitive meanings is crucial since indeed the teachers are "institutional messengers" and the students are given a knowledge filtered by their teachers' personal meanings and choices.

3.2 Methodologies of qualitative research

We carried out a mixed-methodological study – qualitative and quantitative (Onwuegbuzie and Leech, 2004).

Our methodology was inspired first of all by two methodologies of qualitative research presented by Neuman and Pirie in the monograph on qualitative research in mathematics education edited by Teppo (1998), even if the subject involved in their researches were students. As Neuman we opted for a phenomenological observation technique and for the construction of descriptive categories. While in this framework the results depend strictly on the model, in our research analyzing data other questions emerged that we didn't consider in the beginning. This is contemplated by Pirie's methodology, that allow to let emerge new questions from data, going on with cyclic analyses and making more and more precise the categorization. Then we grounded our investigation on prototypical methodologies used in the OSA, in particular those referred to epistemic analysis of mathematical objects' complexity, teachers' cognitive configurations and didactical suitability.

"Consistent with this assertion, and expanding on Rossman and Wilson's (1985) work, Greene, Caracelli, and Graham (1989) categorized the following five general purposes of mixed-methodological studies: (a) triangulation (i.e., seeking convergence and corroboration of findings from different methods that study the same phenomenon); (b) complementarity (i.e., seeking elaboration, illustration, enhancement, and clarification of the findings from one method with results from the other method); (c) development (i.e., using the findings from one method to help inform the other method); (d) initiation (i.e., discovering paradoxes and contradictions that lead to a re-framing of the research question); and (e) expansion (i.e., seeking to expand the breadth and range of inquiry by using different methods for different inquiry components). As observed by Greene et al. (1989), every mixed methodological study can be classified as having one or more of these five purposes. In recent years, the advantages of mixed methods research have been increasingly recognized." (Onwuegbuzie and Leech, 2004).

3.3 Research methodology

In our research 107 Mathematics high school in-service teachers in Italy have been involved. Teachers were asked to answered an online questionnaire designed to investigate:

- teachers' formation (master degree, training courses attended)
- teachers' knowledge (configurations of objects they associated to R set)
- the practices chosen by teachers involving elements or subsets of R or objects traditionally used in the constructions of real numbers like inequalities, Q etc.
- the semiotic representations of subsets of rational and real numbers they consider best in order to address a goal.

The teachers were all Italians, coming from different regions and characterized by different features in relation to mathematical background, formation as teachers, years of experiences as teachers in the high school, kind of school in which they taught.

All the teachers we contacted were certainly high school Mathematics teachers since the users could answer the questionnaire only if they had been invited and were asked to register using an e-mail address.

25 teachers participated in a pilot study, that we used to test the questionnaire clarity and effectiveness for our goals; other 117 teachers were asked to answer the questionnaire, but only 84 completed at least the first part concerning knowledge, 66 the whole questionnaire. Other 4 teachers were interview in focus group without asking them to answer the questionnaire before, in order to check eventual significant differences between the teachers who had the occasion of reflect at home on the topic while answering the questions and teachers who were interviewed without a previous phase.

In the first part the teachers were asked to answer direct open questions (the question intent corresponded to the request) about knowledge and goals, while in the second part the questions aimed at making come into light preferences and orientations in an indirect way.
Using the Schoenfeld's categories (2010) some of the questions about knowledge have been planned in order to give information about resources, the other questions concerned the goals and the orientations.
Teachers were asked to comment videos about a construction of using a graphical method, (http://www.youtube.com/watch?v=jk08WkwqT Q), an applet titled "Correspondence between real numbers and points of a line" (http://www.youtube.com/watch?v=kuKTyp_b8WI) and a video-recorded lesson carried out by a teacher about different ways to solve and to represent the solution of inequalities (http://www.youtube.com/watch?v=UEBK5DfPxvk). Teachers were asked to indicate preferences about didactical materials and students' answers - in order to investigate orientations by means of semi-guided questions.
After answering the questionnaire the teachers were interviewed in focus groups (3 or 4 members) in which we guided a discussion on questionnaire answers in order to make them explicit their personal choices and the reasons of their choices and to investigate their general orientations concerning the didactical transposition of real numbers and students' difficulties.
We defined some *a priori* categories standing on the literature review and the cognitive analysis we presented in the previous Chapters and on the criteria of didactical suitability we presented in the research framework. We established as a criterion for collocating a teacher in a category the presence of sentences that unequivocally let us deciding if she belonged or not to the category. Some further categories emerged after the first data analysis and other categories were created. Only significant categories are reported and for some categories that could results ambiguous we reported also exemplar sentences (in Italian, the original language in which the teachers answered).

3.4 DMK analysis of the methodology

In this Paragraph, inspired by the analysis proposed in Pino-Fan, Godino & Font (2014), we present the analysis of the Didactical-mathematical knowledge (DMK) of the in-service teachers, proposed by Godino (2009) and framed in the OSA (Godino, Batanero & Font, 2007). DMK was elaborated standing on the models proposed by other authors about teachers' mathematical, didactical and pedagogical knowledge in a wide sense .

There could be significant differences between teachers' declared practices and teachers' real practices and also there are other dimensions of suitability of the didactical actions but in this particular case we conjecture that unsuitable choices in the designing of teaching sequences vanish all the attempts to make effective the didactical actions and furthermore that deficiencies or unawareness in this phase imply unavoidably bad choices concerning the other dimensions.

We used OSA's methodologies and analytical tools to analyze the practices the teachers report from their didactical past experience concerning real numbers and continuous sets or declare to do in their classrooms usually, depending on the kind of students they are working with. In particular we will analyze the configurations emerging from the different practices they declare to plan or to have already experienced and we will compare the configurations they would need in the teaching sequences they propose with the ones we can hypothesize they use standing on their statements and reports.

The questionnaire

The version of the questionnaire we used to investigate the teachers' didactical-mathematical knowledge (DMK, Godino, 2009) is the result of an elaboration of a previous pilot version we tested with 20 in-service teachers and that we asked researchers and teachers to evaluate.

We asked to answer online the definitive version of our questionnaire to 107 in-service teachers. The teachers could use books and other resources, answer the questionnaire using a lot of time and re-opening the questionnaire at any time after the conclusion of the questionnaire. This way we gave the teachers the time to rethink at the mathematical object and at their teaching-learning experiences.

Not all the teachers answered the whole questionnaire but anyway we took in account their answers for quantitative and qualitative analyses, even if the *case-studies* we used for our conjectures and conclusions are based on completed questionnaires and interviews.

The 107 in-service teachers were all Italian but had very different backgrounds from the point of view of formation (PhD in mathematics, Master Degree in Mathematics, Teachers training courses, Master Degree in STEM disciplines different from Mathematics, like Physics, Engineering, Statistics or combination of these like PhD in Mathematics who attended teacher training courses and so on). Also they had many different features like different ages, different teaching experience, different regional provenience, different socio-economics contexts in which they had taught, different kind of high school (technical, humanistic, scientific, classical, professional high schools).

The reasons why in the first part of the investigation we decided to ask different teachers' to answer the questionnaire were merely three:

- in Italy there are many possible paths that allow to teach Mathematics in the high school and, being the mathematical object we are analyzing very complex, the teachers' mathematical knowledge concerning this object could be very different. In order to investigate teachers' choices and to propose possible teachers' training courses concerning this topic we needed to have a wide panoramic on different possibilities.
- Since we were going to correlate teachers' DMK (Godino, 2009) and their declared choices, a research problem we consider interesting in this case is to understand how far teachers knowledge influence teachers' choices. We conjectured in the very beginning that, given the nature of this mathematical object, in some aspects intuitive and formally very articulated, the choices proposed by the teachers about some of the partial meanings could be very similar, in spite of their personal knowledge.
- Teachers with different background or teachers who work in very different context may have different goals and so they may plan differently the teaching sequences or have different opinion about the effectiveness of teaching practices.

The questionnaire was composed by 18 questions. The first questions concerning teachers' knowledge concern the mathematical aspect.

In the first question we asked the teachers information about their background: which Master degree they obtained, which certificated teachers' training courses they attended, in which kind of high school they teach actually and since how many years they are teaching.
The aim of this first question was to identify the teachers and so to interpret better their answers.

In the second question we asked the teachers information about their learning experiences concerning real numbers, giving the opportunity to choose more than one possibility between: at school, at the University in a Calculus course, at the University in other courses, in a teachers' training course, in educational books and in the original books as a self-taught.
In the third question we asked the teachers what are the main features of the set of real numbers in an open format in which the teachers could answer freely.

In the fourth question we posed a crucial question concerning the way the set of real number can be constructed starting from Q as an enlargement. The formulation of the question was explicitly elaborated in order to recall a very widespread practice of introduction of real numbers when the teachers introduce quadratic equations that have no rational solutions or geometric magnitudes whose lengths, in respect of a given unity, needs to be expressed with a not rational root square. Many textbooks and teachers we had talked with reported the usual discourse concerning real numbers as the set in which it's possible to find the root squares of numbers - not always making explicit positive numbers. The answer to this questions allowed us to discriminate very soon between teachers who had studied and understood at least one real numbers set's construction. Indeed, as we explained before, the large heterogeneity of the paths that permit to the people to become mathematics teachers in the high didn't guarantee that the teacher had neither a good

common knowledge concerning real numbers. If the teachers as students learnt that the set of real numbers could be constructed by means of an enlargement of Q with roots (algebraic extension of Q) and some transcendental numbers, maybe as a teacher she would re-propose the same scheme, being this partial meaning her meaning of real numbers.
Even if a written answer doesn't allow to figure out complete profiles of the teachers, the way we expressed the question after our pilot study permitted to us to differentiate the teachers in categories by means of their written answers.

In the fifth question we explored more in depth the teachers' knowledge about the differential and topological structure of R and its relation with the structure of Q, asking if it was possible to define a limit point in Q or if we need R. The aims of these question were indeed more than one. This question permitted us also to classify teachers in relation to their didactical goals and to understand the role they give to real numbers in the Calculus, that is the field in which, depending on the planned didactical sequences, the properties of real numbers could be really essential. In fact we asked to the teachers if the limit point, usually introduced in the high school in order to talk about the limits of functions in the boundaries of their domain, need the properties of real numbers to be introduced or is an independent object, that we can define in a field with less properties. The teachers we interviewed in the pilot study and the textbooks we analysed introduced indeed real numbers a lot of time before the limit points. We consider this point crucial. In fact the didactical sequences in which the introduction of real numbers arrives very early, indeed the whole meaning comes to be identified with a partial, very poor, meaning. This can deal to the following kind of conflicts or deviations from the original meaning:

1. the students may not be able to fit the old meanings of real numbers with the concept of limit point, crucial to pose the problem of the construction of a complete set, without a specific discourse in which the different branches in the epistemic meaning tree;
2. to quote R as the set who contains all the limit points of sequences of Q, R with high probability R must appear as a unitary entity, maybe represented with a line or simply indicated with the symbol R; in both the cases the choice is quite problematic since as we observed in Par. 1.2 the line is not usually interpreted as dense by the students, nor truly composed by infinite elements and the visual representation is very often the only one associated to R as a unitary object. Since we need topological and differential local configurations to talk about dense sets, necessary to introduce limit points, there is a concrete risk to use partial meanings of real numbers that are not complex enough to address this topic; this fact could cause cognitive conflicts.

The two following questions concern the teacher's relation with the national curricula and possible functional goals of the introduction of real numbers. This last point is crucial since, while the real numbers should be part of the curricula, often they are introduced to solve other problems, of different nature, or to define other concepts rather than being the direct goal of a learning sequence. Since this could be very relevant for analyzing teachers' practices concerning real numbers but maybe the teacher would have take it for granted, we listed explicitly possible contents in which the properties of real numbers could be considered necessary.

Once established which was the teachers' aware knowledge concerning real numbers and what were the teachers' goals related to real numbers, we went on showing the teachers some videos reporting very frequent teaching practices concerning real numbers. Two of these (D8 and D9) were explicitly aimed at making emerge partial meanings of real numbers, the last one should use and represent subsets of real numbers, at least in the the original intention. We know from the literature that students', and sometimes teachers', interpretation of the properties of subsets of real numbers in the "algebraic" and graphic register are quite different from those we can expect standing on the institutional meaning. Also through this question we could explore the way the teachers perceive the relation between the "algebraic" and the graphic representations of the intervals, two different partial meanings of real numbers historically connected in a very complex and interesting way.

The last questions concerned tasks assigned to the students and the degree of acceptance of the correctness of the solutions that the students proposed. Each of the solutions could provide us relevant information concerning the representation of intervals of numbers and the association of personal meanings to them by the teachers. In this questions teachers were expected to choose the solution they considered right and to say why the other were wrong. Asking the teachers to comment on the correctness we ask indeed them to interpret the signs and so to associate their personal meanings to them. Furthermore we ask them to put in the clothes of a student who answered that way, in order to see how the teachers correct the students, i.e. if they take in account their personal meaning or correct the students referring to their own meanings.

The detailed Didactic-Mathematical Knowledge (DMK) analysis of the questionnaire is proposed in Par.

4. Data analysis

4.1 **Analysis of the epistemic meaning of real numbers by means of the OSA's tools**

In the following paragraph we will present the holistic meaning of real numbers. This scheme was created following an economic principle: to use the minimum objects and processes that were necessary to create a new configuration. This is the reason why this scheme doesn't represent exactly the historical evolution, or better the historical systems of practices, and doesn't follow a strictly chronological order, even if the chronological evolution is obviously important. Making our choice we were aware that the evolution of mathematical concepts is not linear and that the socio-cultural contexts influenced deeply every mathematician's work. For instance let's consider the cases of the famous epistolary relations between mathematicians: Eratostene and Archimedes, Cantor and Dedekind. When we will not connect two configurations or systems of practices attributed to mathematicians who collaborated in their life we focus only on the configurations of objects and processes really necessary to them to develop their partial meaning of real numbers.

To explicit our choices we report an intermediate scheme that resumes the main steps of the historical evolution of real numbers we took in account (attached in the end of the book)

4.1.1 Systems of practices and configurations in the history of real numbers: the real numbers epistemic meaning

The mathematical object that we call “Set of the real numbers” is very complex and is the results of a very articulated evolution that took an amount of partial steps, scattered all along more than 2000 years. We resumed part of these partial steps in Par 1.1.1.

What characterize the set of real numbers is the great quantity of kind of practices, processes and fields of investigation that in the end flew into the unique denomination “Set of the real numbers”. Furthermore, as we stressed in Par 1.1.1, the relation between the different meanings - the intuitive and the formal ones, the static and the dynamics ones, the practical and the theoretical ones - is sometimes problematic and considered an open problem also by eminent mathematicians.

The risk that we perceived when we decided to face this topic, from a didactical point of view, was the fact that the teachers could not be aware of this complexity and could move backward and forward without taking care of the different natures and histories of the partial meanings. Also, as we stressed, not all the partial meanings related to the continuum, the line, the motion – that are a very important component of the holistic meaning of the real numbers - flew into the configuration “Set of the real numbers”. After the diatribe Newton-Leibniz, that we didn’t deepen, some other important debates emerged concerning the relations between the intuition and the Calculus, the objects of study of Mathematics and Physic, the rigor of the bases and the utility of the infinitesimal procedures.

In this Paragraph we present a possible analysis of the configurations of objects and processes that characterize the holistic meaning of real numbers but selecting only the configurations and the practices that may concern real numbers in the high school; in the further one we present possible cognitive configurations based on the analysis of *conceptual metaphors* and *conceptual blending* by Lakoff & Nunez (2000) , synthesized for our aims in Par. 2.3, and the correspondent analysis based on the connecting strategies presented in Font (2007).

This scheme is the result of a selection that we consider relevant for our analysis and that permit to us to answer the first particular question we posed:

PQ - 1.1 What is the epistemic meaning of real numbers?

In order to carry out hereafter the analysis of the connecting strategies we will make distinctions since the very beginning between domains that can be connected using *grounding* and *linking metaphors.* In the first case, standing on the definition itself, the source domain should be very simple and experienced (grouping objects, container schemata, trace a line, ..). We propose the same primitive embodied mechanisms both for epistemic and cognitive analysis, while we operate distinctions in the research tools between the epistemic meaning, analyzed in terms of historical systems of practices and configurations, and the cognitive meaning, described in terms of metaphor and dualities that allow to picture possible cognitive configurations and their inner cognitive structure. At the side of every configuration we wrote the prevailing connecting strategy.

- **Primitive elements as "fundamental embodied cognitive mechanisms"**

1) Gr1 := Fictive motion

2) Gr2 := Container schemata

3)Gr3:= Objects collection

4) Gr4 := Measuring stick

5) Gr5 := Object construction

6) Gr 6 := Motion along a path

1. **First level: Primitive problems and emerging configurations of objects and processes (1st level's partial meaning)**

P1: *To measure and compare homogeneous magnitudes (Gr4)*
CE1: <u>Commensurable and incommensurable magnitudes</u>
[Eudoxus' theory of magnitudes] & [Measures as comparios the unit] & [Pythagoreans problem of incommensurability of linear magnitudes]

P3: *To characterize the continuum in terms of atomic discrete entities (Gr5)*
CE3: <u>The line is composed by points</u>
[Unlimited divisibility of a segment] & [Points are the atomic results of divisions] & [The line is composed by points] & [Characterization of the continuum and relation between contiguous and continuous] & [Infinitely small distances between contiguous parts of a segment] & [Euclidean geometry's relations between points and lines]

P3': *To identify positions on a line (Gr6)*
CE3': <u>Points are positions on a line</u>
[Points are results of cuts of the segments] & [In a cut a point can lie on one of the two parts or the cut may generate two endpoints, distinct but not far]

P4: *To describe trajectories as result of motion (Gr6)*
CE4:= Line is a trajectory
[Segments are traces of mobile points] & [Points are extremes of extendable segments] & [No stops in the motion implies no breaks in the segments and no gaps]

P5: *To solve equations and inequalities*
CE5: Equations
[Babylonians tables] & [Diofanto's equation] & [Methods for 2nd order equations or more]

P6: *To operate with quantities in contextualized problems of real life (addition, subtraction, multiplication, division, power, exponential, logarithms) (Gr3)*
CE6: Arithmetic
[Operations algorithms] & [Modeling of real life problems with operations] & [Procedures and methods to simplify computation with great numbers] & [Positional systems with Indo-arabic digits and decimal representation]

P7: *To solve "Classical geometrical problems"*
CE7: Geometrical constructions
[Cube duplication] & [Angles trisection] & [Circles rectification] & [Line and compass constructions] & [Regular polygons' construction] & [Aureum section]

P8: *To find a relation between finite and infinite (BMI)*
CE8: Potential and actual infinity
[Infinite as unlimited and unbounded] & [Zeno's paradox and infinite paths that generate a convergent process] & [Infinity definition] & [Cusanus' "infinite finite", curves as limits of polygonals] & [Galileo's composition of curves of infinite infinitely small linear elements] & [Cavalieri's indivisible elements don't recreate the whole figure]

P9: T*o describe physical dependent variations in space and time (Gr1, Gr6)*
CE9: Variations
[Phenomena may be described in term of quantitative time and space variations] & [Graphic representation of phenomena in time and space] & [Speed as a tool to differentiate motions with the same trajectory] & [Acceleration as speed variation rate]

2. Second level: New problems and configurations of objects (2nd level's partial meaning)

P2.1: *To model physical phenomena in terms of variable's variation and dependence using mathematical models*
C2.1 := CE4 & CE9 : Continuous functions for modeling physical phenomena (supposed to be continuous)

[Semiotic articulation]

[D'Alembert wave's function] & [Arbogast's definition of continuous variation] & [Study of bounded or asymptotic variations corresponding to bounded variations of an independent physical variable] & [Interpolation of data through naturally continuous curves associated to phenomena that are considered continuous (volume's variations, pressure's variations, ...)] & [Continuous variation of speed and acceleration]

P2.2: *To describe motion, segments, 2D figures, volumes using infinitesimal quantities*
C2.2 := CE3 & CE8 & CE9 : Infinitesimal methods [Semiotic articulation]
[Galileo's cinematic problem of describing curves by means of infinitely small linear variation] & [Cavalieri's infinitesimal method to calculate areas] & [First integration methods] & [Torricelli-Barrow theorem] & [Continuum is a limit of "sums" of infinite, infinitely small, elements]

P2.3: *To associate points of the line to geometrical constructions and algorithmic procedures*
C2.3 := CE1 & CE3' & CE7 : Points of a line are representations of geometrical constructions [Metaphor]
[Segments' endpoints represent ratios with the unity] & [Linearize surfaces or volumes in order to compare them] & [The line is composed by endpoints of segments that are multiples of the unit, obtained by divisions or linearized geometrical constructions]

P2.4: *To describe incommensurable magnitudes by means of ratios of commensurable magnitudes*
C2.4 := CE1 & CE8: Incommensurable magnitudes approximation [Semiotic articulation]
[Archimedes' approximation methods] & [Eudosso's exhaustion method] & [Archimedean axiom] & [Incommensurable magnitudes can be approximated with couples of commensurable ones] & [Measures are limits of more and more precise measuring processes]

P2.5: *To express operations in a general form using symbols (Viète)*
C2.5 := CE5 & CE6 :Viète's Algebra [Generalization]
[Modern algebra] & [Generalization of numerical operations] & [Manipulations of symbols to simplify computations] & [Manipulations procedures] & [Generic expressions]

P2.6: *To describe curves in terms of functions*
C2.6 := CE3' & CE4 & CE9 : Natural continuous functions [Metaphor]
[Dependent variations (functions) are represented by curves] & [Positions on a curve connect two different values] & [The motion of a point represent the variation of the value assumed by a variable] & [Curves discontinuities as interruptions/breaks correspond to holes in the function domain] & [Flat neighborhoods] & [Intersection of trajectories and tangents] & [Continuous functions are represented by graphics that we can divide in contiguous parts everywhere]

P2.7: *To construct a bijection between numbers that results from operations and their representations*

C2.7 := CE1 & CE6 : <u>Equivalence classes in respect of arithmetical operations [Semiotic articulation]</u>
[Equivalence relations between procedures: equivalent ratios and proportions, equivalent segment as results of different steps of addition or subtraction of two segments] & [Set of magnitudes that are "quotient" of equivalent procedures]

3. Third level: New problems and macro-configurations (3rd level's partial meaning)

P3.1: *To create a bridge between numerical and geometrical generic procedures*
C3.1 := C2.5 & C2.3 : <u>Hybrid continuum (Viète and Descartes) [Metaphor & Generalization]</u>
[Symbols represent at the same time numbers (quantities) and magnitudes] & [Every general procedure represent at the same time geometrical and numerical procedures] & [New numbers must represent incommensurable magnitudes like the diagonal of a unitary square, that have special properties and are impossible to represent in the decimal usual representation and by means of fractions, like $\pi, \sqrt{2}, e$] & [Distinction between rational and irrational numbers] & [Irrational numbers may be results of arithmetical or geometrical procedures] & [Some irrational numbers are not possible to construct geometrically] & [Parallelism between equations and construction of segments using line and compass] & [Distinction between algebraic and transcendental numbers] & [Cartesian geometry] & [R contains all the numbers generated algebraically and geometrically]

P3.2: *To characterize set of numbers in terms of properties in respect of operations*
C3.2 := C2.5 & C2.7 : <u>Numerical sets are algebraic structures [Generalization]</u>
[Properties of generic procedures] & [A subset of all the possible number may satisfy operations properties] & [Different kind of algebraic structures] & [Bombelli's complex numbers] & [Distinction between N, Z, Q] & [Real numbers are numbers that are not imaginary] & [Z, Q and C as "quotients" in respect of equivalence relations defined by means of one operation] & [Operations with roots] & [Impossibility to define R as quotient defined by generic numerical operations]

P3.3: *To estimate the rate of change of a flowing variable known the flowing variable and viceversa (Newton, Taylor, MacLaurin)*
C3.3 := C2.1 & C2.2 & C2.6: <u>Netwon's differential Calculus [Semiotic articulation]</u>
[Varying flowing quantities and their rate of variations are represented by motion] & [Fluxions and fluents] & [Linear infinitesimal variations] & [Taylor's and MacLaurin's series development of analytical functions]

P3.4: *To treat analytically in the Cartesian geometry infinitesimal variations of curves, in particular for the problem of tangents and subtangentes (Leibniz)*
C3.4 := C2.2 & C2.6 & C3.1 : <u>Leibniz's differential Calculus</u> [Semiotic articulation]
[Infinitesimals are variables] & [Comparison between infinitesimal quantities is possible] & [High-order infinitesimals exist] & [Infinitesimal differences' ratios make sense] & [Leibniz's point as a "differential cloud"] & [Operations with infinitesimal quantities] & [Products of infinitesimals are not absolute zeros] & [Infinitesimals can be neglected when infinitely small with respect to other quantities]

4. **Fourth level: Configurations of sets of real numbers (meanings of real numbers)**

P4.1: *To construct the set of real numbers and to put it in relation with the line*

C4.1 := C2.3 & C2.4 & C2.2 & C3.1 & C3.2 : R is the set of Cauchy-Weierstrass-Cantor contiguous classes [Generalization]

[Cauchy's iterative static description of limit processes] & [Cauchy's infinitesimal sequences and Cauchy-convergence of rational series] & [Some Cauchy's rational sequences are convergent but no to a rational number] & [Definition of Cauchy's complete set] & [Irrational numbers are limits of rational sequences] & [More than one rational sequence converge that the same number] & [Quotient of R respect of infinitesimal Cauchy's sequences] & [R is the set of all the possible classes of Cauchy's sequences whose difference is infinitesimal i.e. converge to 0] & [Weierstrass' flat intervals and intervals' convergence to a point] & [Cantor's contiguous classes and creation of irrational contiguous classes] & [Operation between contiguous classes] & [R is the Cauchy-complete field of contiguous classes ordered like rational numbers] & [Cantor's postulate of continuity of the line]

P4.2: *To construct the set of real numbers and to put it in relation with the line*

C4.2 := C2.7 & C3.1 & C3.2 : R is the set of Dedekind's rational cuts [Generalization]

[Some numbers are not rational, necessity of creating the new numbers starting from Q like the other equivalence relations lead to Z and Q] & [Dedekind's completeness i.e. existence of the supreme of every subset in the subset itself] & [Q is not complete] & [Points on the line are results of cuts that leave the dividing element in the left part] & [The point may correspond to a rational number or not; if it doesn't the cut creates an irrational number] & [Dedekind's equivalence relation between rational sections, numbers are equivalence classes] & [R is the complete field of Dedekind's rational cuts classes] & [Operations are derived by Q-operations] & [Dedekind's postulate of continuity of the line]

P4.3: *To mathematize physical variation using the functions and the Calculus*

C4.3 := C3.3 & C3.4: Physical integral-differential Calculus [Metaphor]

[Irrational numbers may represent positions or instants] & [Irrational numbers are finite sums of smaller and smaller increments i.e. series] & [Continuous trajectories are limits of infinitesimal polygonals] & [Curves-trajectories can be "mathematized" locally in small neighborhoods of space-time by means of series and linear differentials] & [Continuous variations are limit of discontinuous infinitesimal linear variations] & [The length of a curve is an integral] & [Continuous functions describe continuous processes but they need to be more than continuous since a variation has a speed and acceleration at least, so the derivatives also need to be continuous]

5. **Fifth level: Further configurations**

P5.1 := *To compare different kind of infinite quantities*

C5.1:= CE8 & C4.1: Cantor's transfinite numbers , set theory and Cohen-Godel's continuum hypothesis [Metaphor]

[Cantor's diagonal theorem] & [Different infinite cardinalities between N-Z-Q and R-C] & [Cantor's set theory and numbers as sets' cardinalities] & [Transfinite numbers] & [Continuum hypothesis]

P5.2:= *To create a system of axioms for the Geometry*

C5.2 := C2.4 & C4.2: <u>Hilbert's axiomatization of the real numbers line [Semiotic articulation]</u>

[Axiomatization of the properties of order; field with the usual operations; Archimedean axiom; Completeness axiom] & [System respecting the Euclidean axioms not extendable] & [R is the only ordered complete field] & [Dedekind's postulate of continuity implies Cantor's one but not *viceversa*] & [Hilbert's completeness axiom leads indirectly to the introduction of limit points, and, hence, renders it possible to establish a one-to-one correspondence between the points of a segment and the system of real numbers] & [The existence of limit points is derivable but not postulated]

P5.3 := *To treat in a precise manner the Leibnizian infinitesimal quantities and to operate with them consistently with the real numbers' properties*

C5.3 := C2.2 & C4.1 : <u>Non-standard Analysis (Robinson) [Generalization]</u>

[Debates about the infinitesimals and the non-equivalence between intuitive and formal conceptions of the continuum and limits] & [Weyl's classification of the different conceptions of the continuum] & [Non-standard analysis reformulates the Calculus using a logically rigorous notion of infinitesimal numbers] & [Reconsideration of Leibniz's infinitely small or infinitely large with the same properties of real numbers] & [Necessity of a set that contains infinitesimal and infinite elements i.e. is non-Archimedean] & [Hyperreal numbers or other non-standard model of real numbers]

4.1.2 Metaphors, representations, dualities, semiotic functions: analysis of epistemic and cognitive connections

In this Paragraph we compare the cognitive analysis proposed by Lakoff & Nunez (2000), that we resumed for our specific goal in Par. 2.1.3, and a possible analysis in terms of dualities and semiotic functions inspired to Font (2007) and Rondero & Font (2015).

As we anticipated, the aim of these comparison is to transform the precious cognitive analysis in a tool to analyze the teachers choices in a didactical perspective.

The OSA analysis may thus inflect the cognitive complexity in didactical terms making it become a tool to analyze the teachers' personal meanings in cognitive terms and to find out how the intrinsic relational nature of mathematics that emerge from the analysis of the metaphors underlying real numbers and the continuum is expressed in the didactical practices declared by the teachers.

There are indeed dualities that concern the metaphors but specify better how they affect the epistemic meaning of real number and the continuum and what roles the dual facets play when the mathematical object are involved in classroom discourses, thought as linguistic games.

A particular case: the "hybrid continuum" and the extensive/intensive duality

The extensive-intensive duality is crucial in the construction of the hybrid configuration of the continuum by

Viète and Descartes, since the problematic aspect of this configuration, that is also its major feature, is the relation between particular and general objects. As Giusti (1990) stressed indeed the innovation that changed radically the conception of the continuum introduced by Viète and Descartes is the double-faced nature of the symbols they used as general objects. The symbol x can represent both the numbers and the (linearized) magnitudes. The general objects is thus the bridge between two independent configurations and the possibility to use algebraic expressions as extensive in relations to different possible intensive configurations is the decisive step towards an hybrid continuum, free from the constraints imposed by the ratios' theory by Eudosso. In the "algebraic discourses" the extensive can represent both numbers and magnitudes.
Let's consider the example proposed by Godino, Batanero & Font (2007) in order to explain the intensive-extensive duality.

"*Extensivo – intensivo (ejemplar - tipo)*. Un objeto que interviene en un juego de lenguaje como un caso particular (un ejemplo específico, p.e., la función $y = 2x + 1$) y una clase más general (p.e., la familia de funciones $y = mx + n$). La dualidad extensivo-intensivo se utiliza para explicar una de las características básicas de la actividad matemática: el uso de elementos genéricos (Contreras y Cols, 2005). Esta dualidad permite centrar la atención en la dialéctica entre lo particular y lo general, que sin duda es una cuestión clave en la construcción y aplicación del conocimiento matemático. "La generalización es esencial porque este es el proceso que distingue la creatividad matemática de la conducta mecanizable o algorítmica (Otte, 2003, p. 187)."

The particular example $y = 2x + 1$ can also be interpreted as a general expression, in which the variables y and x can both represent numbers and magnitudes:

1. $y=3$, $x=1$: $3=2+1$
2. y=segment_1, y=segment_2, 1=the unit, 2=scalar factor

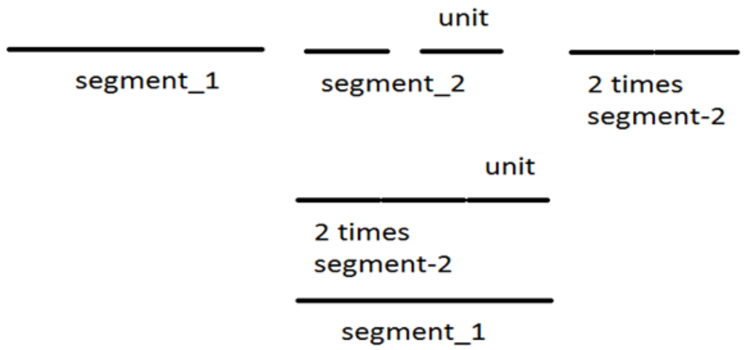

Fig.

Different aspects of measure in the didactical practice in the high school

M1) Measure as a practice in Physics

M2) Measure as a theory in Mathematics

M3) Observed data and interpolation

4.2 Analysis of the institutional meaning of real numbers by OSA tools

4.2.1 Systems of practices concerning real numbers in the national curricula

(First 2 years on 5)

a. The student will acquire an intuitive knowledge of real numbers, with particular attention to their geometrical representation on a line.

EPI:

P2.3: To associate points of the line to geometrical constructions and algorithmic procedures

COG:

Necklace model (features from partial meanings in researches)

Approximations and constructions: the computer and finite numbers

Representations: decimals, fractions and segments

b. The proof of the irrationality of $\sqrt{2}$***and of other numbers will be an important occasion of conceptual detailed study.***

EPI:

P3.1: To create a bridge between numerical and geometrical generic procedures

C3.1 := C2.5 & C2.3 : Hybrid continuum (Viète and Descartes)

COG:

Necessity of the absurd proof

Coordination of different representations

c. The study of irrational numbers and of their possible expressions will provide a significant example of algebraic computation's application and an occasion to face the issue of approximation.

EPI:

P3.2: *To characterize set of numbers in terms of properties in respect of operations*
C3.2 := C2.5 & C2.7 : Algebraic structures and numerical sets

P2.4: *To describe incommensurable magnitudes by means of ratios of commensurable magnitudes*
C2.4 := CE1 & CE2 & CE9

COG:

Non-whole numbers

Exact numbers vs approximations (mia tesi)

Decimal representation of irrational numbers

Finite numbers

d. Lo studente acquisirà la capacità di eseguire calcoli con le espressioni letterali sia per rappresentare un problema (mediante un'equazione, disequazioni o sistemi) e risolverlo, sia per dimostrare risultati generali, in particolare in aritmetica.

EPI:

P2.5: *To express operations in a general form using symbols (Viète)*
C2.5 := CE5 & CE6 :
COG:
Difficulties to give sense to symbolic expressions
Letters, difficulties in conversion from verbal expressions to symbolic expressions (conversion, Duval, 1993; Ferrari,) but also in the symbolic register itself .

e. Particular attention will be given to the Pythagoras' theorem so that its geometrical aspects and its implications in Number theory (introduction of irrational numbers) are understood, stressing merely the conceptual aspects.

EPI:

P3.1: To create a bridge between numerical and geometrical generic procedures

C3.1 := C2.5 & C2.3 : Hybrid continuum (Viète and Descartes)

[Symbols represent at the same time numbers (quantities) and magnitudes] & [Every general procedure represent at the same time geometrical and numerical procedures] & [New numbers must represent incommensurable magnitudes like the diagonal of a unitary square, that have special properties and are impossible to represent in the decimal usual representation and by means of fractions, like$\sqrt{2}$, π , e] & [Distinction between rational and irrational numbers] & [Irrational numbers may be results of arithmetical or geometrical procedures] & [Some irrational numbers are not possible to construct geometrically] & [Parallelism between equations and construction of segments using line and compass] & [Algebraic and transcendental numbers] & [Cartesian geometry]

COG:

Conversion between numerical and geometrical representation of numbers

Approximations of irrational numbers used instead of the number

Conversion between numerical and geometrical representation of numbers

f. The student will study some kinds of functions: f(x) = ax + b, f(x) = |x|, f(x) = a/x, f(x) = x^2 ***both in terms strictly mathematical and to describe and solve applicative problems.***

EPI:

P2.6: *To describe curves in terms of functions*
C2.6 := CE3 & CE4 & CE10

COG:

The domain is not explicit

The continuous graph is immediately associated to the analytical function

To find some points and then to trace a line

Graphic representations to coordinate with tabular and numerical ones

(3rd and 4th years on 5)

g. The study of the circumference and of the circle, of the number π, of the contexts in which exponential growths involving the number e appear will allow to examine in depth the knowledge of real numbers, regarding in particular the transcendental numbers.

EPI:

P6: *To operate with quantities in contextualized problems of real life (addition, subtraction, multiplication, division, power, exponential, logarithms)*
CE6:
P3.1: *To create a bridge between numerical and geometrical generic procedures*
C3.1 := C2.5 & C2.3 : Hybrid continuum (Viète and Descartes)

[
COG:

Approximation of irrational numbers may vanish the attempt to conceptualize transcendental numbers

Confusion between angles and magnitudes (Radiants or meters)

Irrational numbers are represented by means of non-decimal symbols or using approximations obtained with the calculator

c. Through a first knowledge of the problem of the formalization of real numbers the student will be introduced to the issue of mathematical infinity and its connections with the philosophical thought.

EPI:

P8: *To find a relation between finite and infinite*
CE8

P4.1: *To construct the set of real numbers and to put it in relation with the line*
C4.1 := C2.3 & C2.4 & C2.2 & C3.1 & C3.2 : R is the set of Cauchy-Weierstrass-Cantor contiguous classes
[Cauchy's iterative static description of limit processes] & [Cauchy's infinitesimal sequences and Cauchy-convergence] & [Some Cauchy's rational sequences are convergent but no to a rational number] & [Definition of Cauchy's complete set] & [Irrational numbers are limits of rational sequences] & [More than one rational sequence converge that the same number] & [Quotient of R respect of infinitesimal Cauchy's sequences] & [R is the set of all the possible classes of Cauchy's sequences whose difference is infinitesimal i.e. converge to 0] & [Weierstrass' flat intervals and intervals' convergence to a point] & [Cantor's contiguous classes and creation of irrational contiguous classes] & [Operation between contiguous classes] & [R is the Cauchy-complete field of contiguous classes ordered like rational numbers] & [Cantor's postulate of continuity of the line]

P4.2: *To construct the set of real numbers and to put it in relation with the line*
C4.2 := C2.3 & C2.7&C3.1 & C3.2 : R is the set of Dedekind's rational cuts
[Some numbers are not rational, necessity of creating the new numbers starting from Q like the other equivalence relations lead to Z and Q] & [Dedekind's completeness i.e. existence of the supreme of every subset in the subset itself] & [Q is not complete] & [Points on the line are results of cuts that leave the dividing element is the left part] & [The point may correspond to a rational number or not; if it doesn't the cut creates an irrational number] & [Dedekind's equivalence relation between rational sections, numbers are equivalence classes] & [R is the complete field of Dedekind's rational cuts classes] & [Operations are derived by Q-operations] & [Dedekind's postulate of continuity of the line]

P5.1 := *To compare different kind of infinite quantities*

C5.1:= CE8 & C4.1: Cantor's transfinite numbers , set theory and Cohen-Godel's continuum hypothesis

[Cantor's diagonal theorem] & [Different infinite cardinalities between N-Z-Q and R-C] & [Cantor's set theory and numbers as sets' cardinalities] & [Transfinite numbers] & [Continuum hypothesis]

COG:

Finite quantities are assimilated to big finite quantities

Potential infinite vs actual infinite

The line is usually perceived as composed by a finite number of small points-balls

Graphic representations are considered intuitive, while the other no, but the intuition doesn't lead immediately to infinite points nor to the problem of incompleteness of Q and to the relation between completeness and continuity

d. The student will learn to study quadratic functions, to solve quadratic equations and inequalities and to represent and to solve problems involving them.

EPI:

P5: *To solve equations and inequalities*

CE5: [Assyrs and Babiloneses tables] & [Diofanto's equation] & [Methods for 2nd order equations or more]

P6: *To operate with quantities in contextualized problems of real life (addition, subtraction, multiplication, division, power, exponential, logarithms)*

CE6: [Operations algorithms] & [Modeling of real life problems with operations] & [Procedures and methods to simplify computation with great numbers] & [Positional systems with Indo-arabic digits and decimal representation]

COG:

Methods for linear equations and inequalities adapted to the 2nd degree in a wrong way

The lack of the domain in the previous study of functions may lead the students to wrong solutions for applicative problems (negative numbers for measures, numbers belonging to Q\Z to represent non divisible quantities, irrational numbers to represent measures in Physics)

Procedures are not supported by conceptualization; this implies problems in the modeling using quaratic equations and inequalities (Boero, Bazzini & Garuti, 2000)

The representations of intervals are a lot and depends on the specific set considered

The treatment and the conversion between representations may be difficult if the representations arised as objects from not connected practices and the students don't know any semiotic function that can create a new, more general, configuration

e. The student will learn to construct simple growth models using the exponentials, in addition to periodic behaviors, also in relation with other disciplines; this will be carried out both in continuous and discrete contexts .

EPI:

P2.6: *To describe curves in terms of functions*

C2.6 := CE3 & CE4 & CE9

P10: T*o describe variations depending on space and time in Physics*

CE10:

COG:

To merge dynamic and static, continuous and discrete variations

Coordination between graphical (continuous & discrete), verbal and analytical expressions of functions to model phenomena (5th year)

a. The student will deepen the study of the fundamental analytical functions, also using examples from Physics or other disciplines; she will acquire the concept of limit, both of functions and sequences, and will learn to compute limit values in simple cases.

EPI:

P3.3: *To estimate the rate of change of a flowing variable known the flowing variable and viceversa (Newton, Taylor, MacLaurin)*

C3.3 := C2.1 & C2.6 & C3.1

P3.4: *To treat analytically in the Cartesian geometry infinitesimal variations of curves, in particular for the problem of tangents and subtangentes (Leibniz)*

C3.4 := C2.2 & C2.6 & C3.1

COG:

Dynamic/intuitive vs static/formal definition of limits and continuity of real functions in the real domain (Tall & Vinner; 1981; Bagni, 1999)

To merge dynamic and static, continuous and discrete variations, using the Calculus formalization (Leibniz-Newton diatribe)

Coordination between graphical (continuous & discrete), verbal and analytical expressions of functions to model phenomena

b. The main goal will be merely to make the students understand the role of the Calculus as a fundamental conceptual tool in the modeling of Physical phenomena of of other natures.

EPI:

P4.3: *To mathematize physical variation using the functions and the Calculus*

C4.3 := C3.3 & C3.4 Physical integral-differential Calculus

Comments

The problem of infinite cardinality is not faced before, but it's not a problem for the previous practices

The systems of practices concerning the real numbers and the continuum are present largely in the national curricula.

The complex relation between Mathematics and Physics concerning the hardly debated concept of “numbers varying continuously” appears at a certain time and is not treated – as it could be – since the very beginning. If we think to the critic of Bolzano to time dependent intuitions in the Calculus, to the diatribe Newton-Leibniz, engaged indeed in this field (Giusti, 2012) or to the intuitionistic reactions to the Calculus' arithmetization, we can hypothesize easily that this node will create some doubts or sudden cognitive conflicts that maybe the teachers didn't expected and may not be able to manage effectively.

Taking in account the Aristotle's approach to the continuum, its generation through motions, its characterization by means of the concept of contiguous and its relation with infinite divisibility that not implies the possibility to re-compose the continuum starting from its small parts, maybe something can be inserted before in order to prepare the students to compare and opportunely distinguish these two identities, somehow incommensurable (Nunez, 2000) and complementary (Bell, 2000).

Also it's remarkable the precocity of the intuitive approach to the representation of the real numbers on a line in the first 2 years, since it's not trivial at all, as the History and the Didactics of Mathematics researchers stressed many times. Maybe these aspects should be discussed more in depth.

From our first epistemic analysis of the national curriculum the ambitions of this curriculum seems to be very high in comparison with the potential poorness of the students' partial meanings in the various steps reported by an amount of researchers (Ch. 1.2.1) and the lack of problems and tasks that lead the students to act on the “conceptual level” in the textbooks (Gonzales-Martin, 2014;). Also the teachers are assumed to be deeply aware of the epistemic meaning of real numbers and of the interdisciplinary connections with other disciplines, while this is not obvious standing on some researches' results (Tall & Vinner, 1981, Gonzales-Martin, 2014; Arrigo & D'Amore, 1999). The role of the teachers in the realization of the attended curriculum seems to be very important and this confirms to us that the investigation concerning the teachers' choice is decisive to create a bridge between the expected students' knowledge about real numbers in the end of the high school and the weakness of their knowledge highlighted by the researchers.

Furthermore the teachers we interview were asked to express their degree of acceptance and agreement with the national curricula about real numbers and the opinions were very different, in particular in respect of the problem of formalization of real number and the problem of the mathematical infinite.

The reasons of this discrepancy we took in account are mainly two:

1) the teachers thought that there systems of practices that are more important than those ones;

2) the teaching-learning processes concerning real numbers in the high school are different in respect of the epistemic ones; maybe the reasons of this difference lies precisely in the students' difficulties or in the teachers' personal meanings of real numbers.

4.3 Analysis of the questions used to evaluate the teachers’ mathematical knowledge

A priori analysis of the tasks proposed to the teachers concerning the mathematical aspect of DMK
Standing on the assumptions of the model DMK the questions posed to the teacher must have two main features: they must provide information about the degree of correctness of the teachers’ personal meaning in respect of the global meaning of the mathematical object (Pino-Fan, Godino & Font, 2011; Pino, Godino & Font, 2014); they must allow the researcher in the phase of data analysis to investigate the different representations, the partial meanings and the possible configurations that the teachers know and may use in

their practice. The choice of the representations was inspired by the works: Font (2000) in the case of derivatives, reported in Pino, Godino & Font (2014); Pino-Fan, Godino y Font (2013).
This last point is very important in our case since, as we highlighted before, there is not a unique final configuration but there are instead more than one meaning and the meanings are not commensurable and synthesizable in a mathematical object
Standing on this observation it's also more significant to investigate the partial meanings associated to the signs since these could be confused or mixed.

