Appendices

Appendix A: The questionnaire (in Italian, original version)

***1. Questa indagine ha lo scopo di esplorare le criticità dell'introduzione dei numeri reali nella scuola superiore e le pratiche didattiche proposte dagli insegnanti.**

Le chiedo innanzitutto di inserire un nome o un Nickname che sarà il suo codice identificativo se deciderà di partecipare anche alla seconda fase dell'indagine e alcune informazioni preliminari. I dati non verranno in alcun modo associati al suo nominativo durante le fasi di pubblicazione della ricerca e non è necessario introdurre il nome ma è sufficiente avere un codice personale che può servire se parteciperà ad un focus group.

Nome e cognome (o Nickname):

Scuola di appartenenza (livello scolastico):

Formazione (Laurea in..... e/o TFA, SSIS, altro):

Anni di insegnamento: (0-5, 6-10, 11 o più):

***2. Ho studiato i numeri reali:**

- [ ] a scuola
- [ ] all'università di Matematica in un corso di Analisi
- [ ] all'università in altri corsi
- [ ] in un corso di formazione per insegnanti
- [ ] sui libri divulgativi e sui testi originali da autodidatta

Altro (specificare)

***3. Le proprietà fondamentali dell'insieme dei numeri reali sono:**

***4. In ogni estensione numerica (N → Z, Z → Q) c'è una operazione "critica" sugli elementi dell'insieme di cui si effettua l'estensione, ad esempio sottrazione, divisione, che porta alla definizione di un nuovo insieme. Come si può effettuare la costruzione di R a partire da Q?**

***5. Secondo Lei si può definire un punto di accumulazione nell'insieme Q oppure è necessario usare i numeri reali?**

***6. Nelle "Indicazioni nazionali per il sistema dei Licei" sono presenti queste frasi:**

(per il primo biennio)
Lo studente acquisirà una conoscenza intuitiva dei numeri reali, con particolare riferimento alla loro rappresentazione geometrica su una retta. La dimostrazione dell'irrazionalità di radice di 2 e di altri numeri sara un'importante occasione di approfondimento concettuale. Lo studio dei numeri irrazionali e delle espressioni in cui essi compaiono fornirà un esempio significativo di applicazione del calcolo algebrico e un'occasione per affrontare il tema dell'approssimazione.

(per il secondo biennio)
Lo studio della circonferenza e del cerchio, del numero π, e di contesti in cui compaiono crescite esponenziali con il numero e, permetteranno di approfondire la conoscenza dei numeri reali, con riguardo alla tematica dei numeri trascendenti. Attraverso una prima conoscenza del problema della formalizzazione dei numeri reali lo studente si introdurrà alla problematica dell'infinito matematico e delle sue connessioni con il pensiero filosofico.

Per ognuna delle seguenti affermazioni indichi digitando il numero corrispondente alla scelta è:

per niente d'accordo (1)
solo parzialmente d'accordo (2)
abbastanza d'accordo (3)
completamente d'accordo (4)

Le situazioni descritte sono sufficienti per comprendere il "problema della formalizzazione dei numeri reali"	
La conoscenza dei numeri reali è fondamentale in matematica	
La conoscenza dei numeri reali è necessaria per risolvere equazioni e disequazioni di secondo grado	

7. La padronanza delle proprietà dell'insieme dei numeri reali è indispensabile per introdurre (può selezionare più risposte):

- [ ] funzione esponenziale
- [ ] funzione logaritmica
- [ ] calcolo differenziale
- [ ] calcolo integrale
- [ ] successioni e serie
- [ ] valore assoluto
- [ ] intervalli di R
- [ ] limiti
- [ ] sistemi di equazioni

Altro (specificare)

***8. Osservi il primo minuto del video al seguente link:**

http://www.youtube.com/watch?v=jk08WkwqT-Q

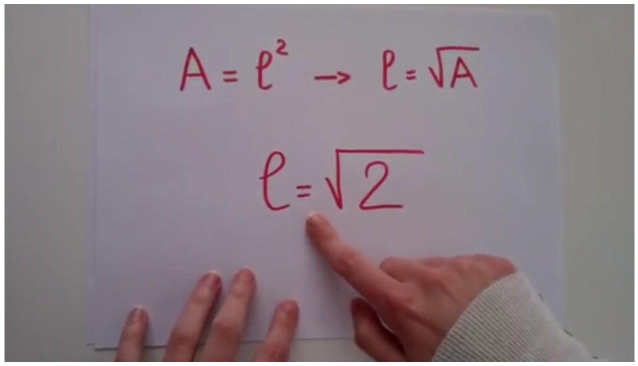
Cosa pensa dei supporti materiali utilizzati (cartoncino, disegni, etc.)? Scelga una o due opzioni

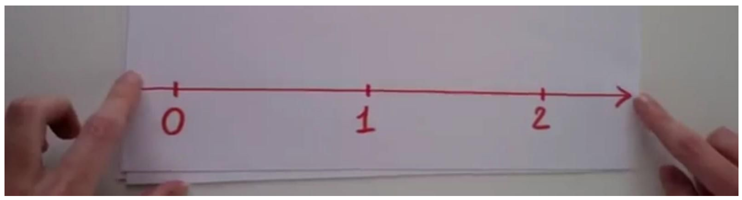
- [ ] aiutano gli studenti a comprendere i numeri reali
- [ ] confondono gli studenti
- [ ] aiutano gli studenti a crearsi una immagine dei numeri reali che potrà essere utile in seguito
- [ ] queste immagini saranno molto utili al momento di imparare a risolvere le disequazioni di secondo grado
- [ ] per l'apprendimento di questi contenuti non sono necessari supporti di questo tipo

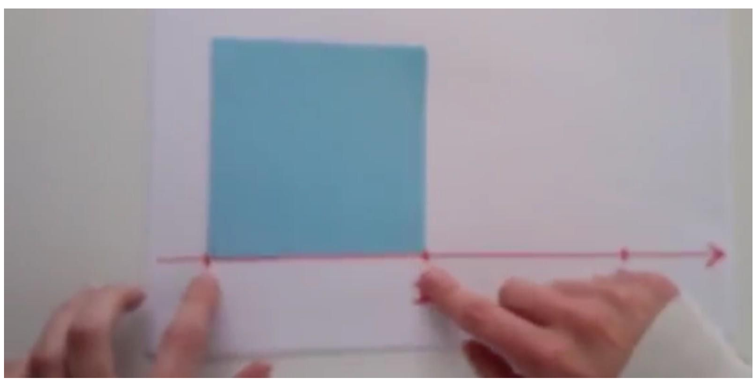
Altro (specificare)

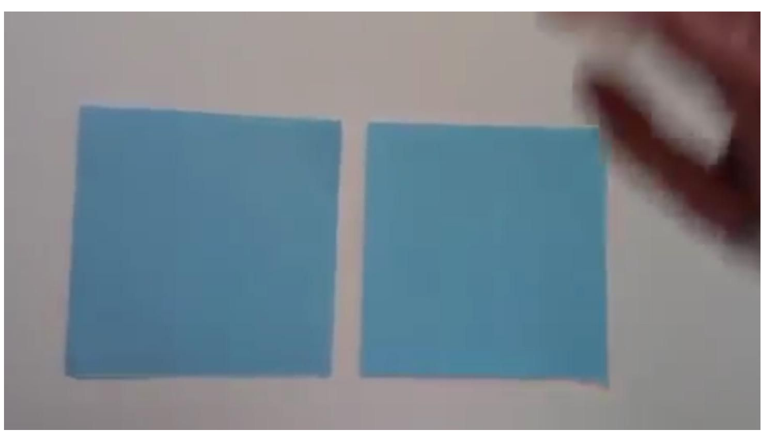
***9. Ha dei suggerimenti (inerenti il linguaggio, i supporti utilizzati, i fini dichiarati, ...) per migliorare il primo minuto di questo video?**

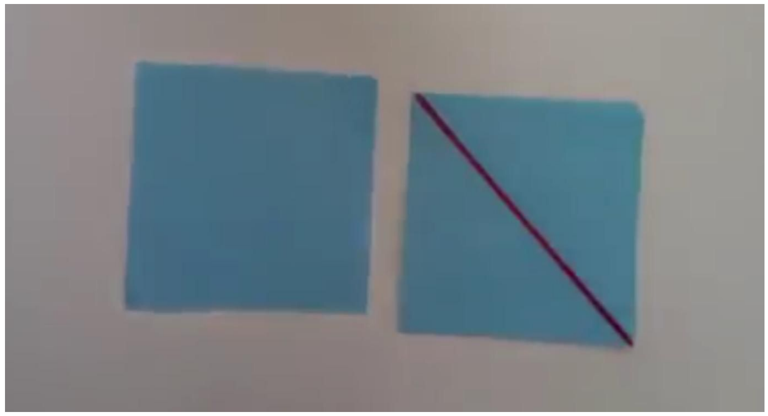
Some frames from the video proposed to the teachers in Q8-9

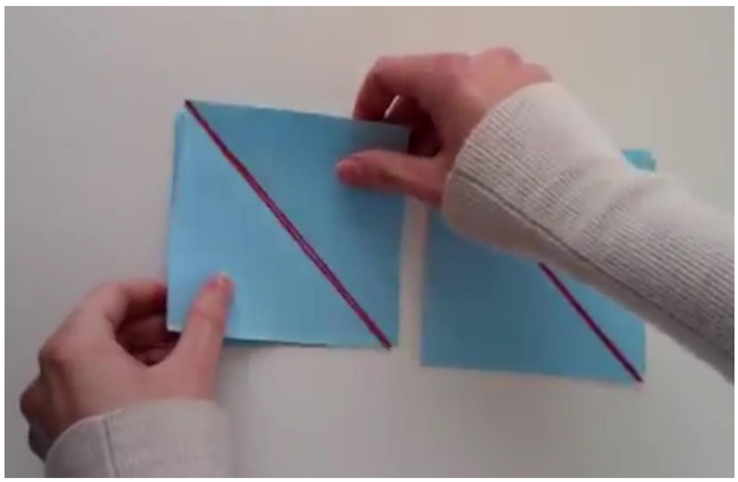

A = ℓ² → ℓ = √A
ℓ = √2

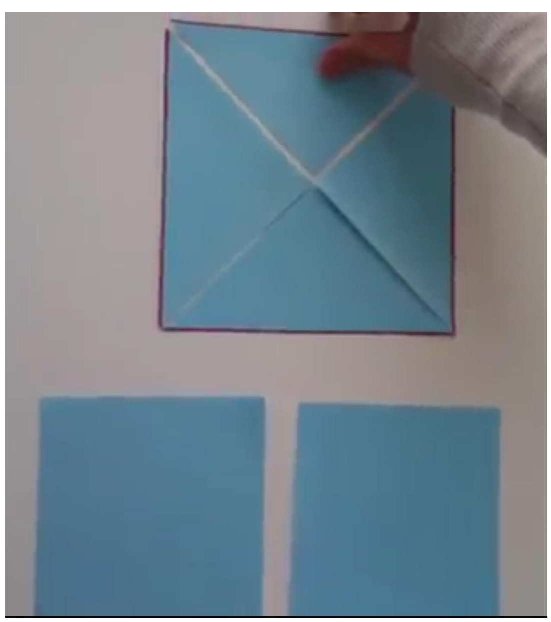

0
1
2

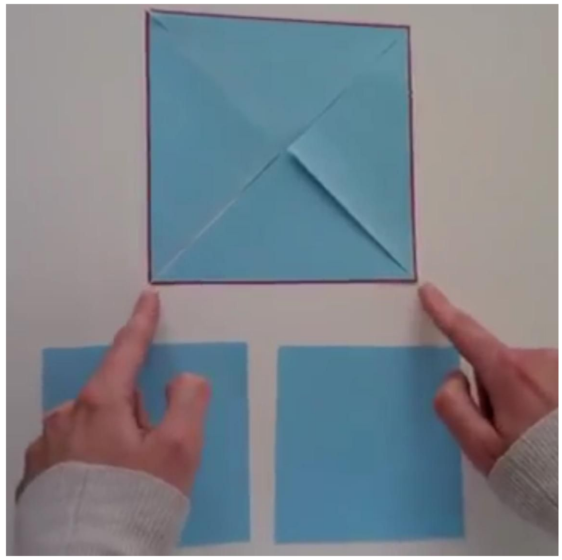


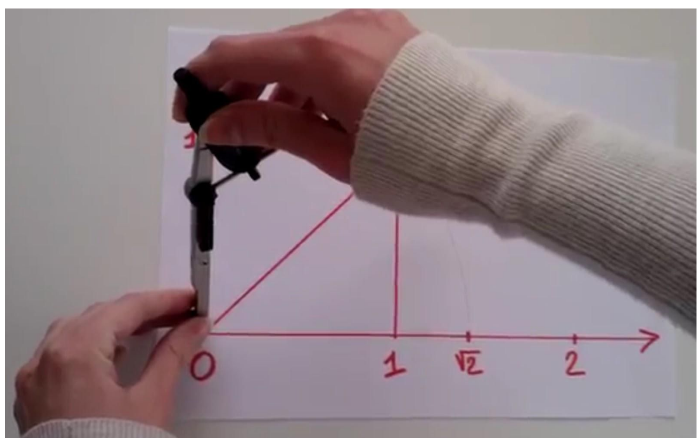


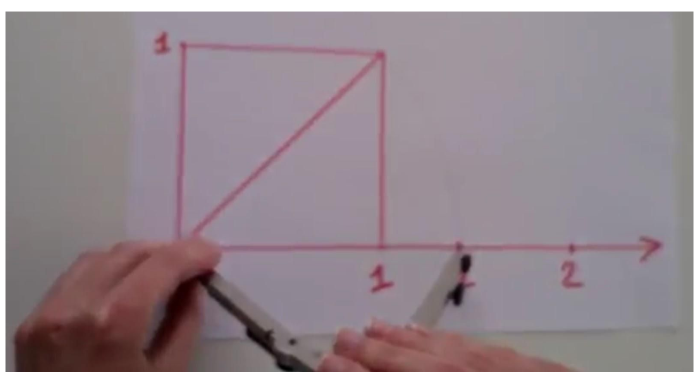


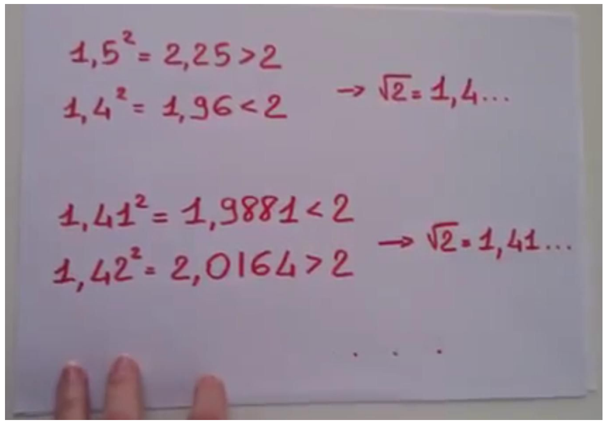


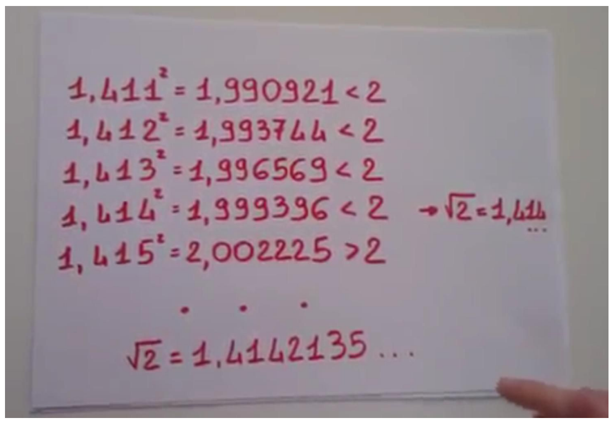


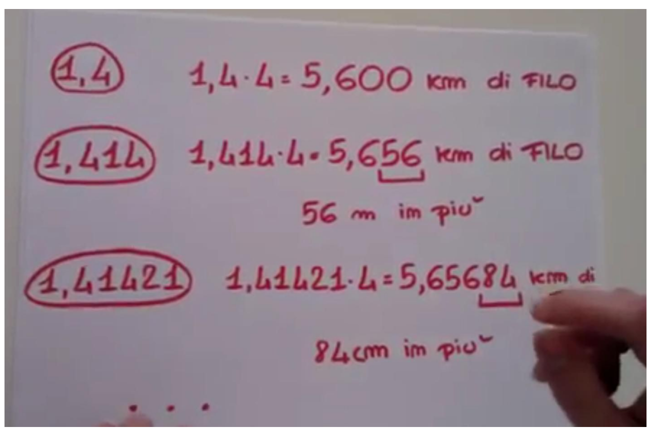


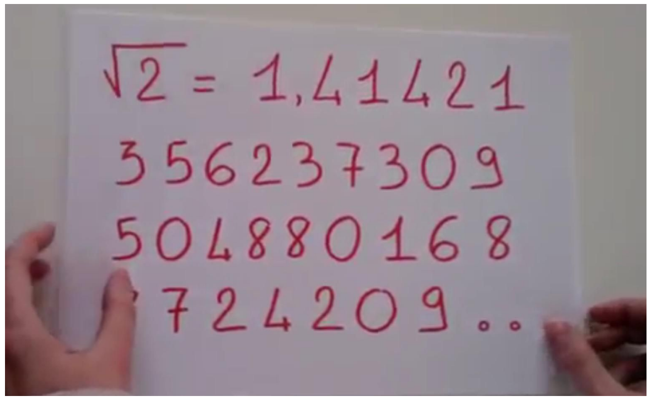

0
1
√2
2

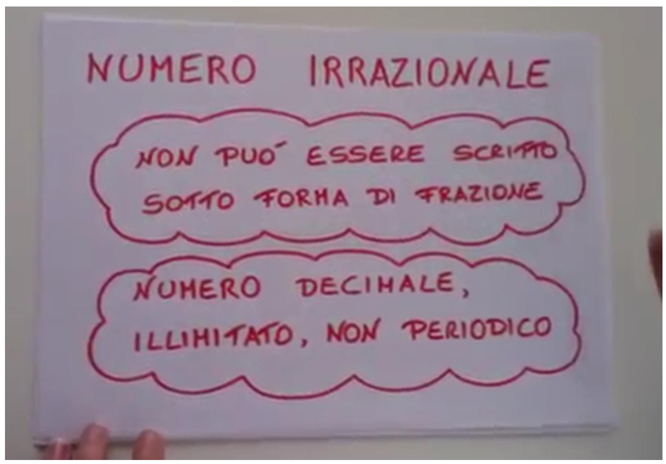

1
1
2


1,5² = 2,25 > 2
1,4² = 1,96 < 2 → √2 = 1,4...
1,41² = 1,9881 < 2
1,42² = 2,0164 > 2 → √2 = 1,41...
. . .


1,411² = 1,990921 < 2
1,412² = 1,993744 < 2
1,413² = 1,996569 < 2
1,414² = 1,999396 < 2 → √2 = 1,414...
1,415² = 2,002225 > 2
. . .
√2 = 1,4142135 . . .


1,4
1,4 · 4 = 5,600 km di FILO
1,414
1,414 · 4 = 5,656 km di FILO
56 m im più
1,41421
1,41421 · 4 = 5,65684 km di
84 cm im più
. . .





***10. Osservi il video al seguente link:**

http://www.youtube.com/watch?v=kuKTyp_b8WI

Il video può aiutare uno studente a comprendere la corrispondenza biunivoca tra numeri reali e punti della retta?

Sì, perché

No, perché

<u>Some frames from the video proposed to the teachers in Q10</u>

***11. Osservi il video seguente, in particolare dal minuto 10:20 al minuto 12:10**

http://www.youtube.com/watch?v=UEBK5DfPxvk

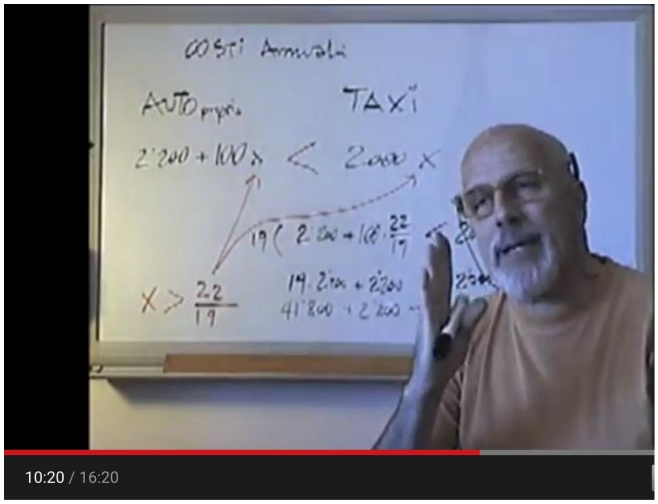
Lei cambierebbe qualcosa nella spiegazione?

***12. Crede che sia opportuna la distinzione tra soluzione algebrica e grafica di una disequazione?**

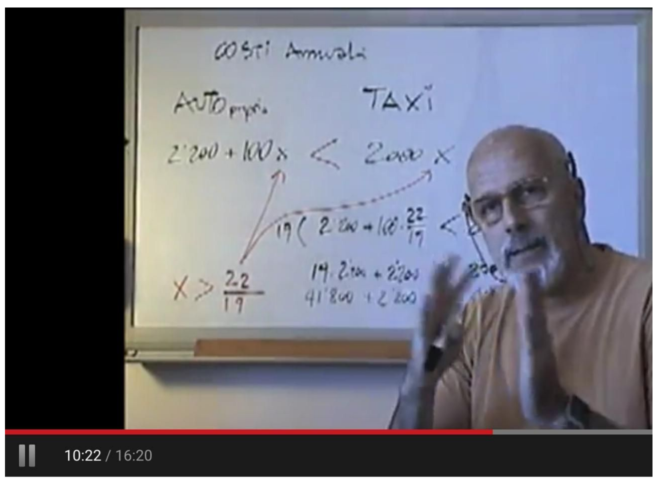
Sì, perché

No, perché

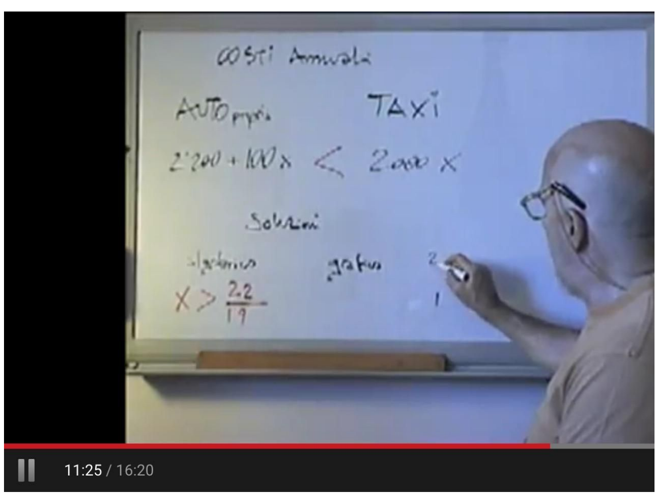
Some frames from the video proposed to the teachers in Q11- Q13



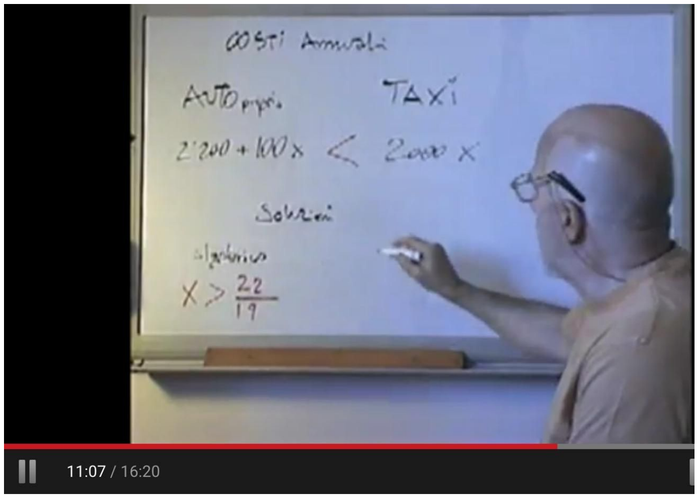
D1grB3 - DisEquazioni: Soluz. Algebrica o Grafica



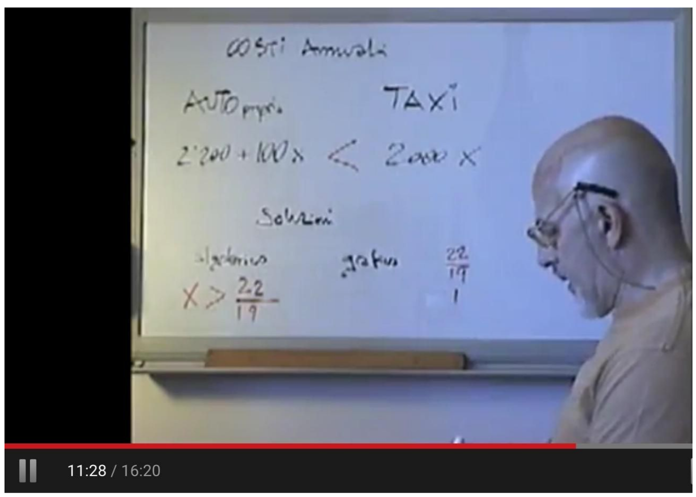
EqD1grB3 - DisEquazioni: Soluz. Algebrica o Grafica



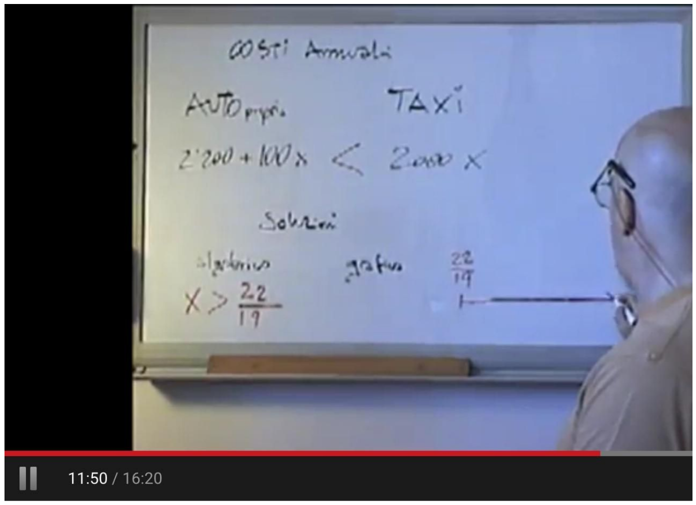
EqD1grB3 - DisEquazioni: Soluz. Algebrica o Grafica



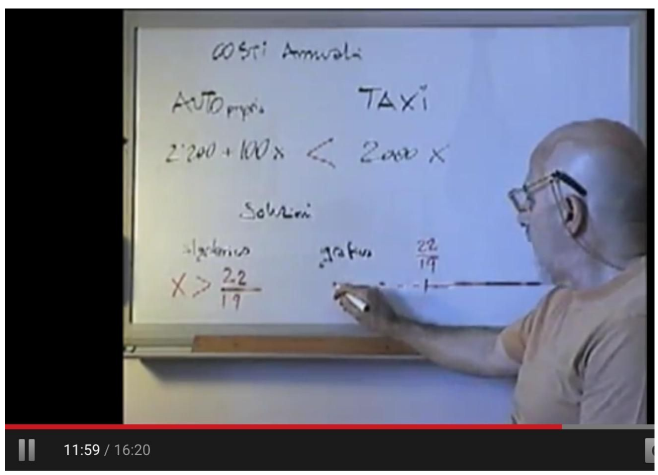
EqD1grB3 - DisEquazioni: Soluz. Algebrica o Grafica



EqD1grB3 - DisEquazioni: Soluz. Algebrica o Grafica



EqD1grB3 - DisEquazioni: Soluz. Algebrica o Grafica



EqD1grB3 - DisEquazioni: Soluz. Algebrica o Grafica

Un libro di testo propone questo esercizio.

13.

$$\begin{cases} |x-1| < 3 \\ |x-4| < 2 \end{cases}$$

Queste sono alcune soluzioni proposte da studenti.

1.

$$|x-3| < 1$$

2.

$$2<x<4$$

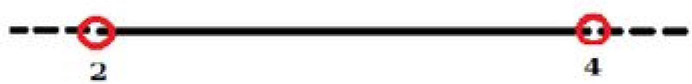
3.

$$x > 2 \wedge x < 4$$

4.

$$[2,4]$$

5.



***13. Quali tra queste metterebbe come soluzione dell'esercizio nel libro di testo? Le n°**

***14. Cosa pensa delle soluzioni fornite dagli studenti?**

a. sono tutte ugualmente accettabili

b. le n° .... non sono da accettare come soluzioni perché

c. è meglio se gli studenti usano tutti la stessa sempre, cioè la n°..........., perché

d. nessuna è accettabile perché

***15. Esprima i suoi giudizi sulle risposte fornite:**

Quali soluzioni sono più adeguate al problema? Perché?