The kind of representations and meanings we explored, in line with our first characterization of the epistemic meaning of real numbers, were those implied in:

1. *common knowledge* (CK), i.e. the representations and practices involving real numbers most used in the italian textbooks and the most recurrent in our pilot study;
2. *advanced knowledge* (AK), i.e.the representations and practices involving real numbers used in the historical steps of evolution of the concept and in the more advanced mathematical practices and in the interdisciplinary significant connected meanings;
3. *knowledge of partial meanings* (PEM) of the epistemic meanings involved in different practices that could be the meaning the students associate to the representations or the partial meaning required in a specific practice or in an historical version of a practice (for instance introducing the root square of 2 using the Pythagoras' meaning of incommensurability requires the partial meaning of numbers as linear magnitudes).

We used before a common set of a-priori categories to label the teachers cognitive configurations, the we went on inverting the direction and creating new a-posteriori categories.

Then in a second phase of the research we associates these categories to the teachers' declared choice and reported experiences in order to evaluate the weight of the personal cognitive configurations in the teaching sequences planning and in the use of the concepts in the practices.

The questions we posed in the different fields were also of a different kind. In particular we investigated:

1. CK and PEM using videos reporting didactical proposals on partial meanings of real numbers and reports of students' answers to tasks we proposed in the pilot study concerning intervals of rational and real numbers.
2. AK using direct, both open and cloze questions, and asking comments on representations of intervals that could be used also in other mathematical contexts. The aim of the cloze questions was to be sure to have information about crucial nodes.

Q1)-Q3) concerns the teachers' formation, the years of experiences as teachers and their background concerning real numbers.

Q4) In every enlargement of numerical sets *($N \rightarrow Z$, $Z \rightarrow Q$)* there was a “critical operation” involving the elements of a set whose properties lead to construct another set (f.i. subtraction, division). How is it possible to construct R strating from the elements of Q?

EPI:

P2.7: *To construct a bijection between numbers that results from operations and their representations*

C2.7 := CE1 & CE6

P3.2: *To characterize set of numbers in terms of properties in respect of operations*

C3.2 := C2.5 & C2.7 : Algebraic structures and numerical sets

P4.1: *To construct the set of real numbers and to put it in relation with the line*

C4.1 := C2.3 & C2.4 & C2.2 & C3.1 & C3.2 : R is the set of Cauchy-Weierstrass-Cantor contiguous classes

P4.2: *To construct the set of real numbers and to put it in relation with the line*

C4.2 := C2.3 & C2.7&C3.1 & C3.2 : R is the set of Dedekind’s rational cuts

COG:

The definition of Z, Q and R requires the notion of equivalence relation, that is usually stressed only in the case of rational numbers, or better, the set of fractions

While for Z and Q an operation is enough to define the equivalence relation, in the case of R a 2^{nd} order logic and a non-algebraic procedure is necessary.

The students are used to work with numbers on the line or with approximations so, in the moment in which the teachers propose to them practices concerning definitions, properties or argumentation about the incompleteness of Q and the necessity of enlarging the set maybe the student will not be able to understand the problem, since they're used to work with the number line and to refer to it with the symbol R, even if their meanings are partial or wrong (Par. 4.2). Furthermore if the teachers' personal meanings don't include the more general and formal aspect of the epistemic of real numbers maybe the problems will be posed in an uncorrect way and many inconsistencies may arise; for instance to introduce R as the set generated to contains all the roots of positive numbers may generate some cognitive conflicts that would require a very careful treatise to be turned into a significant and generative practice.

Some potential cognitive conflicts we can hypothesize *a priori* are:

1) once introduced the problem of incommensurability using geometrical constructions involving square roots, the students should create a new configuration in which new numbers have always a geometrical construction; in this perspective it would not be easy to connect the decimal representation, that is a product of a further important and not trivial step, to the partial meaning

because the irrational numbers created as series of rational numbers have nothing to do with geometrical constructions;

2) once introduced the square roots algebraically and associated the irrational numbers to the roots, maybe it would not be easy to explain the difference between algebraic and transcendental numbers

The numbers are maybe the most complex mathematical objects in terms of quantity of different representations and of relations between them. Enlarging numerical numbers the textboks and the teachers sometimes use time after time different kind of representations depending on the kind of enlargement:

a) From N to Z the numbers are represented usually in the positional decimal representation

b) From Z to Q usually the register of fraction is used.

c) From Q to R sometimes using the graphic representations of numbers and *reductio ad absurdum* involving arithmetical aspects of irrational numbers, sometimes using the decimal representation to synthesize all the differences (rational/irrational, algebraic/transcendental) in one, unique, representation.

Each of the representations is linked to epistemic and institutional different systems of practices and complex historical transitions – let's f.i. think about the long-term debate about the relation between numbers and magnitudes.

Furthermore, since focusing on the point c) is the goal of our question, at the basis of the graphic and the decimal representation of numbers there are also different epistemic partial meanings:

1) Graphic representation of real numbers: geometrical procedures and numbers, potential construction of the set of real numbers adding new elements

P3.1: *To create a bridge between numerical and geometrical generic procedures*

C3.1 := C2.5 & C2.3 : Hybrid continuum (Viète and Descartes)

2) Real numbers are all the possible decimal numbers (series of infinite elements, some periodics and some not)

P4.1: *To construct the set of real numbers and to put it in relation with the line*

C4.1 := C2.3 & C2.4 & C2.2 & C3.1 & C3.2 : R is the set of Cauchy-Weierstrass-Cantor contiguous classes

On one hand this observation doesn't imply anything certain in itself since the way the objects are used in the practices depends on the specific context, the teachers, the previous experiences shared by the teachers and their students and also on the role the teachers use them in the linguistic game (see the Par. 2.1.2 Dualities); on the other hand this could be very probably a source of difficulties for the students since the partial meanings that underlie to the graphical and the decimal representations are different and also positioned at different levels of generality (3/ 4).

Q5) In your mind is it possible to define a limit point in Q or is it necessary to use real numbers?

EPI:

P4.1: *To construct the set of real numbers and to put it in relation with the line*
C4.1 := C2.3 & C2.4 & C2.2 & C3.1 & C3.2 : R is the set of Cauchy-Weierstrass-Cantor contiguous classes

Q8) Look at the first minute of this video: http://www.youtube.com/watch?v=jk08WkwqTQ

EPI:

P2: *To approximate magnitudes*
CE2: [Archimedes' approximation methods] & [Eudosso's exhaustion method]

P2.3: *To associate points of the line to geometrical constructions and algorithmic procedures*
C2.3 := CE1 & CE3 & CE7 : [Segments' endpoints represent ratios with the unity] & [The line is composed by endpoints of segments that are multiples of the unit or obtained by divisions or linearized geometrical constructions]

Q9) Look at this video: http://www.youtube.com/watch?v=kuKTyp_b8WIIl . Can it help the students to understand the bijection between real numbers and the line's points?

EPI:

P4: *To describe trajectories as result of motion (no breaks)*
CE4: [Segments are traces of mobile points] & [Points are extremes of extendable segments] & [No stops in the motion implies no breaks in the segments and no gaps]

P2.1: *To model physical phenomena in terms of variable's variation and dependence using mathematical models*
C2.1 := CE2 & CE4 & CE8 : [D'Alembert Wave's function] & [Arbogast's definition of continuous variation] & [Study of bounded or asymptotic variations corresponding to bounded variations of an independent variable]

Q10) Look at this video, from the minute 10:20 to the minute 12:10 http://www.youtube.com/watch?v=UEBK5DfPxvk . Do you think the distinction between the graphic and algebraic representations?

EPI:

P5: *To solve equations and inequalities*

CE5:

P2.5: *To express operations in a general form using symbols (Viète)*
C2.5 := CE5 & CE6 :

P2.3: *To associate points of the line to geometrical constructions and algorithmic procedures*
C2.3 := CE1 & CE3 & CE7

COG:

The solutions are presented as two different objects emerging from different practices. This choice may lead the students to keep on considering them different in the future.

The set of all the possible numbers is presented wit the line but it's not explained how to recognize a number that is bigger or lower in other representations.

The functional aspect of inequalities is not stressed (Boero, Bazzini & Garuti, 2000). The processes involved are very poor.

The representations can be both useful, but the conversion between them is not obvious and not adequately stressed.

The last questions concern all the representation of intervals. We decided to investigate in depth this object since it reunites in itself many of the crucial issue we highlighted before: it’s a conceptual blending in which graphical, numerical and topological aspects of real numbers and the continuum are extraordinarily tangled. Asking the teachers to comment on possible students’ solutions and to choose some of these proposed representations as representations of the solution in a textbook we pursued the goal to understand how the intervals are presented by teachers and in particular which connection strategies they activate when they talk about interval as systemic entities to their students and if these connections lead a unique global meaning of intervals.

These are the texts of fictive exercises we asked the teachers to comment:

13.

$$\begin{cases} |x-1| < 3 \\ |x-4| < 2 \end{cases}$$

Queste sono alcune soluzioni proposte da studenti.

1.

$$|x-3| < 1$$

2.

$$2<x<4$$

3.

$$x > 2 \wedge x < 4$$

4.

$$[2,4]$$

5.

Fig.

1. Quali sono gli elementi di questo insieme?

$$\{x \in \mathbb{Q} / x^2 < 5\}$$

Queste sono alcune soluzioni proposte da studenti.

1.

2.

$$-\sqrt{5} < x < \sqrt{5}$$

3.

$$[-\sqrt{5}, \sqrt{5}]$$

4.

$$]-\sqrt{5}, \sqrt{5}[$$

5.

$$|x| < \sqrt{5}$$

Fig.

4.3.1 Groups of significant questions

Q7 – Q10

In Q7 we asked the teachers to declare in which system of practices they consider R necessary, asking them to choose between some proposals.

There are more than one reason why the teachers may have answered that R is necessary. We synthesized them in some possible a-priori categories:

1. The teachers know that Q is dense and that the density is the needed feature but thinks that not all the limit point of rational sequences are rational numbers so introduce before the set that contain all these points before introducing limit points;
2. The teachers know that Q is dense and that the density is the needed feature but are used to use limit points in the particular case of limits of rational sequences and recreate the ambient in which they studied limit points without making new choices;
3. The teachers don't know the density is the needed feature and are used to use limit points in the "scholastic Calculus" that concerns real function in the real domain.

Analyzing the teachers belonging to the category K4_B_a we detected great differences in their profiles. This is interesting since it's a counterexample to the assumption that a different background necessarily leads to different choices. In the Chapter 3 we will discuss this topic about all the categories of teachers and choices we created.

In this Chapter we comment specifically on the choice a., that is particularly interesting if it's compared to the choice of introducing real numbers by means of a particular irrational number, usually the root square of 2.

Some teachers who were interviewed while they were answering the questionnaire and other teachers who commented on the answer in a written form explained this choice saying that not all the limit points are rational so they need R to introduce this concept.

To put light on this complementary choices, we will define:

1. "whole to single" choices, i.e. choices that put the whole before the single element, constructing before the "new world" in which an element find a place thanks to common properties we define before and then an example of possible element;
2. "single to whole" choices i.e. choices that put the single element before the whole, using the single element, critical and intrinsically characterized by properties that are then extended to create a "new world", as a particular element in which recognize properties that will be common to the future elements of the whole set.

An aspect of this distinction may be reframed in the OSA's duality extensive-intensive but there are other possible nuances in the case of real numbers.

In this context we are interested in comparing the different teachers orientations in relation to the practices:

1. constructing real numbers starting from Q;
2. defining limit points.

We first of all show prototypical examples of “single to whole” and of “whole to single” choices and then we will comment on the particular case of teachers who make inconsistent choices interpreting a priori a possible cause for the inconsistency.

Before starting we have to remark that inconsistencies at this level are not surprising, in particular in the case of real numbers and continuity, as Tall & Vinner explained in depth yet in 1981. As the authors interpreted using the constructs concept image and concept definition and the cognitive conflict, a person can perceive an inconsistency if a conflict is presented to her in a form in which she can become aware of it.
We are now re-describing and re-interpreting the apparent inconsistency between the choices using a more complex set of tools that allows us to go beyond the difference between concept image and concept image definition and to extend adequately these categories in order to make the analysis a tool for training the teachers and to provide solutions to the “solve” the inconsistency that we can previse to lead the teachers to unsuitable choices.

Root square of 2

1. “Whole to single”: *Let’s consider the set of real numbers. Some of its elements are rational while some other elements are not rational. We can provide an example of this kind of not rational elements: the square of 2. This number represents the diagonal of a 1-edged square.*
2. “Single to whole”: *Let’s construct a 1-edged square and let’s draw its diagonal. We can project this segment on a line putting the left endpoint of the diagonal in 0. We can prove that there are no rational numbers equal to the root square of 2, so a not rational number exists. Let’s now consider the set obtained joining Q the set of this new kind of numbers, that we call irrational numbers.*

Limit point

1. “Whole to single”: *Let’s consider the set of real numbers. Its rational elements are limits of rational Cauchy’s sequences, but not all the rational Cauchy’s sequences converge to rational points. Let’s define a limit point as the point whose circular neighbourhoods minus the point itself always contain at least a point, whatever their radius is. Some limit points of Q are not elements of Q. The set that contains all the limit points of Q is R.*
2. “Single to whole”: *Let’s define a limit point as the point whose circular neighbourhoods minus the point itself always contain at least a point, whatever their radius is. Q is dense so all its elements are limit points in Q, but some limit points of Q are not elements of Q. Let’s consider the rational Cauchy’s sequence defined by the condition: “The square of the element is lower than 2”. It converges but not to a rational point and this point has all the features we require. We can construct a set that contains all the limit points of Q: the set of real numbers R.*

In the first kind of introduction R is given, it appears in the discourses in an extensive form; in the second case we use the discourse to define and sometimes construct the set of real numbers.

What can be problematic in the first case is that R is a very complex mathematical object and that different kind of practices may lead to quite different meanings.

In other words if we recall a partial meaning of real numbers by means of the sign R the teachers should first of all check the students' personal meaning since they are not sure it is the needed meaning. Also they should take care of the relations the students are able to establish between objects emerged in previous different practices.

For instance we could introduce the set of real numbers as the set containing all the rational and irrational numbers and then prove that the root square of 2 is irrational ("whole to single") as an example of irrational numbers, taking care of clarifying the existence of a semiotic function that connects fractions, used in the proof of the irrationality of the root square of 2, to decimal numbers.
This new configurations is possible to construct since the partial meaning used are coherent and sufficient, but this is again a partial meaning of real numbers and the required meanings are very simple: a student must know that a rational number is a number that we can represent by means of ratios between whole numbers and must know what a prime number is.
With these "poor materials" we can obtained a quite poor meaning of real numbers: a set composed by numbers that we can represent in a decimal form, that we can compare and that we can't always represent by means of fractions. Nothing more.

Let's now consider the limit points. In this case if we wish to introduce the concept of limit points basing our definitions on a previous knowledge of real numbers we need much more. As we anticipated the crucial property of a set in which we can define a limit point is the density. The property of density is not easy to represent and the objects emerging from the teachers' practices all along the school years may have properties different in respect of the desired one (Bagni, 1999). Also the students need to know that a not rational point on the line can be around by rational numbers whatever the radio of the neighborhood we consider is, so they need to know the differential structure of the points of the line. Furthermore, even if we explain to them what a rational Cauchy's sequence is, it's not trivial to prove that an irrational number can be thought as limit of a rational sequence. Historically the considerations that lead to construct the set of real numbers was in the reverse order, i.e. the real numbers was thought by Cauchy as limit points of the sequences that didn't converge and then he defined a the property of Cauchy-completeness, that is the discriminant property to distinguish Q and R.
Two main remarks are so necessary:

1. the teachers can't count easily on previous knowledge of real numbers when they introduce the limit points of Q because the student would need a very complex partial meaning of the set of real numbers that include its differential structure; for instance partial meanings like that presented before for the root square of 2 are not enough.

2. if the teacher introduce the set of real numbers as Dedekind cuts, or use dynamic representations for the limits, we already showed that the partial meanings don't include the differential nor the topological properties of R, so they're not suitable even if they are complex configurations. This observation is crucial since yet in the textbooks' analysis we found attempts to introduce the real numbers only focused on the features of their single elements, in particular in the first years. This is easy to understand since the students' own poor partial meanings and this introductions aim at finding a new place to numbers emerging from algebraic or geometrical problems, but it's important to make the teachers aware that this partial meanings are not sufficient to introduce limit points.

There are many reasons to consider the approach "whole to single" more suitable in the case of the root square of 2 rather than in the case of the limit point, nevertheless we saw more rarely this approach in the answers concerning the first rather than the other.

Let's now consider the other approach ("single to whole"). In this case the introduction of square of 2 is more critical and potentially problematic than the other.

When a teachers introduce real numbers using the root square of 2 he is indeed using a particular example that may lead to a different general elements of real numbers. First of all this generalization from an algebraic number to a general irrational number don't include the distinction between algebraic and transcendental numbers.
Then other differences lies in the kind of process that, stemming from the particular element, drive to the whole set.
First of all the generalization of the properties of the root square of 2 can be carried out in different ways depending, time after time, on the practices and the partial meanings that emerged from the practices the teacher choose to introduce it. For instance possible paths are:

1. the root square of 2 is introduced as a case of incommensurability between segments;
2. the root square of 2 is introduced as the solution of a quadratic equation;
3. the root square of 2 is presented as an irrational numbers and then is approximated more and more by means of couple of rational numbers, one greater and one lower;
4. concrete examples of problem-situation involving diagonals or circles are described, an "irrational problem" is posed and then the root square of 2 is used to indicate a concrete object or a measure of a concrete object.

Starting from these four starting point we stress different features of the elements of the new set we are creating. This observation are important also in the analysis of the group of questions Q8-9.

There are many reasons to consider the approach "single to whole" more suitable in the case of in the case of the limit point rather than in the case of the root square of 2, nevertheless we saw more rarely this approach in the answers concerning the first rather than the other.

Furthermore in our pilot study we didn't observe a coherence in the teachers answers in the two cases we analyzed, so the preference for one approach or the other doesn't seem to be a strong orientation but rather a context-dependent contingency, maybe based on didactical traditions. Of course both the two directions are necessary to construct the meaning of real numbers and a teacher can't be considered inconsistent if once decide to move from single to whole and once from the whole to the single element. What we are discussing here is the possibility of realizing such a "generalization" or on the contrary "exemplification" when the mathematical object is the set of real numbers, why the teachers sometimes consider necessary to know real numbers in advance and to use it in the extensive meaning and sometimes not and how far the teachers are aware of the fact the some paths are possible and other are unsuitable.

In the case of real numbers the semiotic functions must be treated with particular attention since the general real numbers is a very complex object, whose nature has been debated for long time and is epistemologically and logically hard to grasp.

Also we didn't observe a strict relation between the richness of the teachers' mathematical knowledge and a reasoned approach to this matters in the pilot study. In the a-posteriori analysis we present a detailed analysis of the interviews in which we detected interesting dynamics reported by the teachers that permit us to understand better the reasons of this apparent inconsistency.

Thinking more closely to the complementarity we analyzed before, we understood that that was indeed only a particular case of a more general complementarity in the teachers' choices about real numbers and the continuum, that we can synthesize as it follows:

1. to introduce R before introducing other objects, using R as extensive in the discourses about the objects quoted before;
2. to introduce the objects necessary to make a configuration of R from a sequence of practices emerge and then construct R as a configuration of objects emerged before.

Teachers may be consistent or not in reference to this choice, as we saw in the analysis of the different approach to the introduction of the root square of 2 and of the limit points.

Q11-12-13-14-15-16-17-18

The questions in this group all concern representations of intervals of real or rational numbers. The analysis of the answers to this group of questions permit us to look for "semiotic profiles", i.e. recurrences or significant orientations that lead the teachers to represent the intervals in particular, the real numbers in general, using a representation, more than one representation, a system of complementary representations and so on. The details concerning the way we collected and analyzed this kind of data is clarified in Par. 4.7.

4.4 **Teachers' formation categorizations**

K1: Teachers' formation

Only PhD in Mathematics: (58, 72), (64)

PhD & Teachers' training: (4),(8), (57, 110, 114)

Only Master in Mathematics : (1, 32, 111) , (2, 62), (5,41), (7), (11, 42), (13), (14, 83, 108), (15, 37, 37), (21, 61), (25), (26), (55, 110) (73), (79), (83, 108); (87); (89); (97)

Master in Mathematics and Teachers' training : (9), (16), (17), (20, 39, 84); (41, 105); (42); (65); (71); (74); (88); (91); (104); (115), (116), (117)

Only Scientific Master not in Mathematics (Physics and engineering): (18), (63), (76), (77), (78), (81)

Scientific Master not in Mathematics & Teachers' training : (10, 40), (46), (53), (92)

Other with teachers' training: (22), (66), (90), (91), (93), (106)

Other: (56), (60), (80), (85), (95), (98), (100), (101), (102)

8 teachers have a quite rare feature for Mathematics high school teachers: they have a PhD in Mathematics; their background in Mathematics is thus very strong and this not usual in the investigations in Mathematics education. If we take a look at the usual categories of people interviewed and tested in the literature review concerning the teaching-learning processes involving real numbers there are a very few studies carried out with teachers with a PhD who are furtherly in-service teachers with some years of experience, or, more generally, teachers with a strong background in Advanced mathematics (AK); the most of the interview people were high school or University students, preservice teachers, the most of which with some elements of confusion, weakness or ambiguity in the disciplinary background.

Moreover some of the teachers we interviewed have also attended training courses and got the National Qualification as Mathematics and Physics teachers in the high school, receiving thus a more or less good formation in Mathematics education.

A distinction would be suitable in general between teachers of different ages, considering that the quality of the research in Mathematics education and in Mathematics education training have changed quickly in the last decades, but in our investigation this variable is not valuable for teachers with PhD because the teachers are equal in age and some of them attended together the training courses.

We have thus the opportunity to show the results of the investigation of teachers' choices of a very group of teachers, interesting in itself but even more in the case of real numbers.

Indeed the previous studies concerning the teachers' choices were oriented to prove more or less the same implicit conjecture, that we resume in a few words this way:

“The teachers' backgrounds concerning real numbers show some element of weakness due to:

- the high complexity of the topic;
- the underestimation of the epistemological and philosophical issues that are so important to become necessary to understand it deeply;
- the poorness of teaching materials and the unsuitability of textbooks.

The teachers' difficulties affect their teaching strategies and often the students difficulties are a sort of heritages of the teachers' wrong personal meanings”.

This was already showed in some researches (Arrigo & D'Amore, 1999, 2002; Gonzales-Martin, 2014), even if some further studies may clarify better what weakness and wrong meanings affect more the teachers' choice, how far this is really a crucial variable in order to improve better teaching courses and consequently better teaching-learning sequences.

Interviewing this little but significant group of teachers we investigated a specular case: what happens if the teachers' choices' suitability doesn't depend on wrong understanding of real numbers? What other factor can we "isolate" if this one disappear? Is the "reverse conjecture" – if the teachers' mathematical background is good, the designed teaching-learning sequences are good - valuable? How much?

We carried out a blended research, interviewing also teachers with PhD (called **'PhD teachers'**), teachers with PhD who attended courses of Mathematics education **('Trained PhD teachers'**), teachers without a PhD but graduated in Mathematics who attended courses of Mathematics education ('**Trained Master teachers'**) and teachers without a PhD but graduated in Mathematics who didn't attended courses of Mathematics education **('Master teachers'**).

Also in the cases in which the background was not guaranteed by a PhD, we investigated the mathematical knowledge in order to interview in the last phase of the methodology – the only qualitative one – only teachers who didn't show wrong meanings, but at least correct partial meanings relevant in the high school.

4.5 Teachers' mathematical knowledge categorizations

In this Paragraph we investigate the part of the questionnaire that permit to us to answer the question:

PQ - 1.2 What is the teachers' mathematical knowledge of real numbers?

4.5.1 Teachers' categories based on the question Q3: the real numbers properties

By means of the question Q3 we investigated the teachers' opinions about the most important features of the set R, that we labeled with:

K2: Teachers' opinion concerning the most important features of real numbers (referred to the epistemic meaning of real numbers we presented in 3.1.3)

Not all the answers were precise or correct but at this level we are interested in a teachers' classification concerning the kind of knowledge they declared to have concerning the properties of real numbers, distinguishing between common knowledge [CK] and advanced knowledge [AK]. We will identify CK with the lower levels in the epistemic meaning scheme (from 1 to 3, labeled with L := lower levels) and AK with the upper levels (4 and 5, labeled with U := upper levels)

K2_A: Topological/differential structure of R (intervals, density) [13] [U]: 2, 7, 11, 15, 18, 20, 47, 65, 66, 72, 83, 101, 111

K2_B: Algebraic structure of R (field) [43] [L]: 1, 2, 8, 17, 20, 22, 39, 42, 46, 51, 53, 54, 55, 56, 57, 58, 59, 62, 63, 64, 65, 66, 67, 70, 72, 73, 75, 79, 84, 87, 88, 90, 93, 98, 100, 101, 104, 106, 110, 111, 116, 115,

K2_C: Relation between Q and R (operations, order, density of Q in R) [10] [U]: 5, 20, 21, 39, 64, 65, 68, 88, 90, 108,

K2_D: Relation between dynamic and static conception of continuity [0] [U]: *No one*

K2_E: Difference between rational and irrational numbers [6] [L]: 20, 72, 87, 90, 92, 97

K2_F: Difference between algebraic and transcendental numbers [2] [L]: 72, 87

K2_G: History of real numbers [1] [U]: 85

K2_H: Construction of R [3] [U]: 63, 85, 88

K2_I: Necessity of a postulate of continuity [0] [U]: No one

K2_J: can represent every magnitude [1] [L]: 97

K2_L: Archimedean axiom [11] [U]: 4, 26, 38, 58, 59, 61, 65, 68, 72, 78, 105

K2_M: Lack of equivalence between Cantor's and Dedekind's postulates [0] [U]: No one

K2_N: Completeness [47] [U]: 1, 4, 5, 8, 9, 10, 13, 14, 16, 18, 22, 25, 26, 29, 32, 36, 37, 40, 41, 42, 51, 53, 54, 55, 57, 58, 61, 62, 64, 65, 68, 70, 71, 72, 73, 74, 77, 78, 87, 88, 91, 92, 93, 95, 108, 110, 116, 115

K2_O: Order [38] [L]: 1, 2, 4, 8, 10, 11, 16, 17, 18, 26, 32, 42, 51, 53, 54, 55, 56, 57, 58, 59, 62, 63, 64, 67, 70, 72, 73, 75, 78, 79, 88, 90, 91, 93, 101, 110, 116, 115

K2_P: Correspondence between R and the points of a line [10] [L]: 2, 7, 11, 39, 63, 81, 87, 90, 92, 108

K2_R: Cardinality [27] [U]: 2, 5, 7, 10, 13, 22, 26, 32, 37, 39, 40, 41, 61, 63, 64, 65, 68, 71, 73, 75, 76, 87, 90, 91, 95, 96, 105

K2_S: Continuous [15] [L]: 4, 14, 17, 22, 38, 39, 59, 60, 63, 65, 72, 76, 77, 78, 83, 96, 111

K2_T: Make possible more operations/all the operations [6] [L]: 7, 11, 32, 34, 53, 97

K2_U: Make possible the differential Calculus [2] [U]: 11, 65

K2_V: Make representable every number in the decimal register [5][L]: 13, 40, 60, 80, 101

K2_Z: R include N, Z, Q and/or is included in C [3] [L]: 20, 87, 89

K2_W: R is metric space [3] [U]: 64, 65, 73

The empty *a priori* categories are also very interesting, since they are crucial elements of the advanced mathematical knowledge and are missing in the teachers' answers. This was confirmed by all the other data of the questionnaires and the interviews.

It's interesting that only a few teacher named some properties that were considered essential in the following of the questionnaire.

We looked with particular attention to these properties or partial configurations are maybe the ones who emerge in the practice as somewhat unconscious and that is maybe disconnected. The teachers can also belong to more than one category.

In the answers to Q3 we identified 8 kind of cognitive configurations that we grouped in cognitive configurations:

- CC1: Topological/differential (K2A, K2N, K2U, K2W) [66;3]

2, 7, 11, 15, 18, 20, 47, 65, 66, 72, 83, 101, 111; 1, 4, 5, 8, 9, 10, 13, 14, 16, 18, 22, 25, 26, 29, 32, 36, 37, 40, 41, 42, 51, 53, 54, 55, 57, 58, 61, 62, 64, 65, 68, 70, 71, 72, 73, 74, 77, 78, 87, 88, 91, 92, 93, 95, 108, 110, 116, 115; 11, 65; 64, 65, 73

- CC2: Numeric – systemic (K2E, K2F, K2H, K2V) [17; 1]

20, 72, 87, 90, 92, 97; 72, 87; 63, 80, 85, 88; 13, 40, 60, 101

- CC3: Numeric - unitary (K2R, K2Z) [30; 1]

2, 5, 7, 10, 13, 22, 26, 32, 37, 39, 40, 41, 61, 63, 64, 65, 68, 71, 73, 75, 76, 87, 90, 91, 95, 96, 105; 20, 87, 89

- CC4: Algebraic – operations properties (K2B, K2T) [49; 1]

1, 2, 8, 17, 20, 22, 39, 42, 46, 51, 53, 54, 55, 56, 57, 58, 59, 62, 63, 64, 65, 66, 67, 70, 72, 73, 75, 79, 84, 87, 88, 90, 93, 98, 100, 101, 104, 106, 110, 111, 116, 115; 7, 11, 32, 34, 53, 97

- CC5: Axiomatic (K2N, K2O, K2L, K2M) [96; 24]

4, 26, 38, 58, 59, 61, 65, 68, 72, 78, 105; 1, 4, 5, 8, 9, 10, 13, 14, 16, 18, 22, 25, 26, 29, 32, 36, 37, 40, 41, 42, 51, 53, 54, 55, 57, 58, 61, 62, 64, 65, 68, 70, 71, 72, 73, 74, 78, 87, 88, 91, 92, 93, 95, 108, 110, 116, 115;

1, 2, 4, 8, 10, 11, 16, 17, 18, 26, 32, 42, 51, 53, 54, 55, 56, 57, 58, 59, 62, 63, 64, 67, 70, 72, 73, 75, 78, 79, 88, 90, 91, 93, 101, 110, 116, 115

- CC6: Line - systemic (K2J, K2L, K2P, K2R) [48; 0]

4, 26, 38, 58, 59, 61, 65, 68, 72, 78, 105; 97; 2, 7, 11, 39, 63, 81, 87, 90, 92, 108; 2, 5, 7, 10, 13, 22, 26, 32, 37, 39, 40, 41, 61, 63, 64, 65, 68, 71, 73, 75, 76, 87, 90, 91, 95, 96, 105

- CC7: Line - unitary (K2S) [16]

4, 14, 17, 22, 38, 39, 59, 60, 63, 65, 72, 76, 77, 78, 83, 96, 111

- CC8: Relations between Q and R (K2C, K2H) [13,1]

5, 20, 21, 39, 64, 65, 68, 88, 90, 108; 63, 85, 88

We propose here a further quantitative and qualitative analysis of the data. The aims of the quantitative analysis are:

- to show the teachers' cognitive configurations' distribution
- to decide if a group could constitute a category or is rather composed by rare cases

The qualitative analysis concerns the comparison between the epistemic meaning and the teachers' configurations and the prevalence of practices placed on lower or higher levels in the epistemic meaning. This is particularly interesting because our previous analysis showed that the relation between the formal and intuitive dimension plays a crucial role in the field of Calculus and infinity.

- Teachers who belong only to 1 to 3 category [46]

7, 8, 9, 13, 14, 15, 16, 17, 18, 21, 25, 29, 36, 42, 46, 47, 51, 53, 54, 55, 57, 59, 60, 62, 66, 67, 70, 77, 74, 76, 79, 80, 83, 84, 85, 89, 92, 93, 96, 97, 100, 104, 106, 110, 111, 115, 116

- Teachers who belong to more than 3 categories [32]

2, 4, 5, 10, 11, 20, 22, 26, 32, 37, 39, 40, 41, 58, 61, 63, 64, 65, 68, 71, 72, 73, 75, 78, 87, 88, 90, 91, 95, 101, 105, 108

45 teachers on 79 listed properties we grouped in at least 3 categories, while 32 teachers listed more than three kind of properties.

Even if this is just a very ambiguous indication, this oriented us in the very beginning in the identification of teachers that perceived the real numbers as something complex and multifaced and teachers that only focused on a few properties. Many other markers all along the questionnaire can confirm or disconfirm this hypothesis. In order to deepen more into the kind of properties the teacher listed we looked at the level of generality of the listed properties, counting the number of L and U properties the teachers proposed

- Teacher who belong only to L categories [14]

17, 46, 56, 60, 67, 79, 80, 81, 84, 89, 97, 98, 100, 106

- Teacher who belong only to U categories [16]

5, 9, 15, 21, 25, 29, 36, 37, 47, 61, 68, 71, 74, 85, 95, 105

- Teacher who belong both to U and to L categories [12+27+11=50]

a. more U than L: 7, 10, 13, 18, 26, 40, 64, 65, 73, 78, 88, 91

b. more L than U: 2, 4, 8, 11, 20, 32, 42, 51, 53, 54, 55, 57, 58, 59, 62, 63, 70, 72, 75, 87, 90, 92, 93, 101, 110, 111, 115

c. equal: 1, 14, 16, 22, 38, 39, 66, 76, 77, 83, 96, 108, 116

12 teachers on 80 listed (15,6%) only properties at a lower level; 16 teachers on 80 (19,5%) listed only properties at an upper level; the remaining 51 teachers (65%) showed an intermediate behavior.

39 teachers (50,5%) are more oriented globally to L properties, while 27 (35%) are more oriented to U properties; the remaining 11 teachers (14,5%) showed an intermediate behavior.

A priori we could expect that:

- the teachers who list all L properties:

a. weren't aware of the U dimension concerning real numbers;

b. thought at a very simple way to introduce real numbers at school, misunderstanding the question.

Both 1a. and 1.b , given the lack of the U dimension, can be categorized as teachers whose knowledge is CK, since, even if they answered thinking at high school practices, some of the U properties should have been listed because they are in the national curricula.

- the teachers who list all U properties:

a. studied real numbers at the University and just remember the most used properties of real numbers;

b. answered following the “definition principle”, using the minimum number of features that characterize R at a formal level;

c. think at real numbers as the formalized fields constructed or axiomatized between the XIX and XX centuries, avoiding to confuse them with their intuitive versions

Even if we can't only base on this answer to conclude something certain, we can hypothesize that teachers who belong to this category have a AK about real numbers, or better, once at least studied the real numbers at the University at a formal level.

3) the teachers who listed both L and U, at different degrees depending on the quantity of properties of on and another kind:

a. studied real numbers at a formal level, but feel the necessity to distinguish between formal and more operational meanings of real numbers;

b. studied real numbers at a formal level, but feel the necessity to distinguish between formal and more intuitive meanings of real numbers;

c. studied real numbers at a formal level, but feel the necessity to list properties at different degrees of complexity, thinking both at the institutional/epistemic meaning and at the students' cognitive partial meanings;

For each of these configurations we provide prototypical examples in the Appendix B.

The teachers' personal meanings of real numbers concerning the main properties of the real numbers set are associated in this way to the epistemic partial meanings of real numbers, paying attention to the position in the generality scale highlighted in the epistemic meaning (1-3 [L]; 4-5 [U]) :

CC1 ↔ C4.1 (R is the set of Cauchy-Weierstrass-Cantor contiguous classes) [U]

CC2 ↔ C3.1 (Hybrid continuum) [L]

CC3 ↔ C3.2 (Algebraic structures and numerical sets) [L] & C5.1 (Cantor's transfinite numbers) [U]

CC4 ↔ C3.2 (Algebraic structures and numerical sets) [L]

CC5 ↔ C5.2 (Axiomatization of real numbers' set) [U]

CC6 ↔ C3.1 (Hybrid continuum) [L]

CC7 ↔ CE4 (Line as trajectory) [L]

CC8 ↔ C4.1 (R is the set of Cauchy-Weierstrass-Cantor contiguous classes) [U] & C4.2 (R is the set of Dedekind's rational cuts) [U]

In order to give an idea of the distribution of the teachers' answers in the 8 categories we report a graphic representation of the answers' frequencies:

The most of the answers (29%) concerns CC5: **Axiomatic configuration.**

Another very represented category (19,5%) is CC1: **Topological/differential configuration.**

An intermediate value of answers belonging to CC3 (**Numeric – unitary configuration**) (9%), CC4 (**Algebraic configuration**) (15%) and CC6 (**Line - systemic configuration**) (15%) was also registered.

A minority of answers belong to CC2 (**Numeric - systemic configuration**) (5%), CC7 (**Line - unitary configuration**) (4,5%) and CC8 (**Relation between Q and R**) (4%).

Globally the 52,5% of answers were U, while the 47,5% were L.

4.5.2 Teachers' categories based on the question Q4: the real numbers set construction

K3: Teachers' declared approach to the construction or axiomatization of a field of real numbers as an enlargement of the set Q

K3_A: Dedekind's cut [14]: 11, 26, 41, 42, 55, 56, 57, 66, 68, 71, 97, 100, 110, 116

K3_B: Cauchy's classes of convergent sequences [10]: 4, 8, 29, 55, 66, 67, 71, 95, 100, 110

K3_C: Cantor's contiguous classes [14]: 2, 9, 21, 55, 57, 58, 62, 63, 66, 71, 72, 73, 74, 77

K3_D: Weierstrass' intervals [0]: *No one*

K3_E: Hilbert's axiomatization [2]: 16, 71

K3_F: Using √2 and/or other radicals, eventually with a historical approach [43]: 5, 7, 10, 11, 13, 14, 15, 17, 18, 22, 25, 26, 32, 34, 37, 38, 40, 46, 47, 53, 56, 58, 59, 60, 64, 65, 72, 74, 78, 79, 80, 83, 85, 88, 89, 90, 91, 92, 93, 98, 104, 111, 115

K3_G: Extending the decimal finite numbers to infinite numbers (axiomatic) [5]: 11, 32, 41, 42, 100

K3_H: Proposing rational problems without rational solutions [3]: 1, 89, 115

K3_I: Union of rational and irrational numbers [7]: 10, 14, 38, 86, 87, 89, 111

K3_L: Ratio between C and r (π) [3]: 17, 93, 106

K3_H2: Incommensurable magnitudes [2]: 17, 105

K3_I_N: Constructing Q starting from R dividing intervals [1]: 39

K3_L2: Q dense in R [1]: 61

K3_M: union of algebraic and transcendental numbers [5]: 64, 65, 74, 87, 93

K3_N: Correspondence with the line (real numbers are all the point of a line) [3]: 75, 90, 105

K3_O: R as the set that contains all the other [1]: 108

K3_Z: don't remember [5]: 20, 39, 76, 84, 101

The same teachers can belong to different categories since some teachers listed more than one approach. We grouped the teachers’ answers in categories:

IC1) Dedekind (Cuts): K3A [14]

IC2) Hilbert (Axiomatic): K3E [2]

IC3) Cantor, Cauchy, Weierstrass (Limit points): K3B, K3C, K3D, K3L2 [25]

IC4) Root square and π (Example of irrational numbers, R is an enlargement of Q): K3F, K3H, K3H2, K3L, K3O [53]

IC5) Union of different kind of numbers (Rational/irrational, Algebraic/Transcendent): K3I, K3M, K3G [11]

IC6) Correspondence with the points of a line: K3N [3]

Some teachers' answers have not been categorized because they were not correct (lack of the first level of epistemic suitability); also 5 teachers declared not to remember how R could be constructed starting from Q. In this second case the teachers are aware that a formal construction exists, but don't remember it. These teachers constitute a new category, to add to those we created before:

3d. studied real numbers at a formal level, but never integrated the previous, maybe intuitive and operational knowledge, with the formal and less intuitive one (this category had already been identified by Tall & Vinner, 1981); as a result they forgot the new aspects, making the old meanings prevailing.

The IC can be put in relation with the epistemic meaning of real numbers:

IC1 ↔ C4.2 (R is the set of Dedekind’s rational cuts)

IC2 ↔ C5.2 (Hilbert’s axiomatization of real numbers' set)

IC3 ↔ C4.1 (R is the set of Cauchy-Weierstrass-Cantor contiguous classes)

IC4 ↔ C3.1 (Hybrid continuum → [New numbers must represent incommensurable magnitudes like the diagonal of a unitary square, that have special properties and are impossible to represent in the decimal usual representation and by means of fractions, like π,])

IC5 ↔ C3.1 (Hybrid continuum → [Distinction between rational and irrational numbers] & [Distinction between algebraic and transcendental numbers])

IC6 ↔ C3.1 (Hybrid continuum → [R contains all the numbers generated algebraically and geometrically])

Furthermore we grouped the categories in couples of related macro-categories.

Keeping on categorizing the configurations in lower and upper levels in the epistemic meaning we can see that an half of the categories is U while the other half is L.

MC1a) Historical/formal approach: K3A, K3B, K3C, K3D, K3E, K3L2 [41] [U]
MC1b) Adapted approach: K3F, K3H, K3H2, K3L, K3O, K3I, K3M, K3N [65] [L]

Looking at the frequencies we can yet observe that the most of the constructions proposed belong to the L-categories (MC1b):

Other interesting categories in our investigation are:

MC2a) Static introduction: *All*
MC2b) Dynamic introduction: *No one*

Confirming what it had emerged in Q2, no approaches outline the dynamic meaning of continuity and the analogy between the segment as trajectory and other meanings of real numbers, like completeness, bijection with the line, the link between continuous variations and intervals and so on. The dynamic configurations are not considered as possible "operations" that can generate R. This could be referred to the Weyl's and Brouwer's classification of the continuum: the intuitive time-dependent, the punctual and the free-choice acceptations of continuity are considered somehow independent at this level.

Also it's interesting to differentiate the teacher who decided to define the real numbers as a whole, in the actual sense, or as sets of elements, or better to characterize every single element (how to recognize or construct the element, its properties) rather than the whole set (the global properties).

MC3a) Intensive (general) introduction: K3H, K3H2, K3L, K3O [47]

MC3b) Extensive (particular) introduction: K3A, K3B, K3C, K3D, K3E, K3F, K3L2, K3I, K3M, K3N [55]

The most of the introductions proposed by the teacher tends to go on this way:

- to provide examples of problems that create a crisis in a previous numbers' model (Q);
- these problems may be algebraic or geometrical;
- in the hybrid continuum (C3.1) every construction ends to be identified with a number/point of a line;
- to appoint these particular cases of numbers that are not rational (i.e. don't correspond to geometrical constructions associated to ratios) to representative of a general element, the irrational number;
- to say R is the set that contains all the possible numbers, the previous and the numbers "like these particular cases"

This procedure, that we analyzed in depth, is critical from the point of view of dualities (Font, Godino and D'Amore, 2006) since when the generic element is created this way, there are no patterns that permit to create a real generic object, but there are only particular examples that reasonably will remain the only content of the expression “irrational numbers”.

MC4a) Unitary introduction: K3E, K3H2, K3G, K3O, K3N [12]
MC4b) Systemic introduction: K3A, K3B, K3C, K3D, K3F, K3H, K3L2, K3I, K3M [93]

Consistently with the previous trend, but much more impressively, the most of the teachers propose constructions of real numbers that stress the features of the elements of real numbers rather than to introduce the whole set as unitary. In the idea itself of construction there is the inner concept of thinking in a systemic way; nevertheless this is a crucial, maybe the most important, issue concerning real numbers, since we have no possibilities to construct it element by element, but on the other side we have not a simple rule like those that permit to construct Z and Q. This aspect will be investigated in depth in the interviews, since an interaction is necessary to conclude something significant.

It's very impressive the high number and the large heterogeneity of teachers who choose the strategy of using the root square as a way to construct real numbers starting from Q: this introduction that we can place at a medium level in the epistemic scheme is something that the most of the teachers have in common, as we will see also in the analysis of the comments on the first video.

4.5.3 Teachers' categories based on the question Q5: the limit points

Q5 completes the set of three questions concerning the teachers' mathematical knowledge. This question also open a new window, the teachers' goals, and is connected with the following one. We will use in this phase the teachers' questions in order to precise better the category concerning knowledge to which the teachers belong. Then we will also use this question and the previous category to go on categorizing teachers' goals, as we have already explained in 2.6

K4: Teachers' conception of limits point in Q and R

K4_A: Q is sufficient [40]
1, 2, 4, 7, 9, 10, 15, 16, 17, 20, 21, 22, 32, 37, 38, 39, 40, 47, 53, 55, 56, 57, 58, 61, 62, 63, 67, 72, 74, 76, 77, 78, 80, 84, 89, 93, 98, 104, 106, 110, 111

K4_B: R is necessary [32]
5, 8, 11, 14, 18, 26, 41, 42, 46, 59, 60, 64, 65, 66, 68, 73, 75, 79, 83, 85, 87, 88, 90, 91, 95, 100, 101, 105, 108, 115,

K4_C: Other [5]
13, 25, 71, 92, 116

4.5.4 Comparison between the mathematical knowledge categories

We matched the categories emerged in the first two steps, in order to better place the teachers in categories and to start to delineate the teachers' profiles.

IC1) Dedekind (Cuts): K3A [14] [U]

Q2.3 56, 97, 100

Q2.4 41, 68, 71

Q2.5a

b 11, 42, 55, 57, 110

c 66, 116

IC2) Hilbert (Axiomatic): K3E [2] [U]

Q2.3

Q2.4 71

Q2.5a

b 55,

c 16,

IC3) Cantor, Cauchy, Weierstrass (Limit points): K3B, K3C, K3D, K3L2 [25] [U]

Q2.3 67, 100,

Q2.4 9, 21, 29, 71, 61, 95,

Q2.5a 73,

b 2, 4, 8, 55, 57, 58, 62, 63, 72, 110,

c 66, 77,

IC4) Root square and π (Example of irrational numbers, R is an enlargement of Q): K3F, K3H, K3H2, K3L, K3O [54] [L]

Q2.3 17, 46, 56, 60, 79, 89, 98, 106,

Q2.4 5, 15, 25, 37, 47, 74, 85, 105

Q2.5a 7, 10, 13, 18, 26, 40, 64, 65, 78, 88, 91,

b 11, 32, 53, 58, 59, 72, 90, 92, 93, 111, 115

c 1, 14, 22, 38, 83

IC5) Union of different kind of numbers (Rational/irrational, Algebraic/Transcendent): K3I, K3M, K3G [12] [L]

Q2.3 89, 100,

Q2.4 41, 74,

Q2.5a 64, 65

b 11, 32, 42, 87, 93, 111,

c 38

IC6) Correspondence with the points of a line: K3N [3] [L]

Q2.3

Q2.4 105

Q2.5a

b 75, 90,

c

K3_Z: don't remember [5]
20 (Q2_5a), 39 (Q2_5c), 76 (Q2_5c) , 84(Q2_3), 101 (Q2_5c)

After two answers we distinguish teachers in terms of stability or ambiguity of their answers. The stability is not necessary a marker for good knowledge and, viceversa, the ambiguity doesn't imply poor knowledge.