Quali soluzioni sono più adeguate perché saranno necessarie per introdurre altri concetti? Perché?

Altro

Un libro di testo propone questo esercizio.

1. **Quali sono gli elementi di questo insieme?**

$$\{x \in \mathbb{Q} / x^2 < 5\}$$

Queste sono alcune soluzioni proposte da studenti.

1.

2.

$$-\sqrt{5} < x < \sqrt{5}$$

3.

$$[-\sqrt{5}, \sqrt{5}]$$

4.

$$]-\sqrt{5}, \sqrt{5}[$$

5.

$$|x| < \sqrt{5}$$

***16. Quali tra queste metterebbe come soluzione dell'esercizio nel libro di testo e perché? Le n° ....**

***17. Cosa pensa delle soluzioni proposte dagli studenti?**

***18. Esprima i suoi giudizi sulle risposte fornite**

Quali soluzioni sono più adeguate al problema? Perché?

Quali soluzioni sono più adeguate perché saranno necessarie per introdurre altri concetti? Perché?

Altro

Appendix B: Full reports of the analyzed interviews

Interview with Teacher 1

I: “How do you introduce real numbers?”

T: “What a hard question!”

I:”In which contexts do you introduce real numbers? How do you explain the students why you introduce real numbers and how do you represent them?”

T:”I would try … It’s hard … anyway quite soon. Not from the first years but very soon. I Think I would introduce them , very easily, showing that not all the numbers are rational. I would start from √2, they has been presented also in the grade 8. The existence of numbers that are not ratios of whole numbers…. thus a quite historical approach.”

I:”Which properties would you like them to know? Why do you introduce real numbers? Where are they necessary? Many teachers say that they didn’t succeed in introducing real numbers since they are too difficult..”

T:”What? Didn’t they succeeded? No, no, no...

I:”Maybe the Cauchy-convergence... ”

T:”For sure real numbers exists as numbers in the reality. I would start quite quickly since it’s sufficient to have used Pythagoras [theorem, nba] to be obliged to speak of real numbers. Also in Algebra [I would introduce real numbers] quite soon since the solutions of inequalities with x Q bother me...”

I:”Why?”

T:”Mmm .. they don’t… it’s very interesting, once introduced the real numbers, to solve inequalities in Q or in N, showing the difference, but in that case we ask to solve them in Q or in N because formally we didn’t introduce the real numbers, but there is a representation like this [trace a segment] of the solutions. So if you represent the solutions this way you can work with R for sure from the graphical point of view because the property that is congenial at an operational level is the correspondence between points of the line and numbers; for them [the students] is this one. They have naturally in their mind R. A student, when thinks at numbers, has in his/her mind a real number, i.e. he/she perceives the numbers in a continuous way, not in a discontinuous manner. He/she doesn’t perceive that there is a hole between a number and the consequent because you think at everything you see in a continuous manner. Without holes, without spaces between a thing and the other, it’s much more natural for them. They use rational numbers because when they were younger, in the very beginning, they had not confidence with numbers that were not rational because in the reality every real numbers is approximated with a rational one. But in my mind in their mind there is already a real number considered as the sequence of the numbers… once they overcome the question of the naturals, passing from N to Q, for them everything is become a bit .. joined … so quite soon it’s possible to highlight the difference between R and Q”.

I:”Why this [a segment] can’t represent Q?”

T:”Because in the middle there are other numbers… this way …√2.. it’s all that.. if you draw them this way you take everything. There, in the middle, there are things that I don’t want. It’s obvious that I can’t draw it!”

I:”Are you saying that you can’t represent it in a graphic way?”

T:”Yes, and I would say this to them quite soon. Another thing that I don’t like very much is that it’s impossible to put in correspondence it [the segment] with Q. Maybe in the beginning I would say it in an informal way, but quite quickly… because I don’t know if the fact that it’s not in correspondence with Q has everything to do with the the fact that it’s continuous in this sense because they are two different things.”

I:”Are you saying that the correspondence is postulated but is not provable? That once constructed R we are not sure it’s in correspondence with the line? The line has not indeed a definition. There is a different nature between the motion, the numbers and the algebraic structure. We have to postulate that a correspondence exists, you can’t say that R complete the line ”

T:”You could put other things in… But I would do this way: I would show that N and Q are in correspondence, that is not hard, while Q and R no.. to show that there is an enormous jump.”

I:”Would you use in the last year [grade 13, nba] some construction of R to introduce the Calculus? Is there something that needs the properties of Q and R?”

T:”The mathematization of a primitive intuition. In my classroom this year, in the Calculus...”

I:”Didn’t you recall R? Did you find difficulties?”

T:”No, because it’s quite natural. Probably I did many examples… for instance.. that emerges a bit.. when we talked about limit points, the book didn’t propose a definition …. instead I proposed a definition .. we talked for some minutes and since they had studied the sequences I tried to make them reflect about the fact that, while when you calculate the limit of a sequence only to the infinite, when you study a function on the real numbers you calculate limits also in finite points, because the only limit point for N is the infinite, while for R… But, in the reality, it’s a thing that only a few students gather .. the straight line topology is so banal… to be honest no… but it’s transformed by the books in a such banal thing that all [the teachers? the books?] always present all the theorems on the neighborhoods, on the intervals.. because the interval at any rate works… so you don’t gather intuitively… The Calculus is so complex for them that, being asked to learn [to use?] new tools, that a discourse these things may be very good for a few students … In fact I presented a lesson about the topology of the straight line, but I had not time enough. But I was very surprised by the fact that in the book there wasn’t the definition of limit point.. there was nothing.. it asks to compute the limit, but didn’t explain why you have to calculate it, everything is taken for granted. They had talked about contiguous classes, they see the continuum [represents simultaneously a limit point on the line]. They see all but that .. in the sense.. the problem of discontinuity. For them there is all. This is the reason why I say that I’m sure that for them these are real numbers [traces a segment] and that, however I take out stuff there other stuff remains here, close, this is sure. They didn’t understand the sense of talking about limit points. I think that for them this is so banal that if you say < I take a limit point and however I choose some points around I find other points >. If they have a strong image of the continuum and not the same for the discontinuum, everything is a limit point, where is the problem?. In my mind for the ‘mean student’ while the image of the continuum is very strong, it’s stranger the image of the discontinuum. For them the Dirichlet’s function is something horrible obviously. Rightly they asked me why in this continuous function there are infinite points of not-derivability and then one asked <What are these things for in real life?> Very sincerely I answered .. [laughs]... then another student said: < Let’s bet that there are no more functions like this>. It will be useful to introduce them very soon because once you have introduced them you can show the strong difference between this [draws a segment and writes R] and this [traces a segment with some points and writes Q]. This [R] is much more natural than this [Q], very natural”

I:"One is from before Christ and the other is from the end of '900"

T:"From a geometrical point of view this is before Christ [Q] but there is a great difference. The Greeks thought that this [Q] was this [R]. The problem of √2, this is crucial.. Many students write ln2 = … and wrote down an approximation. Why? Because this is what he read in the calculator and then he wrote a further approximation. There is a strong identification and it's not only a matter of symbols because when I correct it they look at me as a fussy person. It's identified, it's not only a symbol but it stems from.. it's like to say that 2 .. you do this procedure and find an irrational. How can you that all are like √2? The problem is that they don't have the concept of approximation."

I: "So they work at the numeric level with Q."

T:"Inevitably! With a Q with 2 or 3 digits..."

I:"Are you saying that there is a difference between the numerical and the geometrical procedures?"

T:"They study Physics from the first year [grade 9, nba], they only work with the calculator, because in Physics they use sinus, cosinus, logarithms ignoring their meaning, then they take the significant digits, rightly..."

I:"But without making explicit what they are doing.."

T:"Exactly! There should be upstream someone who explain quite soon which is the essential difference between the real numbers and their practical uses. My students in the end of the 5 years [9-13 grades, nba] don't know it. I would expect them to know this at least: that they have understood the difference between a real number and one of its approximations. A lot of them.. it's something subtle for them, but it shouldn't be a subtle thing! This is what should emerge as the strongest thing, i.e. the difference between a real number and its approximations… don't forget that there is an infinity of approximations.."

I:"To distinguish the continuum and the physical measures?"

T:"They don't know the great difference between continuum and discrete, interpreted precisely as the number and the approximation."

I:"Should we introduce it before?"

T:"Yes! In the same way we should do … the discrete set N, Z .. otherwise they don't see the difference."

I:"So you are suggesting to start from all the numbers and then to select different kinds of numbers? "
T:"Yes."

Interview with Teacher 2

Commenting the questionnaire:

T:"Properties? What can they be useful for? I need them for the measure (I use the properties) [Silence] In this point I have a doubt: it's necessary to use the real numbers for the limit point? For me it's necessary choose R, but at the same time between two rational numbers there is a rational number. R is necessary."

Looking at the first video:

T:"This was nice. Introduction… Considering that there are many problems concerning reality that make sense."

Looking at the second video:

T:”The real numbers doesn’t appear! No!”

Looking at the third video:

T:”Between 2 and 4… between -2 and 6… The only correct one.. but.. this not wrong… this one NEVER … I don’t like the hatching in the second. are they all acceptable? No, in this one the endpoints are wrong. We could say < It helps for reasoning>. This way it seems that all the students have to choose that one, but I like very much also this one : [2.4]. No one is acceptable in the last one. In all of these there are holes in the middle, aren’t there? Since there is Q.. ”

I:”What problems do you choose usually to introduce real numbers? What do you introduce exactly and when? Also please tell me every time which representation do you use, for instance, the line.”

T:”My first approach to real numbers is in the first lesson about numbers; I always see it as enlargement. I say <There is N, then there is Q, then there is R.> and I say < Which is the necessity?> and then I introduce also C at the same time. At least I give an idea and I always say … In the first year we work essentially with Q, we introduce a part of the real numbers in the second year. In the meanwhile I say why we passed from N to Z and I introduce them as necessary to close the operations in the sets, in this case in respect of subtraction, then in respect of division, square and I say that there are numbers that are not included anymore in the sets. I prove this in the first class, it depends on the classrooms. Usually I do this proof, first of all to introduce the *reductio ab absurdum* since in this case it’s easy to understand and also I need it to say to them: < You can’t take everything for granted but it’s necessary to prove in a Scientific high school.>. It depends on the classroom, if the students follow me, if I don’t waste too many time, sometimes I proposed it only to a student: < Look in internet for this proof and then explain it to the classroom>. In the first year I usually do it also when I talk about Euclidean geometry, thus not exactly in the beginning when I talk about segments. I say: < It’s incredible how a finite thing has in itself something infinite> and I enjoy it very much because I say that in my mind the segment is the paradigm of life, in the sense that we can see the man’s life in the same way, as much as our desire … our desires are infinite. They can say other things. I show why I need segments and I say that there is a correspondence. In fact then I say : <It’s true here but also it’s true between 0 and 1. But I could also do it in Q.> I say: < If you take 0 and 1 there is always a point between 0 and 1. If I talked about an half part I stop at Q, but when they understand that there are other things in the middle, that there is always a point in the middle, that is always the half part and they can go on infinitely, this gives to me the idea of limit point in a certain sense. This is the reason why I had that doubt before, because this is an open discourse. I could also consider in Q a limit point, that is 0, because there is always something that stays… but at the same time there are also points that are not… there is also the square of ½, for instance. If I think at the square of ½ there is something..”

I:”So more in the field of the theory of numbers rather than in Geometry”

T:”Yes because I .. they.. they understand that the segment that doesn’t.. because I say: < We don’t have a so subtle pencil.. This thing already shocks them. When I ask: Finite or infinite? they all think at a finite thing. The line no, but the segment.. because they see it as limited… and so this is a concept that open their mind very much. They propose many questions on this topic.. and I like it very much because it’s a first approach to the infinite.”

I:”Some teachers say that it’s not worthwhile to talk about real numbers in the secondary school..”

T:”It’s difficult to think to work without.. this things are anyway said a bit imprecisely, a part from the proof, but in the second year.. If you want you can do something also in the fifth, since with the limits… but a few times. In the second year I recall it < Do you remember?>, because if I have $x^2 = 4$ there are no big problems, but if we arrive at $x^2 = 3$. What is there? Furthermore I recall this discourse in the first year with the Pythagoras’ theorem”.

I:”Once again with a geometrical approach?”

T:”Yes”

I:”So you carry out two parallel discourses.”

T:”Yes, always. Also with the Pythagoras’ theorem, I work very much also with an historical approach. and here I say precisely that it’s necessary, formally, even if Pythagoras didn’t name it $\sqrt{3}$ or $\sqrt{2}$. How we call it is not so interesting for us… But here it’s really necessary. ”

I:”And the transcendental numbers?”

T:”Okay… In the first and second here no, also , if I have a second year class in which I didn’t work enough on the circle I’ve not even introduced . In other classes I turned out to introduce .. we did a very nice work about the approximation of like is presented in the “Museo del Calcolo” in Pennabilli. I like so much the question of the introduction of ε because it gives precisely the idea of a little thing that is so close to x0 and allows us to see what the function does in that point. To quantify this , $\varepsilon > 0$, that is real, gives exactly the idea of the fact that we get closer and closer because this tends to 0. As representation…..”

I:”With the sequences.. how can we represent Q and construct R? It may seem forced but when I draw the segment if there are only point s of Q or R I can’t see the difference..”

T:”They’re so small that I don’t see them. So I could use this representation [traces a segment] and say that they are of Q, like you did here but with x in Q. We don’t have a way to represent Q”

I:” Here, if one would suggest the extension from Q to R starting from this point, how could he/she do?”

T:”I see it well, graphically it’s understandable that there are many real numbers here inside the segment so this is not exactly Q this one, there’s something more. But you can’t take it out doing some white small empty spaces so sincerely I should reflect. I don’t know how to see another different representation of Q, do you understand?”

I:”Every time we try to represent Q we fall down into the representation of real numbers… but if we want to enlarge Q to R, how can we do?”

I:”In fact.. at this level.. going beyond the square… this opens the door! The question of the correspondence is very important, it has the same kind of infinite.. it’s a bit difficult,, it depends on the classrooms”.

Interview with Teacher 3

T:" So ... How was it? Which mark did I get? [he laughs]"

I:"10! You have been very metacognitive! You tried to look at the questions from an high perspective and to think."

T:"The seventh question.. Which was it?"