In this phase we want to distinguish for instance the teachers who listed U properties and report U properties to construct it from the teachers that listed L properties but report U properties; in the first case the teacher confirm a U knowledge of real numbers, or better shows to have studied al least once formally real numbers and think that real numbers are a formal construction; in the second case a teacher could have studied real numbers formally but think that the important properties of real numbers are the most operational (2) or intuitive (3) one. The number of teachers we decided to analyze in this phase, standing on the first criteria of epistemic suitability, is 76.

Teachers L / L (teacher L who chooses only L constructions) [7] : COMMON

17, 46, 60, 80, 79, 89, 98, 104, 106

Teacher L / U (teacher L who chooses also U constructions) [4] : NOT FORMAL

56, 97, 100, 67

Teacher U / U (teacher U who chooses at least one U constructions) [10] : ADVANCED

68, 71, 9, 21, 29, 61, 95

Teacher U / L (teacher U who chooses only L constructions) [9]: SIMPLIFIERS

5, 15, 25, 37, 47, 74, 85, 105; 41

Teacher ML / L (teacher ML or intermediate who chooses the most of L constructions) [17]:

1, 14, 22, 34, 38, 83, 32, 75, 90, 87, 93, 111; 53, 59, 90, 92, 93, 108, 115

Teacher ML / U (teacher ML or intermediate who chooses the most of U constructions) [6]:

16, 55, 57, 66, 110, 116

Teacher ML / I (teacher ML or intermediate who chooses the same number of L and U constructions) [10]:

2, 4, 8, 11, 42, 58, 62, 63, 72, 77

Teacher MU / U (teacher MU who chooses the most of U constructions) [1] :

73

Teacher MU / L (teacher MU who chooses the most of L constructions) [11]:

7, 10, 13, 18, 26, 40, 64, 65, 78, 88, 91

The teachers who declared not to remember who R can be constructed starting from Q are all L or ML teachers.

6 L-teachers only quoted L-practices: we hypothesize that with high probability these teachers are limited to a common knowledge (CK) of real numbers; instead 4 L-teachers are more oriented to U-practices concerning the introduction of real numbers. We hypothesize that these teachers' knowledge is partially AK but they consider the main properties of real numbers at school the L ones.

There are no intersections between teachers U/U and teachers U/L, so the U teachers result divided in 2 disjointed sets; this permit us to create a first couple of complementary categories.

Only one teacher MU quoted U introductions; maybe the teachers MU have studied the properties of real numbers and remember the properties but their personal meanings are L; this is not rare if we look t the interviews analyses (that we will present later), since the formal properties are traditionnaly quite often associated to partial meanings that are considered more useful and intuitve: the terms the mathematicians use to define very formal properties like the density, the completeness, the infinite cardinality are used without reaching a proper level in the epistemic meaning.

The students ML are distributed in the three categories with a higher frequency for L: L (16), U (6) and intermediate (10); 10 teachers on 63 keep on balancing U and L meanings.

Now we will put together the information from Q2 and Q3 and this answers.

	LL	LU	UU	UL	MLU	MLL	MLI	MUU	MUL	DON'T
K4_A	17, 80, 89, 98, 104, 106	56,67	9, 21, 61	15, 37, 47, 74,	16, 55, 57, 110,	1, 22, 32, 38, 53, 93, 111	2, 4, 58, 62, 63, 72, 77		7, 10, 40, 78	20, 39, 76, 84
K4_B	46, 60, 79	100	68,95	5,41, 85, 105	64, 65, 66, 88	14, 83, 90, 108, 115	8, 11, 42	73	18, 26, 87, 91	101
K4_C			71	25	116	92			13	

Tab. 1

We consider LL/K4_B teachers that are used to contextualize everything in the field of real numbers (intended in its personal partial meaning) rather than decide time after time; indeed, to state that real numbers are limit points of R so to define a limit point R is necessary as a container, the teachers should have an AK that didn't show before. Even if they were only referring to classroom practices and were adapting them to the students' personal meanings, the profile that contains these teachers is characterized by a **low awareness of the complexity of the epistemic meaning of real numbers**. This will be put in relation with the category R NECESSARY and R BEFORE in the following Paragraph.

For the teachers who showed a relevant presence of answers U we can hypothesize, standing on the teachers' comments and also on our further interviews, we hypothesize that the choice B is reasoned and is interpretable using the dichotomy “whole-single” we introduced in the commentary in Par. 2.6.

4.5.5 Comparison between the teachers’ evaluated knowledge and the formation’s categories

We compare now the teachers' categories concerning real numbers and their formation, recalling here the categorization by formation, in order to answer the question:

PQ - 1.3 What relations between teachers' formation and their mathematical knowledge of real numbers emerged?

K1: Teachers' formation

Only PhD in Mathematics: (58, 72), (64)

PhD & Teachers' training: (4),(8), (57, 110)

Only Master in Mathematics : (1, 32, 111) , (2, 62), (5,41), (7), (11, 42), (13), (14, 83, 108), (15, 37, 37), (21, 61), (25), (26), (55, 110) (73), (79), (83, 108); (87); (89); (97)

Master in Mathematics and Teachers' training : (9), (16), (17), (20, 39, 84); (41, 105); (42); (65); (71); (74); (88); (91); (104); (115), (116)

Only Scientific Master not in Mathematics (Physics and engineering): (18), (63), (76), (77), (78), (81)

Scientific Master not in Mathematics & Teachers' training : (10, 40), (46), (53), (92)

Other with teachers' training: (22), (66), (90), (91), (93), (106)

Other: (56), (60), (80), (85), (95), (98), (100), (101), (102)

These result show that the teachers we interview is very heterogeneous, as we hypothesize in the beginning. What we didn't expect was the high complexity of this scheme, that don't divide the teachers in profiles depending on the teachers' formations, as we could expect, but on the contrary is much more articulated.

To find out regularities we had to regress one step before:

Only PhD in Mathematics:(58, 72), (64)

PhD & Teachers' training: (4,8), (57, 110)

Only Master in Mathematics : (1, 14, 32, 83, 108, 111), (2, 62), (5, 15, 37, 41), (7, 13, 26), (11, 42), (21, 61), (55, 110), (25), (73), (79), (87); (89); (97)

Master in Mathematics and Teachers' training : (9), (16, 116), (17, 104), (20, 39, 84); (41, 74, 105); (42); (65, 88, 91); (71); (115)

Only Scientific Master not in Mathematics (Physics and engineering): (18, 78), (63, 77), (76)

Scientific Master not in Mathematics & Teachers' training : (10, 40), (46), (53, 92)

Other with teachers' training: (22, 90, 93), (66), (91), (106)

Other: (56, 100), (60, 80, 98), (85), (95), (101), (102)

Even if there are more recurrences, it doesn't seem that the formation is a good marker in order to create teachers' categories. In fact all the teachers belonging to the same category related to formation are distributed in the mathematical knowledge categories quite equally. This is an interesting result because *a priori* we considered probable that the mathematical knowledge could be quite homogeneous in teachers who had similar formation while this is very different from what we observed: it seems that the mathematical knowledge of real numbers escapes from this kind of categorization.

4.6 Teachers' didactical goals categorizations

4.6.1 Teachers' goals and real numbers: categories based on the question Q7

G1: *Introduce R to work with:*
G1_A: Exponential function
5, 7, 10, 11, 14, 15, 16, 18, 21, 22, 25, 26, 32, 34, 38, 40, 41, 42, 47, 58, 59, 61, 62, 63, 64, 65, 66, 67, 68, 72, 73, 74, 75, 76, 77, 79, 80, 83, 84, 85, 87, 89, 90, 91, 92, 93, 100, 104, 105, 106, 116

G1_B: Logarithmic function
5, 7, 10, 11, 15, 16, 18, 22, 25, 26, 32, 34, 37, 40, 41, 42, 47, 58, 59, 61, 62, 63, 64, 65, 66, 67, 68, 72, 73, 74, 75, 76, 77, 79, 80, 83, 84, 85, 87, 90, 93, 95, 105, 116

G1_C: Differential calculus

4, 5, 11, 14, 16, 17, 22, 26, 32, 34, 37, 39, 40, 42, 46, 47, 53, 55, 56, 58, 59, 63, 64, 65, 66, 67, 72, 74, 75, 76, 77, 80, 83, 84, 85, 87, 88, 89, 90, 91, 105, 108, 110, 111, 116, 115,

G1_D: Integral calculus
4, 8, 10, 11, 13, 14, 16, 22, 26, 32, 34, 37, 39, 40, 42, 46, 53, 55, 58, 59, 62, 63, 64, 65, 66, 67, 68, 72, 73, 74, 75, 76, 79, 80, 83, 84, 85, 87, 88, 89, 90, 91, 104, 105, 108, 110, 111, 116, 115

G1_E: Sequences and series
2, 5, 8, 13, 14, 25, 26, 38, 42, 46, 55, 56, 59, 61, 64, 65, 68, 71, 72, 74, 76, 77, 80, 88, 91, 93, 95, 100, 110, 116,

G1_F: Absolute value
13, 76, 79, 116

G1_G: R - intervals
5, 7, 8, 10, 11, 13, 15, 16, 17, 18, 20, 21, 22, 32, 34, 38, 39, 40, 41, 42, 47, 53, 55, 58, 60, 61, 63, 64, 65, 67, 68, 71, 75, 76, 77, 79, 80, 83, 84, 85, 87, 88, 89, 92, 93, 95, 100, 101, 104, 105, 106, 108, 111, 116, 115

G1_H: Limits
2, 5, 8, 10, 11, 13, 14, 16, 17, 21, 22, 25, 26, 32, 34, 37, 38, 40, 41, 42, 47, 53, 55, 56, 59, 60, 61, 63, 64, 65, 66, 67, 68, 71, 72, 73, 74, 75, 76, 77, 79, 80, 83, 84, 85, 87, 88, 89, 90, 91, 92, 93, 95, 104, 100, 101, 105, 106, 108, 111, 116, 115

G1_I: Equation systems
5, 34, 55, 60, 61, 75, 76, 79, 84, 85, 87, 89, 90, 100, 104, 106, 116, 115

G1_L: Other
1, 29, 40, 57, 93, 115

We posed particular attention to Limits and Sequences and series, since it's another information to compare with the question concerning the necessity of using R for introducing the limit points.

Categories

GO_R_1) R is a prerequisite for quite all the mathematical objects and practices / once introduced R, it is obvious the numerical domain is R if there are no other indications / R is necessary for the graphics [41][L/U] : **R FOR ALL**

5, 10, 11, 14, 16, 22, 26, 32, 34, 37, 40, 42, 47, 58, 59, 62, 63, 64, 65, 66, 67, 68, 72, 73, 74, 75, 76, 77, 79, 80, 83, 84, 85, 87, 89, 90, 91, 104, 105, 116

GO_R_1’) R is not necessary at all [2] [U] : **NO R**

1, 20, 57

GO_R_2) R is necessary only for advanced mathematics (Calculus) [7] [U] : **R FOR U**

4, 13, 88, 108, 110, 111, 115

GO_R_2) R is necessary only for common mathematics/functions' graphics [12] [L]: **R FOR L**

7, 15, 18, 21, 25, 38, 41, 61, 92, 93, 95, 100

The most of the teachers consider R as necessary for introducing quite all the first 4 mathematical objects and systems of practices we listed, both advanced and not.

Two very different aspects of R are necessary in the two cases: the advanced practices need the U properties of R, while to introduce functions this property are not necessary, even if R is considered as the natural domain for the functions since they are supposed to have a continuous graph.

Following the synthetic principle we used to analyze the epistemic meaning of real numbers, the upper meanings of R aren't necessary to introduce the functions with a continuous graph, because standing on the usual practices exponential and logarithmic functions are assumed to be intuitively continuous because there are no reasons to consider a discontinuous behavior rather than proved to be continuous analytically.

An interesting issue concerns the teacher in the first category, both from the point of view of epistemic and cognitive suitability: are they aware of the two different meanings and of their problematic relation? Do they take care of the U properties when they use R in the Calculus' practices? Do they on the contrary try to use other partial meanings, maybe not related to infinitesimals, to the differential and topological structure of R? How do they use intervals and points, the two different kind of "elements" that compose the continuum?

Also it's interesting to investigate better the category of teachers who consider R necessary only in the L-practices, but not for the U-practices. In particular it would be interesting to compare this category with the previous ones in order to characterize better the teachers' goals and the properties they consider important at school and their knowledge of real numbers.

This help us in distinguishing the teachers that are aware of the U-practices but that are oriented to:

1) reconstruction (re-create situations and propose problems through which mathematical objects emerge)

2) simplification (to use objects as cultural static entities to take at school in a simplified version)

For instance to use R as domain always, even when it's not necessary, can be considered a marker of the second kind of approach: the teachers may know that the issue of continuity is anything but intuitive but prefer to simplify; what is particular is that these teachers don't renounce to say that the domain is R, even if they know that the students think at R as a Q enlarged with some known irrational numbers (Hybrid continuum 3.1). Maybe the same teachers avoid to deepen the questions concerning the differential/topological meaning of R when they work with Calculus.

Comparing these categories with the previous:

1) **R FOR L**

(7, 18)[MU/L]; (15, 25, 41)[U/L]; (21, 61, 95)[U/U]; (92, 93) [ML/L], (100)[L/U]

The most of these teachers of these category (8/11) indeed are teachers belonging to the categories U or MU; 5 of these showed an inclination to practices also in the previous practices.

These teachers are teacher with a U-background that decide to simplify and reduce R to its L-properties at school.

2) **R FOR U**

(4) [ML/I], (13, 88) [MU/L], (108, 111, 115) [ML/L], (110) [ML/U]

In this case the most the teachers are ML; also the teachers that were MU were the ones also oriented to L-practices.

3) **R FOR ALL**

(5, 37, 47, 85, 105)[U/L], (10, 26, 40, 64, 65, 91)[MU/L] , (11, 42, 58, 62, 63, 72, 77) [ML/I], (14, 22, 59, 75, 83, 32, 87, 90) [ML/L] , (16, 66, 116) [ML/U]; (79, 80, 104)[L/L], (67), (68), (73), (74), (76, 84) [DON'T]

Quite all the teachers proposed a L-construction of real numbers, and with this choice they confirm their choice.

Nevertheless we have to operate distinctions between the teachers who selected quite all the possibilities for a matter of “comfort”, as we stressed before, and teachers who feel sure if they use R: they got convinced that it is important to use always R, maybe because their teachers did the same and they are not completely aware of the reason why.

The first kind of teachers, once introduced intuitively (CE4) or in a hybrid form (C3.1), for habit always introduce new concepts in R; maybe they identify R both with the L-partial and with the U-partial epistemic meanings and maybe time after time refer to a different partial meaning without explaining it to the students.

In other words we are here paying attention to differentiate naive approaches due to a lack of AK and multifaceted approaches, due to AK, even if maybe not related to complex and well organized meanings.

Since we hypothesized, standing on our pilot study, that it's not a suitable choice to use R dealing with L-partial meanings and then with U-partial meaning without a deep reflection about the relation between the line and the numbers, dense and complete, discontinuous, discrete and continuous sets, formal and intuitive conceptions, we think it's important to introduce further categories that substitute R FOR ALL and that permit to understand better the reason why a teacher is oriented to this choice.

For instance it's interesting to notice that only a little part of the teachers talked about construction of R as the set of all the points of the line or a set generated by continuous variations; being this the partial meanings at the basis of the intuitive approach to the functions with a continuous graph associated to the numbers, we can hypothesize that this are used at school in order to simplify but the connections between R and the continuous graphs are not clarified explicitly.

- **R HABIT (AK): Teachers who show a multifaceted knowledge of R for functions and Calculus**
- **R NECESSARY (CK): Teachers who showed a weak knowledge of R for functions and Calculus**

We hypothesized that the previous categorization could help us to distinguish between these two kind of teachers, but we realized that the information were not enough and there were more variables to take in account. However we decided to investigate the existence of this two categories in further interviews.

What we can see with our questionnaire is instead: do the teachers see R as a starting point or as a point to reach in the end, or maybe never, in the high school? This issue is linked to the possibilities “Sequences and series” (SS) and “Limits” (L) and the categories concerning the possibility to define the limit point in Q or in R. In fact these are all tools to construct R as long as mathematical objects that has properties that we can't always satisfy in Q. We put together here the two groups of data in order to create two categories:

L: 10(A), 11(B), 16(A), 17(A), 21(A), 22(A), 32(A), 34(A), 37(A), 40(A), 41(B), 47(A), 53(A), 60(B), 63(A), 66(B), 67(A), 73(B), 75(A), 79(B), 83(B), 84(A), 85(B), 87(B), 89(A), 90(B), 92(C), 104(A), 101(B), 105(B), 106(A), 108(B), 111(A), 115(B) [34]

SS: 46 (B), 110 (A) [2]

SS&L: 2 (A), 5(B), 8(B), 13(C), 14(B), 25(C), 26(B), 38(A), 42(B), 55(A), 56(A), 59(), 61(A), 64(B), 65(B), 68(B), 71(C), 72(A), 74(A), 76(A), 77(A), 80(A), 88(B), 91(B), 93(A), 95(B), 100(B), 116(C) [28]

64 teacher on 76 choose at least one between Sequences and series (SS) or Limits (L). In this group we look for the teacher who answered consistently with the question concerning the limit points (K4_B) and denote this category with R BEFORE ALL; the other teachers showed an apparent inconsistency, in particular the teachers SS&L who chose K2_A before, because they stated that it's possible to define a limit in Q, while they state know that the properties of real numbers are necessary to introduce limits and sequences, basic element of the limit point configuration (CE2, C2.4, C3.4). It would be interesting and not inconsistent a teacher who knows that it's possible to restrict to Q, but however decide to use R before at school, since this will be a “pure didactical” choice.

- **R BEFORE ALL [26]**

L: 11(B), 41(B), 60(B), 66(B), 73(B), 79(B), 83(B), 85(B), 87(B), 101(B), 105(B), 108(B), 115(B)

SS: 46 (B)

SS&L: 5(B), 8(B), 14(B), 26(B), 42(B), 64(B), 65(B), 68(B), 88(B), 91(B), 95(B), 100(B)

- **Q SUFFICIENT & R NECESSARY [32]**

L: 10(A), 16(A), 17(A), 21(A), 22(A), 32(A), 34(A), 37(A), 40(A), 47(A), 53(A), 63(A), 67(A), 75(A), 84(A), 89(A), 104(A),106(A), 111(A) [19]

SS: 110 (A) [1]

SS&L: 2 (A), 38(A), 55(A), 56(A), 61(A), 71(C), 72(A), 74(A), 76(A), 77(A), 80(A), 93(A [12]

4.7 Teachers' didactical orientations categorization

4.7.1 Teachers' categories based on the question Q8-9-10: the "reality" of irrational numbers

O15: Consider the first video as a good tool [L] [3.1]

O15_A: 1, 5, 7, 8, 10, 11, 13, 17, 18, 20, 21, 26, 32, 38, 41, 46, 47, 55, 59, 60, 61, 63, 67, 72, 73, 77, 78, 83, 84, 85, 87, 88, 89, 90, 91, 92, 93, 95, 100, 104, 106, 108, 116, 115

O15_C: 1, 2, 4, 7, 8, 11, 13, 14, 15, 16, 17, 18, 20, 21, 22, 25, 26, 32, 38, 40, 47, 55, 57, 58, 59, 60, 61, 62, 63, 64, 66, 67, 68, 71, 72, 73, 74, 75, 76, 78, 79, 80, 83, 84, 87, 88, 89, 91, 92, 93, 95, 100, 101, 104, 105, 106, 108, 110, 111, 116, 115

O15_D: 56, 63, 65, 85, 87, 101
O15_E_: Other 32, 115

O16: Consider the first video as a bad or not useful tool [U]
O16_B: 36, 42, 53, 65
O16_E: 4, 20, 34, 36, 37, 57

This video is rich of many interesting elements that concern some crucial debates about real numbers; for a complete analysis see Par 2.6.3.6. the video presents a geometrical problem, an arithmetical proof and a projection of a segment onto a line, called "line of the real numbers", so here the author put together the hybrid approach (3.1) and the Line-Systemic configuration (CC6).

As we stressed before the most of the teachers didn't mention the CC6 configuration before. Other elements are interesting: the problem presented is concrete (a cord that bounds a garden); the line is anticipated to be the "line of real numbers"; then an approximation method is proposed and also irrational numbers are used to represent physical measures with km as unit.

We think that there are too many critical elements and that this concrete approach, maybe useful to catch the students' attention, may vanish in the future the teachers' attempts to create a linguistic game in which the students have to discuss about real numbers, since this is not a concrete problem and in the so-called applied problems of real life, irrational numbers are really not necessary at all and are substituted by their approximations. The same happens in Physics dealing with measures and errors. Irrational numbers emerge analyzing the Nature (golden ratio, π, e) or the musical scales but this are far from being concrete problems in the sense of this video. A very interesting approach to the issue of irrationality, faithful to the historical resources and to the partial meaning to which this problem belongs, is presented in Recchiuti (2015), a Master thesis concerning the role of the History of mathematics in the introduction of irrationality. The author reports a very good reaction in the classroom and stresses the potentiality of carrying out argumentation practices. This seems much more interesting since it's consistent with the theoretical demands and the request of rigor that followed through the evolutions of real numbers.

Nevertheless the most of the teachers seems not to notice these elements, not to consider them as important and misleading, or on the contrary, consider this elements good.

The large agreement concerning it lead us to analyze in depth the reasons of this appreciation and the kind of teachers that disregarded it.

First of all we notice that both the teachers who listed only L-practices or only U-practices indifferently think this video helps the students to understand real numbers. Only one teacher R FOR U show some doubts and critical comments. This is a bit surprising for the R FOR U and for the */U and suggests to us that at school concrete and intuitive approaches are considered good by quite all the teachers' categories; this makes emerge two different interpretations:

- the teachers, looking at the teaching materials, only "see" what they want the students to learn, defoliating respect of the other elements (interesting for cognitive suitability);
- the teachers, whatever is their knowledge, at school tend to choose concrete and intuitive approaches (interesting for the intertwining between epistemic and cognitive suitability)

1) **R FOR L 's [12]** (O15)

(7, 18)[MU/L]; (15, 25, 41)[U/L]; (21, 61, 95)[U/U]; (92, 93) [ML/L], (100)[L/U] (O15)

2) **R FOR U's [7]** (O15)

(4) [ML/I](O15&O16), (13 (O15), 88(O15)) [MU/L], (108(O15), 111(O15), 115(O15)) [ML/L], (110)(O15) [ML/U]

The teachers who are critical are:

42(RFA), 53(RFU), 65 (RFA), 20 (NOR), 34(RFLIMITS), 37(RFA), 57(NOR)

This group of teachers is very interesting for our analysis since maybe renounce to traditional intuitive approaches and look for other paths, recognizing the different nature of the practices that lead to R as a cultural object, both to the partial meaning C3.1 or to the upper C4.1 and C4.2.

Looking better to the categories to which the teachers belong we observe that all these teachers quoted at least some U-properties. Also it's interesting that some of these teachers proposed an introduction using incommensurable magnitudes and root squares but don't agree with this approach. This is exactly what we expected from a group of teachers since the problem itself may be used treat important topics concerning about real numbers but avoiding concretizations.

NO CONCRETE R: 42(RFA/MLI), 53(RFU/MLL), 65 (RFA/MUL), 20 (NOR/LU-DONT), 34(RFU/MLL), 37(RFA/UL), 57(NOR/MLU)

4.7.2 Teachers' categories based on the question Q10: the "flowing correspondence" between the real numbers and the points of a line

The video show without audio-comments a point flowing on a line and some rational finite numbers appear time after time while the point is moving forward. The title of the video is quite ambitious: correspondence between real numbers and points of the line. This topic may be faced at many levels of generality and the continuous flow is one of these most primitive (CE4, C2.1, C2.2). Indeed even if also Newton based his Calculus on flowing magnitudes this was not a base for a U-definition of real numbers and its relations with the line, in particular for properties like density and completeness.

Also we have to remind that no teachers mentioned the dynamical aspects of continuity talking about the properties of real numbers. Looking at the teachers' answers we can observe that quite an half of the teachers think that this video could be useful:

Some teachers highlighted positive and negative aspects.

Thus it's interesting to explore the reasons why so many teachers said this would be a good material to introduce the correspondence between line and real numbers and why some teachers don't agree.

Q13: Video 2: Yes, because...

Video 2: *Yes, because*...(No explanation)

21, 56, 61, 74, 76, 80, 101, 116

O17_A: Makes evident the association between numbers and segments / visualization numbers and points

7, 20, 38, 62, 87, 92, 100, 115

O17_B: The flow of the endpoint is very effective to show the order of real numbers: 8, 10, 108

O17_C: The students have a graphic vision of the problem: 13, 58, 60

O17_D: It's intuitive: 25, 67, 90

O17_E: Makes evident, avoiding not useful words, the completeness/density of R, we could not do the same with the other sets: 71, 83, 22

O17_F: To propose examples is always good: 74, 116

O17_G: Thinks that renforcing conceptions of real numbers through visual images of continuity is a good choice: 32, 67

Q13: Video 2: No, because...

O23: Thinks that density and continuity can't be distinguished by means of graphic representations (other problems are necessary): 1

O24: The second video may confuse the students since the numbers represented are only rational and positive numbers: 2, 10, 16, 34, 40, 41, 42, 47, 55, 79, 64, 65, 68, 72, 91, 93, 110, 111

O25: The flow of the slider of the second video would represent the correspondence with real number (A) if there were no numbers on the line that may suggest a partition in steps (B)

O25_A: 4, 10, 16, 66, 64, 65, 68, 72, 91, 93

"*Lo studente non può vedere ad esempio π o √2*"

O25_B: 4, 5, 10, 22, 90, 93, 66, 115

"Lo slider ha passo uguale a 0.1 e sembra che i numeri razionali ricoprano la retta reale"
"Dà un'impressione di continuità che non compete alle variazioni che si leggono in corrispondenza della posizione del punto"

O26: A zoom of a small part of the line that become longer may suggest the correspondence between real number and points of a line: 4, 115

O27: The second video is not enough, an explanation is necessary: 5, 17, 26, 53, 63, 73, 95

O28: The scope is not clear: 14, 37, 46, 77, 78

O29: It's impossible to see the correspondence in this way: 15, 22, 57, 84, 85

O30: The dynamicity may confuse the students: 18, 22

O31: The necessity of sampling should be motivated : 40

O32: The visive intuition is not enough to make the student understand the deep questions concerning the correspondence: 53

O33: The graphic is not effective: 59, 101

O34: The movement should be continuous instead of going on step by step since the line seems to have empty spaces: 66

O35: Two representation at the same time may confuse the students: 101

O36: It seems that length and numbers are the same thing / lack of unit: 105, 110

"Cercherei di non confondere tra numeri e punti"

O37: The infinity of real numbers between 1 and 2 doesn't appear:106

Macro-orientations about flow and real numbers

*OF) The visualization of a flow helps the students to have intuitions about real numbers properties(*O17E, O17B, O34, O17, O17A, O25, O36*)*

OF_2_A) It's possible to visualize density and completeness by means of a flow: O17E

OF_2_B) It's possible to visualize the order of real numbers by means of a flow : O17B, O34

OF_2_C) It's possible to visualize the correspondence between real numbers and points of a line by means of a flow: O17, O17A, O25, O36

*OF') Visualization of a flow don't help the students to understand real numbers properties (*O30, O36, O37, O31, O23, O29, O32)

OF_2'_A) It's impossible visualize infinite points in a segment: O37, O31

OF_2'_B) It's impossible to visualize density and completeness by means of a flow: O23

OF_2'_D) It's impossible to visualize the correspondence between real numbers and points of a line by means of a flow : O29, O32

Comparison between previous categories and the categories emerged in Q13

Yes, because (no explanation)

(21,61) [UU], (56) [LU], (74)[UL], (76, 101), (80)[LL], 116[MLU]

O17_A: Makes evident the association between numbers and segments / visualization numbers and points: (7)[MUL], (20), (38, 87, 92, 115)[MLL], (62)[MLI], (100)[LU]

O17_B: The flow of the endpoint is very effective to show the order of real numbers (8)[MLI], (10)[MUL], (108)[MLL]

O17_C: The students have a graphic vision of the problem: 13[MUL], 58[MLI], 60[LL]

O17_D: It's intuitive: 25[U/L], 67[LU], 90[MLL]

O17_E: Makes evident, avoiding not useful words, the completeness/density of R, we could not do the same with the other sets: 71[UU], (22, 83)[MLL]

O17_F: To propose examples is always good: 74[UL], 116 [MLU]

O17_G: Thinks that reinforcing conceptions of real numbers through visual images of continuity is a good choice: 32[MLL], 67[LU]

Quite all the teachers in this category are teachers with profiles containing U-properties; the most of them showed to be oriented towards L-practices in the first categorization; looking at the reasons why they would choose this video we confirm that these teachers are oriented to try to simplify the approaches to real numbers; in particular it emerge that the way all of them try to do it is by means of visualization, also of properties like density and completeness.

Through the flowing point they try to avoid to formalize at any cost.

This category is for us: **INTUITIVE SIMPLIFIERS** and is associated to OF. This teachers are convinced that very formal properties have an intuitive dimension that can substitute every formalization.

To comment on this category we can quote again Fletcher (Par. 1.1.1).

"Newton based his ideas of limits and differentiation on intuitions of motion; other mathematicians based their ideas of continuity on spatial intuition. These kinematic and geometric conceptions fell into disfavour in the nineteenth century, as they had failed to provide satisfactory theories of negative numbers, irrational numbers, imaginary numbers, power series, and differential and integral calculus (Bolzano, 1810, preface). Dedekind pointed out that simple irrational equations such as $\sqrt{2} \cdot \sqrt{3} = \sqrt{6}$ lacked rigorous proofs (1872, §6). Even the legitimacy of the negative numbers was a matter of controversy in the eighteenth and nineteenth centuries (Ewald, 1996, vol. 1, pp.314–8, 336). Moreover, Bolzano, Dedekind, Cantor, Frege and Russell all believed that spatial and temporal considerations were extraneous to arithmetic, which ought to be built on its own intrinsic foundations" (Fletcher, 2007).

The real numbers as complex results of geometrical, algebraic and arithmetical processes belong to a somehow counterintuitive thread in the mathematics, as it was confirmed by the critiques we reported briefly in Par 1.1.1 and inserted as a further development in Par 3.1.3.

Focusing on the other teachers we can observe that the main reasons why the video is considered unsuitable are exactly the contrary of those proposed by the INTUITIVE SIMPLIFIERS. This is particularly interesting.

In fact the two main teachers of objections are:

- **semiotic-representative**: the only visualization is not enough to introduce such a complex issue; the coordination with other registers and in particular with representations in the verbal register would be necessary;
- **goal-oriented:** the scope is not connected with the video and is not clear; a mediation and a discourse that pose the problem is necessary.

We name this group SEMIOTIC COMPLEXIFIERS.

SEMIOTIC COMPLEXIFIERS : 5, 17,26, 53, 63, 73, 95, 14, 37, 46, 77, 78, 15, 85, 22, 57, 84, 18, 22, 40, 59, 101, 66, 105, 110, 106

INTUITIVE SIMPLIFIERS: (21,61) [UU], (56) [LU], (74)[UL], (76, 101), (80)[LL], 116[MLU], (7)[MUL], (20), (38, 87, 92, 115)[MLL], (62)[MLI], (100)[LU], (8)[MLI], (10)[MUL], (108)[MLL], 13[MUL], 58[MLI], 60[LL], 25[U/L], 67[LU], 90[MLL], 71[UU], (22, 83)[MLL], 74[UL], 116 [MLU], 32[MLL],

	INTUITIVE/SIMPLIFIERS	SEMIOTIC/COMPLEXIFIERS
BEFORE	100 108 60 83	5 26 73 95 14 37 46 85 101 66 105
Q SUFF/R NEC	21 10 67 22 32	17 53 63 22 84 40 106

The most of the INTUITIVE SIMPLIFIER doesn't belong to the categories **"R before all"** or **"Q sufficient & R necessary";** maybe these two categories are in different ways "structured" while the other teachers follows the principle of lower formality and maximum degree of intuitiveness, escaping from the epistemic structures.

For all the four crossed categories it's interesting to deepen the ways they decide to approach the different problems-situations concerning the real numbers. First of all we explore their relation with the categories concerning the first video, in particular with the teachers who didn't declare that the video was a good tool.

NO CONCRETE R/SEMCOMP: 53(RFU/MLL), 37(RFA/UL)

No teachers are NO CONCRETE and INTUITIVE.

1) R FOR L 's distribution (O15)

(21)[U/U][QI]; (95) [U/U][BS], (100)[L/U][BI]

2) R FOR U's distribution (O15)

108(O15)[MLL][BI]

There are no significant relations between the categories concerning the first and the second video.

We go on analyzing the last questions, all concerning the representation of intervals. The first question concern the third video, while the other the teachers' preferences and comments about some students' answers.

4.7.3 Teachers' categories based on the question Q11-12-13-14-15-16: the representations of intervals of rational and real numbers

a. Categories based on the third video (Q11)

O38: The third video is not suitable to introduce inequalities

O38_A: The problem presented in the third video isn't useful: 1, 41, 72

O38_B: The language used in the third video isn't good: 1, 7, 88

O38_C: The solution presented is not the graphic solution (a procedure) but instead a graphic representation: 2, 4, 13, 53, 84

O39: The graphic representation allows to visualize better the solutions: 4, 5, 53, 63, 66, 72

O40_A: There are not two solutions but only two representations of the solution: 4, 7, 10, 14, 17, 18, 19, 21, 22, 26, 37, 38, 42, 57, 65, 68, 78, 84, 88, 91, 93, 95, 101, 105, 110, 115

O40_B: It's not clear that the solution of the inequalities are intervals, too many attention is posed to equations and it seems that intervals can only be represented in the graphic register: 10

O40_D: Prefer explicitly the graphic solution even if the representations are equivalent: 38, 53

O40_E: The graphic representation is intuitive: 10

O40_F: The algebraic approach helps in the "limit cases": 10

O40_G: The graphic representation makes visible the infinite quantity of solutions: 11, 34, 53, 64

O40_H: The graphic representation makes the students understand the meaning: 13, 53, 73, 87, 90

O40_I: The graphic representation is useful in the practice: 15, 47, 57, 63, 72, 73

O40_L: It's better to coordinate the registers: 16, 25, 26, 60, 65, 74, 75, 80, 83, 89, 101, 106, 107, 111, 116, 115

O40_M: the graphic representation is not a prove of the solution: 21

O40_N: The graphic solution makes understand the intervals: 32, 62, 66

O40_O: Should emphasize the starting point: 32

O40_P: It's not clear if the methods are equivalent: 40

O40_Q: The methods are not the solutions: 55

O40_R: The graphic solution is always better; it's necessary: 63, 85

O40_S: The choice of a method depends on the goals: 67

O40_T: The algebraic solution is exact, the other is approximated: 68

O40_U: The graphic solution is part of the solution procedure: 73

O40_V: The graphic solution is a "representation" of the algebraic one: 77, 87, 92

O40_Z: The graphic representation makes concrete the example: 79

O40_J: The graphic representation is not the solution and too visualization obstructs the abstraction: 95

INTERVALS OF REAL NUMBERS

RI1: Absolute value, same register, used for limits: 55, 65, 66, 72, 80, 88, 93, 108

RI2: "Endpoints" inclusion with <: 1, 2, 4, 5, 7, 8, 10, 11, 14, 15, 16, 17, 20, 21, 22, 25, 26, 32, 34, 39, 40, 41, 42, 47, 53, 55, 57, 60, 63, 65, 67, 68, 72, 73, 75, 76, 77, 78, 79, 80, 83, 84, 85, 87, 88, 89, 90, 91, 92, 93, 95, 105, 106, 110, 111, 116, 115

RI3: < and or/and representation: 1, 20, 22, 42, 55, 57, 65, 80, 90, 93, 105, 106,

RI4: "Endpoints" inclusion with parenthesis [a,b]: 40, 61, 62, 63, 66, 68, 83, 85, 88, 95, 108, 116, 115

RI5: Graphic representation: 1, 4, 5, 14, 16, 17, 26, 32, 41, 42, 53, 57, 62, 63, 64, 65, 73, 74, 80, 87, 91, 93, 105, 106, 110, 115

INTERVALS OF RATIONAL NUMBERS

QI1: Graphical representation: 1, 13, 17, 116,

QI2: $-\sqrt{5} < x < \sqrt{5}$

1, 2, 7, 10, 11, 15, 16, 21, 22, 25, 32, 116

QI3: [-√5 ,√5]

16

QI4:]-√5 , √5 [

1, 8, 11, 15, 16, 17, 22, 32, 116,

QI5: |x| < √5

1, 4, 11, 13, 21,

O41_B: Different intervals' representations are not equivalent: 57, 90

O41_C: Some representations are more suitable than others in specific practices: 1, 2, 8, 10, 17, 33, 34, 47, 62, 72, 79, 85, 87, 88, 91, 108, 115

O41_D: Some representations of intervals are immediate/clear/explicit/intuitive or on the contrary, implicit: 4, 5, 7, 8, 10, 11, 17, 20, 26, 32, 34, 37, 42, 53, 62, 64, 66, 73, 74, 79, 91, 95, 110

O41_E: Only some of the proposed solutions are representation of intervals: 1, 2, 7, 15, 22, 34, 67, 68, 73, 75, 79, 80, 83, 84, 85, 87, 88, 89, 91, 92, 95, 105, 106, 108, 110, 111,

O41_F: Some representations are not "finished"/are the problem and not the solution: 5, 7, 10, 11, 15, 22, 32, 39, 40, 41, 53, 57, 62, 68, 77, 84, 91, 95, 115

O41_G: The synthetic representations are better: 10, 40, 63, 78, 79, 95, 115

O41_H: The representation are complementary and the meaning is a configuration of the different objects: 57, 75

O41_I: Every representation has specific features: 67, 65

O41_L: The graphic representation is not formal enough/"represents" the algebraic one: 83, 90, 95, 13

O41_M: The graphic representation of intervals can be useful to introduce upper and lower endpoints in the real numbers: 87

O42_A: The parenthesis [,] represent inclusion also in the case of rational numbers: 1, 4, 10, 11, 13, 14, 17, 21, 22, 32, 34, 40, 42, 57, 60, 62, 63, 67, 68, 72, 74, 79, 80, 85, 87, 88, 89, 91, 92, 94, 108, 116, 115

O42_B: The domain of inequalities should be indicated only in the case of rational numbers: 4, 8, 10, 20, 26, 32, 41, 57, 62, 63, 64, 65, 68, 72, 73, 78, 80, 83, 84, 85, 94, 111, 115,

O42_E: Segments "are" intervals of real numbers: 2, 4, 5, 8, 16, 22, 26, 32, 39, 42, 53, 55, 64, 65, 68, 73, 77, 78, 83, 93

O42_F: The usual representations of intervals of real numbers can't represent intervals of rational numbers: 2, 5, 8, 14, 20, 22, 26, 39, 41, 53, 55, 64, 65, 73, 77, 78, 83, 84, 90, 101, 105, 110,

O42_G: All the representations are good: 7, 47, 61, 66, 76

O42_H: In the representation of intervals of rational numbers irrational numbers can't be used: 21, 87

O42_I: A draw is not a set: 62

INTERVALS REPRESENTATIONS’ CATEGORIES

CIR_1) To coordinate adequately registers to represent the solutions of inequalities (intervals) is useful and important/ a lack of good coordination between verbal and other representation makes a didactical practice unsuitable: O38_B, O40_A, O40_F, O40_L, O40_P

1, 7, 88, 4, 7, 10, 14, 17, 18, 19, 21, 22, 26, 37, 38, 42, 57, 65, 68, 78, 84, 88, 91, 93, 95, 101, 105, 110, 115, 10, 16, 25, 26, 60, 65, 74, 75, 80, 83, 89, 101, 106, 107, 111, 116, 115, 40

CIR_2) There is a hierarchy in the intervals' representations

a. The graphic representation is better/more intuitive, more synthetic O39, O40_D, O40_G, O40_H, O40_I, O40_N, O40_R, O40_Z, O22

4, 5, 53, 63, 66, 72, 38, 53, 11, 34, 53, 64, 13, 53, 73, 87, 90, 32, 62, 66, 15, 47, 57, 63, 72, 73, 32, 62, 66, 63, 85, 79

b. The algebraic representation is better/more precise, more formal: O40_T: 68

c. The graphic representation "represents" the algebraic one: O41_L : 83, 90, 95, 13

d. The algebraic representation "represents" the graphic one: O40_V: 77, 87, 92

CIR_3) The intervals representations are equivalent: O40_A, O40_P, O42_G: 4, 7, 10, 14, 17, 18, 19, 21, 22, 26, 37, 38, 42, 57, 65, 68, 78, 84, 88, 91, 93, 95, 101, 105, 110, 115, 40, 7, 47, 61, 66, 76

CIR_3') The intervals representations are not equivalent: O41_B: 57, 90

CIR_5) A segment can represent the infinite real solutions of an inequality: O39, O40_G, O42_E

11, 34, 53, 64, 4, 5, 53, 63, 66, 72, 2, 4, 5, 8, 16, 22, 26, 32, 39, 42, 53, 55, 64, 65, 68, 73, 77, 78, 83, 93

CIR_5') A segment can't represent the infinite real solutions of an inequality: O40_M, O40_J, O42_I : 62, 21, 95

O7: A segment on the number line is a representation of a subset of R: 22

CIR_6: The usual representation of real numbers intervals can represent a subset of Q: O42_A, RI5&QI1, RI4&QI3, RI1&QI5, RI2&QI2: 1, 10, 11, 13, 14, 17, 21, 22, 32, 34, 40, 42, 57, 60, 62, 63, 67, 68, 72, 74, 79, 80, 85, 87, 88, 89, 91, 92, 94, 108, 116, 115, 1, 17, 16, 1, 2, 7, 10, 11, 15, 16, 21, 22, 25, 32, 116

CIR_6': The usual representation of real intervals can't represent a subset of Q: O42_E, O42_F, O42_H

2, 4, 5, 8, 16, 22, 26, 32, 39, 42, 53, 55, 64, 65, 68, 73, 77, 78, 83, 93, 2, 5, 8, 14, 20, 22, 26, 39, 41, 53, 55, 64, 65, 73, 77, 78, 83, 84, 90, 101, 105, 110

4.6.3.3 Teachers' semiotic profiles

GENERAL SEMIOTIC CATEGORIES OF ORIENTATIONS

GSO_1) **No need of mediation:** A representation can be intuitive/immediate/clear, whoever the interpreter of the sign is: O17A, O17B, O17D, O17E, O39, O40_E, O40_G, O40_H, O40_N, O41_D, O42_E : 7, 20, 38, 62, 87, 92, 100, 115, 8, 10, 108, 13, 58, 60, 71, 83, 22, 4, 5, 53, 63, 66, 72, 11, 34, 53, 64, 10, 32, 62, 66, 4, 5, 7, 8, 10, 11, 17, 20, 26, 32, 34, 37, 42, 53, 62, 64, 66, 73, 74, 79, 91, 95, 110, 2, 4, 5, 8, 16, 22, 26, 32, 39, 42, 53, 55, 64, 65, 68, 73, 77, 78, 83, 93

GSO_1') **Mediation is necessary**: A graphic representation is not enough, language or other representations are necessary to give sense to it := O7, O23, O27, O28, O33, O40_M : 22, 1, 14, 37, 46, 77, 78, , 17, 26, 53, 63, 73, 95, 21, 59, 101

"Il concetto deve essere mediato necessariamente dall'insegnante"

GSO_2) A **graphic representation is always useful** and more intuitive than the others (ostensive, hierarchy): O22, O17_C, O17F, O40_D, O40_H: 13, 58, 60, 74, 116, 38, 53, 13, 53, 73, 87, 90

GSO_3) The **coordination between different registers may confuse the students:** O35: 101

"Presenta simultaneamente più entità matematiche senza focalizzare l'obiettivo"

GSO_4) S**ome representations are more suitable** than others in specific practices: O41_C, O41_M: 1, 2, 8, 10, 17, 33, 34, 47, 62, 72, 79, 85, 87, 88, 91, 108, 115

"Io preferirei porre l'accento sul fatto che rappresentano la stessa cosa, e che se ne userà l'una o l'altra a seconda del contesto e della convenienza"

GSO_5) Some signs are not representations since they're not finished processes / **Some signs associated to intervals represent the task and not the solution:** O41_E, O41_F: 1, 2, 7, 15, 22, 34, 67, 68, 73, 75, 79, 80, 83, 84, 85, 87, 88, 89, 91, 92, 95, 105, 106, 108, 110, 111,

5, 7, 10, 11, 15, 22, 32, 39, 40, 41, 53, 57, 62, 68, 77, 84, 91, 95, 115

GSO_6) The **synthetic representations** are better: O41_G: 10, 40, 63, 78, 79, 95, 115

GSO_7) **Different representations are complementary** / the meaning is a result of configurations: O41_H, O41_I, GSO_4: 67, 65, 57, 75, 1, 2, 8, 10, 17, 33, 34, 47, 62, 72, 79, 85, 87, 88, 91, 108, 115

"La soluzione grafica porta a trovare la soluzione algebrica per es. in sistema di disequazioni"
"Si tratta di due rappresentazioni distinte di un concetto matematico, che si arricchiscono proprio in virtù della loro diversità"

GSO_8) There is a **hierarchy between the representations:** CIR_2 : 4, 5, 53, 63, 66, 72, 38, 53, 11, 34, 53, 64, 13, 53, 73, 87, 90, 32, 62, 66, 15, 47, 57, 63, 72, 73, 32, 62, 66, 63, 85, 79, 68, 83, 90, 95, 13, 77, 87, 92

a. The graphic representation is better/more intuitive, more synthetic O39, O40_D, O40_G, O40_H, O40_I, O40_N, O40_R, O40_Z, O22

b. The algebraic representation is better/more precise, more formal: O40_T:

"Per passare dalla presentazione grossolana della grafica alla soluzione raffinata dell'astrazione algebrica"

c. The graphic representation "represents" the algebraic one: O41_L :

"a mio avviso non esiste una "soluzione algebrica" ed una "soluzione grafica" della disequazione; la seconda è una "rappresentazione" convenzionale della prima"

"non sono due "soluzioni", ma l'espressione della stessa soluzione sotto due diversi punti di vista. Devono imparare a vedere la soluzione grafica come espressione grafica dell'algebrica."

d. The algebraic representation "represents" the graphic one

GSO_9) **Different representations are equivalent**: CIR_3)

"Sono diverse rappresentazioni di uno stesso insieme numerico"
"No, la soluzione è unica, abbiamo diversi modi per rappresentarla"

GSO_10) **A representation <u>is</u> the object / the representation is univocal:**

"la 1), 3) e 4) rappresentano intervalli della retta reale"

"nessuna perché l'insieme in cui cercare le soluzioni non è R"

Intersecting these categories we create 4 profiles:

<u>ABSOLUTE MEANING</u>

5, 11, 32, 39, 7, 68, 92, 77, 83, 110**: GSO_1 & GSO_5**

4, 16, 42, 55, 64, 66, 20, 93, 100, 71**: GSO_1**

38, 13, 58, 60, 74**: GSO_1 & GSO_2**

65**: GSO_1 & GSO_7**

<u>COMPLEX SEMIOTIC APPROACH</u>

2, 62, 108**: GSO_1 & GSO_4 & GSO_5 & GSO_7**

87, 34**: GSO_1 & GSO_2 & GSO_4 & GSO_5 & GSO_7**

79, 115**: GSO_1 & GSO_2 & GSO_4 & GSO_5 & GSO_6 & GSO_7**

8, 72**: GSO_1 & GSO_4 & GSO_7**

10**: GSO_1 & GSO_4 & GSO_5 & GSO_6**

62, 91: **GSO_1 & GSO_5 & GSO_7**

<u>MEDIATION AND COORDINATION</u>

1**: GSO_1' & GSO_2 & GSO_5**

14, 46, 21, 59**: GSO_1'**

101**: GSO_1' & GSO_3**

<u>GLOBALLY INCONSISTENT</u>

22**: GSO_1 & GSO_1' & GSO_5**

53**: GSO_1 & GSO_1' & GSO_2**

63, 78**: GSO_1 & GSO_1' & GSO_6**

17**: GSO_1 & GSO_1' & GSO_4**

26, 37, 77**: GSO_1 & GSO_1'**

73**: GSO_1 & GSO_1' & GSO_2 & GSO_5**

95**: GSO_1 & GSO_1' & GSO_5 & GSO_6**

<u>OTHER</u>

90, 116**: GSO_2**

15, 41, 84, 80, 89, 105, 106, 111**: GSO_5**

57, 67, 75, **: GSO_5 & GSO_7**

40: **GSO_5 & GSO_6**

33, 47**: GSO_4 & GSO_7**

85, 88**: GSO_4 & GSO_5 & GSO_7**

4.8 **From teachers' profiles to the teaching sequences concerning real numbers: the epistemic, institutional and personal dimensions of some teachers' choices emerging in the interviews**

4.8.1 *Case studies*

4.8.1.1 PhD teachers

Our first qualitative analysis of the teachers' answers to the questionnaire concerned the PhD teachers, i.e. high school teachers with a PhD in Mathematics. The aim of this analysis was to look for recurrences in the choices or in the emerging orientations of the teachers with an advanced background in Mathematics asked to teach high school students. We omit in this case the analysis of the teachers' knowledge, focusing only on their remarkable comments on the videos that we proposed and on the signs produced by the students.

Even if this is only a preliminary study on the topic, based on the questionnaire, we report anyway the results in order to orient possible future researches with some qualitative categories. Three of these PhD teachers were instead interview more in depth and their interviews are reported and commented in the following Paragraph.

We analyzed them using our research framework concerning difficulties

The results are resumed in the following tables.

T.	C1	C2	C3
4		a. 2 and 5 (Q14) are those that better give, in the most immediate manner, the idea of interval of real numbers. b. 1,2,3,4 (Q15), make me think at a continuous interval of solutions. I would avoid 1 (Q15) because the reference to real numbers is too explicit. d. 3 (Q15) is the only sign in which it's relevant the fact that the inequality is to solve in the set Q.	a. We visualize better the set of the solutions if we represent them on a line. b. 2 and 5 (Q14) are those that better give the immediate idea of intervals of real numbers.
8	The 4 (Q14) (without the intervals' endpoints) will be necessary when the topology of the real line and the limits of functions will be studied	a. The flow of the indicator on the line shows very efficiently the succession of real numbers. b. All the solutions (Q15) don't take into account that solutions are to be searched in Q and not in R.	
57			a. Yes, because it gives a graphic representation of the solution (Q12) (the solution is considered the algebraic representation , *author entry*). b. Materials (squared papers, cords used in Q8) can be useful merely because they allows to visualize geometrically the approximation.

72	a. This (answer 5, Q 14) will be useful when students will have to solve inequalities only by graphic methods. b. 1 (Q14) will help with the concept of limit.		a. Yes, the graphic solution (Q12) helps students in interpreting geometrically and visualize the set of the solutions. This will be more and more useful when then will have to solve inequalities only using graphic methods.
64		a. This way (Q10) rational numbers seem to be in correspondence with the line . b. The exercise (Q15) asks a subset of rational numbers, while students' solutions are subsets of real numbers.	The graphic solution (Q12) explains, through the image, more in depth the infinite cardinality of the solutions.
110		Solutions (Q15) are unacceptable because usually these expressions indicate intervals of real numbers, not rational.	

Tab. 1

T.	C4	C5	C6
4		The reference to R is too explicit (Q15, answer 1).	I'll use a fixed scale on the line and the slider that flow along the line.
8	I'll put the interval in solution 4, intersecting it with Q.	2 and 5 (Q14), since the others are written in a manner a bit implicit.	The flow of the indicator on the line (Q10) shows very efficiently the succession of real numbers.
57	a. 4, eventually 1, (Q15) making explicit the conventions (without hatching). Anyway we have to specify that x is in Q. b. 3 (Q15) is not acceptable because the symbol used indicate the inclusion of endpoints (not rational).	a. 1 (Q14) is unacceptable because an explicit solution is not given. b. 5 (Q14) represents in a more explicit manner the solution even if the indicated set is the same.	
72	2, 4, 5 after having pointed out that only rational numbers have to be considered.	2, 3 and 5 (Q14) because those are the most explicit.	
64	One right answer could have been Q intersect with the answer 4.		This way it seems that there is a bijection between rational numbers and the line! (Q10)

110		2 and 5 (Q14) indicates in a clearer manner the set of the solutions.	

Tab. 2

Analyzing teachers' preferences some categories emerged:

- 4, 8, 64 and 110 attributed to the representations of subsets of R an absolute meaning , coinciding with their own advanced personal meaning (C2).
- 4, 57, 72 and 64 think that the visualization of real numbers by means of graphic representations makes evident their properties (C3).
- 8, 57, 72 and 64 use the same conventions for representing open and close intervals of real numbers and of rational numbers (C4).
- 4, 8, 57, 72 and 110 believe that a representation can be explicit or implicit or clear absolutely (C5).
- 4, 8 and 64 think that the flow of a mobile point on a line can represent the correspondence between points and numbers (C6).

We list below three answers whose rarity is interesting in order to compare orientations with literature results:

- Only the teacher 4 takes in account explicitly in one case students' possible difficulties of students in the process of interpretation of signs.
- Only the teacher 8 affirms that algebraic representation of intervals helps students in conceptualizing interval more than the graphical one.
- Only the teacher 57 affirms that it's necessary to use different representation with their different meanings in order to show the potentiality of each representation.

The orientations listed before are explicitly indicated as criteria to choose or avoid to use some representations:

- 4, 8, 64 and 72 belong to M2 i.e. choose representations on the strength of personal interpretations of signs.
- 4, 57 and 72 belong to M3 i.e. teachers that choose graphic representations because visualization helps students.
- 4, 8, 57 and 72 belong to M5 i.e. teachers that choose representations that they think to be explicit and avoid representations that they think to be implicit.
- Only 4 and 8 refer to students in order to justify a preference.

From this preliminary analysis it emerges that the most of these teachers (4 or 5 on 6) tends to attribute to the signs an absolute meaning and to choose the representations on the base of their expert interpretations. Furthermore they consider the representations immediate, clear, explicit, somehow "meanings bearer", also in case of very complex meanings, whose conceptualization is the results of the combination of metaphors and a lot of consequent blends.

In the last two decades in Mathematics education it have been stressed the role of Semiotics in the Mathematics teaching-learning processes from many perspectives; for an overview a good reference, even if not very recent, is the book edited by Radford, Schubring & Seeger (2008). From all the perspectives, even very different, the teachers' position towards the issue of signs and representations would be considered ingenuous and not fruitful in the field of Mathematics education.

In fact also the theories that neglect the anthropological dimension of mathematical objects and the crucial role of interpretation, mediation, socialization and personalization of meanings would reject the "immediacy" of signs as meaning-makers.

We can thus consider ingenuous sentences like the following from all the points of view:

"2 and 5 (Q14) are those that better give, in the most immediate manner, the idea of interval of real numbers."

"2 and 5 (Q14) indicates in a clearer manner the set of the solutions."

"The flow of the indicator on the line (Q10) shows very efficiently the succession of real numbers"

"In the exercise (Q15) a subset of rational numbers is required, while students' solutions are subsets of real numbers."

"We visualize better the set of the solutions if we represent them on a line"

"The graphic solution (Q12) explains, through the image, more in depth the infinite cardinality of the solutions.".

These orientations can be considered ingenuous in general, but their unsuitability is even more evident in the case of real numbers, considering the fact that the teachers listed, answering the questions about knowledge, very formal properties that can't be considered immediate in any case.

We analyzed these sentences in order to evaluate their cognitive suitability.
We neglected in this case, considering this analysis only a preliminary study, to deepen how far the use of the flow to show a correspondence between points and numbers isn't correct from the point of view of the connection between the systems of practices: we just remark that the "dynamic thread" in the epistemic meaning (Par. 4.) is disconnected from the configurations concerning the line as a complete set of points, rational or irrational, whose cardinality is "infinitely infinite" in the sense of transfinite numbers introduced by Cantor. The correctness at the epistemic level we attributed to the teachers' knowledge is thus to intend in the academic sense: the teachers proposed correct configurations in respect of the "classical approach" to the Calculus, that we identify here with the tradition described by Lakoff & Nunez (2000) that we resumed in Par. 2: "The heritage of the Weierstrass metaphor can be easily recognized in the following metaphors, that are accepted and never discussed, as Lakoff & Nunez stress:

1. The spaces are sets of point
2. The points on a line are numbers
3. The functions are ordered couples of numbers
4. The continuity of a line is numerical absence of holes
5. The continuity of a function is conservation of proximity."

If we pass to the cognitive level we can't avoid to take care about the potential didactical implications of the teachers' orientations emerged in the answers, that make suppose a low didactical and meta-didactical knowledge (Par.).

We quoted before many perspectives on the role of Semiotic in Mathematics education, but we use here the OSA tools to investigate the semiotic features of these sentences since this is part of our research framework.

In particular, considering the kind of data we are analyzing, we will restrict the analysis to some criteria of cognitive and ecological suitability:

CS1: Previous knowledge (the students own the necessary previous knowledge to study the new topic; the expected meanings to reach are possible to achieve in all their components):

This is the most critical aspect of the teachers' sentences. The teachers indeed consider the flow of a point on a line as an action that makes correspond "every stage" – whatever it means - with "every real number" included between the number corresponding to the starting point and the endpoint.

This very complex meaning is not only supposed possible to achieve just showing a point flowing onto a line without posing the problem of the relation between the two domains (Lakoff & Nunez, 2000), but, on the contrary, the identification is considered "immediate" – term etymologically really unfitting thinking at the studies concerning the crucial role of the semiotic mediation (Bartolini Bussi & Mariotti, 2008).

To make clearer our analysis we report here some excerpts of the cognitive analysis:

6. The natural continuous line is not clearly distinguished from the line as set of points, that indeed are not point but elements
7. The real numbers are not clearly distinguished from the points of the line
8. The real line is not a line
9. What is usually called continuum is not distinguished from a naturally continuous line

Every attempt to do characterize the continuum in terms of discrete entities is necessarily metaphoric, since we are trying to grasp something in terms of something completely different, namely its opposite, and we have to be aware that many aspects of the continuum are lost.

Some teachers believe instead that a student drawing a segment is drawing a "subset of real numbers", or are convinced that "The graphic solution (Q12) explains, through the image, more in depth the infinite cardinality of the solutions". The students' personal meanings are not taken in account, or, in a better perspective, the teachers are so used to the metaphors that they are ingenuously convinced they are immediate.

Also, about the identification between the motion and the correspondence Numbers-Points another important remark referred to Par. 2 is necessary: the metaphor *Arithmetic is motion along a path* is associated to numbers by means of a metaphor that associate movements from a point-location to another to operations, while in this case the movement can't be associate to operations without compromise irremediably the properties of the points as numbers; in fact every step from a point to another, even infinitesimal, would contradict the property of continuity ; what it would be numerically preserved is another property, the completeness as gaplessness. If the step is infinitesimal, considering the result in a BMI of a limit process, to a "true" actual infinitesimal distance correspond two numbers that we can't distinguish. This metaphor is thus unsuitable and would require a very refined intertwining of metaphors and blending that is certainly not immediate nor could be considered a personal meaning owned by the students without a specific dissertation about infinity and the continuous that explain BEFORE how motion, points, numbers, elements are related each other by means of metaphors and THEN why the motion could – in a very complex conceptual blending – represent a metaphorical correspondence.

Arithmetic is motion along a path.

Source domain **Motion along a path**	Target domain **Arithmetic**
Physical segments (consisting of ultimate elementary parts)	Numbers
Act of moving along a path	Operations
A point-location along the path	The result of an operation
The origin of the path	Zero
Further from the origin than	Greater
Closer from the origin than	Less
Moving from a point-location A to a point as much as the distance of a point B from the origin	The addition A + B
Moving towards the origin from a point-location A as much as the distance of a point B from the origin	The subtraction A – B

We formulated relying on the cognitive analysis by Lakoff & Nunez, the following hypotheses: the orientations concerning the suitability of the identification between segments – maybe generated by motion along a path – and real numbers were not right, in particular since they were aimed at making more clear, evident, immediate, explicit and so on the properties of real numbers. These orientations are indeed not supported by empirical data, as we stressed in the analysis of students' difficulties, quoting in particular Bagni (2000).

Quite all the teachers affirm that visualization is important in the didactical transposition of real numbers and the graphics representation of intervals (segments) as solutions of inequalities is preferred by all the teachers. Even in one case a teacher affirm that the graphic representation let more than the other visualize the infinity of the solutions. When teachers avoid the graphic representation "line" the reason is that graphically we can represent subsets of R and not of Q. Overstepping this last debatable sentence - how to represent for instance Dirichlet's function if not with a straight line? - we will focus the problem from the point of view of results about visualization in didactics of mathematics. According to Bagni (1998) visual techniques can be useful, but it must be controlled by teacher very carefully. Its inattentive use can cause difficulties for learners, up to generate incomplete learning and sometimes being misleading (Duval, 1994). In particular Bagni (2000, p. 4) stresses the inadequacy of graphic representations of properties like density and continuity, so important for the set of real numbers. In a research carried out in the high school in Italy he observed that some students, were "tempted" to use graphical methods to answer questions concerning density and continuity "because the habit of using graphical methods in the didactics of mathematics in the high school [...] but while the difference between discrete and continuous sets is graphically perceivable, we cannot affirm the same for the difference between dense and continuous. [...] The graphical methods turn into being misleading >>. Furthermore "the attempt to apply directly graphical representations to the understanding of the difference between dense and continuous sets is doomed".

Many researchers showed that the students' perception of the line is anything but dense and complete set of points. Arrigo e D'Amore (1999, 2002) named *dependence* the belief that the cardinality of points of a segment get bigger when the segment become longer (Tall, 1980). Also they observe that the "visive inclusion" of a segment in a longer one, that can be associate to the Euclidean notion "The whole is bigger than one of its parts", affects negatively the students construction of meaning. We can say, using OSA terminology, that there is a practice (operation of inclusion between segments) that becomes a part of the meaning of the object 'set of real numbers' (system of practice a student associate to numbers, identified through other practices with the point of a line) that is not compatible with the desired meaning of 'set of real numbers' (infinite set). Furthermore students often don't become aware of the true issue of continuity because of an intuitive model, also visual: the "model of the necklace" (D'Amore, 1999; 2002) i.e. the image of a segment a sequence of little balls linked by a cord (Arrigo e D'Amore, 1999; 2002; Tall, 1980; Gimenez, 1990; Romero i Chesa & Azcárate Giménez, 1994). This model, incompatible with density and continuity, was showed to be a very widespread intuitive model (Fischbein, 1985) of a line both for high school and university students and for primary school's teachers interviews (Sbaragli, 2006). Being a strong intuitive model for teachers, it is very used in the primary school and this is one of the main reason why, in absence of a reflection, this is transmitted from one generation to another. High school teachers' interpretation of the line as a continuous sets is far from being similar to those of the colleagues teaching in the primary school and maybe teachers with a PhD in Mathematics are not aware neither of the current interpretation of the line as a necklace in the primary school nor of the fact that this is the previous model onto which they try to construct the dense and complete field of real numbers. Maybe this is the reason why they consider even "too explicit" (Teacher A, C5) the reference to R if we use a segment to represent numbers. Anyway the "expert interpretation" prevents the teacher to frame correctly the initial students' cognitive meanings.

CS4: High cognitive request: Relevant cognitive and metacognitive processes are activated

The cognitive request are even too high form an epistemic point of view, instead the metacognitive processes are almost absent and this is exactly the reason why these choices are unsuitable from a cognitive point of view.

ECS2: Intra and inter-disciplinary connections (the contents are related with other mathematical contents (with advanced mathematics and the other curricular contents) and with contents belonging to other disciplines (extra-mathematical contexts or other educational steps))

The correspondence between real numbers and positions along a path is crucial in Physics, at least in Classical mechanics, that was deeply influenced by Newton. It's not casual that the mathematization of motion was formalized in terms of fluxions and fluent variables, as much as it's quite natural in Physics to consider the motion direction.

Nevertheless also in this perspective an "immediate" identification is not suitable because in Physics the point is identified by a number that is metaphorically grounded to the measure stick rather than to object constructions and elements. Are measures real numbers? This topic is not trivial, as we stressed before, because, as usually, the answer lie on the perspective. The identification between positions and real numbers may be accepted adopting an aware theoretical perspective, i.e. the line is a mathematical model and the numbers don't represent "measured positions" in the concrete sense of measures realized with a stick. Some Mathematics teachers with a Master degree in Physics that we interviewed stated vehemently that it's senseless to talk about real numbers because they are not necessary, in particular in Physics, being measures in any case rational and including unavoidable errors. The reverse perspective, from "measured position" to "continuous models" is not only problematic but, in a certain sense, impossible; only by means of assumptions based on the "continuum myth" (Fabbrichesi & Longo; 2006) we can think to complete continuously the model of a given phenomenon; in other words we have to SUPPOSE it's continuous. Even if it's acceptable in the Classic Physics, for Modern Physics the paradigm is completely overturned.

EC4: Didactical innovation: Innovation based on researches and reflections is taken in account.

No researches or reflections seem to be taken in account by the teachers standing on their written answers.

4.8.1.2 Analysis of some relevant teachers' interviews

After a first categorization based on the questionnaire, we carried out a qualitative study of some teachers' choices in order to answer the question:

PQ - 1.5 What systems of practices concerning real numbers do the teachers declare to prefer and to choose?

A group of teachers who had participated in the first part of the research, answering the online questionnaire, was interviewed in small focus groups (3-4 participants for every group), following the scheme we used in the pilot study.

The teachers' interviews were carried out following an interview protocol but in a semi-guided form, that was elaborated after the pilot study. The protocol contains general open questions concerning the system of practices the teachers implement to introduce real numbers and explain their properties, but also crucial questions that address topics that we consider core problems. These problems lie in the intersection between historical systems of practices and configurations, partial meanings of real numbers that are quoted in the national curricula (institutional meaning) and recurrent system of practices and configurations of objects emerged in previous analyses of "traditional scholastic practices" reported in the textbooks (Tall & Vinner, 1981; Bronner, 2000; Merenluoto & Lehtinen, 2002; Gonzales-Martìn, 2014) and quoted by the teachers in the pilot study.

Some of the teachers we interviewed in focus group have been interviewed also individually in a further session of interviews. This further step was added thanks to two main reasons:

1. some of the teachers answers in focus groups had to be clarified more;
2. some teachers didn't participate very much in the focus group discussions;

We will report just the analyses of the interviews we decided to use for some specific case studies. Some of these were realized in focus groups while other are individual interviews.

This phase of the methodology was designed in order to answer the second question:

GQ - 2 Are the teachers' choices epistemically, cognitively and ecologically suitable?

Some of the interviews we carried out have not been analyzed because their analyses wouldn't have added something relevant in respect of the teachers' answers in the questionnaire.

Other interviews' analyses have been discarded because in this second phase of the research our analyses aimed at stressing merely the creation of significant categories in the cases of teachers who showed a good understanding of real numbers. Since the characteristic issues regarding real numbers are very complex, as it was also shown by many authors and was confirmed by our analyses, some teachers didn't have the occasion during their formation paths to deepen the study of real numbers and sometimes their personal meanings of real numbers didn't satisfy the first condition of epistemic suitability, i.e. the correctness, and the second one, i.e. the representativeness.

On the contrary we went on analyzing teachers' interviews in the cases in which teachers' personal meaning was a configuration, or even a collection, of correct but partial meanings, and even when different systems of practices were inconsistent from an higher perspective but couldn't be discovered as inconsistent by the teachers because of a lack of critical problems that could lead them to productive cognitive conflicts (Tall & Vinner, 1981).

We also report the focus group interview with the four teachers who didn't participate in the first part of the research, i.e. who didn't answer at home the questionnaire, thinking and reflecting on their own on the topic before the interview, in order to compare their reactions and those of the teachers who participated in all the steps. Our hope was to evaluate our research methodology through a comparison with another simpler one, that only consists in focus groups, in order to find out its strength and weakness points.

These choices were oriented to make the results of the analysis and of the categorization useful in order to design in the future teachers training programs and teaching-learning sequences useful for face the problem in a suitable way; in this perspective we were interested in the teachers' profiles that satisfied at least the minimum requests of mathematical knowledge and epistemic suitability.

Also we posed problems that had the potentiality to change teachers' orientations by means of cognitive conflicts and stimulated them to reflect better on the relations between historical-institutional problems, epistemic partial and general meanings, their practices, the students' behaviors and results from the literature review concerning students' difficulties. We observed interesting dynamics in the teachers' interviews and during further meetings with the teachers involved in the research only in the cases in which teachers had shown a knowledge and declared choices suitable enough in the previous part of the research. These happenings reinforced more our decision to focus our attention only on the cases in which a change or significant dynamics could be somehow prototypical for further studies on teachers' training.

The teachers' interviews analysis consists in the following 6 steps:

1. Teachers' interviews' synthesis step by step
2. Teachers' declared choices
3. Teachers' personal meaning
4. Teachers' reports of students' difficulties with real numbers
5. Teachers' orientations about teaching-learning sequences concerning real numbers

Every description is presented both in a narrative form, following the chronological order in which every configuration or orientation appeared and taking in account the interaction with the interviewer and the eventual groupmates, and in a synthetic form oriented at creating the teachers' configuration to compare with the epistemic ones. In the second phase the codes assigned to every configurations and the practices, processes and objects listed to characterize every partial configuration will be used. In order to allow a better follow the analysis a detached paper reporting the epistemic meaning structures is attached in the end of the book. The interviews, translated in English, are reported in the Appendix A.

Teacher 1: (Real numbers are much more intuitive than rational numbers)

1. *Teachers' interview synthesis step by step*

Step 1 (answering to I1): Historical approach, real numbers are numbers that are not ratios of whole numbers

Step 2: Real numbers exist in the reality

Step 3: Pythagoras' theorem oblige to speak of real numbers

Step 4: Algebraic real numbers are necessary to solve inequalities

Step 5: The solutions of inequalities are segments

Step 6: First of all it's necessary to solve inequalities in R using the graphic representation of segments or half-lines

Step 7: R is the set of the points of the line

Step 8: Real numbers are continuous, are perceivable in a continuous way

Step 9:The segment has no holes, every point has precedent and consecutive points

Step 10:Rational numbers are approximations of real numbers

Step 11:In the reality the numbers are approximated

Step 12:In the number line passing from N to Q everything becomes "a bit jointed".

Step 13:If a segment is result of a movement to draw it, it's full

Step 14:Obviously it's impossible to represent Q graphically because the segment is continuous.

Step 15 (interacting with I):Q is equipotent to N and that R is not. There in an enormous jump.

Step 16 (interacting with I):R is the mathematization of a primitive intuition.

Step 17 : Definition of limit points. Differentiation between limit of a sequence, that we calculate only to ∞, while we can calculate the limits of real functions also in a finite number: is the only limit point for the sequences

Step 18 : R is a topological space

Step 19: Calculus main theorem are formulated in topological spaces

Step 20: Interdisciplinarity with Physics: approximations are identified with numbers and put in the line because of the necessity of using transcendental functions that they don't know and calculate using the calculator without being aware of the process.

Step 21: Difference between real numbers and their practical use through approximations, that are respectively continuous and discontinuous sets.

Step 22: Starting from R, stress the existence of N, Z, Q as discrete subsets of R.

2. *Teachers' declared choices*

a. She introduces irrational numbers with the example of $\sqrt{2}$ "showing that not all the numbers are rational. I would start from $\sqrt{2}$, they has been presented also in the grade 8. The existence of numbers that are not ratios of whole numbers…. thus a quite historical approach"

b. Also in Algebra she would introduce real numbers quite soon since "the solutions of inequalities with x Q bother me"
c. "[…] you can work with R for sure from the graphical point of view because the property that is congenial at an operational level is the correspondence between points of the line and numbers"
d. "I would do this way: I would show that N and Q are in correspondence, that is not hard, while Q and R no.. to show that there is an enormous jump"
e. In the Calculus uses "the mathematization of a primitive intuition" of continuity rather than introducing more formal conceptions
f. "when we talked about limit points, the book didn't propose a definition …. instead I proposed a definition .. we talked for some minutes and since they had studied the sequences I tried to make them reflect about the fact that, while when you calculate the limit of a sequence only to the infinite, when you study a function on the real numbers you calculate limits also in finite points, because the only limit point for N is the infinite, while for R…"
g. She introduces the Dirichlet's function as example of function derivable nowhere
h. "It will be useful to introduce them very soon because once you have introduced them you can show the strong difference between this [draws a segment and writes R] and this [traces a segment with some points and writes Q]."
i. She operates "at the numeric level with a Q with 2 or 3 digits"
j. She would start from all the numbers and then to select different kinds of numbers to distinguish between irrational numbers and the rational decimal approximation that the students are used to compute.

3. *Teachers' personal meaning*

a. Real numbers are more intuitive than rational numbers
b. Operations with real numbers and real numbers should be clearly separated
c. The inequalities solved in Q are disturbing
d. The continuous line represents the real numbers
e. The formalization of real numbers is a mathematization of a primitive intuition

4. *Teachers' reports of students' personal objects and/or difficulties with real numbers*

a. In the students' minds R is a line
b. The correspondence between numbers and points is innate, it is very natural for them
c. When a student think of a number, he thinks of a real number
d. The students perceive the numbers in a continuous way
e. Would say to the students that Q is impossible to represent graphically
f. When the students are too young they study rational numbers but they have no confidence with the decimal approximation of magnitudes
g. In the students' minds there is already a spontaneous conception of real number as a sequence of numbers
h. Only a few students understand the discourse about limit points for sequence and for real functions
i. The Calculus is so hard for the students to study that a deep discourse concerning real numbers could be useful only for some students
j. The students see the continuum, the contiguous classes
k. For the students everything is full, there is everything in the segment. If they take off something it's obvious that I find something more. The meaning of accumulation point is impossible to understand for them.
l. The students have a strong idea of continuity but they don't have a similar idea of discontinuity.

5. Teachers' orientations about teaching-learning sequences concerning real numbers

a. Real numbers should be introduced very soon, in order to clarify that numbers and approximations are different.
b. It's interesting to solve inequalities in Q or in N only after introducing them in R
c. We can't avoid to ask them to solve inequalities in R only because we're not able to introduce them formally
d. It's useful to work on the graphic representation as a good substitute of more formal representations of real numbers
e. It's easy to show that Q is equipotent to N and that R is not.
f. The line's topology is trivial, or better, it's not trivial but it's usually presented in the textbooks in a trivial way

Teacher 2: (The relation between finite and infinite in the segment is a paradigm of life)

1. Teachers' interviews synthesis step by step

Step 1: R is the set of limit points of Q

Step 2: The representation of Q has holes, R has no holes. Segments are subset of R.

Step 3 (answering I1): R is an enlargement of Q, in the chain
N Z Q R C.

Step 4: Z is necessary for subtraction, Q is necessary for division. Some numbers are not included in Q.

Step 5: Proof of irrationality

Step 6: "Geometrical numbers": numbers are endpoints on a line of segments constructed through geometrical procedures

Step 7: A finite thing can contain infinite points

Step 8: Between 0 and 1 there are always points (limit points), but this is also true in Q (density of Q)

Step 9: Dividing we construct Q, Other things are in R.

Step 10: There is always a point in the middle, between two points.

Step 11: Points have no dimensions

Step 12: R is necessary for limits

Step 11: When you solve equations coming from geometrical problems like those that contain Pythagoras' theorem, you need real numbers

Step 13: is a geometrical number that can be approximated by means of geometrical procedures

Step 14: , infinitesimal quantity, is a real number

2. *Teachers' declared choices*

a. "My first approach to real numbers is in the first lesson about numbers; I always see it as enlargement. I say <There is N, then there is Q, then there is R.> and I say < Which is the necessity?> and then I introduce also C at the same time. At least I give an idea […] to close the operations in the sets, in this case in respect of subtraction, then in respect of division, square"

b. "I say that there are numbers that are not included anymore in these sets", like √2. "Usually I do this proof, first of all to introduce the *reductio ab absurdum"*

c. In the first year she introduces Q, in the second year a part of R.

d. "I usually do it also when I talk about Euclidean geometry, thus not exactly in the beginning when I talk about segments. I say: < It's incredible how a finite thing has in itself something infinite> and I enjoy it very much because I say that in my mind the segment is the paradigm of life, in the sense that we can see the man's life in the same way, as much as our desire … our desires are infinite."

e. It's important to go on with parallel discourses between Geometry and Algebra in relation with numbers and infinity: "I show why I need segments and I say that there is a correspondence. In fact then I say : <It's true here but also it's true between 0 and 1. But I could also do it in Q.> I say: < If you take 0 and 1 there is always a point between 0 and 1. If I talked about an half part I stop at Q, but when they understand that there are other things in the middle, that there is always a point in the middle, that is always the half part and they can go on infinitely, this gives to me the idea of limit point in a certain sense."

f. She introduces the density saying: < We don't have a so subtle pencil >.

g. "If I have $x^2 = 4$ there are no big problems, but if we arrive at $x^2 = 3$. What is there? Furthermore I recall this discourse in the first year with the Pythagoras' theorem".

h. π must be introduced geometrically, in a laboratorial way "we did a very nice work about the approximation of like is presented in the "Museo del Calcolo" in Pennabilli. I like so much the question of the introduction of ε because it gives precisely the idea of a little thing that is so close to x0 and allows us to see what the function does in that point. To quantify this , $\varepsilon > 0$, that is real, gives exactly the idea of the fact that we get closer and closer because this tends to 0."

i. She presents the passage from Q to R graphically, between rational numbers there are irrational numbers

j. "If you want you can do something also in the fifth year, since with the limits… but a few times. The infinitesimals are necessary to introduce limits, so close to x0…"

3. *Teachers' personal meaning*

a. Segments are paradigms of life: finite things contains infinite points

b. The definition of limit points in Q or R is an open problem

c. We don't have a way to represent Q

d. I should reflect on representations

4. *Teachers' reports of students' difficulties with real numbers*

a. Only for good students the proof of irrationality of 2

b. The students pose many questions and are interested when a teacher talks about finite and infinite quantities

c. The students accept that the line is infinite, but don't accept easily that a segment is infinite

6. Teachers' orientations about teaching-learning sequences concerning real numbers

Real numbers are necessary for the measures

Teacher 3: (We have to give to the students strong bases and to present Mathematics as an history of humans' cultural conquests)

1. Teachers' interviews synthesis step by step

Step 1: R is necessary as domain of functions

Step 2: Zeno's paradox anticipate the topology of the line

Step 3: R is a topological space

Step 4: The relation between R and the line is not trivial

Step 5: To create a 1-dimensional object using 0-dimensional objects is a problem

Step 6: The line without a point is not visualizable, but also the line, as long as the points.

Step 7: The relation between continuous (intuitive) and discrete objects is very complex

Step 8: [Irrational numbers have infinite digits without repetitions] & [Rectification of the circle through geometrical approximation procedures] are practices to treat in parallel, these are two aspects of the same discourse

Step 9: Numerical-geometrical approach to exhaustion using Archimedes' methods

Step 10: R is a field, extension using historical approaches from N to Z and to Q and then using
2.

Step 11: Algebra's fundamental theorems

Step 12: Mathematics was adapted to R and not viceversa, R is the only example we can propose of so many objects: metric spaces, complete fields, and so on.

Step 13: Algebraic symbols are representations of objects, like segments or numbers. Numbers are abstract ideas, Pythagoreans.

Step 14: Q is the set of fractions; fractions are relations between magnitudes, commensurable magnitudes

Step 15: Decimal representation of Q is good for Physics

Step 16: In the Calculus we use intuitive variations/intervals rather than static representations

Step 17: Hyperreal numbers

3. Teachers' declared choices

a. An approach he uses "is the field, or better the closure in respect of the operations. This helps me first of all as an historical approach, a construction of the numbers from the natural, rational numbers and then I start talking about Pythagoras and I never stop"

b. He uses Zeno's paradox in the third year to anticipate the definition of limit and the line's topology "This becomes in the fourth year, when you approach the definition of limit, when you talk about the topology of the line you to talk about neighborhoods and the basic concept that is interesting to me is that this representation of the real numbers by means of a line is not a trivial thing. Once I asked a student <Is there a bijection or are they the same thing? How can you assure that there is a bijection?>.

c. The topic "Mathematics and music" is very useful to introduce the concepts of logarithms and exponentials, but also all the enlargement of numbers sets, using the musical scale: natural, rational, geometrical irrationals, not geometrical numbers, complex numbers.

d. "When Pythagoras start to use fractions, rational numbers.. for instance Odifreddi talks about ratio, reason, connected with the word *logos*, that is one of the ancient principles, while I prefer to stress the analogy between ratio and relation. […] "

e. To approximate numbers using excess and deficiency approximations he uses" Archimedes' method with approximations, for instance with π, using inscribed and circumscribed polygons. They know what a perimeter and an area are, that is a number included by… that this are contiguous classes if we want, they are classes of rational numbers and you can express a perimeter.. then strange polygons, in which also irrational numbers appears, I don't talk about contiguous classes but they understand that these are two sequences that approximate the number, one in excess and one in deficiency, and that the difference can be reduced as much as they want."

f. He introduces the geometrical and not geometrical numbers "this [the incommensurability of the diagonal an the edge of a square] don't shock them as much as the fact that we can't represent the cubic root of 2. […] I draw the diagonal of a square, then I report the segment on the line, while in the cube's duplication, the cubic root of 2 is not representable using the ruler and the compass"

g. He introduces the Algebra's fundamental theorem

h. Solve polynomial equations: One of the most important problem in the history was to solve polynomial equations, Tartaglia, Gauss… If they studying polynomial functions don't know that they can find 1, 3, 5 zeros and that if the grade is odd one is surely real … also this means to have understood the real numbers

i. Use Geogebra to create a coordination between algebraic and geometrical aspects of real numbers

j. He uses x to represent segments or numbers but "I always stress this is a representation.[…] when I indicate a letter, or a number on the axes I don't say: < This is the number three, this the number four, ...> and so on but I say this is always a representation of the number. Also ' Pythagorically' speaking it's an abstract idea, something that comes from elsewhere, I realize a representation first of all because this is one… also I use Geogebra, think at the name, from a Cartesian point of view, as you were saying, it's true they are joint but there also two separated representations, there are two windows, the algebraic one and the graphic one. I try to use always the both of them"

k. Task in Geogebra: to zoom in enlarging a point, in order to show the draw is not the object

l. Represents Q by means of fractions

m. Defines limit points but work with limits without using it

n. Task: Ask the students what kind of numbers are e, π, sin π, ln 3

o. Ask the students: "Could you assure there is a correspondence between the line and the numbers?"

p. "I'll try to make them understand that the fact to find always, every time I take a point on the line, a real number this seems to me already a conquest. Also I like very much to talk about neighborhoods with holes, without their centers, that is very frequent in my tests about the limits, but also it would be enough to talk about open limited intervals. To make them understand that you can approach one or the other endpoints but without reaching it it's an important thing."

q. Using Geogebra and zooming in the line we can deal students' to reflect on the fact the line is not its draw, since it has a grower thickness

r. "exponentials and logarithms are useful for ... You do it usually in the middle of the fourth year [grade 12, nba] while I introduce them in the end of the year because immediately after I start introducing the limits and these are prototypical real functions with real domain.

s. "I highlight the difference between the power function and the exponential function" proposing a Task: to ask which is the domain x^2, 2^x, x^x

t. He introduce contiguous classes once, but then he talks intuitively.

u. "I also avoid to work with the sequences and I immediately shift to the limits of functions because to talk about function from N to R, as sequences are, try to create short circuits between the continuous and the discrete"

3. Teachers' personal meaning

a. A question at the University about the fact that we are not sure there are no holes in the line conditioned his way to talk about real numbers and the line to his students

b. Every time we take a point it's a conquest to say it's a real number

c. We can't visualize the line without a point

d. A draw is different from an object, it' only a representation.

e. The properties of R are shocking

f. Teachers are quite obliged to define limit points or contiguous classes but it's unuseful

g. Continuous is not the limit of discrete, it's another thing.

h. It's important to distinguish geometrical numbers and numbers we can't construct by means of geometrical procedures

i. Dense and continuous may have the same representations, the continuous is a support for discrete sets

j. Real numbers, infinity and the Algebra's fundamental theorem are conquest of the humanity.

k. To solve polynomial equations and to study polynomial functions.

l. Q is a set of relations, this is its best geometrical representation, not the points on the line

m. The existence of real numbers as results of processes doesn't imply their geometrical construction

4. Teachers' reports of students' difficulties with real numbers

a. Functions like 2x, x irrational and xx are in the hyperuranium for the students

b. The students have doubt when they reflect on the the fact that 0-dimensional objects create a 1-dimensional object

c. The students create misconceptions when a teacher merges discrete and continuous.

d. When he was at school he approximated 2 with 1,4; 1,41; 1,414 reducing as much as he want the difference. He thought to have understood but then he realized he didn't understand the deep meaning.

e. The students understand the difference between geometrical and not geometrical numbers, but they are shocked when a number is not geometrical, like 32

f. The students are not used to think in discrete sets

g. Limit points' meaning is related to the properties of Q, examples are always in Q and never in R.

a. It's very important to present to the students questions and problems that lead humanity to wonderful conquests

5. ***Teachers' orientations about teaching-learning sequences concerning real numbers***

a. Properties of real numbers are not necessary to introduce exponentials and logarithms but are necessary to complete the functions
b. Exponential and logarithmic functions are prototypical real functions in the real domain
c. Formalization of real numbers is not useful from a didactical point of view, even in the Scientific high schools
d. It's important to analyse deeply the relation between the line and the numbers
e. The graphic representation is important since it's synthetic, but it can't ben the only one; it' necessary to coordinate analytical aspects, limits, inequalities.
f. Avoid to talk about sequences in N and then limits of functions since this passage creates a short circuit between continuous and discrete, it can create misconceptions.
g. We have to construct strong bases and not to refer to everything to do too many things.
h. We must not introduce everything but only something that is useful for some goals.
i. To know real numbers means also to study polynomial function and to solve polynomial equations.
j. It's important to coordinate numerical aspects, algebraic aspects and graphic/geometrical aspects
k. The axiomatic approach are not effective because students can't understand anything by means of them; indeed the mathematics was adapted to R and not viceversa; R is the example of all this axioms and all the mathematical definitions.
l. Limits can be introduced without defining limit points
m. It's important to defoliate concepts to give strong bases

Teacher 4: (Only a very few of real numbers can be taught in the high school; we teach a *pseudo*-mathematics)

1. ***Teachers' interviews synthesis step by step***

Step 1: Real numbers are in correspondence with the points of the line

Step 2: Irrational numbers can be approximated using rational numbers

Step 3: The problems that lead to R are geometrical

Step 4: The most important thing is to represent the total and linear order of R using the line, that is different from the axiom of the order

Step 5: R has an algebraic structure that must be consistent with that of Q in terms of properties of the operations

Step 6: R is approximate with Q

Step 7: R is a set whose elements are identified by their properties and not by their decimal representation

Step 8: Q is dense in R, ordered a consistent way in respect of Q, complete

Step 9: Actual infinity of R vs potential infinity of the limits' approach

Step 10: Algebraically we work with Q, but we use the line to represent the numbers

Step 11: In the limit processes with function there is a continuous variation towards the point [dynamic]

Step 12: A continuous function is a function that makes correspond to small variation of x in a small interval, small variation in the functions' image interval

Step 13: Intervals of Q and R can be identified graphically, but it's full and I know it's R

Step 14: The segment is a set of points that correspond to rational and irrational numbers; the rational points are constructed dividing and multiplying, the irrationals we use at school by means of geometrical constructions

Step 15: /can be rational numbers but not all the limit processes lead to rational numbers, like 2

Step 16: Infinite rational steps don't imply something is rational

Step 17: The relation between continuous and discontinuous is complex: limits of functions and separated steps are not the same

Step 18: Intervals contains other intervals, smaller and smaller

Step 19: There is a parallelism between correspondence and postulates, R is in correspondence with the line

Step 20: The existence of infinite points is a geometrical axiom; the Archimedean axiom concerns the line and not only the abstract numerical set

Step 21: The domain we take for granted is R

2. *Teachers' declared choices*

a. Introduce real numbers as points of the line, taking the correspondence for granted: "To do what we have to do we can use the preconception of the correspondence between real numbers and line. We can take it for granted and act consequently […] what can you say to the poor student who trusts you? Rational numbers can be enriched in such a way that you can put as much numbers as the line's points and in the right order. You could… density, completeness, continuity are in the ideas world. They're used without being explicit."

b. "I would rely very much on geometry to go beyond the rational numbers. In Algebra it's not necessary but with the radicals, the operations.. […] In mathematics we need to use numbers that aren't rational. In the first year [grade 8, nba] .. radicals. If they are square we can represent them very well geometrically. If the radicals are not squares you can insert them in a consistent order with that of rationals and you don't need the structure of real numbers."

c. "There are points that we can't refer to rational numbers, i.e. we say that to make it correspond to Q, in the correspondence between R and the line, you could begin to see which points correspond to Z, choosing 0 and 1, using the compass and the ruler going on putting fractions, divisions of segments. Q is consistent with infinite points onto the line ordered like the rational numbers. But then there is Ippaso, who says that not all the real numbers, i.e. all the lengths, correspond to numbers that we can construct this way. So, for instance, the length of the diagonal with ruler and compass is not one of these infinite points for a finite number of step from the unitary segment. It's possible to construct, not rational but from the rationals. All the numbers like this..."

d. He indeed uses the rational approximations of irrational numbers, but pretend to work in R: “You live continuously with the double truth: what we write down formally is in the real numbers set, what we use are drawings and calculations”

e. Introduce $\sqrt{2}$ for its property i.e. the number whose square is 2

f. “The Calculus is graphic-centered, indeed the way real numbers are imagined is the real line, with an abuse of language, that in my mind makes a few damages at this level but may have many advantages. We need a representation of the total order as linear order, it's not interiorized as the axiom of the order. In R we need the order and then the operations with R are compatible with the operation in Q”

g. He uses the properties: Q is dense in R, ordered a consistent way in respect of Q, complete, without making them explicit: “I introduce the limit of a real function in the real domain in every point x0 of R. […] The properties... What is useful to know to work with Calculus? 1. Q is dense in R. 2 The order in Q and R are compatible". 3. Complete.”

h. He introduces continuous functions using limits in a dynamic form with the ε/δ formalism: “many things are used without making them explicit. Limits exists and are unique, the upper extreme exists.[…] a function that tends to x_0, a limit *L* that is not a value of the function. You choose an interval that goes from $l + \varepsilon$ to $l - \varepsilon$, [he draws the interval and the axes] then the idea is to have for every choice of this ε a δ here in the bottom, $x_0 - \delta$, $x_0 + \delta$ in an interval centered in x_0 [he draws a segment] and all the image of the *f* lies inside the chosen interval”

i. “Every time I speak about intervals I mean 'without holes' i.e. including rational numbers, i.e. the points that correspond to rational and irrational numbers. It would be important to consider a R interval

j. Real number are the decimal numbers: “A way to introduce them in a sly manner is as decimals.. they’re sequences, better, series, but we don’t say this to the students! Some of these series have no limits”

k. To describe the properties of the line you go on reducing distances until the thickness of the pencil

l. If a student asks how is it possible to find always a number between the two given numbers (talking about density) he would answer “that he must divide with the imagination, and thus he has to imagine to make a zoom and to go closer and closer so that every interval is expanded and contains other intervals, that contains other intervals”

3. Teachers’ personal meaning

a. As human beings we only use Q, so we “play the play” of approximating R using Q.

b. To construct a rigorous construction is a finesse; Euler didn’t have the real numbers but he did so many wonderful things

c. R is important for the properties of irrational numbers, not for its decimal representation

d. The segment is a subset of R because I can draw it full

e. If I see a segment after I can’t establish if this represent Q or R

f. You always leave with the double truth: the set of real numbers, that is an abstraction, but you operate with finite representations, draws and computations

g. Maybe numbers don’t exist, they are only human inventions

h. R is the line and you have to make an act of faith that Q can be completed to R

i. The sequences must converge to something

j. The humans are limited, they can go imagining only until the thickness of a pencil; it would be sufficient Q to carry on this discourse

k. Metaphorically an irrational number is like the border of an abyss: we approach t something that doesn’t exist; teachers don’t understand very well too what happens; we get used at a certain point

l. The formalization was useful to clear Dedekind’s conscience, but it’s useful for anything

m. We don't have a symbol for Q intervals; we can't express a Q interval using irrational numbers as endpoints

4. ***Teachers' reports of students' difficulties with real numbers***

a. The correspondence between numbers and points is innate and sufficient, it's a preconception and we can take it granted
b. What makes intelligible to the contemporaries the limits is to be potential
c. Persons thinks algebraically in terms of rational numbers but see them onto a line as real numbers
d. The students see the approach in a continuous variation towards a point when they talk about the limits
e. To understand the continuity is necessary a counterexample taking off a point
f. It's very hard for the students to understand the relation between Q and R
g. Only smart students are annoyed if you don't deepen the problems
h. Decimals are sequences, better series, but we don't say this to the students
i. The students have an intuition of the line as small ball, one close to the others
j. To deepen theoretical questions creates more doubts

5. ***Teachers' orientations about teaching-learning sequences concerning real numbers***

a. To be rigorous with real numbers is not necessary to do what we have to do in the high school
b. The existence of a correspondence between points and numbers is sufficient
c. To go beyond rational numbers we need Geometry; in Algebra it's not necessary; then he changes his idea thinking at radicals and functions.
d. In Mathematics we need not rational numbers since the 8th grade with radicals because the students see geometrical constructions connected to irrational numbers.
e. You can insert the irrational numbers between the rational being consistent with the order of Q without considering the algebraic structure of real numbers.
f. The Calculus is graphic-centered, the functions are identified with their graphic representation so R is the line.
g. To construct R formally has a few advantages and many prejudices.
h. The axiom of the order is something formal
i. What we need are the properties of numbers like $\sqrt{2}$ not their decimal representation
j. We can omit historical passages without losing something in terms of meaning; we just can choose what to do and what avoid to do in order to simplify Mathematics for the students
k. It's important to consider full intervals for the Calculus
l. You have to construct R equal to R for computations and order, that can contain Q nd irrationals
m. We teach a pseudo-Mathematics, you have to decide what to say to the poor students who trust you
n. Density, completeness, continuity are used without being explicit
o. You need real numbers to talk about sequences and series
p. The adults created a wonderful Mathematics, I can only show to the students some hints
q. We have to make Mathematics easier, to make it more understandable

Teacher 5: (The students have many difficulties with real numbers: maybe we simplify too much)

The teachers' interviews analysis consists in the following 8 steps:

1. *Teachers' interviews synthesis step by step*

Step 1: N Z Q R

Step 2: R is the union of rational and irrational numbers, the set of all the numbers existing in nature

Step 3: The diagonal of the unitary square is 2; we need a set that contains all the numbers

Step 4: Between 0 and 1 there infinite points

Step 5: The conventions used for the representations of intervals of R are worthful also for Q

Step 6: R is necessary for equations, inequalities, functions; quite for everything.