I:" It was the one in which you were asked to say when the properties of real numbers are necessary"

T:"Yes, Yes! That one. I already wrote in the answers.. It's clear that they're important but in the different things I did about the topic Mathematics and music, when you work with the notes as a discrete set and you work with rational Numbers, fractions.. You may show to the students... Wait.. I show you a slide, I showed them a keyboard, the order of the notes is clearly that of the keys, they have an arithmetic progression while the frequencies grow by multiplications. If you say that every octave the frequency is doubled, you may show that at every octave, you have a power in base 2 while the logarithm is the exponent you raise it to. So you don't need.. I'm speaking only of an introduction, as a Concept, it would not be possible ... The logarithmic function is another thing, but as an introduction, you were talking about introduction, weren't you? Did I write these things? "

I:"Yes, you did. Also you added the exponential function, while you said that to introduce the Integral and differential Calculus, as long as to introduce limits and intervals, you need R. I agree with you, this was a sort of provoking question since it's not necessary to introduce the properties of R to introduce exponential and logarithmic functions. Of course this is my opinion. Every teacher has her own logic and it's important that every teacher follows her fil-rouge but I only observe that to talk about the exponential function in the real domain is such a problem... A deep knowledge of real Numbers is needed in this second case. When you raise to $\sqrt{2}$ you don't have concrete examples, but however you choose a method, contiguous classes, cuts, limits, ... You say <I know I can go on with the exponents approximating it more and more accurately, the exponent tends to the square root of 2 and so... While the concept of exponential doesn't need the properties, the exponential function on R is not exactly trivial. On the contrary sometimes you can find people who believe that R has to come before every other content."

T:"No no no! On the contrary in my mind precisely exponentials and logarithms are useful for ... You do it usually in the middle of the fourth year [grade 12, nba] while I introduce them in the end of the year because immediately after I start introducing the limits and these are prototypical real functions with real domain. Furthermore what you are saying is also more true because I highlight the difference between the power function and the exponential function, where the X is at he exponent and then I present X raised at X... With this we really go in the hyperuranium! X raised at X drive them crazy!

I:"I can imagine...."

T:"I ask them which is the domain ... These are for sure some challenges"

I:"Very good.. Fundamentally the aim of this conversation is to fix some things that seemed to me to emerge in your questionnaire and to ask you something about the way you introduce the real Numbers, what strategies do you use, in what moments you decide to introduce them, what interdisciplinary examples you propose, since you were talking about history and philosophical issues. Let's start from the fundamental properties of real Numbers, you wrote: operations, order and completeness axioms"

T:"Yes"

I:"Of course there was no room for deeper questions, but I ask you now If, over all about the completeness, in the moment you construct real numbers you ... Since you said you would also take in account the historical happenings, how you would present this completeness? There could be many approaches, you for instance talked about axioms that leads more towards the axiomatic approach than to the construction of real Numbers. How do you try to explain to the students what is this completeness, thinking at what they need."

T:"Yes , it's clear that to formalize in a very hard way .. For the students, also of the fifth year [grade 13, nba] Who won't attend a scientific course at the University I don't think that a too formal didactical approach is not 100% effective. There is the risk that I cause them some brains imbalances! I begin in the third year [grade 11] when they study the Zeno's paradox I start with a first idea. This becomes in the fourth year, when you approach the definition of limit, when you talk about the topology of the line you to talk about neighborhoods and the basic concept that is interesting to me is that this representation of the real Numbers by means of a line is not a trivial thing. Once I asked a student <Is there a bijection or are they the same thing? How can you assure that there is a bijection?>. To say that infinite point generate a set with dimension 1, that something with dimension 0 could generate something with dimension 1 do shock them. We don't enter the labyrinths of the topology or metrical spaces. When I was at the University I was shocked by the Geometry's Assistant professor who said: < On this line there are no holes, even If I'm not so convinced.. > Ahah it was wonderful! Maybe I won't say this... But I'll try to make them understand that the fact to find always, every time I take a point on the line, a real number this seems to me already a conquest. Also I like very much to talk about neighborhoods with holes, without their centers, that is very frequent in my tests about the limits, but also it would be enough to talk about open limited intervals. To make them understand that you can approach one or the other endpoints but without reaching it it's an important thing. the line without a point, for instance.. it's important that they understand that is something that we can't visualize, we have to reconstruct it as a mental idea. Graphically, using Geogebra for instance, the line without a point is not different from the line with that point."

I:"Yes, if you take out a point, graphically, it's like to take out an infinity of points.."

T:"Exactly. It's a conceptual conquest, the history of mathematics… in the history this conquest took 2000, 2500 years."

I:"Well.. and.. working without a 'mean' class what do you do in practice usually, how do you… which strategy do you use? You're one of the small part of the teachers I interviewed that is aware of the limitations and the problems due to the use of the graphic representation. Also the books usually the line without a point is represented with a line with an empty circle."

T:"Well, it's clear that the graphic representation is a synthetic representation that is important, as I wrote in the questionnaire, but it can't be the only representation! You need the analytical one, the inequalities, the limits, that also are linked to inequalities.. An aspect that could be associated to Zeno are the contiguous classes, for instance, but I propose it only a little and reluctantly. I propose it only.. because I'm obliged! Only to take in account the history of mathematics… I try to talk about it once and stop! I also avoid to work with the sequences and I immediately shift to the limits of functions because to talk about function from N to R, as sequences are, try to create short circuits between the continuous and the discrete, and coming back and so on, I noticed that often provokes some misconceptions… I like to say to them an irrational number, or a transcendental one, has infinite digits without repetitions and this helps me when… we organize for instance the pi-day and I always try to recall its history .. also in this occasion to say that the number has infinite digits, or that the we can't rectify the circle, or that we can't square the circle.. these are two approaches to the same problem ..."

I:"And.. about the misconceptions you were talking about before.. what are the most interesting? It

seemed to me that you were saying - and I found it very interesting - if I try to propose a formal approach, I have to present it but in the end it seems that the students understand less than before. What happened in your classrooms when you tried to do something like that?"
T:"Once I tried… Wait… before I tell you this.. when I was in high school I remember that my teacher, trying to present to me √2, presented to me the sequence with the contiguous classes, 1 ; 1,4; 1,41 and so on, and the fact that I can reduce as much as I want this difference…. I understood it, but I needed two years of University to understand the deepest meaning! I like, maybe because I graduated with a thesis in Numerical Analysis, the numerical-geometrical approach for instance starting from the exhaustion, that is the Archimedes' method with approximations, for instance with π, using inscribed and circumscribed polygons. They know what a perimeter and an area are, that is a number included by… that this are contiguous classes if we want, they are classes of rational numbers and you can express a perimeter.. then strange polygons, in which also irrational numbers appears, I don't talk about contiguous classes but they understand that these are two sequences that approximate the number, one in excess and one in deficiency, and that the difference can be reduced as much as they want. Also in this case this approach is something that yearns more for a basic concepts, an impression, for a platonic idea, their idea of number with infinite digits… I don't know.. the Cauchy's sequences, you can take the neighborhoods of a point, as small as you want, the points are as close as you want... you don't call them Cauchy's sequences but … they see it, they see it by mean of inequalities that we solve numerically or that we solve algebraically.."
I." You are saying that there are properties of real numbers that could cause difficulties if we introduce them formally but all these properties are not so necessary in the high school, so we introduce only some things that are functional to the some goals, aren't you?"
T:"Exactly"
I: "And you said that one of the most important thing is that its graphical version can contain all the numbers, so discrete and continuous functions can be contained in the same representation. You said: why to study functions with discrete domains? The continuum is a kind of support for the discrete, it's a wonderful world in which you can do what you want. Let's think about the discrete problem presented this year in the final exam."
T:"This was a double-edge sword, because the students are not used … maybe I decide not to present the sequences, I work with it in the project Mathematics and reality, and here I can spend more time on this topic, but the final exam's problem was misleading. I draw on another question you posed in the questionnaire. Another approach I use is the field, or better the closure in respect of the operations. This helps me first of all as an historical approach, a construction of the numbers from the natural, rational numbers and then I start talking about Pythagoras and I never stop "
I:"Because it concerns the music!"
T:"What I presented in my papers and conferences about the music is precisely starting from the musical scale from the point of view of the enlargement of numerical sets. You begin talking about naturals, then Pythagoras switch to rational numbers, and 2 that comes out … and then roots that you can't represent geometrically, and this is the Bach's semitone, the waves' equations, the logarithm I showed before, until the Fourier's series and the complex numbers. So.."
I:"Do you present this topic to teachers or to students or the both of them?"
T:"There is also a video… here you can find a way to reach R. I also always introduce, even if it's not in

the national curricula, the Algebra's fundamental theorem. It's the minimum to do it! Joint to the Arithmetic' fundamental theorem.. is one of the conquests of humanity! They can't finish the high school without these results is unconceivable! To know that you can decompose every polynomial in C … "

I:"Yes. In particular if you say that you want to create a field in which you can define the square root of every number..."

T:"Of course! One of the most important problem in the history was to solve polynomial equations, Tartaglia, Gauss… If they studying polynomial functions don't know that they can find 1, 3, 5 zeros and that if the grade is odd one is surely real … also this means to have understood the real numbers."

I:"So you are saying that there are also all the numerical and algebraic aspects of the real numbers, not only the graphic one."

T:"Yes, exactly! the axioms.. ok.. you say the field is ordered, Archimedean and so on… so what? Of course! You are happy.. an example of field, this [R], an example of vector space, this [R], or R^n, but it's the rest of mathematics that then was adapted to the properties of R and not vice versa. Since it has some properties more and more of its properties has then been found."

I:"A question, that you suggested. In my thesis I got aware that many conceptions of continuum exist and that real numbers are only one of the possible aspects of continuity, a sort of attempt to make static on one hand the processes' continuity, also in the physical sense (to fill all the gaps, to pass through a segment, ...) but also the other process that is that of divisibility. Divide here, divide there, you always find something. Another aspect is that of equations. There is thus a plurality of meanings, as you were saying, not only the differential, or the topological aspect, but also to understand that this set also allow to solve numerical problems. So I decided to study more the practices linked to continuity in the history and what emerged is that to be honest a rift last in time between the numerical and the geometrical approach, that the Greeks had showed since the very beginning and that had been a bit obscured by Descartes and Viète when put together numbers as solutions of equations and the geometrical construction, making a sort of outrage to the Greek's deny to talk about numbers and magnitudes as they were the same things, put a symbol in the place of a number, that can represent a magnitude or a number - as nowadays we do continuously, as it was normal - but to do this requires to put together two worlds that are a bit different. At school this is done in a quite tacit manner and it's not usual to reflect about the meaning of putting numbers on a line and to say that points and numbers are the same thing. What I want to ask you. You turn out in showing the most of this different aspects. Do you find some of this aspects more difficult for the students to learn? which strategies could you suggest to other teachers?"

T:"First of all when I indicate a letter, or a number on the axes I don't say: < This is the number three, this the number four, ...> and so on but I say this is always a representation of the number. Also ' Pythagorically' speaking it's an abstract idea, something that comes from elsewhere, I realize a representation first of all because this is one… also I use Geogebra, think at the name, from a Cartesian point of view, as you were saying, it's true they are joint but there also two separated representations, there are two windows, the algebraic one and the graphic one. I try to use always the both of them."

I:"If you use the both of them… it's not so frequent.."

T:"For instance with my students in the third year, try to enlarge the point. What is this a point or a circle? What is this? Euclid says that the point has no dimension… what are we doing? but we are not representing the point, we're representing one of its representations… the point itself, that is a geometrical

entity… you can't draw it, you only can represent it! It's like the graph of the line, or the graph of the curve .. to visualize better it on Geogebra you can increase the thickness of the line.. but the line have dimension 1, you can't make it larger!"
I:"So you are saying that your students feel the need of understanding better, if you create the occasion, if you stimulate them and then you leave them free to ask.."
T."Yes.. but I want that a thing is clear. I don't think this is a trivial question."
I:"Sometimes it seems that this is lost time"
T:"No, no.. furthermore in my school I'm one of the few teachers that use it often. Geogebra to do, for instance.. I'm working on circumferences in the third class. I proposed this very simple example. The systems with three unknowns.. the linear systems you studied are also useful for this. Now open Geogebra, use the tool! Draw a point. what happens? Nothing. Draw two points. What happens? Geogebra draws a line. Draw another point. What happens? Geogebra draws a circle. This takes 10 seconds, but it's very significant. Also I prepare some Geogebra files to do many other things. For instance I show to them the Euler's line, one of the most important post-Euclidean geometry results. I think we should do it. It's wonderful… it takes a few time. Both of the approaches are important… What is important is to give the right weight to the kind of approach you are you are using. You have always to stress that you are always representing! Also the triangle was not a triangle, was a representation.. the triangle isn't a draw, it's abstract. I link for instance Donald duck - Mathemagic land. It's wonderful!"
I:"I agree with you. I propose you last question concerning the questionnaire. It concerns the core question of the last part. It concerns Q, its representations and the fact to enlarge Q to R. The idea .. it seems that the most frequent difficulties concern the irrational numbers, infinite digits, radicals.. but I conjecture that the students have more troubles working with Q in some representations rather than working with R. A colleague for instance said that, if you reason from a topological point of view, R is a very complex structure but if you think at high school mathematics R is intuitive and Q is not intuitive. What do you think about it? This remind me what you were saying concerning the discrete sets. Q is not discrete but it's more complicate to think from some points of view. The problem that it seems to me that is emerging, if I want to construct R starting from Q, following a construction, intuitive or one of the formal classical one, I need before to know what is Q, what a Q interval is. Do you turn out to work with the students on Q and to construct R starting from q or is the process much softer than a 'Hankel enlargement'?"
T:"I do this: I have one or two lesson on the numerical chain from an historical point view. Indeed I'm reflecting on what you are saying… It's not trivial to represent Q but you could represent Q in the extensive form, as a set, you wrote is as a fraction between whole numbers, or a whole and a natural number"
I:"In the numerical register, in the world you were talking about before."
T:"I do this, also in my conferences about music and mathematics. When Pythagoras start to use fractions, rational numbers.. for instance Odifreddi talks about ratio, reason, connected with the word *logos*, that is one of the ancient principles, while I prefer to stress the analogy between ratio and relation. When you put two numbers one onto the other, you are creating a ratio but also a relation. They understand that there is a reason in the harmony of the spheres, like Pythagoras said, and also there is an attempt before Tolomeus and then reconsidered in the '500 to simplify some fractions that came out from the model, like 15/8. This

implied the simplification of the musical scale. So the cognitive process that is the reduction of fractions to minimal terms becomes not only a technical procedure but also an... aesthetic necessity.. it's not a trivial thing. I think I have problems with them when they say… is it possible that these numbers came out in the end of the exercise? But this don't shock them as much as the fact that we can't represent the cubic root of 2. about the representation of Q… I don't know… more with Cantor, you can come up to the discourse of density.. the representation of Q is a fact very connected to the geometric representation… two edges that are commensurable"

I:"Maybe you are saying that the representation of Q is linked to the geometrical constructions but it's anyway something abstract, aren't you? The problems may emerge when … also with the teachers.. you say Q is dense in R. Between two rational numbers there is always a real number.. but the way we say this can be also misleading. For instance a teacher expressed the idea in a manner that suggested that there is a rational number, then there is an irrational numbers, then another rational and so on. In the moment in which you represent a number with a point, that indeed is a segment's endpoint, you don't represent the two magnitudes you are comparing but you somehow obscure the unity and what you really do is to identify a number with a point. to visualize the density of this set of point is not exactly trivial..."