Step 7: The number line is important since we have to represent 2in the first year

Step 8: If we don't introduce R, the irrational number would remain punctual. We need R to complete the line as a continuum

Step 9: In R the consecutive of numbers doesn't exist

3. *Teachers' declared choices*

a. She introduces the sets chain: "I start introducing all the numeric sets, in the end I don't present R with the contiguous classes, as one could do in the University or in a Scientific high school, I work in a very simple manner. I define all the numbers, natural numbers and so on, then I say, guys, there are also the irrational ones, I show that not all the numbers are possible to express in form of a fraction, then I introduce the irrational numbers."

b. She uses as example of real number because they already know it: "I talk with them of the famous π, I show that not all the numbers are rational using the rules, and then I say: < Guys, there are the numbers.. the *matryoshka*. There is this big *matryoshka*, that are the irrational numbers, then there are the rational numbers; joining this two we obtain R, that are all the numbers that exist in the Nature> , in a very simple manner"

c. Introduce the problem of the diagonal of the unitary square without proof and put into the line

d. The fraction are all the finite or periodic decimal numbers; some numbers that exist in the nature are not rational so we construct the set that contains all the numbers: she says: < Guys, there are limited rational numbers that I transform this way, there are the periodic numbers, and there is a set numbers with whom I can't do this. They are unlimited and not periodic. Is there this number? Is there π? Not all the numbers can stay here, so let's construct the set of all the numbers that exist in the Nature and we call them real numbers.>

3. *Teachers' personal meaning*

a. R is the set of all the numbers existing in the nature

b. We can't represent intervals of Q if the extremes are irrational numbers

c. R is necessary quite for all in Mathematics, in particular for real functions in the real domain

4. *Teachers' reports of students' difficulties with real numbers*

a. The students have many difficulties with real numbers

b. The students don't understand that between 0 and 1 there are infinite numbers
c. The students don't read the information about the set of numbers in which they have to solve equations and inequalities
d. The students forget everything
e. The students' problems concern rational numbers
f. The students always ask how many points they must draw to trace the line

5. Teachers' orientations about teaching-learning sequences concerning real numbers

a. It's very hard to introduce real numbers
b. Introduce real numbers in a very simple manner because she teaches in a Professional school; if she would have taught in a Scientific high school she would have used contiguous classes
c. It's impressive that in the Scientific high schools the problems are the same and also the teaching sequences are the same
d. Maybe our error is not to deepen enough, to simplify too much or to omit explanations
e. To use rational functions, or to us N, Z, Q with inequalities or functions is too complex, she never does it

Teacher 6: (The students forget everything: they remember only the last thing you teach, so they use real number and forget the rational ones)

1. Teachers' interviews synthesis step by step

Step 1: Real numbers contains new elements: the roots and some other numbers like these

Step 2: A new set is created when an operation opens the path and create the exigence to extend the previous set

Step 3: The diagonal of the unitary square is an irrational numbers; after Ippaso the exigence of creating a new set emerge

Step 4: Between two numbers there are always infinite numbers

Step 5: The concept of infinitesimal can also be rational and sometimes it's 0, sometimes it's not

Step 6: A segment is full, it represents R

Step 7: R is necessary because we need algebraic numbers

Step 8: The concept of continuum and real number are known and they are both necessary in the Calculus

2. Teachers' declared choices

a. He introduces the numerical sets to create a set closed respect of the operations: "there are already the roots, things like that. In general I feel the need to reconsider them, when numbers like e, π, come out. So usually what I do is to discuss about the sets. I start with the natural numbers set, then in the moment in which we have operations that we can't execute in this set the exigence to introduce a new set emerge."
b. He introduces R as the set that contains the roots

c. He introduces intuitively the real numbers: “We always work with real number but with a quite intuitive approach, because if you approach it from another point of view… also the foundation of irrationality, you have to make choices, priority choices, you can’t lose one week…”
d. He introduces the density of numbers and of the line at the same time
e. Represent the numbers using a line
f. Task: Take a segment in Geogebra and enlarge it more and more: “So many times to understand such a thing there are the computers, if I draw two point on Geogebra, if I zoom the page they see that these two points are close, if I reduce or enlarge, between them there is already room. You take the points that are in the middle and I reduce more and more there is always a room so in theory between these two points there is always something. Then Achilles and the turtle…”
g. He introduces infinitesimals for teaching limits: “..I say that in the limit 0 is not 0 really”
h. He introduces limits in a intuitive dynamic way
i. He introduces contiguous elements in Geometry: “When you use Geometry, many times you introduce the postulates, the contiguous elements and so on, but don’t think that many students understand these things.”
j. In the last years before the Calculus he takes for granted they know the continuum and the real numbers: “trust in the fact they have studied them. Not only some real numbers.. π is a transcendental number, is irrational… I can’t show the Taylor’s series! I’ve also proposed it in the previous years in particular courses. You say.. you go on with a sequence, you arrive at the fiftieth digits, we are human.. […] We are talking about students of the fourth or fifth year, in which the concept of real numbers is already known. […] They have difficulties, but the concept has already been introduced. When there are roots, we have to talk about the real numbers.. so when we talk about intervals, an interval is a set of real numbers, the concept has already been introduced in a certain sense. We, in the last three years, I take for granted that it has already been introduced”
k.
l. Rational problems are particular cases

3. Teachers’ personal meaning

a. Mix operations and geometrical procedures, identifies numbers and segments

4. Teachers’ reports of students’ difficulties with real numbers

a. The students don’t understand the infinitesimals and never understood that between 1 and 1,0001 there are infinite numbers
b. The students understand that in a segment there are infinite points
c. The students don’t understand that they can divide a segment in parts and then divide it again and always find again infinite numbers, in particular when the points are very close
d. The students are more impressed by the graphical representation so I use the graphical representations for the numbers
e. The density presented through zoom in Geogebra are understood but they don’t leave a trace
f. The students write x <
g. 5, in analogy with the equations
h. The students have an elementary knowledge because after some years they study only for the mark, not to learn really. They forgot, they are not able to use what they learn
i. The students have many difficulties when a teacher formalizes
j. A neighborhood have always infinite points: it’s obvious, also his son understand
k. Potential infinity in form of recurrent procedures is easy to understand, while the infinity itself (infinite quantity) no

l. The problem for the students is working in Q, not in R, they are used to work in R and not used to work in Q
m. A student can operate with real numbers even if she doesn't know rational numbers
n. The students don't know that real numbers have no consecutive numbers
o. The students focus only the last thing on the last thing they study; they forget Q because you have introduced R as last.
p. The students have an intuition of the correspondence between numbers and points, for them the line is continuous

5. Teachers' orientations about teaching-learning sequences concerning real numbers

a. Real numbers are introduced in the first years of the high school
b. R is necessary to introduce functions and to solve equations
c. Infinitesimals are difficult to teach
d. R and the continuum are necessary in the Calculus, but the teachers don't need to recall them since they are known
e. The problem is the lack of time, not the fact that the students don't understand
f. A teacher must make choices, she can't do everything in a few hours
g. A teacher trusts the fact that the students knows real numbers, because they have studied them, even if the most of them have difficulties
h. At school R is used intuitively, using the continuum
i. Takes for granted that a neighborhood has infinite points
j. Takes for granted the students know R intervals
k. The textbook don't help the teachers: there are no exercises in Q or Z

Teacher 7: (We are responsible for the students difficulties with rational and real numbers)

1. ***Teachers' interviews synthesis step by step***

Step 1: There are infinite numbers like the root square of 2

Step 2: R is Q joined to irrational numbers

2. ***Teachers' declared choices***

a. She introduces real numbers as a joining of rational and irrational numbers
b. She says to the students that there are infinite numbers like $\sqrt{3}$, π, ….
c. Task: she asks the students to invent irrational numbers in the decimal register: "An activity that I propose is: < Let's invent some irrational numbers.> […] <Let's invent number with an infinity of digits, 0,123456.. a sequence.. numbers that has an infinity of not periodic digits, it's a way to lead them to touch a bit.. "
d. She places the new numbers on the line and ask the students to construct other real number geometrically "This is just a first approach, to present the .. between 1 and 2 where $\sqrt{2}$ is placed, with the ruler you can measure 1, and $\sqrt{2}$? How can you do it? It has an infinity of digits. You can realize the geometrical construction. Just now I'm going to divide them in small groups and to ask them to construct other real number using Geogebra."

3. ***Teachers' reports of students' difficulties with real numbers***

a. My students would not have read x Q

b. The students have many difficulties with quadratic equations
c. The students solve the quadratic equation associated to quadratic inequalities and linear inequalities: this is the reason why they write $x<\pm\sqrt{5}$
d. The students have difficulties also to understand algebraic numbers
e. The choices depends on the kind of schools
f. For the students is very difficult to learn limits

4. ***Teachers' orientations about teaching-learning sequences concerning real numbers***

a. What do we do at school to make them doing such things?
b. Maybe we don't work enough in Q

Teacher 8: (The students get confused when we formalize: real numbers are intuitive)

1. ***Teachers' interviews synthesis step by step***

Step 1: Real numbers had already been constructed by Archimedes

Step 2: N Z Q R

Step 3: Convergence of intervals on a rational point or on a "void space" using Q, you have to construct a number

Step 4: is not of the previous kind, you have to construct it in a different way

Step 5: You can find in the nature in so many beings

Step 6: $\sqrt{2}$ is the diagonal of the unitary square

Step 7: R is necessary for exponentials to complete the graphic

Step 8: the topology of the line is necessary to introduce the limits

Step 9: R is completed also with transcendental numbers

Step 10: R is in correspondence with the points of a line

Step 11: On the line using R you lose the concept of order, you can't establish the consecutive number

Step 12: The separating elements of the line must exist

Step 13: R is necessary to solve quadratic equations

Step 14: e is a limit, while is different

Step 15: Hyperreal numbers: a number that has a strange circle around

Step 16: The model of continuity doesn't work for everything real

2. ***Teachers' declared choices***

a. Introduces R saying that there are N, Z, Q and then we need to go on enlarging because there is a numbers that are not rational, like √2, that can be constructed using intervals converging to a void onto the line, even if there are also numbers like π that are different: “start from N, then you had to construct Z, Z contained something that was equipotential to N. Then from Z you had to construct Q; then there was something more like √2, so you … I remember there was a proof… the convergence of that intervals that seemed to arrive there but the convergence was on the void, there was nothing. So you have to construct something. Then there was another way to construct π. π … how was it? Where does it come from? It’s transcendent. You told them some stories, the classical approach in the third year with root square of 2, that is the classical absurd proof, with the bisection method, that is given two rational numbers there is always a number in the middle but in some cases there isn’t even if the intervals converge, so what I remember from Pini is that intuitively you can see it, while I show that you can’t obtain π this way, it’s another kind of number that is there, inside, and that you need because otherwise the ratio between circle and ray is not there. It comes out from this kind of stuff.”

b. He says the line is complete adding to Q not only roots but also transcendental numbers

c. He uses real numbers but without formalizing: “You use them but you don’t formalize With the graphic is surely easier. The graphic to represent R but also C is surely the easiest. On an axes the imaginary, the real on the other. Absolutely, absolutely, this is the best register.”

d. He introduces √2 as the diagonal of the unitary square projected on the line

e. He has a phenomenological approach to the exponential, f.i. growth rhythm , that doesn’t need real numbers, but you need real numbers in the middle

f. He introduces intuitively the topology of the line and its density in order to introduce limits

g. The enlargement from Q to R is realized using the roots

3. *Teachers’ personal meaning*

a. R is an Archimedean set

b. He got confused at the University when his Professor formalized real numbers

c. You can use numbers without formalizing

d. You can complete with continuity but there is the problem of

e. R is a set in correspondence with the line

f. Nature is analogic, is discrete

g. Irrational numbers aren’t always between two rational numbers

h. R is a tool, we need to learn to use real numbers

i. Hyperreal numbers are strange, he’s not sure they exist

j. The model of continuity doesn’t work for everything real

4. *Teachers’ reports of students’ difficulties with real numbers*

a. He thinks that to formalize confuses the students

b. He tries to make them understand the concept of limit using / as rarely as possible

c. The students see the density of the line, they understand it; it’s very intuitive

d. The separating element 2emerge in the practice of dividing segments: there must be a number whose square is 2

e. The students have a vague sensation of what R is

f. It’s important that 2 exists, not what it is

5. *Teachers’ orientations about teaching-learning sequences concerning real numbers*

a. He uses real numbers without formalizing
b. He uses R for everything; he introduces R before every other thing
c. Introduce 2as a diagonal of a square projected with a compass on the number line
d. The exponentials need R to complete the graphic
e. You need real numbers, a bit of topology of the line, to introduce limits
f. When you introduce the Calculus they already know real numbers that they studied in the first years
g. To say you can't establish the consecutive number is necessary for the limits
h. The new books simplify much more than the previous ones
i. We must take care of the goals we have to reach with these students
j. He doesn't talk about the different cardinalities even if it's in the book
k. The graphic representation for R is the best one, teachers should work more with it

Teacher 9: (The real numbers are important to do beautiful things in Geometry)

1. ***Teachers' interviews synthesis step by step***

Step 1: Irrational numbers

Step 2: Geometrical irrational numbers are different in respect of the irrationals found as limits

Step 3: The line is necessary for real numbers

Step 4: Real numbers are separating elements of contiguous classes

Step 5: Operations with real numbers (roots,)

Step 6: The Euclidean geometry's axioms contribute to create the properties of R as the set of points of the line

Step 7: Real numbers emerge in geometrical problems

Step 8: Irrational numbers are numbers we can't express using fractions; it's connected to Geometry with commensurable and incommensurable magnitudes

2. ***Teachers' declared choices***

a. She introduces irrational numbers using the diagonal of the unitary square: "There are numbers that are not rational, there isn't a fraction that express them. In Geometry you connect the discourses because you present commensurable and incommensurable segments."
b. She introduces real numbers in order to use roots in Geometry
c. She introduces real numbers using contiguous classes
d. She presents operations involving irrational numbers
e. She prove in the Euclidean geometry that between two point there is always a point and that every point has a precedent, without associating numbers to the points: "You have to take in account that in the first two years they study Euclidean geometry. When I introduce the line, you know that there are the axioms. When they are in the first year. It's the first proof they see. You show that every point has a precedent, so.. that concept that between two points you have always a point …"

f. She introduces irrational numbers: “because historically this was the evolution. It’s also incredible that they got it so fast. This discovery.”

3. Teachers’ personal meaning

a. It’s impressive the Ancient got so fast so good results
b. Real numbers are necessary in Geometry
c. The line has geometrical properties that in a second time

4. Teachers’ reports of students’ difficulties with real numbers

a. A students of her would answer a real number is a separating element for contiguous classes
b. The students “defoliate”, go at the core the matter

5. Teachers’ orientations about teaching-learning sequences concerning real numbers

a. Teachers only remember what they teach usually; the other things are vague and they have to recall them in their mind studying again
b. In the first years you teach irrational numbers, you don’t go on; she doesn’t know what her colleagues do in the further years
c. The Euclidean geometry’s axioms contribute to create the properties of R as the set of points of the line
d. Textbooks simplify very much, both for R and for Q
e. You can do a lot of nice things in Geometry with real numbers, merely using Pythagoras’ and Euclid’s theorems in which roots are necessary

Teacher 10: (I understand my students: I studied the formal aspects of real numbers and I came back to the trace of a segment)

1. Teachers’ interviews synthesis step by step

Step 1: Real numbers as separating element of contiguous classes

Step 2: R is the line; the examples of irrational numbers are geometrical

Step 3: R is not necessary for the series; N is enough.

Step 4: Exponential functions are defined in R

Step 5: Density of the line

Step 6: Dynamic conception of continuity; continuous segments can be traced with a pencil

Step 7: Formalization is not necessary

Step 8: Limit points as points we can approach without reaching them

Step 9: Irrational numbers are represented with other symbols (π,e,..)

Step 10: R is necessary to give an idea of C

2. *Teachers' declared choices*

a. She introduces the line, the diagonal at the unitary square and the Pythagoras' theorem or real numbers as separating elements of contiguous classes, depending on the students
b. To introduce the exponential functions "You present 2 raised at N, 2 raised at Z, 2 raised at Q and then you give 2 raised at √2 and you explain that this becomes a matter of separation between one and the other, the way you can define it."
c. She introduces the density of the line saying there are not consecutive numbers
d. To represent the correspondence between points and numbers she traces the segment and stop: "The graphic, because it's intuitive and as she were saying you have to start from the Geometry, that was the crisis".
e. She uses the roots to enlarge Q, to say that other numbers exist, she doesn't go on.
f. She uses the history of Mathematics to show that Mathematics is in never-ending evolution
g. She says that it's different to work with the pen and with the computer "because the computer shows pixels"

3. *Teachers' personal meaning*

a. When she was attending the training courses, when the Professor talked about real numbers she did the same: she repeated the difficult lesson and then she forgot everything, going on drawing a continuous line
b. The Professor said these things are fundamental for a teacher, she got ashamed
c. Formalizations are due to a specific way to study at the University; you must tell the history of Mathematics but you don't need stuff like this at school

4. *Teachers' reports of students' difficulties with real numbers*

a. Only gifted students understand real numbers as separating elements of contiguous classes
b. Only two students understand the formalization; for the other f.i. if they write $\pi/3 + k\pi$ they don't know what k is
c. The students thinks at R as a continuous trace, also with the functions. They think at a small ball that moving leave a trace
d. The students don't pose the problem of connecting the point using a continuous line in the exponential; they don't care about the meaning of 22or of the existence of empty spaces
e. Only a few students understand, she reassures the others saying that everything works and it's OK
f. The students understand the limit points very intuitively
g. To the students who wants to know more or attend Mathematics courses at the University she answers the day after, after having restudied
h. The students' mistakes may be due to an excess of precision

5. *Teachers' orientations about teaching-learning sequences concerning real numbers*

a. If you have gifted students you can introduce real numbers as separating elements of contiguous classes, otherwise you introduce the line, the diagonal at the unitary square and the Pythagoras' theorem
b. R is not necessary for the series; N is enough. for the other things R is necessary
c. Drawing a segment may represent the correspondence between numbers and points
d. The computer is different in respect of the pencil because it shows pixels

e. She knows that real numbers exist, like the golden ratio and so on, but when she teaches it's another thing
f. She never introduces function from N into N
g. To represent irrational numbers she avoids to use the decimal representation, uses symbols
h. If you go too in depth you risk that the students make mistakes
i. The graphic representation is the best because the crisis was geometrical and it's intuitive

Teacher 11: (Interdisciplinarity and real numbers: we lose many opportunities to distinguish between continuous and discontinuous entities)

1. ***Teachers' interviews synthesis step by step***

Step 1: Motion is described by means of functions more than continuous (C2 at least), the dynamic representations requires continuous derivatives

Step 2: Real numbers may be necessary to complete functions continuously

Step 3: Qualitative variations of functions don't need real numbers and limits

Step 4: In Physics the teachers anticipate and use limit processes before the Calculus

Step 5: For the theorem of comparison between limits real numbers are necessary

Step 6: The Calculus is full of dynamical concepts with a static theory

Step 7: Sequences and functions defined in isolated points give a new idea of continuity

Step 8: The continuum is a limit of the discrete: here we miss something about real numbers

Step 9: The exponential functions is enlarged to conserve good properties also extending the domain, leaving aside the original definition and then it's completed continuously

Step 10: You need real numbers to extend continuously functions that are expressed analytically

2. ***Teachers' declared choices***

a. Introduce R to complete exponential functions continuously
b. Many concepts we need in the Calculus are introduced before, also in other disciplines, in particular Physics
c. She ask the students to reason on variations, intuitive derivatives, before the Calculus
d. She introduces limit processes in Physics before introducing limits and the Calculus
e. She presents parametric functions, with a parameter that can make a function discontinuous in one point: "The line with a hole always comes out... when you study the hyperbolic curve with parameters <For which values of k..? The classical homographic function in which you are asked to say what you obtain varying the parameter."
f. She says to the students that a function is obviously continuous in isolated points
g. She introduces the sequences. Would like to ask the students if a sequence is continuous or not
h. She introduces the asymptotic behavior
i. Task: She would now ask the students to draw $f(n)=n^2$. Maybe they would trace a continuous graphic

j. Introduce the exponential function in the third class, before the limits and the Calculus: here she miss something about real numbers
k. She completes the functions continuously, from the discrete to the continuum, adding numbers in the functions' output corresponding to new numbers in the domain
l. To introduce the exponential function she recalls before the power's definition with natural numbers at the exponent (multiplying n times), proving their properties; then she says that to conserve the properties for other inputs the mathematicians had to renounce to the original definition; she misses the exigence to extend declaring only that her aim is to study the real exponential function. You define the exponential function to conserve its properties transforming the way you want the input value
m. She proved many times before that $\sqrt{2}$ is irrational

3. Teachers' personal meaning

a. The theorem of comparison for limits is the apotheosis of the necessity of real numbers
b. R is necessary to complete functions continuously
c. Sequences are natural; the students study discrete sets since the first year of primary school
d. She constructs the exponential function step by step, expanding the domain, and then completes it continuously
e. Functions associate input to outputs
f. The exponential function associate sums to products, this is its essential property, this becomes indeed its definition; thinking better this is true only until we use Q
g. We extend exponential function metaphorically continuously; from Q to R the previous scheme crashes
h. You have to invent irrational numbers to complete functions, it's good since we trace lines but we need to express them by means of symbols, we can't calculate them.
i. Asking to complete the functions continuously you take for granted they are continuous innately; what does it mean? It's a chaos, I must re-think to everything

4. Teachers' reports of students' difficulties with real numbers

a. The students are used to see trajectories more regular than we need for continuity
b. Classical Physics, in particular Mechanics, contributes to give to the students a too regular image of continuity
c. The students are ready to understand variations and derivatives even in the fourth year, before the Calculus and the limits; it's important to talk about qualitative variations and to compare variations at least as much as to talk about continuity
d. The too simplified models of function make the students wonder and ask why they have to consider limits in flat domains
e. A student asked: <If a function is continuous in a set, is this continuous also in its subsets?>. She always stress they have to complete continuously and maybe this is the reason why they wonder if the function was continuous before or in isolated points, that instead is obvious
f. The students' functions' outputs are never pathological
g. The students always ask what they have to do in the connection point between discontinuous functions
h. The students may react bad when they got aware there had been historical debates and that also the mathematicians may find something difficult to accept. They say: < If even they didn't understand, how can I understand?>

i. The students in the primary school think naturally in discrete sets, but when we study the graphic of function they think that continuity is natural while restrictions of the domain are perceived as not natural
j. Maybe keeping on working on discretely the students would never fell it's continuous

5. Teachers' orientations about teaching-learning sequences concerning real numbers

a. Teachers are used to present too regular trajectories for continuity
b. We must take care of variations not only of punctual continuity
c. Before the Calculus we can prepare better the students; in particular we should find not regular functions and talk about them instead of regularize functions also when they are not regular, like we do modelling phenomena in Physics
d. It's easy to present function with holes using parametric functions
e. If we present too regular functions, then we have continuously to motivated choices about the domains that are different from R
f. It's important to work on sequences to deconstruct too simplified ideas of continuity
g. To introduce the complexity using the historical debates, like Berkeley's critics to infinitesimals, is a double-edged sword
h. Introduce the asymptotic behavior very carefully; the teachers would need the help of an expert to plan this part.
i. Continuous functions, expressed analytically, must be completed continuously adding irrational numbers in the input
j. If we extend adding we never reach continuity

4.7.1.3 Teachers' profiles and teachers' choices suitability: how are they related?

In this Paragraph we sum up the results of the two steps of the research in order to answer the question:

PQ - 1.5 What are the relations between teachers' mathematical knowledge, orientations and goals and their declared choices?

We will not analyze the didactical suitability of the teachers 5-6-7 because, as 2 of the 3 teachers themselves recognized, their choices were evidently unsuitable from all the points of view we are analyzing. The teachers had not the occasion to reflect previously on the topic answering the questionnaire and maybe the teachers who answered the questions of the written questionnaire had benefits from the reflection.

However we keep on with our research choice to analyze only the sequences that are interesting from some point of view. These teachers didn't show wrong personal meanings of real numbers but in the designing of the sequences they weren't effective, in particular because of their common trend towards a simplification motivated with a, maybe excessive, attention to cognitive suitability. As one of the teachers recognized, maybe simplifying too much they complicate things.

We thus decided to go on analyzing only the teaching sequences proposed by Teachers 1,2,3,4,8,9,10, 11.

- **Teacher 1 (4)**

Formation: Master in Mathematics , PhD in Mathematics, National qualification as a Mathematics and Physics teacher in the high school (TFA 049), Qualification as Associate professor of Topology

Years of experience as a teacher: 5

Studied real numbers: at the University in a course of Calculus and at school

The fundamental properties of real numbers are: continuous, complete, ordered, Archimedean

In every numerical set enlargement (N → Z, Z → Q) there is a "critical operation" involving the elements of the set to enlarge, like subtraction and division, that lead to a new set. How is it possible to construct R starting from Q?: By closure in respect of the concept of "limit" of sequences of rational numbers: starting from the power inverse operation you show that new numbers "arise" that have the following feature: they can be estimated from the bottom and from the top in a more and more precise (i.e. with a decreasing difference) using rational numbers. At this point you can make the "closure" of Q in respect of this operation, i.e. you consider all the numbers that you can obtain this way.

Is it possible to define a limit point in Q or is it necessary to use real numbers? Yes, f.i. 0 is a limit point of 1/n

<u>Categories and significant sentences in the questionnaire:</u>

CC1 ↔ C4.1 (R is the set of Cauchy-Weierstrass-Cantor contiguous classes) [U]

CC5 ↔ C5.2 (Axiomatization of real numbers' set) [U]

CC6 ↔ C3.1 (Hybrid continuum) [L]

CC7 ↔ CE4 (Line as trajectory) [L]

IC3) Cantor, Cauchy, Weierstrass (Limit points): K3B, K3C, K3D, K3L2 [25]

IC3 ↔ C4.1 (R is the set of Cauchy-Weierstrass-Cantor contiguous classes)

MC1a) Historical/formal approach

MC3b) Extensive introduction

MC4b) Systemic introduction

K4_a) Q is sufficient

Teacher ML / I (teacher ML or intermediate who chooses the same number of L and U constructions)

GO_R_2) R is necessary only for advanced mathematics (Calculus) [7] [U] : **R FOR U**
(G1C, G1D)

O15_C (The first video may help the students to create an image of real numbers that could be useful in the future) & O15_E (The supports used is not necessary)

> *"L'incipit è molto buono (problema del cartoncino), e anche la parte che riporta il problema sulla retta reale; non mi piace la frase "dalla geometria sappiamo che": per me la definizione di radice quadrata è il lato del quadrato che ha area pari al radicando, quindi non è per "una regola geometrica"; non mi piace neanche il riferimento alla calcolatrice: questo problema è nato molto prima della calcolatrice e la calcolatrice non lo risolve."*

OF) The visualization of a flow helps the students to have intuitions about real numbers properties

O25_A: The flow of the slider of the second video would represent the correspondence with real number (A) but in this case the flow is not suitable.

> *D10:* ***Osservi il video al seguente link: http://www.youtube.com/watch?v=kuKTyp_b8WIIl video può aiutare uno studente a comprendere la corrispondenza biunivoca tra numeri reali e punti della retta?***
>
> *"No, perché lo slider sale necessariamente con una scansione (cioè ha un passo) visibile, quindi non dà l'idea che ad esempio tra due numeri qualsiasi e "vicinissimi" ce ne stanno infiniti. In effetti lo slider mostra solo numeri razionali. Piuttosto userei una scala fissa sulla retta e il cursore che scorre lungo la retta. Oppure un' effetto zoom che va sempre più in piccolo facendo vedere che per quanto piccolo vai resta sempre un segmento lunghissimo se zoommi."*

O39: The graphic representation allows to visualize better the solutions

O40_A: There are not two solutions but only two representations of the solution

> *D11:* ***Osservi il video seguente, in particolare dal minuto 10:20 al minuto12:10 http://www.youtube.com/watch?v=UEBK5DfPxvk Lei cambierebbe qualcosa nella spiegazione?***
>
> *Probabilmente non distinguerei tra soluzione algebrica e grafica, ma direi semplicemente che visualizziamo meglio l'insieme delle soluzioni se le rappresentiamo sulla retta*
>
> *D12:* ***Crede che sia opportuna la distinzione tra soluzione algebrica e grafica di una disequazione?***
>
> *No, perché non sono due "soluzioni" sono due modi diversi di scrivere una soluzione*

RI2: "Endpoints" inclusion with $<$

RI5: Graphical representation
QI5: $|x| < \sqrt{5}$

O41_D: Thinks that some representations of intervals may be immediate/clear/explicit/intuitive

O42_A: The parenthesis [,] represent inclusion also in the case of rational numbers

O42_B: The domain of inequalities should be indicated only in the case of rational numbers

O42_E: Segments "are" intervals of real numbers

CIR_1) To coordinate adequately registers to represent the solutions of inequalities (intervals) is useful and important/ a lack of good coordination between verbal and other representation makes a didactical practice unsuitable
CIR_2) There is a hierarchy in the intervals' representations
a. The graphic representation is better/more intuitive, more synthetic

CIR_3) The intervals' representations are equivalent

CIR_5) A segment can represent the infinite real solutions of an inequality

CIR_6'): The usual representation of real intervals can't represent a subset of Q la n° 4 non è accettabile perché 2 e 4 non sono soluzioni del sistema; la 1 è corretta quindi accettabile, ma credo che per uno studente sia molto difficile interpretarlo come intervallo centrato in 3 di raggio 1; inoltre non mi sembra (ma forse sono io che non lo vedo) che ci si arrivi naturalmente, mi sembra molto forzata

> *D15:* ***Esprima i suoi giudizi sulle risposte fornite:***
> *Quali soluzioni sono più adeguate al problema? Perché?*
> *2 e 5: sono quelle che meglio rendono in maniera più immediata l'idea intervallo di numeri reali; mi piacerebbe molto anche la soluzione]2,4[*
>
> *D16:* ***Quali tra queste metterebbe come soluzione dell'esercizio nel libro di testo e perché? Le n° ....***
> *nessuna, ma la migliore è la 5; le 1,2,3,4,mi fanno pensare ad un intervallo continuo di soluzioni e dato che la disequazione è in Q non è così (in effetti credo che non abbia senso risolvere disequazioni in Q, eventualmente ha un senso farlo in Z)*
>
> *D17:* ***Cosa pensa delle soluzioni proposte dagli studenti?****b. le n° .... non sono da accettare come soluzioni perché*
>
> *la 1 la eviterei perché è troppo esplicito in riferimento ad R; nella 2, 3, 4 aggiungerei esplicitamente l’intersezione con Q*
>
> *D18:* ***Esprima i suoi giudizi sulle risposte fornite***
> *Quali soluzioni sono più adeguate perché saranno necessarie per introdurre altri concetti? Perché?*
>
> *La 3 è interessante perché è l'unica "scrittura" in cui è rilevante il fatto che la disequazione sia in Q*

ABSOLUTE MEANING: GSO_1) **No need of mediation**

Didactical suitability

EPS1: Errors: There were no wrong practices in the teachers' interview.

EPS2: Ambiguity:

a. In the Calculus uses “the mathematization of a primitive intuition” of continuity rather than introducing more formal conceptions

b. “when we talked about limit points I proposed a definition .. we talked for some minutes and since they had studied the sequences I tried to make them reflect about the fact that, while when you calculate the limit of a sequence only to the infinite, when you study a function on the real numbers you calculate limits also in finite points, because the only limit point for N is the infinite, while for R…”

c. Dirichlet’s function as example of function derivable nowhere

d. “It will be useful to introduce them [real numbers] very soon because once you have introduced them you can show the strong difference between this [draws a segment and writes R] and this [traces a segment with some points and writes Q].”

EPS3: Processes richness: a. “when we talked about limit points I proposed a definition .. we talked for some minutes and since they had studied the sequences I tried to make them reflect about the fact that, while when you calculate the limit of a sequence only to the infinite, when you study a function on the real numbers you calculate limits also in finite points, because the only limit point for N is the infinite, while for R…"

b. Dirichlet's function as example of function derivable nowhere

c. "I would do this way: I would show that N and Q are in correspondence, that is not hard, while Q and R no.. to show that there is an enormous jump"

d. She would distinguish between irrational numbers and the rational decimal approximation that the students are used to compute.

EPS4: Representativeness: The most significant partial meaning is the following, but she talks about it as something that should be done, not as somethings that she does really.

a."It will be useful to introduce them very soon because once you have introduced them you can show the strong difference between this [draws a segment and writes R] and this [traces a segment with some points and writes Q]."

The teacher is very well prepared in Mathematics – PhD teacher but the practices she declared to propose are not representative; the cause seems to be the fact that she's convinced that quite everything concerning is intuitive and innate so she thinks to present configurations that are significant but, not presenting the systems of practices and the connection strategies, in the end she don't succeed in presenting significantly the complexity of real numbers. She tries to represent complex configurations avoiding to enter their complexities and trusting the tudents intuition and innate predisposition to "think continuously". She refers indeed to grounding metaphors considering them much more advanced than they result to be to the students without the suitable problems and connections; in other words in order make real numbers simple she tries to connect everything directly to ground, avoiding intermediate partial configurations.

She only proposes the line, the roots and the decimal approximations but their connections are considered innate and intuitive.

a. The students perceive the numbers in a continuous way

b. In the students' minds there is already a spontaneous conception of real number as a sequence of numbers

c. The students see the continuum, the contiguous classes

d. It's easy to show that Q is equipotent to N and that R is not.

The teacher don't seem to take care of the connections between the different representations and configurations because she 's convinced that these are natural and innate. The teacher showed in the interview to be interested to the students' cognitive configurations and to be aware that using merely the line their conceptions are not suitable for grasping some concepts like the limit points and so on, but attributes the difficulties to the fact that real numbers are not introduced as soon as possible distinguishing immediately between the numbers and their approximations.

Cognitive suitability

CS1: Previous knowledge: In the first introduction of irrational numbers she proposes, using numbers as segments, she identifies points and numbers without expliciting the relation between measures of segments and numbers she takes for granted that the students are sued to it and maybe she's right, even if she doesn't problematize this relation enough.

In the further steps she takes for granted that the students' personal meanings are sufficient while the our cognitive analysis disconfirms her assumptions.

a. in Algebra she would introduce real numbers quite soon since "the solutions of inequalities with x Q bother me" : the students seeing some irrational numbers on the line as extremes of full segments are lead to think that it is possible using the numbers they already know. It's thus a choice that instead of lead the students to grasp representative meanings of real numbers lead them associate wrong configurations to real numbers. The teacher doesn't understand that this choice is connected to the problems that she observed in the students' learning processes concerning the limit points, reported in this sentence:

"*They see all but that .. in the sense.. the problem of discontinuity. For them there is all. This is the reason why I say that I'm sure that for them these are real numbers [traces a segment] and that, however I take out stuff there other stuff remains here, close, this is sure. They didn't understand the sense of talking about limit points. I think that for them this is so banal that if you say < I take a limit point and however I choose some points around I find other points >. If they have a strong image of the continuum and not the same for the discontinuum, everything is a limit point, where is the problem?"*

b. "you can work with R for sure from the graphical point of view because the property that is congenial at an operational level is the correspondence between points of the line and numbers"

Also in this case she declared to use R intuitively as a full segment, but the students only knows ome numbers so they go on convincing theirselves that the numbers they know "are" (implicit metaphor) all the point of the line.

c. "N and Q are in correspondence, that is not hard, while Q and R no.. to show that there is an enormous jump"

Another time the teacher proposed suddenly a very hard configuration, that we placed in the last level on 5, believing that the students may have graphical intuitions of trasfinite numbers: the enormous jump should be showed using the Cantor's diagonal theorem, that involves decimal representations with infinite digits and very formal procedures. Also as we stressed in the literature review even after several attempts to explain that different infinities exist, the students remain often convinced that the infinities are always in biunivocal correspondence, but furthermore in the graphic register she declared to use almost in every occasion the literature showed that for the students is very hard, sometimes impossiblem to conceptualize the infinite cardinality and the density of real numbers.

d. when we talked about limit points I proposed a definition .. we talked for some minutes and since they had studied the sequences I tried to make them reflect about the fact that, while when you calculate the limit of a sequence only to the infinite, when you study a function on the real numbers you calculate limits also in finite points, because the only limit point for N is the infinite, while for R…"

e. Dirichlet's function as example of function derivable nowhere

Once again the teacher suddenly introduce configurations that are cognitively disconnected from the rest. Indeed if the students associate natural continuous configurations to continuous functions and also to limits, as she declared, the Dirichlet's function, that is one the "monsters" born as consequences of the Cauchy-Weierstrass's definition of continuity, is impossible to evaluate as continuous or not. The teachers remembered that the students were attonished, and this is quite natural standing on our considerations.

CS4: High cognitive request:

As we observed before in the case of PhD teachers, we confirm here for this PhD teacher that to a high cognitive request is not associated a metacognitive request that make the students able to support the high request with suitable partial meanings. In particular in this case the search for immediate intuitivity clashes very much with the cognitive request.

Ecological suitability

ECS1: Adaptation to the national curricula: The teacher's choices are all included in the national curriculum but the philosophical reflection of infinity and the "first formalization" of real numbers.

ECS2: Intra and inter-disciplinary connections:

The teachers referred explicitly to interdisciplinary issues concerning Physics, in particular measures. She observed that a reason why the students are not prone to face theoretical problems about numbers is the use of numbers they are used to do during the Physics lessons. Indeed she says they are used to approximate with decimal numbers with 2 or 3 digits considering the numbers equal to their approximations. She proposed a way to connect the practices in the two fields anticipating in Mathematics an introduction of real numbers before the moment in which the students get used to use finite decimal numbers as if they were the only important numbers.

EC4: Didactical innovation: Innovation based on researches and reflections is taken in account.

The teacher didn't mention didactical innovations or researches, even she had attended a teachers' training course for one year.

- **Teacher 2 (117)**

Formation: Master in Mathematics, National certifications as Mathematics and Physics high school teacher

Years of experience as high school teacher: 18

Studied real numbers: at the University and in teachers' training courses

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Didactical suitability

EPS1: Errors: There were no wrong practices in the teachers' interview.

EPS2: Ambiguity:

a. "I say <There is N, then there is Q, then there is R.> and I say < Which is the necessity?> and then I introduce also C at the same time. At least I give an idea […] to close the operations in the sets, in this case in respect of subtraction, then in respect of division, square"

The teacher introduces R immediately, in the first lesson, proposing it as an element of the "sets chain" and explaining that the enlargement are due to the arithmetical exigence to 'close the operations'. She knows it's not so simple for R and that the root square si not enough to complete all the set, but in this discourse it emerge a sort of virtual unitary object, the “sets chain” that is the solution to the problem of the closure of operations. We will re-analyze this sentence also from a cognitive point of view, in relation with the placement of this configuration in the epistemic meaning.

b. She introduces the density saying: < We don’t have a so subtle pencil >.

This sentence, reported by the teacher, suggests that if we had a subtler pencil, we will have reached the dimension of a point. In Lakoff & Nunez (2000) we could place this configuration in an intermediate step of the definition of infinitesimals without reaching the actual infinite “final configuration” of the BMI. She interrupts this way the BMI processes towards an actual infinitesimal, that is instead what she wants to use in the following: “To quantify this , $\varepsilon > 0$, that is real, gives exactly the idea of the fact that we get closer and closer because this tends to 0.”

d. She presents the passage from Q to R graphically, between rational numbers there are irrational numbers

It's impossible to complete R adding points to the “rational line”. It's possible to characterize the known irrational numbers in terms of rational numbers – sequences (C4.1) or cuts (C4.2) – but not starting *tout-court* from the line adding points. The processes are numerical and, even more important, second order logic propositiona are necessary; also this process would require an actual infinity of axioms of existence. The processes that lead to R as a complete set operate on generic elements in order to generalize the procedures but it's not a finite process of creation. Since it would be impossible to create infinite irrational numbers with the continuum power-one-by one. Furthermore, as Dedekind stressed many times, the arithmetization process of real numbers doesn't happen on the line, but the attempt is precisely to “emancipate” R from a line, towards a numerical construction.

EPS3: Processes richness: Some of the teacher's proposals are interesting from this point of view:

a. “I usually do it also when I talk about Euclidean geometry, thus not exactly in the beginning when I talk about segments. I say: < It’s incredible how a finite thing has in itself something infinite> and I enjoy it very much because I say that in my mind the segment is the paradigm of life, in the sense that we can see the man’s life in the same way, as much as our desire … our desires are infinite”

She poses the question of infinity not in a concrete form, but approaching the problem from a point of view that was really relevant in the History of mathematics and infinity: two of the crucial observations about the relation between finite and infinite in relation to the segments are indeed of this kind. Let's think to the Cusanus' intuition about the curve as limit of a polygonal composed by infinite infinitesimal elements - not correct from a strictly mathematical point of view but very valuable in Physics and in the theory of measure- and to the Cantor's intuition about the existence of biunivocal correspondences between lines and segments, or between segments and squares and so on. All of them referred to metaphysical problems that, indeed, inspired them in the development their theories of infinity that are still relevant. This approach is closer to the epistemic practices than concrete approaches like that we presented in the second video.

b. “It’s important to go on with parallel discourses between Geometry and Algebra in relation with numbers and infinity: “I show why I need segments and I say that there is a correspondence. In fact then I say : <It’s true here but also it’s true between 0 and 1. But I could also do it in Q.> I say: < If you take 0 and 1 there is always a point between 0 and 1. If I talked about an half part I stop at Q, but when they understand that there are other things in the middle, that there is always a point in the middle, that is always the half part and they can go on infinitely, this gives to me the idea of limit point in a certain sense.”

The teacher carry out a double-sided system of practices, mapping properties and operations in the geometrical and in the arithmetical domains. With some imprecisions that we have alreasy stressed this choice is very relevant for the teaching-learning of real numbers.

c. π must be introduced geometrically, in a laboratorial way “we did a very nice work about the approximation of like is presented in the “Museo del Calcolo” in Pennabilli.

This approach to irrational numbers, even if partial, is certainly valuable from an historical-epistemic point of view.

EPS4: Representativeness: The relevant practices listed and commented in the previous point are almost representative of the complexity of real numbers, even if some important configurations are not taken in account and substituted by configuration that the teacher tried to use in the practices with the role of configurations of further levels of generality, not without avoidable potentially negative effects at the cognitive level (see. Ambiguity 2)

Cognitive suitability

CS1: Previous knowledge: The teacher seems to go on slowly and to take care of the students' personal meanings.

The most critical aspect of their choices concerns the concept of density; she declared:

“I say: < If you take 0 and 1 there is always a point between 0 and 1. If I talked about an half part I stop at Q, but when they understand that there are other things in the middle, that there is always a point in the middle, that is always the half part and they can go on infinitely, this gives to me the idea of limit point in a certain sense.”

She introduces however the density saying: < We don’t have a so subtle pencil >.