T: "In my mind the error is to try to represent Q on the line, in the sense that the more effective geometrical representation of Q is the one I told you before. I don't want to represent a fraction as a number, if I'm not looking for solutions of equations of course, 22/7 on the line but I represent it, if I need...or the approach, for instance, but it's more used in Physics... you can transform in the decimal representation and this way clearly the form is that of a real number, etc etc "

I:"Maybe the decimal representation… but when we have to enlarge Q or we stay in the numerical register, but you often usually return to the graphical example. You say that the best way to represent Q is numerical or with an abstract idea of ratio"

T:"The commensurability, Euclid, Pythagoras. I always talk about the problem of the cube's duplication… wait. all of them know the history of the square root of 2, Ippaso… but the cubic square of 2 is not π, is not , it's there, in the difficulty chain, before you present the square root of 2, 3, 4, 5, you present it as a sequence, you always add a unit more.. for the square roots of whole numbers… how can I represent it? I take a compass, I draw the diagonal of a square, then I report the segment on the line, while in the cube's duplication, the cubic root of 2 is not representable using the ruler and the compass.. the fact to say that there are numbers that they can compute using the calculator. I ask which is the value of the cubic root of 2? They immediately answer… this is 1, … and I say < No! This is ABOUT 1,...>. It's important that they know that there are many numbers that exists and that we can't represent geometrically, because Q, all the rational numbers can be represented geometrically, as ratios between edges, two whole numbers, you put them in relation.. this is something that you can see somehow. So I won't… it's true that at the University you are asked to know what density is … to explain, to prove, but I prefer.."

Interview with Teacher 4

I:"In your opinion in which contexts is necessary to use real numbers, when it's necessary to define them at school?"

T:"It's not important to define them in a rigorous way. To do what we have to do we can use the preconception of the correspondence between real numbers and line. We can take it for granted and act consequently. Beyond the rational numbers, excluding π, a teacher can't be subtle and realize a rigorous construction. One is content with the use of an approximation. Euler didn't have a rigorous definition but he made so many good things. I would rely very much on geometry to go beyond the rational numbers. In Algebra it's not necessary but with the radicals, the operations.. but there also functions.. In mathematics we need to use numbers that aren't rational. In the first year [grade 8, nba] .. radicals. If they are square we can represent them very well geometrically. If the radicals are not squares you can insert them in a consistent order with that of rationals and you don't need the structure of real numbers. Real function with real domain.. the graphic.. the Calculus is graphic-centered, indeed the way real numbers are imagined is the real line, with an abuse of language, that in my mind makes a few damages at this level but may have many advantages. We need a representation of the total order as linear order, it's not interiorized as the axiom of the order. In R we need the order and then the operations with R are compatible with the operation in Q. We, as human being, can solve operations only with rational numbers, rather in Z so we play this play to approximate R with Q continuously."

I:"Why don't we use only Q?"

T:"Because √2 is not in Q! "

I:"Why don't use its approximation in Q?"

T:"Because.. in some cases... for instance we don't know √2, but we know one of its properties. When we use this number or when we make it appear in some expressions or something like that we use the fact that it has a property, i.e. if it’s raised to the power 2 it's 2, but this is important because the most of the real numbers doesn't have a property like this one and we don't use them at school."

I: "And in the Calculus? Do we use them?"

T: "Yes, in the Calculus yes!"

I: "When?"

T: "If one wants to work seriously with the Calculus he has to introduce real numbers in a rigorous way."

I: "For instance, in the last year, when you have to introduce real functions in the real domain? What about the graphic of the function? And what about limit points, limits, integrals? Do we need R? Which properties?"

T: "I introduce the limit of a real function in the real domain in every point x_0 of R. I'll think now to how I would go on. The properties... What is useful to know to work with Calculus? 1. Q is dense in R. 2 The order in Q and R are compatible. 3. Complete."

I: "Complete?"

T: "Yes.. many things are used without making them explicit. Limits exists and are unique, the upper extreme exists. Is there all the construction of real numbers - whatever it is - in the national curriculum? Before or after the Calculus? [reads again the question concerning the national curriculum] How can we talk about transcendental numbers without... Mmm.. a preliminary understanding of the problem of formalization. But.. who wrote this? Elements of the approximated computation. The mathematical infinity? The potential infinite, the actual infinite: a number as big as we want or an infinite number.... The definition of limit avoids the actual infinite, the reason why it's acceptable for a modern eye is exactly this: it doesn't imply that there is the infinite but that we can go beyond a limit that a person establish (potential infinite)"

I: "What does R is necessary for in this discourse?"

T: "To be honest a person thinks algebraically in terms of rational numbers but he sees them in terms of real numbers i.e. of the continuity of the line, images them onto the line. I have always to add that a limit point ... one sees this approaching to the limit point without reaching it but in a continuous variation. To know what does this continuity mean or its rigorous definition is left..."

I: "What do you mean with continuity? Passing through, approaching without leaving back holes? Graphically how would you do it? Think about the limit. What is there of R in this representation?"

T: "Mmm.. a function that tends to x0, a limit l that is not a value of the function. You choose an interval that goes from l + epsilon to l - epsilon, [he draws the interval and the axes] then the idea is to have for every choice of this epsilon a delta here in the bottom, x0 - delta, x0 + delta in an interval centered in x0 [he draws a segment] and all the image of the f lies inside the chosen interval. What am I using of R and why this representation? I'm sure that it's real with R domain ... but also tracing Q lead to fill it..."

I:"Let's imagine we are working with R"

T: "Yes because in the definition of limit it's useful to have a counterexample, the function jumps here, so here you understand that for every interval of this value, doesn't matter how small it is around x0, there is a part that goes beyond. Every time I speak about intervals I mean 'without holes' i.e. including rational numbers, i.e. the points that correspond to rational and irrational numbers. It would be important to consider a R interval."

I:"How can we represent the difference?"

T: "There are points that we can't refer to rational numbers, i.e. we say that to make it correspond to Q, in the correspondence between R and the line, you could begin to see which points correspond to Z, choosing 0 and 1, using the compass and the ruler going on putting fractions, divisions of segments. Q is consistent with infinite points onto the line ordered like the rational numbers. But then there is Ippaso, who says that not all the real numbers, i.e. all the lengths, correspond to numbers that we can construct this way. So, for instance, the length of the diagonal with ruler and compass is not only one of these infinite points for a finite number of step from the unitary segment. It's possible to construct, not rational but from the rationals. All the numbers like this... "

I:"Why do you need it to be full to define limits?"

T:"Besides the definition of limit the fact that is rational doesn't change anything, the definition goes on be founded because Q is dense in R, but this is not convincing.. or is it? It can be an idea."

I:"So why don't we use only the rational numbers?"

T: "Because if $x_0=\sqrt{2}$ what can we do? Not all these procedures lead to rational numbers, do we want to evaluate functions only in the rational values? No.. even if then it's a rational approximation. You live continuously with the double truth: what we write down formally is in the real numbers set, what we use are drawings and calculations."

I:"If they don't work with R, which is the set of numbers with which they work?"

T:"R is something that exists, because since the first year we use x in R, then R is the line, it's a numerical set that exists and has properties equal to Q for operations and order, so it's very compatible with it but has more elements, it's a very enlarged Q. It's very hard to get the students understand this! Q is quite nothing in comparison. We add a few elements maybe numbers don't exist really, they're only human inventions, it's strange. They're strange. You can say R is the line. Is R the line?"

I:"A postulate is needed."

T:"In the pseudo-mathematics that is taught the postulate ... between rational numbers and points (that you can see starting from the Euclidean sets) ..."

I:"The ratios?"

T:"Ruler and compass… Talete's theorem… Somehow.. what can you say to the poor student who trusts you? Rational numbers can be enriched in such a way that you can put as much numbers as the line's points and in the right order. You could… density, completeness, continuity are in the ideas world. They're used without being explicit."

I:"Can this annoy the students?"

T:"Only the smart students. For instance: rational or irrational? When Euclid talks about rational or irrational numbers starts from magnitudes.. definitions of ratio that also includes the rational numbers (Q is dense in R, approximated by ratios of internal)."

I:”Rational sequences that tend to irrational numbers?”

T:”A way to introduce them in a sly manner is as decimals.. they’re sequences, better, series, but we don’t say this to the students! Some of these series have no limits.”

I:”So you think sequences and series are necessary to introduce real numbers or the contrary? you wrote that the properties of real numbers are necessary to introduce sequences and series.”

T:”Yes, because otherwise...”

I:”Would you introduce before R using the draw of the line?”

T:”With the draw and the act of faith that Q can be completed to R. R is this [draw an oriented line, 0 and the unit]. Let’s take a sequence. Why do I need R to say that this is a sequence and this a series? The problem is in the limit, or in the sum of the series. In the sense that you can define the sequences as sequences of rational numbers but quite all of them don’t converge to something. It’s a bit difficult to explain in what sense.. Thinking to the fact that the students has in their mind an a-theoretical, intuitive you said.. a visual intuition with some properties the idea to have some little balls, one close to the other.. I take a sequence that behave like this, more and more small, you do this until you don’t reach the thickness of the pen [he laughs]. <You are a limited being and for you the convergence is equal to the thickness of the pencil!> It would be sufficient also a set smaller than Q..”

I:”What you were saying is particular. In the Cauchy construction if the number to whom the sequence converges is rational well, if it doesn’t converge to a rational I invent a number that was not given before, it’s given later.”

T:”Yes, but.. you have to give sense to the Cauchy’s sequence s that’s not so natural. The convergent sequences converge to something, but quite all of them don’t converge to rational numbers. But if R doesn’t exist , or R is this one [indicate the previous ordered line] how can we…? In a way we should communicate the concept of convergence.”

I: ”In the limit there is a concept of convergence, even if potential. It’s an approaching”

T:”In the definition of limit you get the static definition.”

I:”You start from a point, that is the limit point, that needs to live in a set before.”

T:”I take this point, limit of a function. A bit an approaching to the border of the abyss, keep the feet … approach something that doesn’t exist … infinite rational paths doesn’t imply to be rational. It’s something that maybe we don’t understand very well too… The problem is that to such a thing - how to say? - we get used at a certain point… the proof for habit [he laughs]. The limit of a function is a thing, you can think to the limit of a sequence as separated footsteps, one separated from the other..”

I:"A play. Imagine what a good student or a student with difficulties in mathematics could ask you. [Silence] For instance what would you say to a student who asks < What does it mean that I approach infinitely a point? The research showed that some students don't accept even the density of rational numbers since they imagine some little pearls. A problem could be < How long can I divide? I should stop at a certain moment."

T:"I will answer that he must divide with the imagination, and thus he has to imagine to make a zoom and to go closer and closer so that every interval is expanded and contains other intervals, that contains other intervals".

I:"So you would change the representation. Did you realize it?"

T:"No. it's always the same."

I:"Before you were approaching the point without reaching it; what you are saying now is that you look both from the right and the left side."

T:"I approach it only from a side but somehow I know that this sequence, moving of smaller and smaller steps, can't go beyond a certain limit, so it makes sense to enlarge the small window in both the directions."

I:"Anyway you would use the graphic register. What about moving to other representations different from the line?"

T:"Mmm. yes.. but.. you should use… but anyway what is the infinite representation of numbers? It's something that.. to be honest, also in the first year, you say <rational numbers are the periodic numbers, the irrational numbers are the aperiodic numbers, but what are they exactly? You suppose that it's meaningful to talk about infinite sequences of digits.."

I:"So you obscure a big truth since the very beginning.."

T:"The adults' world has already invented a beautiful mathematics, these are some hints that I show to you but be calm, everything is good, everything is consistent. This approach is very used..."

I:"Imagine to borrow something from the history of mathematics, for instance the debate between Dedekind and Cantor, or the Berkeley's objections towards infinitesimals quantities. We are not the only ones to whom these questions cause distress…"

T:"It's true. Berkeley were used to say the incremental ratio, divided by h, $h \neq 0$, then we consider $h = 0$."

I:"Also Euler had to face this controversy; also Dedekind had the problem to define the so-called continuum in order to make it intelligible and to found the Calculus on a fixed conception of continuity. There are three introductions to the book in which he defined the continuity and the relation with the irrational numbers. I think that in particular a comment is interesting, concerning Dini's interpretation of his work, in which Dedekind makes explicit his intention to emancipate from the theory of magnitudes and from the line, in order to create numbers as a pure form of the thought. He tries to give up with the visual representation. Don't you think that since the problem existed in the history of mathematic and is in a certain sense still open we could use the historical debates to give a sense to the feeling of uncertain of the students dealing with real numbers?"

T:"In practice is it useful for anything? It was useful for the purpose of a clear conscience for Dedekind but it's not useful at all."

I:"In what sense useful?"

T:"To make mathematics easier. Do this make the definitions more understandable? It's a justification.. he turns out in justifying a posteriori or a priori that somehow logically could live without it."

I:"So you won't talk about the postulate of continuity? Every constructive procedure doesn't end up in the line. a postulate is necessary..."

T:"Isn't there a parallelism between algebraic and geometrical postulates that allows us to do it?"

I:"No. We must postulate the continuity of the line. Also the Dedekind's and Cantor's postulates are not equivalent. You start from Q but you're not sure, you can't prove that there is a correspondence with a line."

T:"Ah no? So we can't prove that the set of all the constructions is in a correspondence with the line, that there are no holes.."

I:"No"

T:"The line is.. is perceptual. No.. it's not perceptual, is stem from… you don't see. But.. what is the line? [he laughs]"

I:"You can think at it as a whole or as a set of points, for instance."

T:"In the Euclidean sense is something that verifies the proportions that defines the line. If I have two points… When he says that.. Isn't an axiom that it has an infinity of points?"

I:"May to make the students aware of these problems give a smaller room to doubts?"

T.": It creates more doubts! There are doubts anyway. What happens between two rationals is not a problem at school, in the theory is not organized much better. The concept of sequence is hidden."

I:"But if you try to show that Q is dense in R you have troubles using the line."

T:"But you can prove that Q is dense in R"

I:”Not using the line, you need other structures.”

T:”The Archimedean axiom concerns the line and this allows to prove that the rational points are dense in the points of the line. The two elements are points… the segment that connect the two points is bigger than a segment1/n and so, starting from 0, the point, the corresponding segment, 1/n .. ”

I:”But if you say 1/n you’re not on the line.”

T:”Yes, because this construction is a unitary segment divided in n parts.”

I:”But if n is very big you must stop at a certain point, when the points are too close.”

T:”If two points are distinct, they are somehow far. Are there in between only irrational points? No, if you take 1/n, n=2, using Archimedes you can repeat it many times and here there must be another point. You can also take it small, it’s enough that 1/n is big enough.”

I:”In previous experiments it’s highlighted that students have unexpected troubles with the graphic representation of limits, limit points and so on [I draws the graphic representation of limits]. To the graphic representation an interpretation is given by the students that is different from the one you’re expecting. You know what to look, you know its… meaning”

T:”You would construct at a theoretical level? Did anyone do it? Was it fruitful?”

I:”...”

T:”Many things at school are given for granted for habit or for inertia, because we sneak off the theoretical crevices.”

I:”What would you say to a student who represent an interval of Q with a segment?”

T:”That usually if a person doesn’t precise the domain it’s given for granted that it’s R. You consider Q R but these symbols are valid in R and not in Q.”

I:”Why?”

T:”These symbols [$-\sqrt{5}$, $\sqrt{5}$] express a relation between elements of R”

I:”What problems does the create?”

T:”It means <or equal>. This is false. The habit is that we use to include, < to exclude.”

I:”In Q it’s not possible to use this convention. Is it?”

T:"The subsets of Q, even not graphically represented, give troubles. There is not a symbol for subsets of Q.. we could say the green segments are rational, the blu are real [he laughs]".

Focus group with teachers 5-6-7

I:"I'm working in my PhD thesis on didactical transposition of real numbers in the high school. I'm interested in this topic since the students don't understand usually real numbers even in the end of the University sometimes."

T5:"Absolutely. "

I:"Does it happen to you too?"

T5: "It's a big problem. Yes."

I:"Also students that attend Mathematics' courses sometimes, in front of some questions, have many doubts. What are you greater difficulties in your classrooms?"

T5:"I work in a Professional Institute, it's a reality complex itself. I start introducing all the numeric sets, in the end I don't present R with the contiguous classes, as one could do in the University or in a Scientific high school, I work in a very simple manner. I define all the numbers, natural numbers and so on, then I say, guys, there are also the irrational ones, I show that not all the numbers are possible to express in form of a fraction, then I introduce the irrational numbers. "

I:"With proof that root square of 2….?"

T5:"Yes, yes. The proof is very difficult for them, I talk with them of the famous π, I show that not all the numbers are rational using the rules, and then I say: < Guys, there are the numbers.. the *matryoshka*. There is this big *matryoshka*, that are the irrational numbers, then there are the rational numbers; joining this two we obtain R, that are all the numbers that exist in the Nature> , in a very simple manner, depending on the the audience. I don't use the contiguous classes and so on. "

T6: "I work in a Scientific high school."

I: "So the situation is a bit different, isn't it? Do you turn out to introduce real numbers?"

T6: "Yes. I work in the first two years and they are already included, because there are already the roots, things like that. In general I feel the need to reconsider them, when numbers like e, π, come out. So usually what I do is to discuss about the sets. I start with the natural numbers set, then in the moment in which we have operations that we can't execute in this set the exigence to introduce a new set emerge. "

T5:"So the *matryoshka* dolls, the dolls.. "

T6: "To introduce the concept of real numbers I use the classical example of the diagonal of the square. I tell us a bit of history, Pythagoras, Pythagoras' history "

T5: "Exactly."

T6: "Then the name of the person who… I don't remember the name … who had revealed the history of the segment. In the end a discourse emerge concerning operations that we can't execute in this set. So the exigence to construct a new set emerges. Then I talk a bit about complex numbers because it arises that this root of -5.. at a certain point you say that there are complex numbers."

T5: "So you do it like I do! You don't present the contiguous classes, all that discourse… so it's not only for the teachers who teach in a Professional school. It's the same for all."

T6: "Yes."

T5:"I present them as rational numbers joint to irrational numbers and I don't prove it with $\sqrt{2}$, I use π that is a number that they know."

T6: "No, from the exigence of create new sets."

T5: "I say: < And now what do we do? Let's enlarge> Yes, yes, I do the same. I'm comforted…"

T6: "I need go further, and to introduce the complex numbers."

T5: "Yes, the last exigence is to introduce the square root of a negative number.. and so there is this particle , that raised at 2 gives -1. Ok, Ok."

I: "What are the students' greatest difficulties with real numbers? Why don't you introduce them as contiguous classes, as complete set, and so on?"

T6: "Many times the concept… the concept.. the greatest difficulties they have is with the concept of infinitesimal quantity. To understand that if I take 1 and 1,0001 I find between these two numbers an infinity of numbers."

T5: "Between 0 and 1 there are 0,1, 0,2, … ten numbers according to my students!"

T6: "So when … to understand that … if I take two numbers, between those numbers I can find an infinity of numbers .."

T5: "No they don't understand this!"

T6: "The concept that a part of an infinite can be again infinite. For instance the concept that has already been introduced in the first two years that a segment contains an infinity of points. When I say that a segment contains an infinity of points it's OK, but when I take 2 points and I draw on a blackboard two very close points they don't understand that.. they have difficulties in understanding that between two points that are so close there are infinite elements."

I: "You work with a graphical representation, is it right?"

T6: "Also, you give also a theoretical justifications, but what impress them more is the graphic representation."

I: "And in the graphic one they have more difficulties."

T6: "Yes. So many times to understand such a thing there are the computers, if I draw two point on Geogebra, if I zoom the page they see that these two points are close, if I reduce or enlarge, between them there is already room. You take the points that are in the middle and I reduce more and more there is always a room so in theory between these two points there is always something. Then Achilles and the turtle…"

I: "Is this accepted by the students?"

T6: "ehmm… yes.. but in the moment I turn out to do it graphically.."

I: "When they study the Calculus, do they recall these concepts?"

T6: "Yes, indeed I talk merely for the Calculus, because the concept of limit, the concept of infinitesimal, all these things.. for instance I say that in the limit 0 is not 0 really."

T4: "We talked about it already in the previous focus group. I was in Bologna. Here we talked merely about the intervals, the inequalities. It was about solving inequalities in R and in Q."

I: "Yes, let's talk about the students' answers I presented in the questionnaire. We asked to represent the solution of this inequality in Q. Let's discuss about it. We presented it because usually this is the way we represent the sets we need to introduce R as Dedekind' rational cuts. We take a subset of Q that approximated. The problem is that the students interpreted it as an hybrid sets including the root square of 5."

T6: "2 and 5 could be considered correct specifying that x belongs to Q; 1 in my mind is not correct because a segment, with the all points represented is not Q, not all the points are in Q. The other could all be correct, acceptable, even that with extremes included if I explicit that the solutions belong to rational numbers."

T5: "Wait! With the extremes included?"

T6: "Yes, if you say x belongs to Q"

T5: "Ok, ok… you specify it, all right. But I don't think they have written x belongs to Q. "

T6: "In the answers it's not explicated. From the second to the last all could be correct, but only if you explicit that the solution must belong to Q."

T5:"Yes, yes."

I: "and what about the graphic representation? What should have they done? How should they represent it graphically?"

T6: "I don't know how to say it… only I could say that also in this case they have to write that the points are of Q."

I: "We should also include you in the discourse [referring to T5]."

T5: "Eh, yes."

I: "This is a crucial point in the teaching experiment. The students in some occasion don't recognize the necessity of constructing a set of real numbers while they are calling real numbers another set they have been using every day for several years. We had the problem to enlarge a set they have already enlarged. How can we explain that Q is different from R."

T6: "My students would have chosen the first, because they don't read x belong to Q."

T5: "This is exactly what I think of my students. For sure they would not have mentioned it at all. They solve $x = \pm\sqrt{5}$. They wouldn't have read x in Q, or even if they read it, they don't take care of it."

I: "Don't they take it in account?"

T5: "Absolutely."

I: "In your mind why?"

T5: "Because at school what do we usually do? "

T6: "I get very angry because they solve it writing $x < \pm\sqrt{5}$."

T5: "No!! But what can we do to make them understand? I say to them the famous concordance or discordance of signs."

T6: "The associated equation."

T5: “The associated equation! I have to repeat it every time! How can we solve this one?”

T5: “Because at school they pass immediately… the quadratic inequalities have many difficulties… for instance $x^2 > 5$, they solve the equations before.. $x^2 > 4$, $x > \pm\sqrt{2}$. And this a frequent error. Doing the inequalities after the equations, they solve them in the same way combining quadratic equations and linear inequalities. ”

T6: “I ask them: < Is this < + 5 or < -5. What is this number? ”

T5: “It’s impressive that we have the same difficulties. At different level we have the same difficulties. ”

T6: “The problem with this kind of tasks is that they write $x < \pm\sqrt{5}$.”

T5: “This is what I was saying. If you ask $x^2 > 4$, they write $x > \pm\sqrt{2}$. ”

T5: “Yes. They use the < as it is a =.”

T5: “Maybe because of the linear inequalities, because you solve them..”

T5: “I’ve so many difficulties.”

T5: “After some months some exercises appears …”

T6: “Another important thing in my mind is the same problem that some adults are showing, i.e. they have an elementary knowledge. After the primary school the students work with airtight containers, they study for the mark, not to learn. Many times… for instance.. in grade 7, they work 4 months with the proportions, they become good in using proportions, and the next year they don’t remember, they don’t care about them anymore!”

T5: “It’s true.”

T6: “In the first year of high school if you talk about proportions they ask you: <What is a proportion?>. They have been working for four months and don’t remember anything.

T5: “It’s true, they don’t remember anything!”

T6: “So.. since at the Scientific high school the quadratic inequalities are presented in the second year , when they arrive at the fourth, fifth year and they have to analyze functions they have to apply these things… they easily do things like $x < \pm\sqrt{2}$ and so on. Probably when they studied inequalities they didn’t do such a thing! The core problem is that they actually study not to learn but for the mark.”

I: “So actually they are not able to use what they studied before.”

T6: “How many times? Of course I’m referring to the mass, to the most of them , there are always exceptions, but an amount of students do this.”

I: “Another question. Since it’s so hard to introduce real numbers, what about not introducing real numbers? What are the activities in which they’re really necessary?”

T6, T5: “Absolutely no.”

T6: “In what sense? To pretend they don’t exist?”

I: “To work in other sets, not in R.”

T5: “No! And what about the real functions in the real domain?”

T6: “And the algebraic numbers?”

T5: “And how can you solve quadratic equations?”

I: “You could introduce algebraic numbers and avoid to introduce real numbers..”

T5: “There are difficulties also with algebraic numbers..”

[Bustle]

I: “To work with algebraic numbers and to introduce completeness is very different. The algebraic numbers are nor complete nor continuous. Sometimes we say we work with real numbers but numerically we introduce only algebraic numbers or rational numbers. To introduce real numbers would mean to construct a field with all the properties: complete, totally ordered, Archimedean..”

T6: “When you use Geometry, many times you introduce the postulates, the contiguous elements and so on, but don’t think that many students understand these things. So what should we do? Delete all Geometry? ”

I: “It was just a proposal. Since it seems that no one understand what real numbers are.”

T6: “Not no one… but when …”

[Bustle]

T5: “Sometimes the students don’t understand what rational, irrational means.”

T5: “Irrational. The distinction between rational and irrational.”

T5: “They don’t understand, they don’t understand… Maybe we should just say it, without deepen.”

T5: “Maybe the error is that we don’t deepen… ”

T5: “Eh.. this could also be.”

T6: “I’ve also lost a bit the motivation. I remember that some years ago I taught also in the first two years I remember that there were these difficulties”.

I: “Some of your colleagues reported their experiences and the most of them told me that when they try to introduce R using contiguous classes, Dedekind’s cuts..”

T6: “The concept of continuum more or less is known. In the last year, teaching the Calculus, I would like to see how do you do without… ”

I: “For instance, standing on the national curricula, in the last year the teachers are asked to formalize R. But many teachers say to me that in the practice it’s quite impossible to do it since the students don’t understand. ”

T6: “They are evolved primates, maybe we should have more [don’t understand, nba].. I have many difficulties… ”

T5: “It also depend on the kind of school.”

T6: “The problem is not that they don’t understand, the problem is the time. We don’t want to fuel an argument, we are teachers that also attend training courses and aim at training other teachers but when the national school reform instead of potentiating the mathematics curricula reduced the number of hours, I don’t know what can I do.”

T5: “In the Liceo Classico with 2 hours a week… “

T6: “Also the concept of irrationality, also the mathematics seen from the foundational point of view. In the Liceo Classico, with 2 hours a week; you have to make choices, priority choices, you can’t lose one week…”

I: “So to work with Calculus it’s sufficient that they know some irrational numbers.”

T6: “You trust in the fact they have studied them. Not only some real numbers.. π is a transcendental number, is irrational… I can’t show the Taylor’s series! I’ve also proposed it in the previous years in particular courses. You say.. you go on with a sequence, you arrive at the fiftieth digits, we are human.. ”

T5: “It’s very hard. It depends on the point of view.”

I: “Is this sufficient.. some irrational numbers are sufficient for the Calculus?”

T6: “We always work with real number but with a quite intuitive approach, because if you approach it from another point of view…”

I: ”Not formalizing.. Ok. Maybe some problems may emerge you have to take for granted that in a neighborhood of a point there are infinite points, finite points.. ”

T6: “It’s possible to do it, with a recurrent procedure… it’s quite understandable. Even my son understood it! You can also use the intervals.. ”

I: ”How? The interval may also be critical, since we should introduce it only in R. But if we have to enlarge Q we try to represent Q-intervals and to construct R starting from Q, that is different from adding some irrational elements.”

T6: “We are talking about students of the fourth or fifth year, in which the concept of real numbers is already known.”

I: “But we have said that the students don’t have this concept.”

T6: “They have difficulties, but the concept has already been introduced. When there are roots, we have to talk about the real numbers.. so when we talk about intervals, an interval is a set of real numbers, the concept has already been introduced in a certain sense. We, in the last three years, take for granted that it has already been introduced.”