Maybe she introduces too soon this concept, maybe dragged by the enthusiasm of anticipating so interesting properties talking about the segments, but she is obliged to use very concrete examples that are not suitable from the cognitive of view. This is confirmed by the literature review and by the theoretical analysis, as we have already stressed. In fact the reference to the thickness of the pencil only contributes to reinforce wrong images of the line that in the literature review have been named *necklace model,* namely not dense , least of all complete.

Anticipating too much the introduction of density and using the line, the teacher maybe make the students construct partially wrong configurations.

CS4: High cognitive request: Relevant cognitive and metacognitive processes are activated.

The processes activated by the sequence proposed by the teacher are not very ambitious from the epistemic point of view, while they are quite high in propertion from the metacognitive point of view. This sequence is thus not very ambitious but very balanced.

Ecologic suitability

ECS1: Adaptation to the national curricula:

An approach to formalization is completely avoided while the problem of infinity is partially faced.

ECS2: Intra and inter-disciplinary connections:

There are no references to interdisciplinary connections with Physics, while there are connections with philosophical issues.

EC4: Didactical innovation: Innovation based on researches and reflections is taken in account.

In the interviews we analyzed no didactical innovations have been mentioned, but there is an interesting reference to a Science museum in which the Archimedes procedure to approximate Pi is presented in a laboratorial way.

- **Teacher 3 (115)**

Formation: Master in Mathematics and Training courses for Mathematics and Physics high schools teachers and Mathematics education researchers
Years of experience as high school teachers: 18

Studied real numbers: at the University in a course of Calculus and as an autodidact in original sources and educational books

The fundamental properties of real numbers are: axioms of the operations, order, completeness

In every numerical set enlargement (N → Z, Z → Q) there is a "critical operation" involving the elements of the set to enlarge, like subtraction and division, that lead to a new set. How is it possible to construct R starting from Q? Historically speaking, it began with the introduction of irrational algebraic numbers (roots) starting from equivalent geometrical considerations. You could start f.i. from the solution of algebraic equations with at least 2 as order – to ask Ippaso da Metaponto how does value of the ratio between diagonal and edge of a square – and then going on, towards the transcendental. I go crazy for the Lindemann-Weierstrass' theorem. Furthermore the Platonic proof (very … real in truth) of the irrationality of √2 is didactically still an excellent deductive tool.

Is it possible to define a limit point in Q or is it necessary to use real numbers? In my mind R is necessary.

Categories and significant sentences in the questionnaire:

K2_B: Algebraic structure of R (field)

K2_N: Completeness

K2_O: Order

CC1: Topological/differential

CC4: Algebraic – operations properties

CC5: Axiomatic

K3_F: Using √2 and/or other radicals, eventually with a historical approach

K3_H: Proposing rational problems without rational solutions

IC4) Root square and π (Example of irrational numbers, R is an enlargement of Q)

MC1b) Adapted approach

MC3a) Intensive (general) introduction

MC4b) Systemic introduction

K4_B: R is necessary

Teacher ML / L (teacher ML or intermediate who chooses the most of L constructions)

GO_R_2) R is necessary only for advanced mathematics (Calculus): G1C, G1D, G1G, G1H, G1I, G1L

R for "Limits" but not for "Sequences and series"

R BEFORE ALL

R FOR U

> ***D7:* La padronanza delle proprietà dell'insieme dei numeri reali è indispensabile per introdurre (può selezionare più risposte):**
>
> *calcolo differenziale*
> *calcolo integrale*
> *intervalli di R*
> *limiti*
> *sistemi di equazioni*
> ***Altro (specificare)** numeri complessi - NB Le proprietà di R sono ovviamente indispensabili per TRATTARE TUTTI gli argomenti proposti, ma quelli che non ho indicato possono essere INTRODOTTI anche senza R (es. exp e log con una tastiera)*

O15: Consider the first video as a good tool

O15_C (The first video may help the students to create an image of real numbers that could be useful in the future) & O15_E (The supports used is not necessary)

> *D9:* ***Ha dei suggerimenti (inerenti il linguaggio, i supporti utilizzati, i fini dichiarati, ...) per migliorare il primo minuto di questo video?***
> *farei vedere un gomitolo di spago*

O17_A: Makes evident the association between numbers and segments / visualization numbers and points

O25: The flow of the slider of the second video would represent the correspondence with real number (A) if there were no numbers on the line that may suggest a partition in steps (B)

O26: A zoom of a small part of the line that become longer may suggest the correspondence between real number and points of a line

> *D10:* ***Osservi il video al seguente link: http://www.youtube.com/watch?v=kuKTyp_b8WIIl video può aiutare uno studente a comprendere la corrispondenza biunivoca tra numeri reali e punti della retta?***
> - ***Sì,*** *fino ad un certo punto, visto che lo step è di 0.1, praticamente "discreto"*
> - ***No, perché*** *forse collega il concetto più alla "distanza" che alla "ascissa". Si potrebbe zoomare con Geogebra mostrando che il segmento rimane sempre pieno.*

INTUITIVE SIMPLIFIERS

O40_A: There are not two solutions but only two representations of the solution

O40_L: It's better to coordinate the registers

O41_C: Some representations are more suitable than others in specific practices

O41_F: Some representations are not "finished"/are the problem and not the solution

O41_G: The synthetic representations are better

O42_A: The parenthesis [,] represent inclusion also in the case of rational numbers

O42_B: The domain of inequalities should be indicated only in the case of rational numbers

RI2: "Endpoints" inclusion with <

RI4: "Endpoints" inclusion with parenthesis [a,b]

RI5: Graphic representation

CIR_1) To coordinate adequately registers to represent the solutions of inequalities (intervals) is useful and important/ a lack of good coordination between verbal and other representation makes a didactical practice unsuitable

CIR_3) The intervals representations are equivalent

CIR_6) The usual representation of real numbers intervals can represent a subset of Q

<u>COMPLEX SEMIOTIC APPROACH</u>

GSO_1) **No need of mediation**

GSO_4) S**ome representations are more suitable** than others in specific practices

GSO_5) Some signs are not representations since they're not finished processes / **Some signs associated to intervals represent the task and not the solution**

GSO_6) The **synthetic representations** are better

GSO_7) **Different representations are complementary** / the meaning is a result of configurations

D11: Osservi il video seguente, in particolare dal minuto 10:20 al minuto 12:10http://www.youtube.com/watch?v=UEBK5DfPxvk Lei cambierebbe qualcosa nella spiegazione?
soprattutto NON direi che si tratta di "due tipi di soluzione" !!!
D12: Crede che sia opportuna la distinzione tra soluzione algebrica e grafica di una disequazione?
No, perché *non sono "due tipi di soluzione" ma due forme diverse della stessa soluzione (=sostanza), che vanno "visualizzate" assieme*

D13: Quali tra queste metterebbe come soluzione dell'esercizio nel libro di testo? Le n°
2 e 5, ma anche la 4 correggendola in]2,4[
D14: Cosa pensa delle soluzioni fornite dagli studenti?
a. sono tutte ugualmente accettabili *forse equivalenti ma non accettabili come forma "finale", sintetizzata*
b. le n° .... non sono da accettare come soluzioni perché*1 ha ancora la forma di una disequazione, 4 comprende gli estremi, 3 è equivalente ma non compattata*
c. è meglio se gli studenti usano tutti la stessa sempre, cioè la n°.........., perché *omologazione? mai!*
D15: Esprima i suoi giudizi sulle risposte fornite:
Quali soluzioni sono più adeguate al problema? Perché? *la soluzione è la stessa, le sue forme che preferisco quelle da me prima indicate*
Quali soluzioni sono più adeguate perché saranno necessarie per introdurre altri concetti? Perché?*2 >> condizioni di esistenza (domini di funzioni); 3 e 4 >> domini come sottoinsiemi di R*

D16: Quali tra queste metterebbe come soluzione dell'esercizio nel libro di testo e perché? Le n° ....
tutte tranne la 3, che comprende gli estremi; ma specificherei in tutte che x deve stare in Q...
D17: Cosa pensa delle soluzioni proposte dagli studenti?
b. le n° .... non sono da accettare come soluzioni perché*3: comprende gli estremi*
d. nessuna è accettabile perché *occorre specificare che x sta in Q*
D18: Esprima i suoi giudizi sulle risposte fornite
Quali soluzioni sono più adeguate al problema? Perché?*1, 2 e 4*
Quali soluzioni sono più adeguate perché saranno necessarie per introdurre altri concetti? Perché?

la 5 è la forma standard di un intorno circolare (di 0), utile per i limiti

Didactical suitability

EPS1: Errors: There were no wrong practices in the teachers' interview.

EPS2: Ambiguity: The teachers' declared explanations and definitions are generally very clear and nursed.

The only poteantially problematic aspects , that we have already analyzed for Teacher 2, is the introduction of real numbers using the enlargements associated to operations:

a. An approach he uses "is the field, or better the closure in respect of the operations. This helps me first of all as an historical approach, a construction of the numbers from the natural, rational numbers and then I start talking about Pythagoras and I never stop"

However in the whole sequence, very complex, this configuration is not definitive and isn't identified with R thus we think this is not anbigous in the global sequence.

b. He introduces contiguous classes once, but then he talks intuitively.

c. He defines limit points but work with limits without using it

He feels the necessity to respect the formal tradition, but he's not convinced at all of the effectiveness of this approach and only implies the formal aspects without using them. This may confuse the students or convince them that the formal aspects are equivalent to the ones they are dealing with.

EPS3: Processes richness: The sequence includes relevant processes from the mathematical point of view (modeling, argumentation, problems solutions)

There are many relevant processes in the teacher's declared sequence:

a. He uses Zeno's paradox in the third year to anticipate the definition of limit and the line's topology "This becomes in the fourth year, when you approach the definition of limit, when you talk about the topology of the line you to talk about neighborhoods and the basic concept that is interesting to me is that this representation of the real numbers by means of a line is not a trivial thing. Once I asked a student <Is there a bijection or are they the same thing? How can you assure that there is a bijection?>.

b. The topic "Mathematics and music" is very useful to introduce the concepts of logarithms and exponentials, but also all the enlargement of numbers sets, using the musical scale: natural, rational, geometrical irrationals, not geometrical numbers, complex numbers.

c. "When Pythagoras start to use fractions, rational numbers.. for instance Odifreddi talks about ratio, reason, connected with the word *logos*, that is one of the ancient principles, while I prefer to stress the analogy between ratio and relation. […] "

d. To approximate numbers using excess and deficiency approximations he uses" Archimedes' method with approximations, for instance with π, using inscribed and circumscribed polygons. They know what a perimeter and an area are, that is a number included by… that this are contiguous classes if we want, they are classes of rational numbers and you can express a perimeter.. then strange polygons, in which also irrational numbers appears, I don't talk about contiguous classes but they understand that these are two sequences that approximate the number, one in excess and one in deficiency, and that the difference can be reduced as much as they want."

e. He introduces the geometrical and not geometrical numbers "this [the incommensurability of the diagonal an the edge of a square] don't shock them as much as the fact that we can't represent the cubic root of 2. […] I draw the diagonal of a square, then I report the segment on the line, while in the cube's duplication, the cubic root of 2 is not representable using the ruler and the compass"

f. Solve polynomial equations: One of the most important problem in the history was to solve polynomial equations, Tartaglia, Gauss… If they studying polynomial functions don't know that they can find 1, 3, 5 zeros and that if the grade is odd one is surely real … also this means to have understood the real numbers

g. He uses Geogebra to create a coordination between algebraic and geometrical aspects of real numbers

h. . He uses x to represent segments or numbers but "I always stress this is a representation.[…] when I indicate a letter, or a number on the axes I don't say: < This is the number three, this the number four, ...> and so on but I say this is always a representation of the number. Also ' Pythagorically' speaking it's an abstract idea, something that comes from elsewhere, I realize a representation first of all because this is one… also I use Geogebra, think at the name, from a Cartesian point of view, as you were saying, it's true they are joint but there also two separated representations, there are two windows, the algebraic one and the graphic one. I try to use always the both of them"

i. He proposes a task in Geogebra: to zoom in enlarging a point, in order to show the draw is not the object

l. Task: He ask the students what kind of numbers are e, π, sin π, ln 3

m. He asks the students: "Could you assure there is a correspondence between the line and the numbers?"

n. "I'll try to make them understand that the fact to find always, every time I take a point on the line, a real number this seems to me already a conquest. Also I like very much to talk about neighborhoods with holes, without their centers, that is very frequent in my tests about the limits, but also it would be enough to talk about open limited intervals. To make them understand that you can approach one or the other endpoints but without reaching it it's an important thing."

o. Using Geogebra and zooming in the line we can deal students' to reflect on the fact the line is not its draw, since it has a grower thickness

p. "exponentials and logarithms are useful for ... You do it usually in the middle of the fourth year [grade 12, nba] while I introduce them in the end of the year because immediately after I start introducing the limits and these are prototypical real functions with real domain. […] I highlight the difference between the power function and the exponential function" proposing a task: to ask which is the domain x^2, 2^x, x^x

EPS4: Representativeness:

The relevant practices listed and commented in the previous point are almost representative of the complexity of real numbers, even if some important configurations are not taken in account and substituted by configuration that the teacher tried to use in the practices with the role of configurations of further levels of generality, not without avoidable potentially negative effects at the cognitive level (see. Ambiguity 2)

Cognitive suitability

CS1: Previous knowledge: the students own the necessary previous knowledge to study the new topic; the expected meanings to reach are possible to achieve in all their components.

The teacher declared to approach very carefully every step and to design the teaching-learning sequences thinking at longitudinal paths.

We remark a particular choice.

r. "I also avoid to work with the sequences and I immediately shift to the limits of functions because to talk about function from N to R, as sequences are, try to create short circuits between the continuous and the discrete"

The teacher is very deeply aware of the intertwining between cognitive, epistemic and didactical isue concerning the passage from discrete to continuous entities. This woud be a crucial point for Teacher 11.

CS4: High cognitive request: Relevant cognitive and metacognitive processes are activated.

The teacher avoid to formalize quite completely and this is not necessary since it seems that he avoids that dimension for a matter of personal reluctance; maybe he is a representant of the profile that inspired Tall & Vinner (1981) when they introduced the notion of concept image definitin and concept definition, stressing that if the formal definition was irremediably in conflict with the concept image the result could be a total refusal of the formal dimension.

Nevertheless many proposals are significant both from the cognitive and the metacognitive points of view: geometrical and not geometrical numbers; problematic relation between numbers and points of a line; historical configurations like Pythagoras' “musical numbers” and Zeno's paradox.

Ecologic suitability

ECS1: Adaptation to the national curricula: the contents, their implementation and evaluation correspond to the curricular indications

An approach to formalization is completely avoided while the problem of infinity is partially faced, as long as the most of the configurations listed in the national curriculum.

ECS2: Intra and inter-disciplinary connections: the contents are related with other mathematical contents (with advanced mathematics and the other curricular contents) and with contents belonging to other disciplines (extra-mathematical contexts or other educational steps)

There are no references to interdisciplinary connections with Physics, while there are connections with philosophical issues.

EC4: Didactical innovation: Innovation based on researches and reflections is taken in account.

In the interviews we analyzed many didactical innovations have been mentioned: the teacher is completely aware of the difference between representations and objects, of the possibility that some practices may lead the students to create wrong configurations that are avoidable and that there are tools like Geogebra that may support the very hard process of creating metaphors between arithmetical, algebraic and geometrical domain in the teaching-learning of real numbers.

- **Teacher 4 (110)**

Formation: Master, PhD in Mathematics and National qualification for Mathematics and Physics high schools teachers
Years of experience as high school teachers: 5

Studied real numbers: at the University in a course of Calculus and at school

The fundamental properties of real numbers are: two operations makes real numbers a field (with characteristic 0); total order, compatible with the operations; complete.

In every numerical set enlargement (N → Z, Z → Q) there is a “critical operation” involving the elements of the set to enlarge, like subtraction and division, that lead to a new set. How is it possible to construct R starting from Q?: Two equivalent constructions: 1) the method of Dedekind’s cuts (separating elements of two sets whose union is Q and that have no maximum or minimum in Q, f.i. : $\{x \text{ in } Q \mid x^2<2\}$ e $\{x \text{ in } Q \mid x^2>2\}$; 2) quotient of set of the Cauchy’s sequences with convergent sequences

Is it possible to define a limit point in Q or is it necessary to use real numbers? Yes, it’s possible to define it also in Q. F.i. the set $\{x \text{ in } Q \mid x > 0\}$ has 0 as a limit point.

<u>*Categories and significant sentences in the questionnaire:*</u>

K2_B: Algebraic structure of R

K2_N: Completeness

K2_O: Order

CC1: Topological/differential

CC4: Algebraic – operations properties

CC5: Axiomatic

K3_A: Dedekind's cut

K3_B: Cauchy's classes of convergent sequences

IC1) Dedekind (Cuts): K3A [14]

IC2) Hilbert (Axiomatic): K3E [2]

K4_A: Q is sufficient

Teacher ML / U (teacher ML or intermediate who chooses the most of U constructions)

GO_R_2) R is necessary only for advanced mathematics (Calculus): **R FOR U**

G1_C: Differential calculus
G1_D: Integral calculus
G1_E: Sequences and series

“Sequences and series” (SS) but not “Limits” (L)

Q SUFFICIENT & R NECESSARY

O15_C: Help the students to create an image of real numbers that could be useful in the future

> ***D9:** **Ha dei suggerimenti (inerenti il linguaggio, i supporti utilizzati, i fini dichiarati, ...) per migliorare il primo minuto di questo video?***
> *Mi lasciano perplesse due cose: 1) di solito si conosce (si misura) il lato di un quadrato e si calcola l'area, non viceversa. Forse si potrebbe enunciare il problema come un problema di "raddoppiamento"... 2) c'è un po' di confusione tra il problema di arrotondare un numero fino ad avere abbastanza cifre decimali per motivi pratici, e quello d stabilire che un numero abbia infinite cifre decimali aperiodiche. Questo secondo problema è astratto e indipendente dal primo. Potrebbe essere che radice di 2 sia periodico di periodo 1000000. Gli esempi dati non permetterebbero di distinguerlo da un numero decimale aperiodico...*

O24: The second video may confuse the students since the numbers represented are only rational and positive numbers

O36: It seems that length and numbers are the same thing / lack of unit

> ***D10: Osservi il video al seguente link: http://www.youtube.com/watch?v=kuKTyp_b8WIIl video può aiutare uno studente a comprendere la corrispondenza biunivoca tra numeri reali e punti della retta?***

- ***No, perché** non è chiara la corrispondenza; è fissato l'origine ma non una scala (il punto corrispondente a 1). Difficile giustificare la posizione dei numeri negativi*

D11:** **Osservi il video seguente, in particolare dal minuto 10:20 al minuto 12:10http://www.youtube.com/watch?v=UEBK5DfPxvk Lei cambierebbe qualcosa nella spiegazione?

Almeno dire "rappresentazione" della soluzione: la soluzione è l'insieme dei numeri che soddisfano la disequazione: essa è rappresentabile in modi differenti, tra cui quello "algebrico" e quello "grafico"

D12:** **Crede che sia opportuna la distinzione tra soluzione algebrica e grafica di una disequazione?

- ***No, perché** No, perché la soluzione è un ente astratto: ciò che cambia è la sua rappresentazione*

D17:** **Cosa pensa delle soluzioni proposte dagli studenti?

- ***d. nessuna è accettabile perché** solitamente le espressioni usate indicano intervalli reali, non razionali*

SEMIOTIC COMPLEXIFIER

O40_A: There are not two solutions but only two representations of the solution

RI2: "Endpoints" inclusion with <

RI5: Graphic representation

O41_D: Some representations of intervals are immediate/clear/explicit/intuitive or on the contrary, implicit

CIR_3) The intervals representations are equivalent

CIR_6’: The usual representation of real intervals can’t represent a subset of Q

<u>ABSOLUTE MEANING</u>

GSO_1) **No need of mediation**

GSO_5) Some signs are not representations since they’re not finished processes / **Some signs associated to intervals represent the task and not the solution**

Didactical suitability

EPS1: Errors:

No wrong practices have been detected in the teachers' sequence.

EPS2: Ambiguity:

a. “Introduce real numbers as points of the line, taking the correspondence for granted: “To do what we have to do we can use the preconception of the correspondence between real numbers and line.”

“density, completeness, continuity are in the ideas world. They’re used without being explicit.”

He indeed uses the rational approximations of irrational numbers, but pretend to work in R: “You live continuously with the double truth: what we write down formally is in the real numbers set, what we use are drawings and calculations”

This approach mixes and confuse very different levels so in respect of our analysis it can be considered globally ambiguous. The situation is however more complex.

The teacher is a PhD teacher. His approach is interesting because he declared many times to he plays this play: he consider a *pseudo*-mathematics the high school mathematics so his constraints is to be consistent with “his mathematics”, formal and correct and to show to the students only 'glimmers of truth'. He thus has always two worlds in his mind: the “true mathematical world” and the “pseudo-mathematical worlds”. Taking care of being coherent with the previous one but avoiding absolutely to make the students participate in this play. His didactical approach is thus very similar to the approaches of Teacher 1 and Teacher 8, even if in respect of the last one the knowledge is really much more advanced.

Analyzing better the whole interview we observed that the teacher was not convinced at all of the formalization so the position he assumed is a bit more complex: he thinks to leave the students away from the “true mathematics” but he's partially convinced that this is not a 'pity' because Dedekind's and other formalizations are indeed ways to clean the mathematicians consciences. Thus he thinks that what really matters of real numbers in mathematics are formal properties and operational practices, but in a totally different and independent way. These plans are not connected and aren't the same mathematical object.

From this perspective this approach is interesting, but the teachers' declared choices are however very classical and potentially unsuitable; the good awareness of the complexity is not transformed in a suitable didactical sequence, as we can see looking at the following choices:

b. He introduces continuous functions using limits in a dynamic form with the ε/δ formalism: “many things are used without making them explicit. Limits exists and are unique, the upper extreme exists.[…] a function that tends to x_0, a limit L that is not a value of the function. You choose an interval that goes from $l + \varepsilon$ to $l - \varepsilon$, [he draws the interval and the axes] then the idea is to have for every choice of this ε a δ here in the bottom, $x_0 - \delta$, $x_0 + \delta$ in an interval centered in x_0 [he draws a segment] and all the image of the f lies inside the chosen interval”

c. Real number are the decimal numbers: “A way to introduce them in a sly manner is as decimals.. they’re sequences, better, series, but we don’t say this to the students! Some of these series have no limits”

d. To describe the properties of the line you go on reducing distances until the thickness of the pencil. We commented extensively this choice in the analysis of Teacher 2's sequence.

EPS3: Processes richness & EPS4: Representativeness

The processes he had in his mind are relevant but his choice to reduce as much as possible the complexity lead him to declare not very relevant processes.

An interesting processes is the discussion about geometrical numbers, that we have already analysed in the Teacher 3's interview.

Cognitive suitability & Ecologic suitability

The teacher profile is very similar to Teacher 8 from the cognitive point of view.

Teacher 8 (83)

Formation: Master in Mathematics

Years of experience as high school teacher: more than 20

Studied real numbers: at the University in a course of Calculus and at school

The fundamental properties of real numbers are: dense, continuous

In every numerical set enlargement (N → Z, Z → Q) there is a "critical operation" involving the elements of the set to enlarge, like subtraction and division, that lead to a new set. How is it possible to construct R starting from Q?: Ratio between the measure of a square and the diagonal

Is it possible to define a limit point in Q or is it necessary to use real numbers? It's necessary to use R

K2_A: Topological/differential structure of R

K2_S: Continuous

CC1: Topological/differential

CC7: Line – unitary

K3_F: Using √2 and/or other radicals, eventually with a historical approach

IC4) Root square and π (Example of irrational numbers, R is an enlargement of Q)

IC4 ↔ C3.1 (Hybrid continuum → [New numbers must represent incommensurable magnitudes like the diagonal of a unitary square, that have special properties and are impossible to represent in the decimal usual representation and by means of fractions, like π,])

MC1b) Adapted approach

MC3b) Extensive (particular) introduction

MC4b) Systemic introduction

Teacher ML / L (teacher ML or intermediate who chooses the most of L constructions)

K4_B: R is necessary

GO_R_1) R is a prerequisite for quite all the mathematical objects and practices / once introduced R, it is obvious the numerical domain is R if there are no other indications / R is necessary for the graphics : **R FOR ALL**

G1_A: Exponential function

G1_B: Logarithmic function

G1_C: Differential calculus

G1_D: Integral calculus

G1_G: R - intervals

G1_H: Limits

“Limits” but not “Sequences and series” (SS)

R BEFORE ALL

O15: Consider the first video as a good tool [L] [3.1]

O15_A(Help the students to create a good image of real numbers) & (O15_CThe first video may help the students to create an image of real numbers that could be useful in the future)

O17_E: Makes evident, avoiding not useful words, the completeness/density of R, we could not do the same with the other sets

INTUITIVE SIMPLIFIERS

O40_L: It's better to coordinate the registers

RI2: "Endpoints" inclusion with <

RI4: "Endpoints" inclusion with parenthesis [a,b]

O41_E: Only some of the proposed solutions are representation of intervals

O41_L: The graphic representation is not formal enough/"represents" the algebraic one

O42_B: The domain of inequalities should be indicated only in the case of rational numbers

O42_E: Segments "are" intervals of real numbers

O42_F: The usual representations of intervals of real numbers can't represent intervals of rational numbers

CIR_1) To coordinate adequately registers to represent the solutions of inequalities (intervals) is useful and important/ a lack of good coordination between verbal and other representation makes a didactical practice unsuitable

CIR_2) There is a hierarchy in the intervals' representations
c. The graphic representation "represents" the algebraic one

CIR_5) A segment can represent the infinite real solutions of an inequality

CIR_6): The usual representation of real intervals can't represent a subset of Q

ABSOLUTE MEANING

GSO_1) **No need of mediation**

GSO_5) Some signs are not representations since they're not finished processes / **Some signs associated to intervals represent the task and not the solution**

D10:* *Osservi il video al seguente link: http://www.youtube.com/watch?v=kuKTyp_b8WIIl video può aiutare uno studente a comprendere la corrispondenza biunivoca tra numeri reali e punti della retta?* *No, perché *i due slider mostrano la stessa cosa, non aggiunge niente*

D11:* *Osservi il video seguente, in particolare dal minuto 10:20 al minuto 12:10http://www.youtube.com/watch?v=UEBK5DfPxvk Lei cambierebbe qualcosa nella spiegazione? *Quella non è la soluzione grafica, che è il confronto tra i grafici i due rette*

D12:* *Crede che sia opportuna la distinzione tra soluzione algebrica e grafica di una disequazione?* *Sì, perché *si perché educa alla flessibilità*

D14:* *Cosa pensa delle soluzioni fornite dagli studenti?
b. le n° .... non sono da accettare come soluzioni perché *5 perché non è espressa con linguaggio formale*
D15:* *Esprima i suoi giudizi sulle risposte fornite:
Quali soluzioni sono più adeguate perché saranno necessarie per introdurre altri concetti? Perché? *3 introduce alla logica*

D16:* *Quali tra queste metterebbe come soluzione dell'esercizio nel libro di testo e perché? Le n° .... *la 2 con aggiunta x appartiene. a Q*
D17:* *Cosa pensa delle soluzioni proposte dagli studenti? nessuna è accettabile perché non si distingue tra Q ed R

Didactical suitability

EPS1: Errors

We found one wrong practice in the teachers' sequence:

"then there was something more like √2, so you … I remember there was a proof… the convergence of that intervals that seemed to arrive there but the convergence was on the void, there was nothing. So you have to construct something. Then there was another way to construct π. π … how was it? Where does it come from? It's transcendent."

The teacher, pursuing the goal of differentiating between geometrical and algebraic numbers, fall into an ingenuous confusion: he confused the difference between geometrical and algebraic numbers with the difference between geometrical numbers and numbers resulting from the Weierstrass process of construction of irrational numbers by means of convergent sequences of encapsulated intervals. Since he had found reading on his own many construction of π but not as a limit, he consider it a "different number"-

Following this discourse π seems not to be a real number.

Nevertheless we decided to analyze his interview because his profile was interesting also for this particular aspect, that we didn't consider and never read in the literature: numbers constructed geometrically may be seen as limits?

EPS2: Ambiguity:

The previous practice is certainly the first element of ambiguity, both for th evidence we discussed before and for the immediate identification of line and numbers, that we explained for Teacher 1 and 2.

Also he uses real numbers but without formalizing: "You use them but you don't formalize with the graphic is surely easier.", confirming the previous. Also he introduces intuitively the topology of the line and its density in order to introduce limits, as the first teacher, so we have already commented this choice.

The teacher has also a problematic approach to the passage from a 'phenomenological approach' to exponentials to the continuous function, that he consider the 'true exponential'. The unsuitability of this choice will be explored in depth in the analysis of Teacher 11's interview, due to the teacher herself.

EPS3: Processes richness:

a. He has a phenomenological approach to the exponential, f.i. growth rhythm , that doesn't need real numbers, but you need real numbers in the middle.

As in the case of π, he tries to pose interesting problems in which transcendental numbers emerged, posing problems to the previous knowledge of irrational numbers.

Nevertheless it's problematic the way the teacher pass from this 'phenomenological approach' to the continuous exponential function, as we said before.

b. The introduction of real numbers by mean of "the convergence of that intervals that seemed to arrive there but the convergence was on the void, there was nothing." is relevant. Also the teacher highlight a very important thing, neglected by the other teachers: "So you have to construct something."

EPS4: Representativeness:

The teacher, expect from the introduction of real numbers by mean of "the convergence of that intervals, has a very poor approach to real numbers, reducing everything to the natural continuity and using the blend *Numbers as a set of point* in a total identification with the naturally continuous line.

Cognitive suitability

CS1: Previous knowledge & CS4: High cognitive request

We may say that the teacher's approach is coherent with a very poor expected knowledge. We found a similar approach in Teacher 1, but there is a string difference, that is the cognitive request, very much lower.

The most critical point is the intuitive approach to the passage from natural continuity to limits and continuous functions, but it seems that everything is constantly flattaned on natural continuity.

The previous knowledge is instead otally unsuitable for the Weierstrass formal approach to real numbers he quoted, but it seems evident that the teacher approaches also this topic in a very 'intuitive' way.

The previous knowledge he expects is thus as poor as the essence of his approach to the topic.

Ecologic suitability

ECS1: Adaptation to the national curricula: the contents, their implementation and evaluation correspond to the curricular indications

A very few of the configurations required by the national curricula are presented by the teacher.

ECS2: Intra and inter-disciplinary connections: the contents are related with other mathematical contents (with advanced mathematics and the other curricular contents) and with contents belonging to other disciplines (extra-mathematical contexts or other educational steps)

The teacher showed a string interest for the topic of the interdisciplinarity with Physics, inspired by one of our sentences. We explained it better in the Paragraph about teachers' self-examination.

EC4: Didactical innovation: Innovation based on researches and reflections is taken in account.

No didactical innovations and reflections are taken in account.

Teacher 9 (85)

Formation: Master degree in Statistics and Economy

Years of experience as high school teacher: 27

Studied real numbers: At school

The fundamental properties of real numbers are: in a second class the real numbers are presented as separating elements of two contiguous classes, after the presentation of the existence of irrational numbers with an historical approach using the Pythagoras' theorem.

In every numerical set enlargement (N → Z, Z → Q) there is a "critical operation" involving the elements of the set to enlarge, like subtraction and division, that lead to a new set. How is it possible to construct R starting from Q?: The problem arises verifying that the rational numbers don't permit to exhaust the numerical sets and thus if I reach Q, using the arithmetic mean of two consecutive numbers, to reach the set of irrational numbers I present the Pythagoras' theorem, that leads me to a number that is not necessarily rational.

Is it possible to define a limit point in Q or is it necessary to use real numbers? It's necessary to use R

K2_G: History of real numbers

K2_H: Construction of R

CC2: Numeric – systemic

CC8: Relations between Q and R

IC4) Root square and π (Example of irrational numbers, R is an enlargement of Q)

K3_F: Using √2 and/or other radicals, eventually with a historical approach

K4_B: R is necessary

Teacher U / L (teacher U who chooses only L constructions)

R FOR ALL

G1_A: Exponential function

G1_B: Logarithmic function

G1_C: Differential calculus

G1_D: Integral calculus
G1_G: R - intervals

G1_H: Limits

G1_I: Equation systems

R BEFORE ALL

“Limits” but not “Sequences and series”

O15: Consider the first video as a good tool [L] [3.1]

O15_A: Help the students to understand real numbers

O15_D: These images will be very useful when the students will have to learn to solve quadratic inequalities

SEMIOTIC COMPLEXIFIERS

O29: It's impossible to see the correspondence in this way

O40_R: The graphic solution is always better; it's necessary

RI2: "Endpoints" inclusion with <
RI4: "Endpoints" inclusion with parenthesis [a,b]

O41_C: Some representations are more suitable than others in specific practices
O41_E: Only some of the proposed solutions are representation of intervals
O42_A: The parenthesis [,] represent inclusion also in the case of rational numbers
O42_B: The domain of inequalities should be indicated only in the case of rational numbers
CIR_2) There is a hierarchy in the intervals' representations
a. The graphic representation is better/more intuitive, more synthetic
CIR_6) The usual representation of real numbers intervals can represent a subset of Q

<u>REPRESENTATIONS FOR SPECIFIC PRACTICES</u>

GSO_4) S**ome representations are more suitable** than others in specific practices

GSO_5) Some signs are not representations since they're not finished processes / **Some signs associated to intervals represent the task and not the solution**

GSO_7) **Different representations are complementary** / the meaning is a result of configurations

D10: ***Osservi il video al seguente link: http://www.youtube.com/watch?v=kuKTyp_b8WIIl video può aiutare uno studente a comprendere la corrispondenza biunivoca tra numeri reali e punti della retta?***
No, perché *no in quanto inutile*

D11: ***Osservi il video seguente, in particolare dal minuto 10:20 al minuto 12:10http://www.youtube.com/watch?v=UEBK5DfPxvk Lei cambierebbe qualcosa nella spiegazione?***
Non la chiamerei soluzione bensì rappresentazione della soluzione in via grafica
D12: ***Crede che sia opportuna la distinzione tra soluzione algebrica e grafica di una disequazione?***
No, perché *no la soluzione viene presentata per via grafica*

D13: ***Quali tra queste metterebbe come soluzione dell'esercizio nel libro di testo? Le n°***
in un biennio va bene la 2, in un triennio è meglio]2,4[
D14: ***Cosa pensa delle soluzioni fornite dagli studenti?***
a. sono tutte ugualmente accettabili *no*
b. le n° .... non sono da accettare come soluzioni perché*4 è errata*
c. è meglio se gli studenti usano tutti la stessa sempre, cioè la n°..........., perché*5 perché è corretta*
d. nessuna è accettabile perché *no*
D15: ***Esprima i suoi giudizi sulle risposte fornite:***
Quali soluzioni sono più adeguate al problema? Perché? *la sol 5 mostra l'intervallo*

c. è meglio se gli studenti usano tutti la stessa sempre, cioè la n°..........., perché *2 è corretta perché siamo in Q*

Didactical suitability

EPS1: Errors:

No wrong practices have been detected in the teachers' sequence.

EPS2: Ambiguity:

The teacher use a completely geometrical approach to rational and irrational numbers in the first two uìyears of high school in which she teaches. Numbers are used to represent magnitudes.

Her knowledge is not advanced but she's coherent in her choices.

The only aspect of ambiguity is the fact that she declared to introduce real numbers as contiguous classes but all inside a geometrical approach, thus quite far from the original one. Her meaning of contiguous classes is to be interpreted in the Aristotle's meaning of contiguous, thus in a very lower level of generality and complexity.

This could be misleading and provides us an information to interpret some reseraches' results: the teachers who are used to teach in the last three years may read in the programs that the students have studied R as set of contuguous class but the meaning is eveyrthing but the epistemic one. Such a simple configuration – contiguous means close, have an extreme in contact with the other – can't support high levels' systems of practices and maybe thi is the reason why the teachers always report that the students seem not to know anything about real numbers even if they studied them in the first years of high school, as contiguous classes or Dedekind's cuts. In the bith of the cases the name used for the configuratiion is misleading at all because partial meanings of a totally different nature are identified unsuitably.

EPS3: Processes richness

a. “There are numbers that are not rational, there isn’t a fraction that express them. In Geometry you connect the discourses because you present commensurable and incommensurable segments.”

She connects the number to the geometrical magnitudes, not identifying them trivially.

b. She prove in the Euclidean geometry that between two point there is always a point and that every point has a precedent, without associating numbers to the points: “You have to take in account that in the first two years they study Euclidean geometry. When I introduce the line, you know that there are the axioms. When they are in the first year.

She insert a proof, even if totally geometric, thus the process quality is high.

EPS4: Representativeness

Only a very few of real numbers is presented but maybe this is sufficient for the scholastic level in which she teaches.

It's interesting the fact that she higlights the historical relevance of the Greek's theory of magnitudes and the fact that it's surprising the quality of their mathematics considering the epoch in which it has been created.

Cognitive suitability

CS1: Previous knowledge:

Coherently with her choice, the teacher proposes practices that are sustainable for the students.

CS4: High cognitive request:

The sequence is not ambitious from this point of view.

Ecologic suitability

ECS1: Adaptation to the national curricula:

The sequence is coherent with the national curricula even if other relevant configurations should be introduced.

ECS2: Intra and inter-disciplinary connections &EC4: Didactical innovation:

No innovation based on researches and reflections or interdisciplinary connections are taken in account.

Teacher 10 (84)

Formation: Master in Mathematics and Training courses for Mathematics and Physics high schools teachers

Years of experience as high school teachers: 15

Studied real numbers: At school, at the University in a course of Calculus and in a teachers’ training course

The fundamental properties of real numbers are: the real numbers contain the neuter element for multiplication and addition.

In every numerical set enlargement (N → Z, Z → Q) there is a "critical operation" involving the elements of the set to enlarge, like subtraction and division, that lead to a new set. How is it possible to construct R starting from Q? In my classrooms I present real numbers starting from geometrical constructions of not commensurable segments like the diagonal of a pentagon constructed starting from the golden ratio of its edge and the construction of the golden rectangular.

Is it possible to define a limit point in Q or is it necessary to use real numbers? Yes, you can define it.

<u>*Categories and significant sentences in the questionnaire:*</u>

K2_B: Algebraic structure of R

CC4: Algebraic – operations properties

K3_Z: don't remember exactly how real numbers are constructed starting from Q

K4_A: Q is sufficient

R FOR ALL

G1_A: Exponential function

G1_B: Logarithmic function

G1_C: Differential calculus

G1_D: Integral calculus

G1_G: R - intervals

G1_H: Limits

G1_I: Equation systems

Q SUFFICIENT & R NECESSARY

"Limits" but not "Sequences and series"

O15_A: Help the students to construct a good image of real numbers

O15_C: These images will be useful in the future

SEMIOTIC COMPLEXIFIERS

O29: It's impossible to see the correspondence in this way

O38_C: The solution presented is not the graphic solution (a procedure) but instead a graphic representation

O40_A: There are not two solutions but only two representations of the solution

RI2: "Endpoints" inclusion with <

O41_E: Only some of the proposed solutions are representation of intervals

O41_F: Some representations are not "finished"/are the problem and not the solution

O42_B: The domain of inequalities should be indicated only in the case of rational numbers

O42_F: The usual representations of intervals of real numbers can't represent intervals of rational numbers

D10: ***Osservi il video al seguente link: http://www.youtube.com/watch?v=kuKTyp_b8WIIl video può aiutare uno studente a comprendere la corrispondenza biunivoca tra numeri reali e punti della retta?***

- ***No, perché*** *No, mi sembra inutile*

D11: ***Osservi il video seguente, in particolare dal minuto 10:20 al minuto 12:10http://www.youtube.com/watch?v=UEBK5DfPxvk Lei cambierebbe qualcosa nella spiegazione?***
Non la chiamerei soluzione, ma al limite rappresentazione grafica della soluzione

D12: ***Crede che sia opportuna la distinzione tra soluzione algebrica e grafica di una disequazione?***

- ***No, perché*** *No, la soluzione è unica ed è un sottoinsieme di elementi dell'insieme considerato per le quali la proposizione è vera. Possiamo scegliere di rappresentarla in vari modi, ma è sempre la soluzione*

D13: ***Quali tra queste metterebbe come soluzione dell'esercizio nel libro di testo? Le n°***
La due è la disequazione equivalente che comparirebbe nel libro di testo

D14: ***Cosa pensa delle soluzioni fornite dagli studenti?***

- ***a. sono tutte ugualmente accettabili*** *La risposta 5 è l'unica che rappresenta la soluzione anche se per via grafica, le altre sono disequazioni equivalenti*

D16: ***Quali tra queste metterebbe come soluzione dell'esercizio nel libro di testo e perché? Le n° ....***
se le soluzioni devono essere date in Q la soluzione 2 potrebbe essere la scelta migliore se si specifica che x appartiene a Q

Didactical suitability

EPS1: Errors:

No wrong practices have been detected in the teachers' sequence.

EPS2: Ambiguity:

a. She introduces the line, the diagonal at the unitary square and the Pythagoras’ theorem using “the roots to enlarge Q, to say that other numbers exist” and she doesn’t go on” or sometimes she introduces real numbers as separating elements of contiguous classes, depending on the students' knowledge quality.

The first approach have been already commented in Teacher 1's interviews analysis.

The second approach could be good consistently with the suitable that are not quoted by teacher so globally we consider this choice ambiguous.

b. To introduce the exponential functions “You present 2 raised at N, 2 raised at Z, 2 raised at Q and then you give 2 raised at √2 and you explain that this becomes a matter of separation between one and the other, the way you can define it.”

As we did for Teacher 8, we refer the analysis of this choice to the analysis of the interview of Teacher 11.

c. She introduces the density of the line saying there are not consecutive numbers

The absence of consecutive number in Q is not related to practices involving the points of a line, thus this introduction requires a good knowledge of the blend of decimal numbers or fractions and the point of a line, that is not trivial.

d. To represent the correspondence between points and numbers she traces the segment and stop “because it’s intuitive and as she were saying you have to start from the Geometry, that was the crisis”.

As we stressed in the analysis of the answers to question 10, for several reasons this choice is misleading.

EPS3: Processes richness & EPS4: Representativeness

The teacher pursue the goal of siplifying as much as possible, thus she avoids all the high cognitive processes but the proof of the irrationality, that we already commented.

The processes are almost poor and globally the sequences is not representative of the complexity of real numbers and the continuum.

The only relevant remark is the difference between the representation of the line with computer and with the pen: in the first case you see the pixels wjìhile in the second case the trace is continuous.

Cognitive & Ecological suitability

CS1: Previous knowledge & CS4: High cognitive request & ECS1: Adaptation to the national curricula & ECS3: Didactical innovation

The teachers' profile is very similar to Teacher 8 so we can identify them using this markers.

Teacher 11 (114)

Epistemic suitability

EPS1: Errors: There are no wrong practices in the teachers' sequence.

EPS2: Ambiguity

a. The teacher introduces real numbers in order to complete the exponential function adding some irrational numbers to the domain in order to reach the natural continuous function.

We will use an excerpt of the teacher's interview in order to comment on this practice because the teacher realized talking with the interviewer that the practice she proposed usually was more complex than she had thought until that moment:

> T: “[talking about sequences, limits and asymptoptic behaviour] These are conceptual nodes that should be approached with the accuracy of a teacher trainer. I'm now thinking to the sequence, this is more spontaneous...”
>
> I: “Maybe they would not represent it graphically.”
>
> T: “Graphically maybe they would trace the parabolic curve continuously.”
>
> I: “It would be interesting to ask them.”
>
> T: “If you think at it they have studied this before, think at the primary school, and this is natural concept. Is natural the extension or the restriction? Now they think at the restriction as not natural, while the construction of the exponential function....”
>
> I: “This could be a critical point, the dialectic relation between extension and restriction.”
>
> T: “Is there a natural direction? The path of the exponential function is an expansion.”
>
> I: “How do you present it?”
>
> T: “I work on the extension by continuity.”
>
> I: “Should we go from discrete to continuous or from the continuous to the discrete?”
>
> T: “I go from the discrete to the continuous.”
>
> I: “But here [I indicates the naturally continuous graph the teacher was drawing in the meanwhile] you go in the backward.”
>
> T: “Here I miss something about the real numbers... I tell you how I did it.. I'm going to do it now in my third class. I reconsider the scholastic history of powers. First of all we consider the variable at the exponent, then we recall the first misconception, i.e. power is multiplying a number by itself n times. Then I ask them the properties and to prove them. I focus on 2 raised at 0. What does it mean? A student once asked me: < My teacher explained to me that I multiply it 0 times, then I mustn't write it down, but since we're working with multiplications when we have nothing, it's 1, the neutral element...”
>
> I: “Sometimes a teacher can explain something to herself and then explain it to the students in the same way. Maybe sometimes I do it teaching Physics [I laughs]”
>
> T: “Me too. [T laughs]. I say here two things concerning powers. We have multiplications, the properties and so on. We reach a point in which we have to renounce at this. I miss here which is the exigence to do this, because you finalize it to study the exponential function in the real domain!”
>
> I: “I think that if you divide a number by itself to conserve the formal properties you subtract the exponents and obtain the conventional expression 2 raised at 0.”
>
> T: “I explain to them that to save the properties the mathematicians were forced to abandon the original conception. Paradoxically you define the exponential functions by means of the properties you want it to have. You transform the input value. ”
>
> I: “You build what you want it to be and then you use the properties as a definition.”
>
> T: “Paradoxically this become its definition. The exponential function is.. to the input sums.. that machine that giving sums returns products. Once understood this they don't forget the properties of logarithms.”
>
> I: “I never saw the exponential this way.”

T: “This becomes the new definition because you define this but what is the sense? We do this to conserve that properties. The properties says to us that we can interpret 2 raised at -n as ½ raised at n.”

I: “Are you saying we need a structure?”

T: “Exactly. We renounce to the nursery rhyme: < A power 2 raised at n is the multiplication...> but you have the features we want. Here returns the concept of extending by continuity, even if it's another kind of continuity.”

I: “Maybe the extension by continuity is strange for them because they never extended by continuity what was obtained by means of this process. I take 2 raised at 0, 3, ½, and so on but these are discrete numbers. To trace the line you have to..”