I: “How much do they know, what do they know about real numbers? An example.. this set and its complementary are or are not a Dedekind’s cut. The students answered <No> because A intersected with B is not empty but instead it’s equal to root square of 5. The most of the students didn’t same to be able to reason in Q, but rather they use a set that is bigger than Q and is not R. Which is the true set they use?”

T6: “Maybe the problem is not in the set of real numbers, but in set of rational numbers.”

T5: “Yes.”

T5: “Absolutely.”

T6: “Because many times the students, many times the set of rational numbers is not introduced, we take it for granted. We talk a lot about natural numbers, we talk a lot about real numbers, we don’t talk about rational numbers. Maybe the problem is Q, but since usually we introduce fractions, in the middle school, sometimes in the primary school, Q is very neglected. Also because the concept of set in that year is hard to introduce.”

I: “Intuitively how do you introduce the real numbers in the first years?”

T5: “In a second class I’m now introducing the geometrical construction of square root of 2, I use the line.”

I: “So you say a not rational number exists, and then..”

T5: “Infinite numbers like this one, root square of 3, Pi…”

T5: “Starting from the fractions I say: < Guys, there are limited rational numbers that I transform this way, there are the periodic numbers, and there is a set numbers with whom I can’t do this. They are unlimited and not periodic. Is there this number? Is there π? Not all the numbers can stay here, so let’s construct the set of all the numbers that exist in the Nature and we call them real numbers. ”

T7: “An activity that I propose is: < Let’s invent some irrational numbers.> Otherwise…”

T5: “Otherwise they are only punctual.”

T7: “<Let’s invent number with an infinity of digits, 0,123456.. a sequence.. numbers that has an infinity of not periodic digits, it’s a way to lead them to touch a bit.. ”

I: “And how would you connect it with the square, the diagonal? How do you pass from this discourse to the other?”

T6: “What she [T5] were saying.”

T7: “This is just a first approach, to present the .. between 1 and 2 where √2 is placed, with the ruler you can measure 1, and √2? How can you do it? It has an infinity of digits. You can realize the geometrical construction. Just now I’m going to divide them in small groups and to ask them to construct other real number using Geogebra.”

I: “So in your mind it’s necessary to work with real numbers.”

T6: “Essentially we work with real numbers, the problem is not to work with rational numbers.”

T7: “And how could we do?”

T5: “To use rational functions in the rational domain? No! No! Much more complicated!”

T7: “The equations and the inequalities are not usually solved in a numerical set. In my mind this is important.”

I: “Could this be important? Don’t you say the inequalities are solved in Q, R, N?”

T5: “No, only in particular cases.”

T6: “Only with particular problems, if we need a rational number.”

T5: “N, Z.. I never do this. I don’t say: < The solution is in Q, the solution is in N>.. it’s normal that it’s R. sometimes I do it to do something different, but generally no. ”

T6: “Only once I found in a book exercises to solve not in R.”

T7: “sometimes there are problems in N. Find that natural number…”

T6: “No I say precisely the equations. Not in all the books.”

T5: “With the equations it could happen.”

T5: “Yes, with but the inequalities no..”

T7: “In my mind when the students are used to work in R, that is a bigger set, that you finally reach… the problem is the contrary, the concrete problem… for instance a discrete problem, a merchant sells some bottle of wine. To see that the function isn’t continuous, but it’s discrete, here it becomes very complicated because they’re used to draw lines instead of only drawing points. In my mind this is a risk we run while the process should be the contrary, if one starts from R, one could say : < Stop! I’m arrived. Why should I come back?”.

T5: “It’s true. The problem may be that one pass from natural numbers to rational numbers, some properties are true in the set of natural numbers, that the students supposed to be valid also in the set of rational numbers. The consecutive for instance.”

I: “This property is often easily transported also into R.”

T5: “This problem of the consequent indeed is.. ”

T7: “The consequent of 0,2 is 0,3. The consequent of ¾ is 4/4.”

I: “So you are confirming that they have problems in Q. ”

T6: “They are used to concentrate on the last thing they are doing. The last thing we do are real numbers, what you did before, the rational numbers, is overcome. What they know relatively well all the real numbers, because in the moment in which … 1 ,2 , 3 the primary’s school teachers taught them..”

T5: “Be careful, sometimes the primary’s school teachers don’t know these things.”

T7: “The concept of relative numbers already is.. the rational numbers… they know this is a subset of real numbers. The problem is not R.”

I: “So they work without knowing in which set they work.”

T6: “No. They know they’re working in the set of real numbers. If you say you are going to work with rational numbers they are in crisis. < Does a set called rational numbers exist?>. They know they it’s a continuum, that there is an association between points and numbers.”

I: “Well, this is not so trivial from a mathematical point of view.”

T6: “They have intuitions. Every point is associated to a number, that I can represent number on a line, that there is this continuum. The rational numbers are forgotten.”

T7: “Do they ask you how many points have they to use? 2,3 4 infinite. Even if say it thousands times they always ask it to you. They are scared, this is a recurrent question.”

T5: “It’s true. Can I use the number of points I want?”

T6: “In the moment itself they understand, as he was saying [T6], after a week they ask you again.”

I: “What do they ask?”

T5: “They draw two points and then the line ends.”

T6: “No, two no. I don’t want.”

T5: “Why not two?”

Focus group with teachers 8-9-10

T8: “I would not have done it that way. This is not the graphic solution. Graphic is when you have the intersection in a point. Y = x, y = 22/17, solution, intersection. If we want to make it graphic. This is not graphic.”

I: “But it’s widespread.”

T8: “It’s awful. You have to put f(x)=g(x). Then <, >, … They have to be precise with the graphics. These are intervals, what does it mean? [..] Was the discussion at the Master interesting?”

I: “Yes, we couldn’t stop it even if it was Saturday. The debate was very heated. A physician asserted that real numbers don't exist and so it's senseless to talk about them. So he animated the discussion in a very interesting way. It was wonderful."

T8: "This was a bit drastic. Yes, if he thinks at the measure in that sense under certain aspects he's right. I've never an infinity of digits, I can have 1,4 1,42 maybe he thinks at this … boh.. He was drastic."

I: "A statement like this animates a discussion."

T8: "Maybe he was wishing to provoke."

I: "Another said to me: < I challenge you to construct real numbers!>"

T8: "Beh, it from the epoch of … Archimedes was already able to do it! Archimedes… it lasts still nowadays. The Archimedean axiom. If you place them in the correct way, given two numbers we always find another number that multiplied by another becomes bigger. He was able, think if we are not!"

I: "It's a debated topic."

T8: "Yes, but how did he study these real numbers? I understand that one can be confused as we were studying these numbers with Pini [his Professor of Calculus at the University]. You had to start from N, then you had to construct Z, Z contained something that was equipotential to N. Then from Z you had to construct Q; then there was something more like √2, so you … I remember there was a proof… the convergence of that intervals that seemed to arrive there but the convergence was on the void, there was nothing. So you have to construct something. Then there was another way to construct π. π … how was it? Where does it come from? It's transcendent. I'm old… I studied Mathematics at the University."

I: "At school what do you turn out to do about real numbers."

T8: "Uh.. only a few. You told them some stories, the classical approach in the third year with root square of 2, that is the classical absurd proof, with the bisection method, that is given two rational numbers there is always a number in the middle but in some cases there isn't even if the intervals converge, so what I remember from Pini is that intuitively you can see it, while I show that you can't obtain π this way, it's another kind of number that is there, inside, and that you need because otherwise the ratio between circle and ray is not there. It comes out from this kind of stuff. Sometimes I look my old notes but … mah… I think they are so complicated."

I: "And does the students understand?"

T8: "√2, yes. It's not a problem. π .. they remain a bit more confused. I've a French book in which you see that π comes out in so many ways… bodies, organisms, leaves… an infinity of stuff. It's really an extraordinary number.. it's called…"

T10: "Let's go on with the interview."

I: "Let's discuss a bit about the questionnaire."

T10: "Well, so we are going to relax"

I: "Yes."

T9: "I work in the first two year, I don't remember irrational inequalities.."

T8: "What?"

T9: "I do the irrational equations, not the irrational inequalities. You present in the second year the irrational numbers, but you don't spend too time. You explain well the .."

I: "We are going to discuss about the real numbers since no one knows how to introduce them in the high school, there are many opinions and difficulties, the national curricula says you should formalize real numbers in the last year, but it’s not trivial.”

T8: “No, no, no. You use them but you don’t formalize.”

I: “I re-pose a question. How do you introduce real numbers? In the questionnaire did you answer thinking at mathematics or thinking at your students?”

T9: “I thought at the students. I work in the first two years, maybe my colleague who work in the last three years may say more. In the beginning of the second year you, depending on the classes...”

T10:”I remember a class with a students, who was a genius, in that class we introduced the separating element, the contiguous classes and things like that. We had the Mereu [textbook]. In other classes in which you have difficulties, you show the line, you show that if you consider a square, you take the diagonal, you use the Pythagoras’ theorem that is known since the previous years, then you stop.”

T8: “We also do the same.”

T6: ”If you have another class, with better students, you can do something more.”

I: “In the question concerning the usefulness of real numbers, exponential, logarithmic. What did you choose?”

T8, T9, T10: “Quite all!”

T10: “Maybe the series can also be introduced using N”

T8: “No, the series…”

T10: “We chose everything.”

T8: “The exponential… for sure! And how could you do?”

I: “Well. And how do introduce it exactly?”

T8: “My approach is quite phenomenological. I present the growth rhythms. I show or the vaccines, how do leaves grow in a tree, and so on. So you have or the history of the merchant who met a man who said to me: < I give you a million, you give me 1 euro, 2 euro and so on for a month. Doing this the real numbers aren’t involved since this is a sequence of whole numbers, but when you go and see what happens in the middle then you need the real numbers.”

I: “So how do you manage expressions like 2 raised at $\sqrt{2}$?”

T10: “I present 2 raised at N, 2 raised at Z, 2 raised at 2, and then you go on with your fantasy! ”

T8: “Yes, you do this way.”

I: “The concept of power is not so easily associated to repeat n times, …”

T10: “You present 2 raised at N, 2 raised at Z, 2 raised at Q and then you give 2 raised at $\sqrt{2}$ and you explain that this becomes a matter of separation between one and the other, the way you can define it. ”

T8: “Yes , you complete. There’s always the problem of π, that you can’t construct this way. The transcendent… is not possible to construct it this way… it’s another thing.”

T9: “It’s another thing.”

T8: “But it arise in a wonderful way when you present the complex numbers with Euler… it’s wonderful. I always do this because I say…”

T9: “When do you do this? In the third, the fourth year?’”

T8: “In the fourth.”

I: “And the limits? You need the real numbers also to introduce the limits?”

T9: “Yes, I chose it.”

T8: “Of course! Yes! You need a minimum of topology of the line.”

I: “And how do you do it?”

T8: “I try to avoid as much as I can the epsilon/delta, because they get confused. The classical definition that a student see usually in the first Calculus course at the University, that is epsilon/delta.. at a certain moment I introduce it, but before I try to make them understand the concept.. quite … the neighborhood, an open interval, what it means. ”

I: “do you introduce the concept of limit point before or after the introduction of R?”

T8: “Yes, they have already R”.

I: “And do you present R?”

T8: “R had already been constructed in the second year! You have to complete Q, then you have already put inside the transcendent numbers. But you don’t formalize, you give a set in a correspondence with the line.”

I: “One of your students how would answer the question < What is a real number?>”

T9: “In the first two years he would answer it’s a separating element of contiguous classes, then I don’t know.”

T8: “Nothing more.”

T10: “We take for granted that they know what a real number is… and that the teacher also know it!”

T9: “Because after, always in the first two years, you execute the operations with the real numbers.”

T8: “The only thing that I stress very much is that you lose the conception of order, i.e. given a real numbers you don’t know what is its consecutive number, you can’t establish it.”

T10: “That there is not a precedent nor a consecutive number”

T8: “That you have difficulties with this.. the only thing I say is that given a real number we don’t know which is its consecutive number. What comes after is vague, it’s the only thing I give because I need it for the limit points, the extremes, but these are always chats very intuitive.”

I: “Do they work on the line? Do they understand?”

T8: “In an intuitive way they see, yes. They understand… they see.”

I: “And what about the infinity of the digits, the decimal representation? Do they understand in this way? Because changing representation something could change. Is it possible to make them understand in this register what is the continuity, the postulate of continuity, …”

T8: “Well, no. I do this at the beginning of the fifth year.”

I: “Think for instance at the video with the slider.”

T9: “I didn’t understand it.”

T8: “He was playing cheating, because you have a slider that is moving on the number line, the other on a small line but indeed it’s the same thing. It’s cheating.”

I: “How would you explain that the line is complete, that the holes are all filled?”

T6: “That way. You take a marker, you draw a line on the blackboard and stop.”

I: “You do something like that in your lessons, don’t you?”

T10: “Yes, while it’s different using the computer because the computer shows pixels while …”

T8: “Yes. Now I understand why that physician was arguing that real numbers don’t exist! Because the nature is not analogic, is digital, discrete!”

T9: “You have to take in account that in the first two years they study Euclidean geometry. When I introduce the line, you know that there are the axioms. When they are in the first year. It’s the first proof they see. You show that every point has a precedent, so.. that concept that between two points you have always a point … inside.. now I’m saying it bad, and you show this in the first year.”

T8: “Wait.. but if you show it this way, this should imply that root square is always between two rational numbers.”

T9: “No, but there you do this without associating numbers to points, but when you show …”

T8: “But…”

T9: “The Euclidean geometry.. they do this! So you show exactly this..”

T8: “But if you do the segment √2 using the roots it seems that here inside a number of that kind there should be. The separating elements emerge.”

T9: “Yes, yes.”

T8: “Yes…”

I: “All the aspects like separating elements, postulates of continuity, cuts, ..”

T8: “Postulate of continuity is given. A few… very few. ”

T10: “Let’s talk about the books. My book of the first two years don’t have the separating elements.”

T8: “There was in other books, also the book of the last three years, Mereu, was much better than this one.”

T10: “Why did we change it?”

I: “So the book is not useful to talk about real numbers?”

T10: “Wait. What do you mean with talking about real numbers? When you are at school. I understand because I studied too that things, the real number exist, the golden ratio and so on but when you work ”

I: “I’m also interested in this.”

T10: “This things we are saying are absorbed by two students. The other students, for instance you write pi/3 + kπ , that k, I have ten students that don’t know what it is, do you understand?”

T8: “Yes”

I: “so when they say real function in the real domain what do the really think at?”