T: “You need the real numbers”

I: “You have to add values that makes it continuous.”

T: “I say <Let's call..>, wait, I'm thinking now. This lead you to 2 raised at x with x rational, but the passage to rational to reals, this scheme cracks.. the powers' properties help you until x rational.”

I: “It's analogous to what Dedekind said about the fact we have to invent numbers. I put here a real number, I have 2 raised at x, but if x is unknown, you invent something that makes continuous the functions.”

T: “I like so much this idea of inventing since its trace remains in the moment in which we have to write that numbers, because we use symbols. To express the irrational numbers you have to use symbols, I can't calculate them in their entirety. For instance root square of 2, Pi, 2 raised at Pi.”

I: “In this moment a teacher could read Dedekind, when he wrote that the construction is a free act of the thought. Also a mathematician talking about irrational ...”

T: “What is not obvious is making them aware that here is a hole! The density of Q makes it not obvious. Overall when you want to represent an infinitesimal visually … it's not easy. I travel again through the powers' properties, they drive me until here, but I say. I know that also 2 raised at root square of 2 exists, so how can we give sense to this?”

I: “Do they know that root square of 2 is not rational?”

T: “Yes, I prove it many times.”

I: “You say: < Make it full>. You put there 2, and then how do you extend it by continuity?”

[...]

I:. “You could ask them to complete a function in an interval rather than in a point. They could understand the need of adding points to complete by continuity but also associate a value to a limit point. If you are on the rational... you create a critical situation in which to have the definition of continuity as limit is necessary.”

T: “**Also if you complete it by continuity you're taking for granted it's continuous innately. What does it mean to complete by continuity? It's a chaos. We have to rethink the previous paths because you arrive..**”

I: “You have a function presented like this and ask them to complete it by continuity.”

T: “It would also be nice.. Wait, now I tell you which my exigence is. First of all to investigate their image of continuity, so this could be an exercise: I give three images and you say if it' continuous and why, because the mathematical continuity is also this one [two pieces defined o intervals]. Do you bet they think that this is continuous? [one pieces]. In the assessment of the next week in the fifth class next week I'll put one question of this kind. Here the function is defined. F is continuous on R. What of these graphics are acceptable? In my mind this [cuspid] is not associated to continuity because they associate it to be smooth, with a unique equation... In fact they take for granted that.. they have a concept of similarity added to not to interrupt the trace.”

I: “Maybe in their mind the function stops here and restart there ...”

T: “The word continuity is used very soon, also in the previous years. We have to investigate where you name continuity.”

In this excerpt it's explained clearly which is the problem and how the teacher recognize that something doesn't work. She understand that we play a metaphorical game when we complete functions by continuity. Indeed we don't complete anything but we suppose the functions naturally continuous in a domain (the curves as traces of points-position moving along a path) and then we identify the naturally continuous domain – the segment – with the blend *The point of the line are sets of real numbers.*

To avoid to work metaphorically is certainly an ambiguous choice.

EPS3: Processes richness:

The teachers' sequence is very rich from the point of view of high processes, focusing in particular on the relation between continuous and discontinuous entities, also in relation with Physics lessons. She's also aware of the fact that static configurations are not equal to the dynamic ones: “you are selling dynamical concepts with a static theory”.

a. Many concepts we need in the Calculus are introduced before, also in other disciplines, in particular Physics

b. She define formally the continuity of functions and explains that functions defined on isolated points are obviously continuous

c. She introduces limit processes in Physics before introducing limits and the Calculus

d. She presents parametric functions, with a parameter that can make a function discontinuous in one point: “The line with a hole always comes out... when you study the hyperbolic curve with parameters <For which values of k..? The classical homographic function in which you are asked to say what you obtain varying the parameter.”

e. She introduces the sequences. Would like to ask the students if a sequence is continuous or not.

f. She proved many times before that $\sqrt{2}$ is irrational

g. She introduces limit points and isolated points both formally and intuitively.

EPS4: Representativeness:

While they processes activated in the sequence are significant, a lot of relevant partial meanings are not declared by the teacher, like a reflection concerning the philosophical issues and the formalization of real numbers.

This is not a carelessness or lack of knowledge but the teacher is convinced that historical debates may discourage the students. This orientation is based on a previous experience, that the teacher reported:

T: “Analysis standard and non-standard. In Physics! For the precision you need that the infinitesimal is measurable, but for the precision of your final result it's more precise when it tends to 0.”

I: “This is the point, but there it was approached from the philosophical point of view.”

T: “It would be interesting because you are selling dynamical concepts with a static theory.”

I: “This is crucial, the discussions are indeed about dynamic/static, concrete/abstract (atomic approach), exist/not exist (if it's null, it doesn't exist, but if it doesn't exist it's not bigger than 0). The ontological issues may interest students.”

T: “The history maybe a support for their trust. I said to a student of mine: < We introduced in these two months concepts that lead to discussion … sometimes I say this to encourage but sometimes I have the opposite reaction, like : < If they didn't understand, how can I understand?>. It's a double-edged sword.”

In further meetings she reconsidered this position and she turned to a more problematic appraoch towards the role of historical sources.

Cognitive suitability

CS1: Previous knowledge:

The previous knowledge wasn't mentioned very much. Nevertheless participating in a teachers' class we could verify that her students are used to discuss about matters concerning the formal approach to the Calculus and intervent in a suitable manner, thus the teachers proposals, indeed high from a cignitive point of view, may be considered consistent with her previous work.

Nevertheless some ambiguities about the relation between natural and formal continuity of the domain of real functions in the real domain emerged. Once again we report an excerpt from the interview:

I: “To make them understand why we don't consider whole intervals we could also provide example of condition in applied problems in which there are constraints. You can wish to consider the function in a restricted domain. Otherwise why do you exclude a point? Think at your student's question this morning. <Why should we take off a point>?”

T: “Yes, in that case I was exemplifying a function .. motivate choices about the domain. ”

I: “What could be other problems? You were saying, to present function that has a domain without some points.”

T: “They today asked me another thing, I took it for granted. If a function is continuous in a set, is this continuous also in every subset? One of my students... I had highlighted so much that if you add points to the domain this can become continuous....”

CS4: High cognitive request & ECS1: Adaptation to the national curricula

The cognitive request is very high concerning the formal aspects of continuity, that she treats with continuous domains or domains with a hole in the same discourses. As we already stressed, the philosophical and historical issues are neglected in strength of an orientation about the potential unsuitability of such a choice.

Ecologic suitability

ECS2: Intra and inter-disciplinary connections & EC4: Didactical innovation

The teacher is aware of innovation based on researches and reflections but in this case her knowledge is so deep and advanced about the Calculus that she seems to follow more her own path rather than taking care of didactical innovations, while she is very attentive to interdisciplinary issues and the didactical implications emerging from the research in Physics education. It seems that she tends naturally to problematize her approach to issues that she considers more problematic for her and to be more sure of her choices in the field she knows better from a mathematical point of view.

4.8.1.3.1 Resume of remarkable relations between profiles and choices

Teacher 1

The teacher is a PhD teacher and attended teachers' training courses.. Her knowledge about real numbers and the continuum is advanced and in the questionnaire emerged her good mathematical resources.
The analysis of her didactical knowledge was more problematic since she, moving to didactical orientations, resulted to belong to the category *Absolute meaning,* in particular *No need of mediation*.
This category is relevant to interpret the didactical suitability of her declared choices, that is critical merely from the cognitive point of view. Also this orientation affected her choices also from the point of view of ambiguity and processes richness, because she "jumps" from grounded meanings to very general expected meanings identifying the configurations in strength of the orientation that complex meanings are innate and intuitive. She doesn't take care of the connections between the configurations and her choices are based on a poor didactical knowledge, compared with the literature review and the cognitive analysis.

Teacher 2

The teacher has a Master degree in Mathematics, 18 years of experience and attended teachers' training courses. Her answers in the questionnaire were always oscillating between mathematical and didactical polarities even when she was asked to answer mathematical questions. She resulted a *semiotic complexifier* and introduces *R before all.*
In the interview her choice resulted to be very equilibrated and she showed to be aware of the necessity of connecting different domains, in particular the geometrical and arithmetical one.
Even if her teaching declared sequence is not very ambitious, it's substantially suitable from the cognitive point of view, except for the choices concerning the density. Thinking "didactically" she declared to consider very much the previous knowledge and to adapt her choices to the students. When she talks about density and infinitesimals she uses concrete examples and refers to the limitation of the pencil. Trying to simplify things she creates potential difficulties since the literature review highlighted the recurrence of wrong and persistent finite and discrete models of the line due to such a choice (necklace model) that makes harder, and not simpler, the conceptualization of density.
Her didactical knowledge appears to be based on personal experiences and conception rather than on research works.

Teacher 3

The teacher has a Master degree in Mathematics, attended teachers' training courses and studied real numbers on his own using original sources. He's also expert of the relations between Mathematics and Music and was invited as a speaker in many conferences about the topic.
This passion for the topic music and mathematics resulted to be very relevant to his profile since he deepened the number theory merely to understand better the musical scales, that he uses also as a didactical expedient.
His declared choices are very interesting and suitable quite from all the points of view, being his sequence ambitious but equilibrated, results of reflections and longitudinal design, interdisciplinarity with Philosophy, attention to the coordination and connection between configurations and also to the problematic metaphor that underlie the blend *The line is a set of real numbers*. Also he's aware of, and problematize at school, the snares that affect the relation between continuous and discontinuous entities, between sequences and continuous functions. He declared to use Geogebra in order to work on different domain at the same time

and reported tasks proposed to the students, not only frontal lessons. He showed a cultural approach towards Mathematics teaching and learning that, standing on our *a priori* analysis, should be highly suitable. In the questionnaire his profile was yet complex since he belonged to the categories: *Intuitive simplifiers and R before all* but also *Semiotic complexifier.* In the interview we could understand better which equilibrium there was between the three categories: the teacher have always in his mind the cognitive suitability, avoid the formal aspects of real numbers as much as possible, since he's convinced it's not useful, and this approach lead them to prefer intuitive approaches, but not ingenuous or trivial approaches from a semiotic point of view. On the contrary he uses copiously the "play of the representations" precisely in order to make the topic intuitive step by step (secondary intuition, Fischbein, 1987). He declared to introduce *R before all* in a particular sense: he intentionally choose to avoid the formal meanings and to deal with intermediate configurations, that are compatible with his didactical goals. He doesn't confuse complexity and formality and is thus not afraid of facing complex systems of practices.

Teacher 4

The teacher is a PhD teacher and attended teachers' training courses.
He's knowledge about the topic is advanced, but he's convinced that in the high school the real numbers should be introduced in a "*pseudo*-mathematical way", i.e. giving just some very intuitive images that the teachers knows to be consistent with the true mathematics. He chooses to avoid completely the formal dimension but, conversely, he chooses to try to avoid to enter the historical issues and to simplify as much as possible. His choices are very traditional even if he's aware also of epistemological issues and are thus potentially affected by almost all the "traditional errors" that we investigated the literature review.
In the questionnaire he resulted to belong to the following categories: *Q sufficient & R necessary* (the category of teachers who knows that R is not necessary to introduce limits points and sequences but decide to use anyway always R in the didactical practices); *Semiotic complexifier*; *Absolute meaning,* in particular *No need of mediation*. In light of this fact we can interpret *a posteriori* her profile this way: the teacher is convinced that the students can't understand very much of real numbers in the high school also that there are two truths, the 'pure mathematical' and the 'operational'; he knows that sometimes he could avoid to use the line to represent numbers and introduce limits and continuous functions but he thinks that "real numbers are imagined as the real line, with an abuse of language, that in my mind makes a few damages at this level but may have many advantages"; being convinced that some representations don't need to be interpreted at a scholastic level, i.e. may be only used without being aware of their complexity he uses representations like decimal numbers, the line, the roots only in order to make operations with them. It's confirmed by a sentences: "to be honest, also in the first year, you say <rational numbers are the periodic numbers, the irrational numbers are the aperiodic numbers, but what are they exactly? You suppose that it's meaningful to talk about infinite sequences of digits.. […] A way to introduce them in a sly manner is as decimals.. they're sequences, better, series, but we don't say this to the students! Some of these series have no limits"
While from the point of view of mathematics advanced knowledge he's very different from Teacher 8 and 10, in the end, as long as Teacher 1, their choices are quite similar.
We can state that from our investigation it emerges that a difference in the mathematical knowledge, even very significant, doesn't imply a difference in the teachers' declared choices and thus that 2 PhD teachers and 2 'Not PhD teachers' are impossible to distinguish from the didactical point of view. It means that other variables are significant.

Teacher 8 & Teacher 10

We can comment together the answers and the profiles of Teacher 8 and 10, because they are very similar.

The both of them are aware that there is an advanced mathematics concerning real numbers but the both of them thinks that it's useless at school; deepening more we discovered that they had not very good experiences as students learning real numbers at the University in a formal way. They remember the formal aspects of real numbers as something too difficult to grasp and formally too confuse, that they forget continuously and introduce sometimes to the smartest students only as a duty. This situation remind to us an historical sentence introduced by Dedekind in the introduction to *Continuity and irrational numbers* and a theoretical tool we presented in Par. 2.1.
Let's start from the history:

"Chiunque possegga il così detto buon senso può comprendere questo scritto; esso non richiede affatto particolari cognizioni matematiche o filosofiche. Ma mi rendo bene conto che alcuni dei lettori difficilmente riconosceranno nelle immagini oscure che io farò sfilare dinanzi a loro quei numeri che li accompagnarono durante tutta la vita come cari ed intimi amici." (Dedekind, 1872).

In mathematics education what happened to the teachers may be read in terms of conflict between the concept definition and the concept image, that impedes the learners to create a concept image definition but rather lead them to consider the formal dimension as extraneous and useless (Tall & Vinner, 1981).

Also Teacher 3 and Teacher 4 felt annoyances dealing with the formalizations, but in the previous cases their relation with R had not been so traumatic to avoid it as much as possible. In this case this deny of the formal dimension lead the teachers to avoid completely complex approaches and to get rid of them in the most pain-free way. Their proposals are all concerning natural continuity and even when it would be necessary to go on they decide to simplify the problems. Nevertheless they decide to use quite always R, but it's to understand that this choice is due to the fact that talking about the numerical configurations of R necessarily things become more complicated.

In the questionnaire T8 had been classified: *R for All, Intuitive simplifier* and *Absolute meaning,* consistenly with this analysis. The profile of T10 was more complex: she had been classified *R for All,* but also *Semiotic complexifier and Q sufficient &R necessary.* We interpret the relation between her profile and her choices this way: the teacher knows some elements of advanced mathematics concerning real numbers and is convinced that in general the students should use many representations and that the teachers should "present it also in order to .. from the view of the culture that is enlarged, of the mathematics that is in evolution, things that are hard to understand because they need definitions that make the students struggle, that also make us struggle, in this sense. Then to go into the deep sense precisely we risk this kind of things to happen … we risk these things to happen if you go too much in depths. They don't have the tools yet". She feels the duty as a teacher to give a cultural dimensions to the study of Mathematics, but in this case she lives again her bad experience and avoid to deepen, justifying her choices with the attention to the cognitive suitability".

Teacher 11

Teacher 11 is a PhD teacher and attended teachers' training courses. She also studied Mathematics education in a further Master. She has the most advanced knowledge in Analysis because this was the core discipline of her PhD. She proposes to her students very significant activities, also dealing with the formal dimension of continuity of functions and sets. She focuses merely the relation between continuous and discontinuous entities, talks about sequences and functions, isolated points, limit points and domains "with holes". In her interview she realized that completing continuously functions is not exactly what she does adding irrational numbers to the function domains – in the particular case the exponential. Maybe she never had the occasion to reflect in depth on this topic. It's interesting what happened in her interview because the teacher, answering provoking answers about the topic, realized on her own that something didn’t work and immediately thought to re-consider her position and to modify her choices, without looking for justifications. The differences between this teacher and the other PhD teachers are:

a. a very advanced knowledge in Analysis
b. a good relation with the formal meanings of real numbers and continuity
c. a background in Mathematics education.

These features makes her very flexible to changes and favorably disposed towards innovations in her teaching-learning sequence.

4.8.1.4 Overall categorization of teachers’ declared choices and profiles

Thinking at the first categorization based on the written answers and taking in account that creating teachers’ global profiles had been an hard work in the first phase, we faced the attempt to find overall categories in a problematic way, opening ourselves to the possibility to find only a few significant categories but on the other side, to find in the *case studies* “exemplar teachers” rather than teachers profiles, that are relevant from a theoretical point of view. The quantitative representativeness of the profiles corresponding to the “exemplar teachers “could be investigate in further studies.

PQ - 1.7 Was any categorization of teachers' profiles possible? If yes, what categories emerged from the investigation?

First we report the recurrences in the teachers’ answers:

1. Only 2 teachers (1 and 3) mentioned that the relation between the line and the real numbers is not an identification but rather that to represent R with a line is an operationally comfortable choice;

2. No teachers mentioned the postulate of continuity as necessary to identify the line and the sets of real numbers (4.1 and 4.2);

3. Only 4 teacher declared to introduce R as the set of contiguous classes, but 2 of the teachers admitted that it’s just a matter of duty but they never use formal properties or formal representations; 3 teachers of these stated that the “problem *par excellence*” that make necessary to introduce R is the graphical representation exponential function as a continuous curve. Indeed they present the exponential as a power with natural exponent in the beginning, giving to it a “sense” (i.e. referring them to elementary actions by means of grounding metaphors) and then going on “adding” elements to the set of the possible value assumed by the exponent in order to reach the continuous curves they propose as the only possible representation of exponentials. Here to make sense of expression like $2^{\sqrt{2}}$ they introduce the contiguous classes. Teacher 11 interacting with the interviewer realized that it’s impossible to complete the curve adding elements.

4. All the teachers declared to start from the introduction of the irrational numbers, and in particular $\sqrt{2}$, in order to explain why it’s necessary to enlarge the set Q; however only 1 teacher declared to explain in depth what are rational numbers also from a geometrical point of view and to clarify that on the line we have not only geometrical numbers;

5. All the teachers, asked in a provoking way to renounce virtually to use real numbers in the high school, answered it’s impossible; going more in depth the most of them justified the necessity to use real numbers referring to Pythagoras’ theorem, equations and/or to continuous functions. It’s important to remind that in all the cases the field of real numbers, in the epistemic sense, is not necessary: in the first two cases algebraic numbers will be sufficient, while in the second case all the teachers but 3 referred to natural continuity.

Thus the teachers talking about real number declare to identify very often the set of real numbers or with subsets of real numbers that has the property not to be rational or, when they try to represent the whole set, to use a segment produced by a continuous motion.

6. The systems of practices proposed by the teachers globally were very traditional, but with interesting exceptions, that we will comment in the following points. Generally the reasons that are common in all the declared teaching sequences are:

a. To enlarge numerical sets in order to create a set that could contain all the operations results and the solution of equations.

b. To present the problem of irrationality using a blended approach that identifies the diagonal of the square and the "number" √2 and proving it using arithmetically arguments

c. To complete continuously the curves associated to algebraic or analytical expressions in form of equation

d. To avoid to work with discrete or dense sets that are not complete solving inequalities

e. To explain what does it means to approach a point in the limits' computation

7. Only one teacher declared to avoid to represent Q on the line and to introduce the continuous in terms of discrete terms or as a "completion" of dense sets.

8. In the most of the teaching approaches the segment as a natural continuum appears suddenly as a representation of subsets of real numbers or as result of dynamic continuous phenomena, without a problematization concerning the relation between the two. Also the differential-topological partial meanings are introduced quite suddenly with a total identification with numbers and points and points and positions. This is not suitable from a cognitive point of view as we stressed many times in the research framework. We can state that, with the proper exceptions, the different configurations seemed simply to be" juxtaposed" , "identified" or sometimes "superposed" rather than being articulated or connected by means of metaphorical mappings and semiotic functions. We think it's not casual that the difficulties reported by the teachers are all referred to configurations that are suddenly introduced without a declared connection between the two.

We report here two examples in order to clarify our position:

1. Teacher in the focus group 5-6-7

> I: "What are the students' greatest difficulties with real numbers? Why don't you introduce them as contiguous classes, as complete set, and so on?"
>
> T3: "Many times the concept… the concept.. the greatest difficulties they have is with the concept of infinitesimal quantity. To understand that if I take 1 and 1,0001 I find between these two numbers an infinity of numbers. So when … to understand that … if I take two numbers, between those numbers I can find an infinity of numbers .. The concept that a part of an infinite can be again infinite. For instance the concept that has already been introduced in the first two years that a segment contains an infinity of points. When I say that a segment contains an infinity of points it's OK, but when I take 2 points and I draw on a blackboard two very close points they don't understand that.. they have difficulties in understanding that between two points that are so close there are infinite elements."
>
> I: "You work with a graphical representation, is it right?"
>
> T3: "Also, you give also a theoretical justifications, but what impress them more is the graphic representation."
>
> I: "And in the graphic representations they have more difficulties."
>
> T3: "Yes. So many times to understand such a thing there are the computers, if I draw two point on

Geogebra, if I zoom the page they see that these two points are close, if I reduce or enlarge, between them there is already room. You take the points that are in the middle and I reduce more and more there is always a room so in theory between these two points there is always something. Then Achilles and the turtle…"

I: "Is this accepted by the students?"

T3: "ehmm… yes.. but in the moment I turn out to do it graphically.."

I: "When they study the Calculus, do they recall these concepts?"

T3: "Yes, indeed I talk merely for the Calculus, because the concept of limit, the concept of infinitesimal, all these things.. for instance I say that in the limit 0 is not 0 really."

[…]

T3: "We always work with real number but with a quite intuitive approach, because if you approach it from another point of view…"

I: "Not formalizing.. Ok. Maybe some problems may emerge you have to take for granted that in a neighborhood of a point there are infinite points.. "

T3: "It's possible to do it, with a recurrent procedure… it's quite understandable. Even my son understood it! You can also use the intervals.. "

I: "How? The interval may also be critical, since we should introduce it only in R. But if we have to enlarge Q we try to represent Q-intervals and to construct R starting from Q, that is different from adding some irrational elements."

T3: "We are talking about students of the fourth or fifth year [K11-12], in which the concept of real numbers is already known."

9. As we have already said, Lakoff & Nunez (2000) didn't face the didactical problem of teaching-learning real numbers as a main topic; nevertheless in many occasions the authors committed themselves with pedagogical problem. In a precedent work (1999) in particular Nunez hypothesized which could be the teachers' approaches to the theme of continuity considering the fact the Weierstrass' discretization seemed to permeate so much the mathematical culture that we are not able to identify it, considering it "natural". We compare here in particular one of the sentences, in which a pedagogical problem was presented:

"[..] students are introduced to natural continuity using concepts, ideas, and examples which draw on inferential patterns sustained by the natural human conceptual system. Then, they are introduced to another concept – *Cauchy-Weierstrass continuity* – that rests upon radically different cognitive contents (although not necessarily more complex). These contents draw on different inferential structures and different entailments that conflict with those from the previous idea. The problem is that students are never told that the new definition is actually a completely different human-embodied idea. Worse, they are told that the new definition captures the essence of the old idea, which, by virtue of being 'intuitive' and vague, is to be avoided. This essence is usually understood as situation-free, that is, independent of human understanding, social activity, and philosophical enterprises."

Thanks to our investigation now we can analyze this well posed problem standing on our empirical results and analysis.

We partially confirm it, but what we observed in our research also disconfirm some statements.

Let's start from the common elements. The "discontinuity" between natural continuity and *Cauchy-Weierstrass continuity* appeared in a very consistent measure also in our interviews and is one of the elements that leaded us to consider partially unsuitable the teachers' declared choices from the cognitive point of view. Also we observed sudden switches from one configuration the other in a process of identification or substitution of the two meanings, that from a cognitive point of view open dangerous scenarios since even the underlying grounding metaphors are different. The reasons why the teachers switch from one to the other have something to do with the intuitiveness.

However the dynamics that lead the teachers to move from one configuration to the other are much more complex.

First of all in our investigation it emerged the recurrent presence of a double-way path from the natural to *Cauchy-Weierstrass* continuity and backward: the teachers don't leave so easily the configurations they consider intuitive, both for a matter of cognitive suitability and because they feel better dealing with natural continuity also when it would be necessary a more formal approach, that is instead omitted or reduced to particular cases.

All the teachers we interviewed more or less expressed the same annoyance to use real numbers and the continuum formally. The authors analyzed the continuity of functions but analogous discourses maybe easily transferred to real numbers since the reason why the teachers face the topological properties of the line, the limits, the density is precisely the fact that they want to talk about continuous functions using a numerical domain for the analytical expressions of the functions themselves. Some teachers stated that the approach was vague but not to enterprise a new path towards a formalization, rather to be "honest" from an advanced mathematically point of view. We report some example to support this argument:

Also the passage to formal continuity is usually not presented as "the new definition [that] captures the essence of the old idea, which, by virtue of being 'intuitive' and vague, is to be avoided", but on the contrary the trend is exactly the opposite one: the formal approaches to continuity are presented quite reluctantly and only when they strictly necessary.

Another important remarks is necessary in order to decline the pedagogical problem in an authentic didactical problem: some teachers weren't convinced at all that the pure essence of continuity was the formal one, but on the contrary they relied on a natural approach to continuity as the most authentic one.

10. 7 teachers "justified" their choices of simplification or their renounces (use of partial meanings) referring to potential or experienced students' difficulties, that the most of them consider absolute and not depending on the specific sequence of practice they choose or an oversimplification. Just 1 of these teachers - who hadn't answered the previous questionnaire – discussing in the same focus group about the students' recurrent difficulties and realizing that in all the cases, Professional school and Scientific high school, in all their classes the most of the students was used to forget everything about rational numbers, quadratic inequalities and irrational numbers, came on their own to conclude that maybe the problem is an oversimplification that leave the students out of the crucial questions.

To sum up: 7 teachers declared to simplify and to prefer intuitive approaches – whatever it means – in order to pursue the cognitive suitability of their choices, basing the evaluation on personal orientations or/and on their interpretations of previous experiences. The teachers who didn't orient their choices to simplification were Teacher 3, Teacher 7, Teacher 9 and Teacher 11. Three of them (3, 9, 11) had a good formation in Mathematics education since they had attended Masters for teachers in Mathematics education (3, 11) or had attended many teachers' training courses (7). In particular Teacher 3 showed a very wide knowledge of the topic from the historical, epistemological and interdisciplinary point of view – and also a fascination for the topic - that leaded him to face from different points of view the topic and to choose really time after time what to do, avoiding to rely in didactical traditions and proposing the problem from a cultural point of view without losing effectiveness from the institutional point of view. We can thus hypothesize that suitable and culturally relevant teaching sequences involving real numbers and the continuum not only don't oblige to waste time or to create avoidable, unbearable, difficulties to the students but may contribute on the contrary to create a fertile cultural substratum on which we can nurture the students' mathematical knowledge.

4.8.2 Teachers' "self-examinations": questions and data to pose the right problems

We observed during our interviews, since the first pilot focus groups, that the discussions were not only a resource for our research but they became occasions for the teachers to reflect on their orientations and their choices and sometimes also to change them.

Since many teachers had interesting profiles and careers – PhD in Mathematics, Mathematics educations teachers' trainers, teachers who spoke in conferences and so on – it seemed particularly interesting to us that many teachers with a very high profile as mathematicians and high school teachers considered interesting the discussions, engaged in long debates and sometimes stated that they never thought at some topics.

Some interesting things happened:

1) a teacher involved in our pilot study while he was attending the course of Teachers' trainer in Mathematics education discussed a dissertation concerning the real numbers and the reality;

2) a PhD teacher asked us to design with her teaching sequences concerning real numbers, to distribute along all the years of the high school in a consistent manner;

3) some topics of this discussion were discussed in interdisciplinary teachers' training courses concerning the discrete and the continuum in Modern Physics in which the teachers we interview were involved;

4) a teacher organized a focus group for their colleagues to discuss the topic.

We realized that something relevant happened during our interviews and we decided to add a further analysis of the interviews in order to make emerge the moments in which the teachers felt that what we were talking about should imply changes in their way of thinking about the teaching-learning of real numbers or simply felt uncomfortable with something that we were saying or themselves were declaring. This is the reason why we added a further research question:

GQ – 3 Could our methodology of research be useful in the teachers' training concerning real numbers and the continuum?

We collect here some excerpts from the interviews and then we will comment on them.

Teacher 1

1. I:"In which contexts do you introduce real numbers? How do you explain the students why you introduce real numbers and how do you represent them?"
2. T:"I would try … It's hard … anyway quite soon. Not from the first years but very soon. I Think I would introduce them , <u>**very easily, showing that not all the numbers are rational**</u>. I would start from the root square of 2, they has been presented also in the grade 8. <u>**The existence of numbers that are not ratios of whole numbers…. thus a quite historical approach**</u>."

   […]

3. T: "<u>**They have naturally in their mind R**</u>. A student, when thinks at numbers, has in his/her mind a real number, i.e. he/she perceives the numbers in a continuous way, not in a discontinuous manner."

[…]

4. I:"Would you use in the last year [grade 13, nba] some construction of R to introduce the Calculus? Is there something that needs the properties of Q and R?"

5. T:"The mathematization of a primitive intuition. In my classroom this year, in the Calculus..."

6. I:"**Didn't you recall R? Did you find difficulties**?"

1. T:"**No, because it's quite natural. Probably I did many examples**… for instance.. that emerges a bit.. when we talked about limit points, the book didn't propose a definition …. instead I proposed a definition .. we talked for some minutes and since they had studied the sequences I tried to make them reflect about the fact that, while when you calculate the limit of a sequence only to the infinite, when you study a function on the real numbers you calculate limits also in finite points, because the only limit point for N is the infinite, while for R… **But, in the reality, it's a thing that only a few students gather .. the straight line topology is so banal… to be honest no… but it's transformed by the books in a such banal thing** that all [the teachers? the books?] always present all the theorems on the neighborhoods, on the intervals.. because the interval at any rate works… so you don't gather intuitively… The Calculus is so complex for them that, being asked to learn [to use?] new tools, that **a discourse about these things may be very good for a few students** … In fact I presented a lesson about the topology of the straight line, but I had not time enough. But I was very surprised by the fact that in the book there wasn't the definition of limit point.. there was nothing.. it asks to compute the limit, but didn't explain why you have to calculate it, everything is taken for granted. They had talked about contiguous classes, they see the continuum [represents simultaneously a limit point on the line]. **They see all but that .. in the sense.. the problem of discontinuity. For them there is all. This is the reason why I say that I'm sure that for them these are real numbers [traces a segment]** and that, however I take out stuff there other stuff remains here, close, this is sure. **They didn't understand the sense of talking about limit points.**

[…]

2. **There is a strong identification and it's not only a matter of symbols because when I correct it they look at me as a fussy person. It's identified, it's not only a symbol but it stems from.. it's like to say that √2 .. you do this procedure and find an irrational. How can you that all are like √2? The problem is that they don't have the concept of approximation.**

[…]

3. I:"Are you saying that there is a difference between the numerical and the geometrical procedures?"

4. T:"They study Physics from the first year [grade 9, nba], they only work with the calculator, **because in Physics they use sinus, cosinus, logarithms ignoring their meaning, then they take the significant digits, rightly**..."

5. I:"But without making explicit what they are doing.."

6. T:"Exactly! **There should be upstream someone who explain quite soon which is the essential difference between the real numbers and their practical uses**. **My students in the end of the 5 years [9-13 grades, nba] don't know it. I would expect them to know this at least: that they have understood the difference between a real number and one of its approximations.** A lot of them.. it's something subtle for them, but it shouldn't be a subtle thing! This is what should emerge as the strongest thing, i.e. the difference between a real number and its approximations… don't forget that there is an infinity of approximations.."

7. I:"To distinguish the continuum and the physical measures?"

8. T:"**They don't know the great difference between continuum and discrete, interpreted precisely as the number and the approximation**."

Teacher 1 interview is like a zoom into her didactical experience as a teacher. In the beginning she states many times the real numbers are intuitive, are innate. Also she says that the concept of real number is natural, as long as the concept of limit point.

Going forward, encouraged by the interaction with I, she gets more and more aware of the problems the students show at a closer view and in the end she states that her students don't know the difference between a number and its approximation.

The interaction with I made emerge potential problems that can lead her to productive cognitive conflicts or better, in this particular case, to become more aware of the real numbers' complexity.

In fact in the beginning she says she introduces real numbers in a very simple way: using $\sqrt{2}$ an proving it's not rational; then she says that the students think naturally to real numbers as a segment; then she states that the students understand easily the limit points but they don't understand the problem that lead to limit points. After this moment she begin reflecting on other difficulties, stimulated by I, and many problems emerge in the students' conceptions of real numbers, reaching in the end the awareness that the students ignore the relation between Q and R and the concept of approximation.

These observations are deeply linked to the fact that the teacher prefers an intuitive approach to real numbers stressing the graphical aspects. This is in our mind also the main reason why her students don't understand the problem of density, intrinsic in the definition of limit point.

Furthermore it emerges evidently that the teacher uses the term "real numbers" to identify the intuitive continuum and that when she list the problems, encouraged by the interviewer, refers to other partial meanings of real numbers.

The teacher about the same number of times says it's hard to talk about real numbers and that's innate, intuitive, understood only by a few students.

The teacher is PhD in Mathematics and knows very deeply the problem of the definition of real numbers starting from Q, also at a topological advanced level. So we could ask: why does she expect that for the students this is natural? The answer emerges from their words when we analyze them with our methodology: she identifies different partial meanings with the holistic meaning and expects the students to be owner of the holistic meaning while their system of practices involve only some of the lower partial meanings of real numbers, that are impossible to connect in a unique configuration at that lower level.

In particular in a passage it's interesting to see this how this discrepancy is connected to the students' difficulties in line 7:

"we talked for some minutes and since they had studied the sequences I tried to make them reflect about the fact that, **while when you calculate the limit of a sequence only to the infinite, when you study a function on the real numbers you calculate limits also in finite points**, because **the only limit point for N is the infinite,** while for R… **But, in the reality, it's a thing that only a few students gather** **.. the straight line topology is so banal… to be honest no… but it's transformed by the books in a such banal thing"**

They had talked about contiguous classes, they see the continuum [represents simultaneously a limit point on the line]. **They see all but that .. in the sense.. the problem of discontinuity. For them there is all. This is the reason why I say that I'm sure that for them these are real numbers [traces a segment]"**

The teacher in a few instants puts together many different partial meaning, at different levels:

1) The comparison between limits in the discrete and in the dense case [C3.4];
2) The comparison between limits of sequences and functions [C4.1];
3) Infinite as limit point in N;
4) The "topology of the line"[C3.4];
5) The intervals of real numbers as segments obtained tracing [CE4]

Going on discussing another important partial meaning, never mentioned before, appeared:

6) The difference and the relation between approximation and real numbers [CE2, C2.4]

The teacher seemed not surprised, but rather a structure appeared in her orientations: the students have an innate sense of continuity that we can use in the most of the procedures and, even if the students don't understand deeply what is happening, we can go on trusting the fact that they don't perceive the problems that should emerge. To deepen is not useful because the students got more confused rather than understand better.

From our point of view the teachers' choices are due to a high attention to the students' intuitions and to their efforts to understand.

She attributes the students' difficulties to a difficulty intrinsic in mathematics or to other practices to which they are used in Physics, but she doesn't seem to see that no connections and no partial configurations could support what she expects them to "understand".

She declared that they had already studied real numbers and that she didn't work in a explicit manner on partial meanings that then she expected them to understand in a few instants. Maybe her knowledge became so much part of her way of thinking that she considers it innate and considers all the meanings resumed in every representation of real numbers, whatever the problems and the systems of practices from which they emerged were.

In our analysis it emerges that the teacher didn't take in account a suitable way the epistemic complexity and this seems the reason why their attempts to go deeper failed; also it emerges that it's exactly because she didn't present suitable systems of practices that the students were not able to understand what she was trying to explain to them about limit points and about approximation.

It results that in order to pursue a cognitive suitability she tried to simplify limiting everything to the intuitions, but doing this she created a "world" in which it was no more possible to present the right problems that would have allowed the students to understand what she was trying to explain.

In this case the teacher had already reflected on her own on the students' difficulties and had explained to herself why the students didn't understand, choosing to trying to make everything as intuitive as possible.

This teacher didn't change her mind in this interview and remained convinced that R is the line and is intuitive also in other further conversations; in the end when we presented our OSA's analysis she declared that it would have been interesting to try to re-design a teaching sequence that take care of different meanings.

Nothing "local" made her reflect, even if she was in front of inconsistencies (real numbers are innate, limit points are easy to understand, no one understood thus I decided not to go on, the students aren't able to distinguish between a number and its approximation). What lead her to reflect was the presentation of the organized scheme that resumed the complexity of the real numbers.

Teacher 2

1) I:"With the sequences.. how can we represent Q and construct R? It may seem forced but when I draw the segment if there are only point s of Q or R I can't see the difference.."
2) T:"**They're so small that I don't see them. So I could use this representation [traces a segment] and say that they are of Q, like you did here but with x in Q. We don't have a way to represent Q**"
3) I:" Here, if one would suggest the extension from Q to R starting from this point, how could he/she do?"
4) T:"I see it well, graphically it's understandable that there are many real numbers here inside the segment so **this is not exactly Q this one, there's something more. But you can't take it out doing some white small empty spaces so sincerely I should reflect. I don't know how to see another different representation of Q, do you understand?"**
5) I:"Every time we try to represent Q we fall down into the representation of real numbers… but if we want to enlarge Q to R, how can we do?"
6) I:"**In fact.. at this level.. going further the square… this opens the door! The question of the correspondence is very important, it has the same kind of infinite.. it's a bit difficult,, it depends on the classrooms**".

The teacher showed good mathematical knowledge and also attention to the students' meanings and interests, to the historical dimension, to the algebraic and geometrical aspects of the previous meanings of real numbers.

The most critical aspect concerned the way she was talking about the graphic representation of real numbers on the line, since he was stressing the insufficient thickness of the pencil. This kind of concrete discourses can be problematic as we stressed many times in this thesis. Thus we decided to pose a provoking question about the possibility to represent graphically Q and in general to represent Q in a suitable way in order to construct R. This questions immediately lead her to reflect about the representation of Q and R and to say she should reflect more about it, since she realized it's a problematic aspect.

It's interesting to see how quickly it happened in respect if the previous case. Maybe this teachers' orientations were not so strong and supported by an "inner argumentation" as it happened in the previous case.

Teacher 4

I: "What does R is necessary for in this discourse?"
T: "**To be honest a person thinks algebraically in terms of rational numbers but he sees them in terms of real numbers i.e. of the continuity of the line, images them onto the line.** I have always to add that a limit point ... **one sees this approaching to the limit point without reaching it but in a continuous variation. To know what does this continuity mean or its rigorous definition is left..."**
I: "What do you mean with continuity? Passing through, approaching without leaving back holes? Graphically how would you do it? Think about the limit. What is there of R in this representation?"
T: "Mmm.. a function that tends to x0, a limit l that is not a value of the function. You choose an interval that goes from l + epsilon to l - epsilon, [he draws the interval and the axes] then the idea is to have for every choice of this ε a δ here in the bottom, x0 - δ, x0 + δ in an interval centered in x0 [he draws a segment] and all the image of the f lies inside the chosen interval. **What am I using of R and why this representation? I'm sure that it's real with R domain ... but also tracing Q lead to fill it..."**
[…] **Every time I speak about intervals I mean 'without holes'** i.e. including rational numbers, i.e. the

points that correspond to rational and irrational numbers. **<u>It would be important to consider a R interval.</u>**"
I:"How can we represent the difference?"

[…]

I:"Why do you need it to be full to define limits?"

T:"Besides the definition of **<u>limit the fact that is rational doesn't change anything, the definition goes on be founded because Q is dense in R, but this is not convincing.. o is it? It can be an idea."</u>**

[..]

I:"If they don't work with R, which is the set of numbers with which they work?"

T:"**<u>R is something that exists, because since the first year we use x in R, then R is the line</u>,** it's a numerical set that exists and has properties equal to Q for operations and order, so it's very compatible with it but has more elements, it's a very enlarged Q. It's very hard to get the students understand this! Q is quite nothing in comparison. **<u>We add a few elements maybe numbers don't exist really, they're only human inventions, it's strange. They're strange. You can say R is the line. Is R the line?"</u>**

I:"A postulate is needed."

T:"**<u>In the pseudo-mathematics that is taught the postulate ... between rational numbers and points (that you can see starting from the Euclidean sets) ..."</u>**

[…]

I: "[…] I think that in particular a comment is interesting, concerning Dini's interpretation of his work, in which Dedekind makes explicit his intention to emancipate from the theory of magnitudes and from the line, in order to create numbers as a pure form of the thought. He tries to give up with the visual representation. Don't you think that, since the problem existed in the History of mathematic and is in a certain sense still open we could use the historical debates to give a sense to the feeling of uncertain of the students dealing with real numbers?"

T:"**<u>In practice is it useful for anything? It was useful for the purpose of a clear conscience for Dedekind but it's not useful at all."</u>**

I:"In what sense useful?"

T:"To make mathematics easier. Do this make the definitions more understandable? It's a justification.. **<u>he turns out in justifying a posteriori or a priori that somehow logically could live without it</u>**."

I:"So you won't talk about the postulate of continuity? Every constructive procedure doesn't end up in the line. a postulate is necessary..."

T:"**<u>Isn't there a parallelism between algebraic and geometrical postulates that allows us to do it?</u>**"

I:"No. We must postulate the continuity of the line. Also the Dedekind's and Cantor's postulates are not equivalent. You start from Q but you're not sure, you can't prove that there is a correspondence with a line."

T:"**Ah no? So we can't prove that the set of all the constructions is in a correspondence with the line, that there are no holes.**."

I:"No"

T:"**The line is.. is perceptual. No.. it's not perceptual, is stem from… you don't see. But.. what is the line? [he laughs]**"

[…]

I:"In Q it's not possible to use this convention. Is it?"

T:"The subsets of Q, even not graphically represented, give troubles. There is not a symbol for subsets of Q.. we could say the green segments are rational, the blu are real [he laughs]".

Teacher 4 is a PhD teacher. He is convinced that we don't need to say everything to the students and that the most of properties have to be taken for granted at school; also he thinks that the students should trust the teachers. Furthermore he's convinced that we have to think in a way and operate in another way. The graphical representation of R as a line seemed to him the most effective, but going on interacting with I he became aware that theoretically there was not exactly a coincidence between line and real numbers but rather a problematic relation. Many times he reflected on this issue and the spirit of the conversation changed. In the beginning he was certain it was a good and univocal representation of R, while he realized in the end that this was not a feasible way to stress the necessity of a complete set for the convergence of the rational series.

Teacher 5

T5: "Then the name of the person who… I don't remember the name … who had revealed the history of the segment. In the end a discourse emerge concerning operations that we can't execute in this set. So the exigence to construct a new set emerges. Then I talk a bit about complex numbers because it arises that this root of -5.. at a certain point you say that there are complex numbers."

T5: "So you do it like I do! You don't present the contiguous classes, all that discourse… so it's not only for the teachers who teach in a Professional school. It's the same for all."

[…]

T5: "I say: < And now what do we do? Let's enlarge> Yes, yes, I do the same. I'm comforted…"

[…]

T5: "Between 0 and 1 there are 0,1, 0,2, … ten numbers according to my students!"

T6: "So when … to understand that … if I take two numbers, between those numbers I can find an infinity of numbers .."

T5: "No they don't understand this!"

[…]

T5: “This is exactly what I think of my students. For sure they would not have mentioned it at all. They solve x equal to + or – square root of 5. They wouldn’t have read x in Q, or even if they read it, they don’t take care of it.”

[…]

T5: “It’s impressive that we have the same difficulties. At different level we have the same difficulties. [The students forget everything.] ”

[…]

T5: “Sometimes the students don’t understand what rational, irrational means.”

T2: “Irrational. The distinction between rational and irrational.”

T5: “**They don’t understand, they don’t understand… Maybe we should just say it, without deepen.**”

T2: “**Maybe the error is that we don’t deepen…** ”

The teacher reports many difficulties and in the beginning she’s convinced that these are due to the kind of school in which she teaches: a Professional school. Discussing with the colleagues she becomes aware of the fact that non only the students’ errors were the same in the Scientific high school but also that her colleagues proposed the real numbers the way she did. Realizing that the problems could not depend on the students’ weakness but on the teachers’ practices, at a certain moment, she had the feeling that maybe trying to simplify without renouncing to talk about the real numbers the teachers could make the topic even more hard to grasp.

The evidence of the uniformity of the students’ difficulties and the contemporary observation of the same kind of declared practices by different teachers lead her to reflect and conjecture that avoiding the complexity could be a wrong approach. In this case it happened in a focus group but we hypothesize that this could happen also reporting some teachers’ comments in the focus group in an individual interview asking other teachers to comment on them.

Teacher 6

T3: “In the first year of high school if you talk about proportions they ask you: <What is a proportion?>. They have been working for four months and **don’t remember anything**.

T2: “It’s true, they don’t remember anything!”

T3: “So.. since at the Scientific high school the quadratic inequalities are presented in the second year , **when they arrive at the fourth, fifth year and they have to analyze functions they have to apply these things… they easily do things like $x <\pm\sqrt{2}$** and so on. Probably when they studied inequalities they didn’t do such a thing! The core problem is **that they actually study not to learn but for the mark.**”

I: “So actually they are not able to use what they studied before.”

T3: “**How many times?** […]”

[…]

I: “You work with a graphical representation, is it right?”

T3: “Also, you give also a theoretical justifications, but **what impress them more is the graphic representation.**”

I: ”And in the graphic one they have more difficulties.”

T3: “Yes. So many times to understand such a thing there are the computers, if I draw two point on Geogebra, if I zoom the page they see that these two points are close, if I reduce or enlarge, between them there is already room. You take the points that are in the middle and I reduce more and more there is always a room so in theory between these two points there is always something. Then Achilles and the turtle…”

I: ”Is this accepted by the students?”

T3: “**Ehmm… yes.. but in the moment I turn out to do it graphically..**”

[…]

T2: “Maybe the error is that we don’t deepen… ”

[…]

T3: “**I’ve also lost a bit the motivation. I remember that some years ago I taught also in the first two years I remember that there were these difficulties**”.

[…]

I: “For instance, standing on the national curricula, in the last year the teachers are asked to formalize R. But many teachers say to me that in the practice it’s quite impossible to do it since the students don’t understand”

T3: “**They are evolved primates, maybe we should have more [don’t understand, nba].. I have many difficulties…** ”

T5: “It also depend on the kind of school.”

T3: “**The problem is not that they don’t understand, the problem is the time.** We don’t want to fuel an argument, we are teachers that also attend training courses and aim at training other teachers but when the national school reform instead of potentiating the mathematics curricula reduced the number of hours, I don’t know what can I do.”

[…]

T3: “You trust in the fact they have studied them. Not only some real numbers.. π is a transcendental number, is irrational… I can’t show the Taylor’s series! I’ve also proposed it in the previous years in particular courses. **You say.. you go on with a sequence, you arrive at the fiftieth digits, we are human..** ”

T5: “It’s very hard. It depends on the point of view.”

I: “Is this sufficient.. some irrational numbers are sufficient for the Calculus?”

T3: “**We always work with real number but with a quite intuitive approach, because if you approach it from another point of view…**”

I: ”Not formalizing.. Ok. Maybe some problems may emerge you have to take for granted that in a neighborhood of a point there are infinite points, finite points.. ”

T3: “It’s possible to do it, with a recurrent procedure**… it’s quite understandable. Even my son understood it!** You can also use the intervals.. ”

I: ”How? The interval may also be critical, since we should introduce it only in R. But if we have to enlarge Q we try to represent Q-intervals and to construct R starting from Q, that is different from adding some irrational elements.”

T3: “**We are talking about students of the fourth or fifth year, in which the concept of real numbers is already known**.”

I: “But we have said that the students don’t have this concept.”

T3: "**They have difficulties, but the concept has already been introduced**. When there are roots, we have to talk about the real numbers.. so when we talk about intervals, an interval is a set of real numbers, the concept has already been introduced in a certain sense. We, in the last three years, **take for granted that it has already been introduced**."

[…]

T3: "Because many times the students, **many times the set of rational numbers is not introduced, we take it for granted. We talk a lot about natural numbers, we talk a lot about real numbers, we don't talk about rational numbers.** Maybe the problem is Q, but since usually we introduce fractions, in the middle school, sometimes in the primary school, Q is very neglected. Also because the concept of set in that year is hard to introduce."

[…]

T3: "**They are used to concentrate on the last thing they are doing. The last thing we do are real numbers, what you did before, the rational numbers, is overcome.** What they know relatively well all the real numbers, because in the moment in which … 1 ,2 , 3 the primary's school teachers taught them.."

[…]

T3: "They have intuitions. Every point is associated to a number, that I can represent number on a line, that there is this continuum. The rational numbers are forgotten.

The teacher realizes that his way to treat real numbers was disjointed from the rational numbers, since the students have an intuition of R (CE4 and C3.1) that is not a result of an enlargement that is consistent with the property of the previous sets.

Nevertheless during the interview his position became more and more defensive and he concluded before that the students were responsible for these failures, because they study for the mark, then he added the problem of time. These elements emerged more and more while in the beginning he never referred to them as problems; in particular when he realized that his approach to real numbers and the students' errors were quite the same that were observed in the Professional school and the colleagues were oriented to rethink about the approach to the introduction.

In this case realizing that things were complex and that maybe something should change in his teaching choices the teacher convinced himself that the cause was external and turned to look for the students' faults.

Teacher 8

T1: "Yes, but how did he study these real numbers? **I understand that one can be confused as we were studying these numbers with Pini** [his Professor of Calculus at the University]. You had to start from N, then you had to construct Z, Z contained something that was equipotential to N. Then from Z you had to construct Q; then there was something more like √2, so you … **I remember there was a proof… the convergence of that intervals that seemed to arrive there but the convergence was on the void, there was nothing. So you have to construct something. Then there was another way to construct π. π … how was it? Where does it come from? It's transcendent.** I'm old… I studied Mathematics at the University."

I: "At school what do you turn out to do about real numbers."

T1: "Uh.. only a few. You told them some stories, the classical approach in the third year with root square of 2, that is the classical absurd proof, with the bisection method, that is given two rational numbers there is always a number in the middle but in some cases **there isn't even if the intervals converge, so what I remember from Pini is that intuitively you can see it, while I show that you can't obtain π this way, it's another kind of number that is there,** inside, and that you need because otherwise the ratio between circle and ray is not there. It comes out from this kind of stuff. Sometimes I look my old notes but … mah… I think they are so complicated.”

I: “And **does the students understand**?”

T1: **“√2, yes. It’s not a problem. π .. they remain a bit more confused**. I’ve a French book in which you see that π comes out in so many ways… bodies, organisms, leaves… an infinity of stuff. It’s really an extraordinary number.. it’s called…”

[…]

T3: “That way. You take a marker, you draw a line on the blackboard and stop.”

I: “You do something like that in your lessons, don’t you?”

T3: “Yes, while it’s different **using the computer because the computer shows pixels while …”**

T1: “Yes. **Now I understand why that physician was arguing that real numbers don’t exist! Because the nature is not analogic, is digital, discrete**!”

[…]

T1: “I understood why the physician said that. **Because the universe is digital, is not analogic. He says the real numbers don’t exist, the model of continuity doesn’t work at all.**”

The teacher showed an interesting in the relations between irrational numbers and real life; in its questionnaire and in the interview as well he talk very often of the necessity to visualize, to give sense, to make real numbers intuitive. He read many books concerning some irrational numbers; in particular π, but from his interview it emerges confusion between the “different nature of π” and the fact that it’s impossible to construct using Cantor’s contiguous classes. Also he reports examples concerning exponential growths, problems of real life but every time he’s asked to express opinions about the necessity of using real numbers he’s categorical: you need real numbers. A mishmash of different critical aspects is presented without an internal consistency. Even if he didn’t seem to be aware of it, in the same sentence he stated that he had doubts about π and he got confused when comparing it with the Cantor’s classes methods and that his students had problems to understand the nature of π. One sentences, reported from a pilot focus group, caught his attention: real numbers don’t exist. He quoted it many times referring to the fact that “Nature is not analogical” and at a certain point he realized that continuous models were not always useful to describe the everyday life and the Nature. We can’t state that this sentence clarified his ideas, that remains also a bit confused about the topic, but for sure it impressed him and lead him to think about an important aspect of the relation between real numbers and Nature.

Teacher 11

T : “ […] I love the Calculus itself and every particular function interests to me but they have too simplified models and always one asks me: < Why do we study functions with a different domain, with a hole in the domain?>”

[…]

I: “We should find not regular functions.”

T: “We should find significant examples that model situations... My students for instance often ask me why we have to consider a domain without a point [...]”

[…]

I: “To make understand that there are the holes you can start from N”

T: “But this way you have only isolated points in the domain. ... the study of a domain without a point is different.”

I: “It's to make them imagine domains that are different from R.”

T: “The line with a hole always comes out... when you study the hyperbolic curve with parameters <For which values of k..? The classical homographic function in which you are asked to say what you obtain varying the parameter. I'm now thinking for the first time at this example. Which is the usual situation. I need this x at the denominator. If this is 0 it's a line, otherwise the line has a hole because you have existence conditions. In this case, I never thought to this, the unknown is the parameter itself.”

I: “Yes.”

T: “Here there is a hole for k=1/3.”

[…]

I: “What could be other problems? You were saying, to present function that has a domain without some points.”

T: “They today asked me another thing, I took it for granted. If a function is continuous in a set, is this continuous also in every subset? One of my students... I had highlighted so much that if you add points to the domain this can become continuous.”

[…]

I: “The question was well posed.. Maybe you think they are thinking at the limits, while probably there are using more intuitive concepts. As if they thinks that if it's a strange case we use limits, if they see it...”

T: “The continuity is this discourse..”

I: “It's like they have different theories, different methods for different problems. You sometimes work formally, sometimes intuitively. They could think these are different problems.”

T: “Not only. You create a visual image showing practical examples, then, if you think at it, you demand them to do the contrary, from the equation recognize if the graph they have to construct is continuous or not. So you work in a register that is what you ask them to produce when they apply the concept.”

[…]

T: “I'm thinking now... their output is never pathological, because they are asked to work with usual elementary functions. In the connection points.. they are not prone, when the input you give has different forms.”

I: “It could be.”

T: “This question are recurrent, they ask often.”

[…]

T: “Also the concept of asymptotic behavior. These are conceptual nodes that should be approached with the accuracy of a teacher trainer. I'm now thinking to the sequence, this is more spontaneous...”

[…]

T: “If you think at it they have studied this before, think at the primary school, and this is natural concept. Is natural the extension or the restriction. Now they think at the restriction as not natural, while the construction of the exponential function....”

I: “This could be a critical point, the dialectic relation between extension and restriction.”

T: “Is there a natural direction? The path of the exponential function is an expansion.”

I: “How do you present it?”

T: “I work on the extension by continuity.”

I: “Should we go from discrete to continuous or from the continuous to the discrete?”

T: “I go from the discrete to the continuous.”

I: “But here you go in the backward.”

T: “Here I miss something about the real numbers... I tell you how I did it.. I'm going to do it now in my third class. I reconsider the scholastic history of powers. First of all we consider the variable at the exponent, then we recall the first misconception, i.e. power is multiplying a number by itself n times. […] I say here two things concerning powers. We have multiplications, the properties and so on. We reach a point in which we have to renounce at this. I miss here which is the exigence to do this, because you finalize it to study the exponential function in the real domain!”

I: “I think that if you divide a number by itself to conserve the formal properties you subtract the exponents and obtain the conventional expression 2 raised at 0.”

T: “I explain to them that to save the properties the mathematicians were forced to abandon the original conception. Paradoxically you define the exponential functions by means of the properties you want it to have. You transform the input value. ”

I: “You build what you want it to be and then you use the properties as a definition.”

T: “Paradoxically this become its definition. The exponential function is.. to the input sums.. that machine that giving sums returns products. Once understood this they don't forget the properties of logarithms.”

I: “I never saw the exponential this way.”

T: “This becomes the new definition because you define this but what is the sense? We do this to conserve that properties. The properties says to us that we can interpret 2 raised at -n as ½ raised at n.”

I: “Are you saying we need a structure?”

T: “Exactly. **We renounce to the nursery rhyme: < A power 2 raised at n is the multiplication...> but you have the features you want**. Here returns the concept of extending by continuity, even if it's another kind of continuity.”

I: “Maybe the extension by continuity is strange for them because they never extended by continuity what was obtained by means of this process. I take 2 raised at 0, 3, ½, and so on but these are discrete numbers. To trace the line you have to..”

T: “You need the real numbers”

I: “You have to add values that makes it continuous.”

T: “I say <Let's call..>, **wait, I'm thinking now. This lead you to 2 raised at x with x rational, but the passage to rational to reals, this scheme cracks.. the powers' properties help you until x rational.**”

I: “It's analogous to what Dedekind said about the fact we have to invent numbers. I put here a real number, I have 2 raised at x, but if x is unknown, you invent something that makes continuous the functions.”

T: “**I like so much this idea of inventing since its trace remains in the moment in which we have to write that numbers, because we use symbols.** To express the irrational numbers you have to use symbols I can't calculate them in their entirety. For instance √2, π, 2 raised at π.”

T: “What is not obvious is making them aware that here is a hole! The density of Q makes it not obvious. Overall when you want to represent an infinitesimal visually … it's not easy. I travel again through the powers' properties, they drive me until here, but I say**. I know that also 2 raised at √2 exists, so how can we give sense to this?”**

I:. “You could ask them to complete a function in an interval rather than in a point. They could understand the need of adding points to complete by continuity but also associate a value to a limit point. If you are on the rational... you create a critical situation in which to have the definition of continuity as limit is necessary.”

T: “**Also if you complete it by continuity you're taking for granted it's continuous innately. What does it mean to complete by continuity? It's a chaos. We have to rethink the previous paths because you arrive..”**

[…]

T: “**The word continuity is used very soon, also in the previous years. We have to investigate where you name continuity**.”

Teacher 11 is a Phd teacher. The most interesting evolution in her didactical orientations happened in a discussion concerning interdisciplinary issues involving Mathematics and Physics. She started from the necessity to make more complex the students images of continuous functions. Going on discussing she reflected more and more on what happens in her classroom and started to think about the students doubt concerning the relation between intuitive continuity and the teachers’ questions about domain without a point, isolated points and so on. Then, talking about the exponential function, that she considered the main reason why we need to talk about real numbers, she realized that completing continuously in the case of exponentials has no meaning: it’s impossible to complete adding irrational numbers; you assume it’s continuous and then “complete”. Also she was reflecting on the relation between practices concerning discrete and continuous procedures all along the years the student spend at school and realized that there is a moment in which a radical change should happen: discrete sets are considered “natural” before, then only continuous sets are considered “natural” while the teachers have to justify why they don’t consider a continuous domain.

The reflections concerning real numbers and domains of continuous functions, that was crucial in the History of Mathematics, is usually taken for granted but passing from discrete to continuous sets or coming back is significantly different. Those are not “inverse operations” and the teacher, very interested and willing, took immediately advantage of this discussion to reflect on this issue in order to find a more suitable longitudinal path.

5. Conclusions and further developments

5.1 Conclusions

Real numbers and real functions of real variables are included in all the national high school curricula of the world and also in all the STEM courses programs at university. Also they are involved in all the STEM disciplines, given the use of continuous functions and intervals, in particular for modeling of the processes involving time and space. Also the dialectic discrete/continuous, typical of Calculus, is very important for Physics (Lesne, 2007).

Many papers addressed since the '70s the topic of students' difficulties in learning real numbers, with particular attention to irrational numbers and cardinality. Also the same researches showed that often pre-service teachers conserve doubts on real numbers and continuity after the Master and the pre-service courses. The teachers we interview were in-service teachers with different backgrounds.

After having analyzed in depth the history of real numbers and the continuum, the literature review concerning the difficulties typical of teaching-learning processes involving real numbers and the continuum and the cognitive analysis of the topic, we carried out an empirical research interviewing the teachers. We asked the teachers to answere a questionnaire designed to investigate formation (master degree, training courses attended) and knowledge (configurations of objects they associated to R set), and to explore the practices they associate to R (practices involving elements or subsets of R or objects traditionally used in the didactical transposition of real numbers like inequalities, Q etc) and the semiotic representations of subsets of rational and real numbers they consider best (semiotic representation of intervals). In Schoenfeld's model's (2011) words, the first questions about knowledge give information about resources, the following questions concerning knowledge are about goals, questions about practices and semiotic representations investigate orientations. 20 of these teachers were interviewed in focus groups (3 or 4 members) in which we guided a discussion on questionnaire answers in order to make them explicit their personal choices and the reasons of their choices. Also 11 teachers were interviewed in a further step.

From the data analysis emerges a variety of different resources of different teachers about real numbers. The lack of awareness of the necessity of a postulate of continuity is constant in the data. Also some of these teachers, not graduated in mathematics, showed a personal epistemological position on the nature - even on the existence - of real numbers and this may affect their goals and orientation.

Some of the declared orientations on real numbers and on the students' difficulties in learning processes, - merely those about students' interpretation of the semiotic representations t - have already been disconfirmed by previous research results. For example the graphic representation of continuity is considered the best one from the most the teachers; on the contrary visualization doesn't allow to perceive the properties both of dense and complete sets.

The teachers' choices observed in this research are often based on personal resources that are in general different from the institutional meaning of real numbers.

We analyzed the teachers' answers using *a priori* categories relevant for the aspect investigated by every single question or group of correlated questions; in some case we added significant *a posteriori* categories that we didn't take in account designing the questionnaire. We grouped the teachers' micro-categories in order to create significant macro-categories.

Some interesting categories emerged and we decided to propose them as possible categories to confirm through a qualitative analysis carried out with individual interview and focus groups. We will list them in the following answering the research questions.

We decided to involve in the last phase of our methodology only teachers who had showed a correct knowledge about real numbers and the continuum in the first step, since the correlation between lack of knowledge of the teachers and students' difficulties have already been proved in other studies.

We choose 8 teachers from our starting group and then we compared their interviews with 3 teachers who didn't participate in the first phase, in order to compare them.

We can't deduce anything certain from this comparison but it emerged in this case a difference in the teachers' awareness of the problems and also in the quality of the discussion, that in the second case fell quite soon into a classical discussion about the students' difficulties. Nevertheless it happened something relevant: 2 of the 3 teachers concluded that maybe simplifying too much and avoiding to talk about the crucial issue concerning real numbers the students may have even much difficulties; indeed all of them agree about the fact that their students forgot everything quite soon, in spite of their attempts to repeat over and over again.

The last phase of the qualitative

We went on analyzing the didactical suitability of the first 8 teachers and in the end with compared the categories and the evaluation of their choices.

Finally, since we had observed interesting dynamics during the interview that could suggest questions or problems to pose to the teachers in order to make them reflect on mathematical, cognitive or didactical issues, we reported some commented excerpts taken from the interviews, that we called 'teachers' self-examinations'.

Now we re-pose the research questions and synthesize briefly the answers, that can be found extensively in the text in correspondence with the questions.

GQ - 1 How can we describe the complexity of the teaching-learning processes involving real numbers and the continuum from an epistemic, cognitive and ecological point of view?

We explored the History of Mathematics, looking for interesting partial meanings of real numbers and the continuum that had been used by the mathematicians themselves in order to develop their theories. Than we studied the research papers and books concerning real numbers and the continuum wrote in a didactical perspective, in order to decide how to connect the different partial configurations in a significant way for our aims. We chose the very detailed cognitive analysis by Lakoff & Nunez (2000) and the older but still influent papers of Tall & Vinner (1981) and the works regarding intuition by Fischbein (1979, 1987). Then we used a model of analysis of configurations proposed theoretically by Font (2007) and applied in Rondero & Font (2014) in a study concerning the arithmetic mean. Using the historical and cognitive information and the connecting strategies we described the epistemic meaning that answer the question:

PQ - 1.1 What is the epistemic meaning of real numbers?

We chose to split the epistemic meaning in 5 level of generality because we had to cover a long path from *grouding metaphors* to very complex configurations like the Non-standard analysis or the Transfinite numbers without losing intermediate step that could be necessary in order to analyze the teachers’ answers. Following this description a teacher or a researcher could see if the teachers’ choices are compatible with the possible epistemic paths and if the connection strategies are suitable from an epistemic point of view. The degree of suitability can also be evaluated using the cognitive analysis applied to the epistemic meaning. The result are described in Par. 4.1.

Standing on the previous result we went on analyzing the teachers answers to the questions from 3 to 5 in order to answer the question:

PQ - 1.2 What is the teachers' mathematical knowledge of real numbers?

We analyzed the answers using the epistemic scheme and created categories based on the configurations that appeared in the teachers’ answers (Par. 4.5):

- CC1: Topological/differential
- CC2: Numeric – systemic
- CC3: Numeric - unitary
- CC4: Algebraic – operations properties
- CC5: Axiomatic
- CC6: Line - systemic
- CC7: Line - unitary
- CC8: Relations between Q and R

We mapped this way the categories and the configurations, labeling them by means of only 2 letters, L and U, representing lower and upper configurations in the epistemic description. L have been assigned to configuration from Level 1 to 3, while U to configurations are Level 4 and 5.

- CC1 ↔ C4.1 (R is the set of Cauchy-Weierstrass-Cantor contiguous classes) [U]
- CC2 ↔ C3.1 (Hybrid continuum) [L]
- CC3 ↔ C3.2 (Algebraic structures and numerical sets) [L] & C5.1 (Cantor’s transfinite numbers) [U]
- CC4 ↔ C3.2 (Algebraic structures and numerical sets) [L]
- CC5 ↔ C5.2 (Axiomatization of real numbers' set) [U]
- CC6 ↔ C3.1 (Hybrid continuum) [L]
- CC7 ↔ CE4 (Line as trajectory) [L]

We observed that the most of the answers (29%) concerns CC5 (**Axiomatic configuration); a**nother very represented category (19,5%) is CC1 (**Topological/differential configuration).** An intermediate value of answers belonging to CC3 (**Numeric – unitary configuration**) (9%), CC4 (**Algebraic configuration**) (15%) and CC6 (**Line - systemic configuration**) (15%) was also registered. A minority of answers belong to CC2 (**Numeric - systemic configuration**) (5%), CC7 (**Line - unitary configuration**) (4,5%) and CC8 (**Relation between Q and R**) (4%). Globally the 52,5% of answers were U, while the 47,5% were L.

A teacher could answer on different level. We observed an high quantity of U answers, that lead us to consider a great part of the teachers potentially aware of the existence of a formal dimension of real numbers.

We went on categorizing the teachers standing on their answers concerning the construction of R:

- IC1) Dedekind (Cuts)
- IC2) Hilbert (Axiomatic):
  IC3) Cantor, Cauchy, Weierstrass (Limit points)
- IC4) Root square and π (Example of irrational numbers, R is an enlargement of Q):
- IC5) Union of different kind of numbers (Rational/irrational, Algebraic/Transcendent)
- IC6) Correspondence with the points of a line

It emerged a great agreement of the teacher with very different about IC4, that is not a way to construct R starting from Q but rather a way to explain that not rational numbers exist. Helped by other macro-categories we observed that:

- looking at the frequencies we can yet observe that the most of the constructions proposed belong to the L-categories;
- confirming what it had emerged in Q2, no approaches outline the dynamic meaning of continuity and the analogy between the segment as trajectory and other meanings of real numbers, like completeness, bijection with the line, the link between continuous variations and intervals and so on. The dynamic configurations are not considered as possible "operations" that can generate R. This could be referred to the Weyl's and Brouwer's classification of the continuum: the intuitive time-dependent, the punctual and the free-choice acceptations of continuity are considered somehow independent at this level;

Also it's interesting to differentiate the teacher who decided to define the real numbers as a whole, in the actual sense, or as sets of elements, or better to characterize every single element (how to recognize or construct the element, its properties) rather than the whole set (the global properties).

- The most of the introductions proposed by the teacher tends to go on this way:

1) to provide examples of problems that create a crisis in a previous numbers' model (Q);
2) these problems may be algebraic or geometrical;
3) in the hybrid continuum (C3.1) every construction ends to be identified with a number/point of a line;
4) to appoint these particular cases of numbers that are not rational (i.e. don't correspond to geometrical constructions associated to ratios) to representative of a general element, the irrational number;
5) to say R is the set that contains all the possible numbers, the previous and the numbers "like these particular cases"

This procedure, that we analyzed in depth, is critical from the point of view of dualities (Font, Godino and D'Amore, 2006) since when the generic element is created this way, there are no patterns that permit to create a real generic object, but there are only particular examples that reasonably will remain the only content of the expression "irrational numbers".

- Consistently with the previous trend, but much more impressively, the most of the teachers propose constructions of real numbers that stress the features of the elements of real numbers rather than to introduce the whole set as unitary. In the idea itself of construction there is the inner concept of thinking in a systemic way; nevertheless this is a crucial, maybe the most important, issue concerning real numbers, since we have no possibilities to construct it element by element, but on the other side we have not a simple rule like those that permit to construct Z and Q. This aspect will be investigated in depth in the interviews, since an interaction is necessary to conclude something significant.

Thanks to Q5 we could add an element, that indeed contributed to make more complex the picture since it emerged that the teachers were divided in two quite equivalent parts concerning the knowledge about the possibility of defining limit points in Q or in R. We understood that it was impossible that some teachers believed that the limit pint can't be defined in Q, thus we hypothesized – and the hypothesis was confirmed in the interviews – that the teacher could reason in an articulated way: some of them are used to see every mathematical practice in the "ambient" R, while other could have thought that not all the limit point of Q are in Q so R is needed. This positions are so different that we tried to catch this difference comparing the answers with those of a further questions (Q7).

From this analysis at this step we only found a category: teachers with a **low awareness of the complexity of the epistemic meaning of real numbers,** i.e. the teachers who listed always L properties and here stated that r is necessary, since quite certainly they are not aware of the problem that underlie the other interpretation of the question**.**

We compared immediately the global results with teachers' formation categories, in order to answer the following question:

PQ - 1.3 What relations between teachers' formation and their mathematical knowledge of real numbers emerged?

Even if there are more recurrences, it doesn't seem that the formation is a good marker in order to create teachers' categories. In fact all the teachers belonging to the same category related to formation are distributed in the mathematical knowledge categories quite equally. This is an interesting result because *a priori* we considered probable that the mathematical knowledge could be quite homogeneous in teachers who had similar formation while this is very different from what we observed: it seems that the mathematical knowledge of real numbers escapes from this kind of categorization.

We went on categorizing the teachers' answers concerning goals and orientations in order to answer this question:

PQ - 1.4 What are the teachers' declared goals and orientations?

finding out these categories:

1) R is a prerequisite for quite all the mathematical objects and practices / once introduced R, it is obvious the numerical domain is R if there are no other indications / R is necessary for the graphics : **R FOR ALL**

1') R is not necessary at all [2] [U] : **NO R**

2) R is necessary only for advanced mathematics (Calculus) [7] [U] : **R FOR U**

2') R is necessary only for common mathematics/functions' graphics [12] [L]: **R FOR L**

The most of the teachers have been placed in the first category; this is very interesting since we know that the properties of real numbers are very complex and it's hard to believe that the teacher may really use all the properties of real numbers, in particular because we avoided to go on analyzing the unaware teachers. Thus we hypothesized that the teachers could be inspired by a didactical tradition and/or could identify lower partial meaning of real numbers with the global meaning of real numbers, or furthermore they could choose to disarticulate the global meaning in order to be effective from a didactical point of view. This result is important in our research and guided part of our interviews, since also many teachers with advanced knowledge declared to pursue this multiplicity of goals introducing R.

From the group of questions concerning the teachers' orientations we extracted merely interesting categories from the semiotic point of view:

GENERAL SEMIOTIC CATEGORIES OF ORIENTATIONS

GSO_1) **No need of mediation:** A representation can be intuitive/immediate/clear, whoever the interpreter of the sign is: O17A, O17B, O17D, O17E, O39, O40_E, O40_G, O40_H, O40_N, O41_D, O42_E : 7, 20, 38, 62, 87, 92, 100, 115, 8, 10, 108, 13, 58, 60, 71, 83, 22, 4, 5, 53, 63, 66, 72, 11, 34, 53, 64, 10, 32, 62, 66, 4, 5, 7, 8, 10, 11, 17, 20, 26, 32, 34, 37, 42, 53, 62, 64, 66, 73, 74, 79, 91, 95, 110, 2, 4, 5, 8, 16, 22, 26, 32, 39, 42, 53, 55, 64, 65, 68, 73, 77, 78, 83, 93

GSO_1') **Mediation is necessary**: A graphic representation is not enough, language or other representations are necessary to give sense to it := O7, O23, O27, O28, O33, O40_M : 22, 1, 14, 37, 46, 77, 78, , 17, 26, 53, 63, 73, 95, 21, 59, 101

"Il concetto deve essere mediato necessariamente dall'insegnante"

GSO_2) A **graphic representation is always useful** and more intuitive than the others (ostensive, hierarchy): O22, O17_C, O17F, O40_D, O40_H: 13, 58, 60, 74, 116, 38, 53, 13, 53, 73, 87, 90

GSO_3) The **coordination between different registers may confuse the students:** O35: 101

"Presenta simultaneamente più entità matematiche senza focalizzare l'obiettivo"

GSO_4) S**ome representations are more suitable** than others in specific practices: O41_C, O41_M: 1, 2, 8, 10, 17, 33, 34, 47, 62, 72, 79, 85, 87, 88, 91, 108, 115

"Io preferirei porre l'accento sul fatto che rappresentano la stessa cosa, e che se ne userà l'una o l'altra a seconda del contesto e della convenienza"

GSO_5) Some signs are not representations since they're not finished processes / **Some signs associated to intervals represent the task and not the solution:** O41_E, O41_F: 1, 2, 7, 15, 22, 34, 67, 68, 73, 75, 79, 80, 83, 84, 85, 87, 88, 89, 91, 92, 95, 105, 106, 108, 110, 111,

5, 7, 10, 11, 15, 22, 32, 39, 40, 41, 53, 57, 62, 68, 77, 84, 91, 95, 115

GSO_6) The **synthetic representations** are better: O41_G: 10, 40, 63, 78, 79, 95, 115

GSO_7) **Different representations are complementary** / the meaning is a result of configurations: O41_H, O41_I, GSO_4: 67, 65, 57, 75, 1, 2, 8, 10, 17, 33, 34, 47, 62, 72, 79, 85, 87, 88, 91, 108, 115

"La soluzione grafica porta a trovare la soluzione algebrica per es. in sistema di disequazioni"
"Si tratta di due rappresentazioni distinte di un concetto matematico, che si arricchiscono proprio in virtù della loro diversità"

GSO_8) There is a **hierarchy between the representations:** CIR_2 : 4, 5, 53, 63, 66, 72, 38, 53, 11, 34, 53, 64, 13, 53, 73, 87, 90, 32, 62, 66, 15, 47, 57, 63, 72, 73, 32, 62, 66, 63, 85, 79, 68, 83, 90, 95, 13, 77, 87, 92

a. The graphic representation is better/more intuitive, more synthetic O39, O40_D, O40_G, O40_H, O40_I, O40_N, O40_R, O40_Z, O22

b. The algebraic representation is better/more precise, more formal: O40_T:

"Per passare dalla presentazione grossolana della grafica alla soluzione raffinata dell'astrazione algebrica"

c. The graphic representation "represents" the algebraic one: O41_L :

"a mio avviso non esiste una "soluzione algebrica" ed una "soluzione grafica" della disequazione; la seconda è una "rappresentazione" convenzionale della prima"

“non sono due "soluzioni", ma l'espressione della stessa soluzione sotto due diversi punti di vista. Devono imparare a vedere la soluzione grafica come espressione grafica dell'algebrica.”

a. The algebraic representation "represents" the graphic one

GSO_9) **Different representations are equivalent**: CIR_3)

“Sono diverse rappresentazioni di uno stesso insieme numerico”
“No, la soluzione è unica, abbiamo diversi modi per rappresentarla”

GSO_10) **A representation <u>is</u> the object / the representation is univocal:**

“la 1), 3) e 4) rappresentano intervalli della retta reale”

“nessuna perché l'insieme in cui cercare le soluzioni non è R”

Intersecting these categories we create 4 profiles:

- **ABSOLUTE MEANING**
- **COMPLEX SEMIOTIC APPROACH**
- **MEDIATION AND COORDINATION**
- **GLOBALLY INCONSISTENT**

A very few of the teachers adopted complex semiotic approaches; in particular a lot of teacher, even convinced of the necessity of using different representations, consider their possible meaning absolute and not to be interpreted; we conjectured that a large quantity of the teachers don’t care about the act that the referential meanings are personal and depend on the person who look at the signs and manipulate it. A group of teacher with PhD, that we investigated in the last part of the analysis, were particularly represented by this category. We observed thus that maybe an expert without a suitable background in mathematics education could not be aware of the fact that the meaning is not absolute; f.i. the segment is not an interval of R and stop, but from all the perspective – historical, epistemic, cognitive – this can represent different meanings.

Nevertheless we were positively surprised by the presence of teachers who are partially aware of the role played by the representations, even if the way they have grasped this problem is to deepen more, as Iori (2015) explained in depth in her PhD dissertation.

After the analysis carried out we went on with two *case-studies*: the investigation of PhD teachers’ didactical orientations based on their written answers and of 11 interviews.

PQ - 1.5 What systems of practices concerning real numbers do the teachers declare to prefer and to choose and to prefer?

PQ - 1.6 What are the relations between teachers' mathematical knowledge, orientations and goals and their declared choices?

We will comment in a unitary form on the result of this qualitative phase, answering **PQ – 2.3** (detailed results are reported in Par. 4.8.)

PQ - 1.7 Is any categorization of teachers' profiles possible using the answers to the questionnaires? If yes, what categories emerged from the investigation?

We listed some categories emerge from the questionnaire that can have a value of our research in themselves but we had to confirm by means of interviews our hypotheses. Since we interview just a few teachers we could corroborate them only in some cases.

2) If the systems of practices associated to a configuration is not rich enough in terms of epistemic representativeness, is not cognitively suitable or isn't well connected with other systems of practices, when the teachers tries to involve students in classroom discourses, the students may not be able to participate in the discourses and to grasp the new configuration. Maybe the choices suitability is related to the teachers profile we identified before.

GQ - 2 Does the didactical suitability depend on the categories we used to describe the teachers' DMK? Which is the relation between them?

PQ - 2.1 Which is the relation between the national curricula, the literature review concerning the potential difficulties and the epistemic meaning?

We answered to this question in Par. 4.2 and we have been able to do it using scheme of the epistemic meaning and the literature review. We resumed some relevant relation between the whole meaning and the Italian institutional meaning. The problem of infinite cardinality is not faced before, but it's not a problem for the previous practices

The systems of practices concerning the real numbers and the continuum are present largely in the national curricula, even if the real numbers are not always quoted. The complex relation between Mathematics and Physics concerning the hardly debated concept of "numbers varying continuously" appears at a certain time and is not treated – as it could be – since the very beginning. If we think to the critics by Bolzano to time dependent intuitions in the Calculus, to the diatribe Newton-Leibniz, engaged indeed in this field (Giusti, 1990) or to the intuitionistic reactions to the Calculus' arithmetization, we can hypothesize easily that this node will create some doubts or sudden cognitive conflicts that maybe the teachers didn't expected and may not be able to manage effectively.

Taking in account the Aristotle's approach to the continuum, its generation through motions, its characterization by means of the concept of contiguous and its relation with infinite divisibility that not implies the possibility to re-compose the continuum starting from its small parts, maybe something can be inserted before in order to prepare the students to compare and opportunely distinguish these two identities, somehow incommensurable (Nunez, 2000) and complementary (Bell, 2000).

Also it's remarkable the precocity of the intuitive approach to the representation of the real numbers on a line in the first 2 years, since it's not trivial at all, as the History and the Didactics of Mathematics researchers stressed many times. Maybe these aspects should be discussed more in depth.

From our first epistemic analysis of the national curriculum the ambitions of this curriculum seems to be very high in comparison with the potential poorness of the students' partial meanings in the various steps reported by an amount of researchers (Ch. 1.2.1) and the lack of problems and tasks that lead the students to act on the "conceptual level" in the textbooks (Gonzales-Martin, 2014;). Also the teachers are assumed to be deeply aware of the epistemic meaning of real numbers and of the interdisciplinary connections with other disciplines, while this is not obvious standing on some researches' results (Tall & Vinner, 1981, Gonzales-Martin, 2014; Arrigo & D'Amore, 1999). The role of the teachers in the realization of the attended curriculum seems to be very important and this confirms to us that the investigation concerning the teachers' choice is decisive to create a bridge between the expected students' knowledge about real numbers in the end of the high school and the weakness of their knowledge highlighted by the researchers.

PQ - 2.2 Are the teachers' choices epistemically, cognitively and ecologically suitable?

PQ - 2.3 May teachers' search for cognitive suitability cause the lack of epistemic suitability in the case of real numbers?

PQ – 2.4 Is any overall categorization of teachers' profiles possible? If yes, what categories emerged from the investigation?

Thinking at the first categorization based on the written answers and taking in account that creating teachers' global profiles had been an hard work in the first phase, we faced the attempt to find overall categories in a problematic way, opening ourselves to the possibility to find only a few significant categories but on the other side, to find in the *case studies* "exemplar teachers" rather than global categories, that are relevant from a theoretical point of view but were hard to find this way because of the high variability that characterized the teachers' profile in spite of the quantity of common features that characterized the 20 teachers interviewed in the last phase. The quantitative representativeness of the profiles corresponding to the "exemplar teachers "could be investigate in further studies.

Nevertheless we found out some aggregations that could inspire further studies:

1. Only 2 teachers (1 and 3) mentioned that the relation between the line and the real numbers is not an identification but rather that to represent R with a line is an operationally comfortable choice;

2. No teachers mentioned the postulate of continuity as necessary to identify the line and the sets of real numbers (4.1 and 4.2);

3. Only 4 teacher declared to introduce R as the set of contiguous classes, but 2 of the teachers admitted that it's just a matter of duty but they never use formal properties or formal representations; 3 teachers of these stated that the "problem *par excellence*" that make necessary to introduce R is the graphical representation exponential function as a continuous curve. Indeed they present the exponential as a power with natural exponent in the beginning, giving to it a "sense" (i.e. referring them to elementary actions by means of grounding metaphors) and then going on "adding" elements to the set of the possible value assumed by the exponent in order to reach the continuous curves they propose as the only possible representation of exponentials. Here to make sense of expression like $2^{\sqrt{2}}$ they introduce the contiguous classes. Teacher 11 interacting with the interviewer realized that it's impossible to complete the curve adding elements.

4. All the teachers declared to start from the introduction of the irrational numbers, and in particular $\sqrt{2}$, in order to explain why it's necessary to enlarge the set Q; however only 1 teacher declared to explain in depth what are rational numbers also from a geometrical point of view and to clarify that on the line we have not only geometrical numbers;

5. All the teachers, asked in a provoking way to renounce virtually to use real numbers in the high school, answered it's impossible; going more in depth the most of them justified the necessity to use real numbers referring to Pythagoras' theorem, equations and/or to continuous functions. It's important to remind that in all the cases the field of real numbers, in the epistemic sense, is not necessary: in the first two cases algebraic numbers will be sufficient, while in the second case all the teachers but 3 referred to natural continuity.

Thus the teachers talking about real number declare to identify very often the set of real numbers or with subsets of real numbers that has the property not to be rational or, when they try to represent the whole set, to use a segment produced by a continuous motion.

6. The systems of practices proposed by the teachers globally were very traditional, but with interesting exceptions, that we will comment in the following points. Generally the reasons that are common in all the declared teaching sequences are:

a. To enlarge numerical sets in order to create a set that could contain all the operations results and the solution of equations.

b. To present the problem of irrationality using a blended approach that identifies the diagonal of the square and the “number” $\sqrt{2}$ and proving it using arithmetically arguments

c. To complete continuously the curves associated to algebraic or analytical expressions in form of equation

d. To avoid to work with discrete or dense sets that are not complete solving inequalities

e. To explain what does it means to approach a point in the limits’ computation

7. Only one teacher declared to avoid to represent Q on the line and to introduce the continuous in terms of discrete terms or as a “completion” of dense sets.

8. In the most of the teaching approaches the segment as a naturally continuum appears suddenly as a representation of subsets of real numbers or as result of dynamic continuous phenomena, without a problematization concerning the relation between the two. Also the differential-topological partial meanings are introduced quite suddenly with a total identification with numbers and points and points and positions. This is not suitable from a cognitive point of view as we stressed many times in the research framework. We can state that, with the proper exceptions, the different configurations seemed simply to be” juxtaposed” , “identified” or sometimes “superposed” rather than being articulated or connected by means of metaphorical mappings and semiotic functions. We think it’s not casual that the difficulties reported by the teachers are all referred to configurations that are suddenly introduced without a declared connection between the two.

9. As we have already said, Lakoff & Nunez (2000) didn’t face the didactical problem of teaching-learning real numbers as a main topic; nevertheless in many occasions the authors committed themselves with pedagogical problem. In a precedent work (1999) in particular Nunez hypothesized which could be the teachers’ approaches to the theme of continuity considering the fact the Weierstrass’ discretization seemed to permeate so much the mathematical culture that we are not able to identify it, considering it “natural”. We compare here in particular one of the sentences, in which a pedagogical problem was presented:

“[..] students are introduced to natural continuity using concepts, ideas, and examples which draw on inferential patterns sustained by the natural human conceptual system. Then, they are introduced to another concept – *Cauchy-Weierstrass continuity* – that rests upon radically different cognitive contents (although not necessarily more complex). These contents draw on different inferential structures and different entailments that conflict with those from the previous idea. The problem is that students are never told that the new definition is actually a completely different human-embodied idea. Worse, they are told that the new definition captures the essence of the old idea, which, by virtue of being ‘intuitive’ and vague, is to be avoided. This essence is usually understood as situation-free, that is, independent of human understanding, social activity, and philosophical enterprises.”

Thanks to our investigation now we can analyze this well posed problem standing on our empirical results and analysis.

We partially confirm it, but what we observed in our research also disconfirm some statements.

Let’s start from the common elements. The “discontinuity” between natural continuity and *Cauchy-Weierstrass continuity* appeared in a very consistent measure also in our interviews and is one of the elements that leaded us to consider partially unsuitable the teachers’ declared choices from the cognitive point of view. Also we observed sudden switches from one configuration the other in a process of identification or substitution of the two meanings, that from a cognitive point of view open dangerous scenarios since even the underlying grounding metaphors are different. The reasons why the teachers switch from one to the other have something to do with the intuitiveness.

However the dynamics that lead the teachers to move from one configuration to the other are much more complex.

First of all in our investigation it emerged the recurrent presence of a double-way path from the natural to *Cauchy-Weierstrass* continuity and backward: the teachers don't leave so easily the configurations they consider intuitive, both for a matter of cognitive suitability and because they feel better dealing with natural continuity also when it would be necessary a more formal approach, that is instead omitted or reduced to particular cases.

All the teachers we interviewed more or less expressed the same annoyance to use real numbers and the continuum formally. The authors analyzed the continuity of functions but analogous discourses maybe easily transferred to real numbers since the reason why the teachers face the topological properties of the line, the limits, the density is precisely the fact that they want to talk about continuous functions using a numerical domain for the analytical expressions of the functions themselves. Some teachers stated that the approach was vague but not to enterprise a new path towards a formalization, rather to be "honest" from an advanced mathematically point of view. We report some example to support this argument:

Also the passage to formal continuity is usually not presented as "the new definition [that] captures the essence of the old idea, which, by virtue of being 'intuitive' and vague, is to be avoided", but on the contrary the trend is exactly the opposite one: the formal approaches to continuity are presented quite reluctantly and only when they strictly necessary.

Another important remarks is necessary in order to decline the pedagogical problem in an authentic didactical problem: some teachers weren't convinced at all that the pure essence of continuity was the formal one, but on the contrary they relied on a natural approach to continuity as the most authentic one.

10. 7 teachers "justified" their choices of simplification or their renounces (use of partial meanings) referring to potential or experienced students' difficulties, that the most of them consider absolute and not depending on the specific sequence of practice they choose or an oversimplification. Just 1 of these teachers - who hadn't answered the previous questionnaire – discussing in the same focus group about the students' recurrent difficulties and realizing that in all the cases, Professional school and Scientific high school, in all their classes the most of the students was used to forget everything about rational numbers, quadratic inequalities and irrational numbers, came on their own to conclude that maybe the problem is an oversimplification that leave the students out of the crucial questions.

To sum up: 7 teachers declared to simplify and to prefer intuitive approaches – whatever it means – in order to pursue the cognitive suitability of their choices, basing the evaluation on personal orientations or/and on their interpretations of previous experiences. The teachers who didn't orient their choices to simplification were Teacher 3, Teacher 7, Teacher 9 and Teacher 11. Three of them (3, 9, 11) had a good formation in Mathematics education since they had attended Masters for teachers in Mathematics education (3, 11) or had attended many teachers' training courses (7). In particular Teacher 3 showed a very wide knowledge of the topic from the historical, epistemological and interdisciplinary point of view – and also a fascination for the topic - that leaded him to face from different points of view the topic and to choose really time after time what to do, avoiding to rely in didactical traditions and proposing the problem from a cultural point of view without losing effectiveness from the institutional point of view. We can thus hypothesize that suitable and culturally relevant teaching sequences involving real numbers and the continuum not only don't oblige to waste time or to create avoidable, unbearable, difficulties to the students but may contribute on the contrary to create a fertile cultural substratum on which we can nurture the students' mathematical knowledge.

3) We observe interesting dynamic interviewing the teachers.

GQ – 3 Could our methodology of research be useful in the teachers' training concerning real numbers and the continuum?

Even if our questionnaire and interviews had not been thought in order to train teachers but only to collect data surely a better methodology should be assembled. Nevertheless we observed changes in the teachers' orientations, moments in which the teachers reflected deeply on mathematical and didactical questions, thus we think that in a further investigation this could be, *mutatis mutandis*, a good starting point.

5.2 Further developments

We think that many further developments are possible and necessary in order to deepen the analysis and have more significant results. The most evident development of this works concerns the analysis of implemented teaching-sequences carried out by teachers we interviewed. We evaluated indeed the potential didactical suitability of their choices basing our considerations on the interviews. Our tested methodology could also be used to used to involve in the research other teachers.

The research framework could also be use in all its potentiality investigating the other facets of didactical suitability and the a priori hypotheses we formulated should be investigated in depth analyzing protocols and videos. Also this phase could contribute to confirm or contradict our hypotheses based on the Schoenfeld's model, i.e. that the goals and the orientations, combined with the teachers' resources determine almost precisely the teachers' choices in the classrooms, answering the questions and solving "didactical problems". If it will be confirmed, we could try to work in the phase of teachers' training in order to provide them of the suitable DMK necessary to carry out complex and suitable practices.

Another interesting development would a more detailed analysis of the cognitive structure underlying the epistemic meaning of real numbers and the continuum and furthermore to decline for every configurations a system of didactical practices suitable for every student once known the student' cognitive profile. This study would permit to extract didactical implications and to propose possible teaching-sequences, potentially suitable. In fact, while for some very complex contents didactical suggestions have been published in PhD thesis and papers, a very few have been published concerning suitable teaching-learning sequences about real numbers and the continuum in the high school.

Going closer to the data of our research we consider our data only partially explored and we are convinced that many other analysis should be carried out.

The first possible development concerns the analysis of the relation between the teachers' declared choices and every category emerged in the questionnaire. In this thesis we focused our attention on the didactical suitability of the teachers' choices just in some cases, but a lot of crossed analysis could be carried out in order in order to make hidden recurrences and implications emerge from the teachers' answers.

A further development, connected to all the previous, could concern the field of teachers' training. First of all in this thesis we collected and analyzed an amount of data concerning the teachers' orientations, goals and resources about real numbers, that could be useful to prepare training materials in order to compensate the different deficiencies that a trainer can expect the teachers to have. Also we compared many different aspects of the topic: an historical overview, an epistemic global structure, the relation with the curricula, the teachers' DMK and finally their choices, a cognitive analysis and a literature review about the learning difficulties. Starting from this elements and our results, carrying out further empirical researches and teaching experiments we could have in a reasonable time good materials to train high school teachers.

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