T10: “They think at a continuous trace and at a small ball, because indeed we never show functions from

N into N. They don't pose the problem to connect the exponential. When you pose the problem : < What is e raised at $\sqrt{2}$ they don't pose the problem. They immediately say: < What are you asking us?> and put all the points, they trace all the line and draw it complete, they don't pose the problem that there could be some empty spaces. You pose the problem, some of them understand that this could be a problem and we should understand what we are talking about, you reassure them, my colleagues know what I am talking about, and they go on more serene, they know.. indeed this is what I also do because when Coen defined the real numbers in the training course, with all that classes and stuff like this, I studied them , I repeated and stop, you can forget it my dear, come back to draw the continuous line… in the sense that it's the same to me too, I confess."

I: "And when you work with the limit, the idea to approach without reaching a point. Is this accepted?"

T8: "No, no, no, no…"

T6: "Mmm.. it's accepted from a point of view.. "

T8: "Intuitive, but…"

T10: "Yes, it's exactly like this."

I: "You're saying that in the end R is always used by no one know what R is, aren't you?"

T10: "Yes."

I: "R is always used but a sense of vague remains."

T8: "Yes, in a precise manner, so defined no."

I: "What about the different cardinality?"

T8, T10: "No!"

I: "Sometimes in the books I read: the power of the continuum, different cardinalities and I think: <Does anyone really talk about this topic?>"

T8: "You always have to take in account where we should take this students. The problem is that they have to.."

T10: "In the previous years this was in the curricula. Now only the students who wants Mathematics courses at the University, and you say: <Wait I go home and I explain it to you tomorrow.>"

T8: "We have to do so many things, they need to know how to use that tool. That numerical set…"

T9: "However the textbooks are much more simplified, the rigorous aspect you had in textbooks now is disappeared. Also in the Scientific high school. Also in the textbook the rational numbers are a bit underestimated."

T8: "That book is worse than the other."

I: "They need to know how to use them…. I'm not so sure we need to introduce the real numbers. What do we really need real numbers."

T9: "You do a lot of nice things."

T8: "$x^2=2$ has a solution."

T9: "We do a lot of Geometry in the first two years, the problems where you use Pythagoras, Euclid, so many roots emerge."

T8: "You need to know that there are numbers that give you the solution. Somehow you write them with a finite number of digits or not. What is important is that it exists."

T10: "A problem is that they see them represented with e, π… these irrational are again different."

T8: "These are different."

T10: "You don't find them in equations."

T8: "Exactly; e emerges as limit, but π, π is not like this.."

T10: "Here these is also.."

I: "What about the idea of enlargement, from Q to R? How do you present it?"

T8, T6: "Using the roots."

I: "After the roots. From N to Z you have the need to enlarge because of the subtraction, from Z to Q because of the division. But from Q to R? You show the roots but.."

T6: "You stop there. The goal you have is that other numbers exist but you don't go on."

T9: "There are numbers that are not rational, there isn't a fraction that express them. In Geometry you connect the discourses because you present commensurable and incommensurable segments."

T8: "Yes."

T10: "If you don't introduce R you can't give an idea of C."

I: "Talking about representation is there a register in which it's better, it's easier to talk about real numbers?"

T8: "The graphical one! The graphical one is surely the easiest."

I: "Do you turn out…?"

T8: "Yes, yes. With the graphic is surely easier. The graphic to represent R but also C is surely the easiest. On an axes the imaginary, the real on the other. Absolutely, absolutely, this is the best register. Indeed I think we should work also more in this register, because they see it a lot of years before, in the first two years."

I: "A student in the graphic register showed a particular conception of the line. I asked: <Does the consecutive number of 0 in Q?>. He answered: < No, because Q is dense>. <And what about 0 in R?> >The consecutive of R exists and is the minimum of the number bigger than 0.>"

T8: "Of which interval?"

T10: "What's the age of the student?"

I: "He's a student of the fourth year."

T10: "He expressed well his thought…"

I: "They had studied it."

T10: "ahh.."

T8: "Superior extreme, inferior extreme… "

I: "In your mind why he said this? If the consecutive number doesn't exist in Q, how can it exist in R? In my mind he reasoned using fractions in the first case and using the line in the second case. He imagines that the line is full of points one after the other, so it's impossible for him to put another point between 0 and the minimum."

T8: "There is another crazy person who does things using the hyperreals, who encircle the points with a strange cloud of other strange points, like a monad, I read something but I never understood what it was. There are those numbers, they exist."

T10: "Yes, they exist. "

T8: "Mm.. they exist.. I don't know."

T10: "I appreciate the historical version of these proposals.

T9: "Yes, in my mind it's useful."

T10."In my mind you have to present it also in order to .. from the view of the culture that is enlarged, of the mathematics that is in evolution, things that are hard to understand because they need definitions that make the students struggle, that also make us struggle, in this sense. Then to go into the deep sense precisely we risk this kind of things to happen … we risk these things to happen if you go too much in depths. They don't have the tools yet."

T8: "You create confusion."

T10: "Maybe we ourselves … when you go the university you are prone to learn in a certain way, they try always…"

T9: "They defoliate, they look for the heart of the matter."

T8: "The better register is the graphic."

T10: "The graphic, because it's intuitive and as she were saying you have to start from the Geometry, that was the crisis, the lead to.."

T9: "Yes, also because historically this was the evolution. It's also incredible that they got it so fast. This discovery.."

T8: "When you tell that there was a secret…"

I: "Me too. I fell in love with this topic during the lessons of the professor Coen, who explained that in Dedekind there is indeed only a little more than what we can read in Euclid."

T9: "It's incredible."

T10: "Coen in his lesson was used to say: < It's incredible that the teachers don't know… > and I thought < Oh my God, what a shame…>. I didn't remember anything of that classes… You say the same things he said, and then you forget them again."

T8: "Yes, yes."

T10: "You forget them again because if you don't teach them every day … he talked about them as something essential.. "

T9: "Or you do like I do. I'm stopped at the first two years, apart from some experiences, the next year I will try the fifth year.. You have to recall everything because.. there is nothing to do.. you forget them. I'm graduated in Statistics. If you ask me the Gaussian, maybe I remember it, but all the rest.."

T10: "Sorry I have to go, thank you."

T8: "I understood why the physician said that. Because the universe is digital, is not analogic. He says the real numbers don't exist, the model of continuity doesn't work at all."

I: “Maybe but the fact that you can reduce the interval as long as you want in the model because you know you can do it.”

T8: “But, for instance, they are only approximations of what really happen. I see my son, when have any computation, he develop in Fourier’s series, Laplace and so on because he’s only interested in a particular interval of precision. He was provoking.”

Interview with Teacher 11 (excerpt from a further interview)

I: “In your mind is there a possible inter-disciplinary approach between functions in physics and in mathematics?”

T8. “My teacher in the training course always said that the concept of continuous function in Physics is always also derivable since they continuous phenomena. The continuity, a continuous dependence in Physics must also be derivable.”

I: “Yes, of course. In the motion we also need speed and acceleration.”

T: “Exactly.”

I: “These things are interesting. You are used to see trajectories that are always more regular than the continuous ones.”

T: “The classical trajectory is a continuous trajectory.”

I: “Not only continuous, more regular, also derivable at least times continuously. So in physics you see the functions before and they are much more regular than the continuous. Obviously we don't say it. ”

T: “They have this kind of concept of function. In my mind this is a misconception that when you teach Calculus you find in all the students, in all the students! They don't understand which is the sense of take a function defined in a set different from R, in R without a point.”

I: “In my mind this could be a connection. In Physics if you don't separate model and reality they see this, the function is identified with the phenomenon. So you associate to a point a function.”

T: “At least in the classical physics.”

I: “So the function coincides with the reality. Maybe if you stress that the trajectory and its mathematical description are something different, that mathematics give a model of that motion, these could unchain mathematics from its constraints to be necessary linked to reality. The trace is not the motion but a way to mathematize it.”

T: “We should stress this is a useful semiotic register”

I: “Very useful.”

T: “A way to see some properties.”

I: “If you turn out to stress this is a model.”

T: “If you work well since the first year, they should be ready to learn the concept of derivatives already in the fourth year.”

I: “Maybe. If you don't drive it to the limit or don't stress the computation.”

T: “I also with my students of the fifth year in Physics, even if they had not studied limits and integral I proved the Gauss theorem for the electric field's flow.”

I: “You already summed infinite infinitesimals!”

T: “The concept of differential ”

I: “In my mind too perfect functions are showed even where there are not all the regularities we listed before. For instance in the Clapeyron's representation of thermodynamics transformations what we know is a list of couple of variables measured in equilibrium states, while the transformations are very often represented by means of lines, in which every point is supposed to be known and always included in a small neighborhood of the previous point. I said this immediately to my students.”

T: “This could be a good example, the students have a too simplified model of function, that generates misconceptions. I love the Calculus itself and every particular function interests to me but they have too simplified models and always one asks me: < Why do we study functions with a different domain, with a hole in the domain?>”

I: “You approximate the phenomena with continuous functions but you never say it.”

T: “Taking it for granted”

I: “There is a list of assumptions the textbook obscures. Clapeyron of course was aware of it!”

T: “The really important thing is the awareness. This is a good example. The students are less and less available to play this game. They don't go beyond with the play of thinking.”

I: “We should find not regular functions.”

T: “We should find significant examples that model situations... My students for instance often ask me why we have to consider a domain without a point... Also I ask the students to find some values but also to evaluate their trend. For instance, what does the sinus do from this point to this one? It grows. I demand them to see that at every number from 0 to $\pi/2$, at every number, that can be thought as the measure in radiant of an angle, to which correspond a number from 0 to 1 and grows. ”

I: “If you want to pass from the points to the line, you need some tools. At least the trend permit to know you don't have to take in account relative maximum and minimum points.”

T: “Then we don't draw it? Then.. how does it grow? Do you see it emerges the concept of derivative? If I take 10°, here, or if I increase of 10° after, they immediately see that here the function grows more here. Many students say to me: < Here it's quite horizontal!>. They knows it.. I work a lot on the concept of inclination. They know that this line … all the points have the same variation speed, so this is not a good model, but it's very hard to lead them to reason this way.”

I: “Maybe posing the problem as an open question...”

T: “Exactly. Then.. I don't know.. there is also Geogebra.”

I: “To make understand that there are the holes you can start from N”

T: “But this way you have only isolated points in the domain. ... the study of a domain without a point is different.”

I: “It's to make them imagine domains that are different from R.”

T: “The line with a hole always comes out... when you study the hyperbolic curve with parameters <For which values of k..? The classical homographic function in which you are asked to say what you obtain varying the parameter. I'm now thinking for the first time at this example. Which is the usual situation. I need this x at the denominator. If this is 0 it's a line, otherwise the line has a hole because you have existence conditions. In this case, I never thought to this, the unknown is the parameter itself.”

I: “Yes.”

T: “Here there is a hole for k=1/3.”

I: “to make them understand why we don't consider whole intervals we could also provide example of condition in applied problems in which there are constraints. You can wish to consider the function in a restricted domain. Otherwise why do you exclude a point? Think at your student's question this morning. <Why should we take off a point>?”

T: “Yes, in that case I was exemplifying a function .. motivate choices about the domain. ”

I: “What could be other problems? You were saying, to present function that has a domain without some points.”

T: “They today asked me another thing, I took it for granted. If a function is continuous in a set, is this continuous also in every subset? One of my students... I had highlighted so much that if you add points to the domain this can become continuous.”

I: “And if I reduce it?”

T: “Sometimes I think that to spend to much time in analyzing these concepts..”

I: “The question was well posed.. Maybe you think they are thinking at the limits, while probably there are using more intuitive concepts. As if they thinks that if it's a strange case we use limits, if they see it...”

T: “The continuity is this discourse..”

I: “It's like they have different theories, different methods for different problems. You sometimes work formally, sometimes intuitively. They could think these are different problems.”

T: “Not only. You create a visual image showing practical examples, then, if you think at it, you demand them to do the contrary, from the equation recognize if the graph they have to construct is continuous or not. So you work in a register that is what you ask them to produce when they apply the concept.”

I: “Yes.”

T: “Their output...”

I: “So in your mind it's also a matter of registers.”

T: “I'm thinking now... their output is never pathological, because they are asked to work with usual elementary functions. In the connection points.. they are not prone, when the input you give has different forms.”

I: “It could be.”

T: “This question are recurrent, they ask often.”

I: “Are there other questions heard today that can give us a feedback from the students? I remember a student who said: < If it's a real function in the real domain it's continuous.> and the said: <I can't imagine another way. < This never comes to my mind>. He said exactly this sentence. Maybe we should not only present function in the real domain in the beginning.”

T: “I introduced the sequences just this year with them.”

I: “Maybe they are used from a lot of years to imagine continuous graphs.”

T: “I should try to give them a sequence and ask if it's continuous or not.”

I: “This could be a critical example. Let's write it down as example.”

T: “$f(n) = n^2$?”

I: “We should find a case in which without the theorems you can't answer, in which the theoretical approach is necessary.”

T: “Limits.. the theorem of comparison is the apotheosis.”

I: “An example in which if they only use the visual approach make mistakes. I have a thesis in which the discussion about infinitesimals is reported.. Leibniz, Berkeley, Newton. If the infinitesimal have or not an extension.”

T: “Analysis standard and non-standard. In Physics! For the precision you need that the infinitesimal is measurable, but for the precision of your final result it's more precise when it tends to 0.”

I: “This is the point, but there it was approached from the philosophical point of view.”

T: “It would be interesting because you are selling dynamical concepts with a static theory.”

I: “This is crucial, the discussions are indeed about dynamic/static, concrete/abstract (atomic approach), exist/not exist (if it's null, it doesn't exist, but if it doesn't exist it's not bigger than 0). The ontological issues may interest students.”

T: “The history maybe a support for their trust. I said to a students of mine: < We introduced in these two months concepts that lead to discussion … sometimes I say this to encourage but sometimes I have the opposite reaction, like : < If they didn't understand, how can I understand?>. It's a double-edged sword.”

I: “Maybe we should recreate the historical critic points. Newton approached a physical problem, Leibniz a mathematics problem. Both of them reach the same conclusions but from different point of view. Euler then intervene considering 0 the infinitesimal, Berkeley criticized it vehemently. This could help the students who don't accept what you're saying, because they have any doubt.”

T: “Also the concept of asymptotic behavior. These are conceptual nodes that should be approached with the accuracy of a teacher trainer. I'm now thinking to the sequence, this is more spontaneous...”

I: “Maybe they would not represent it graphically.”

T: “Graphically maybe they would trace the parabolic curve continuously.”

I: “It would be interesting to ask them.”

T: “If you think at it they have studied this before, think at the primary school, and this is natural concept. Is natural the extension or the restriction. Now they think at the restriction as not natural, while the construction of the exponential function....”

I: “This could be a critical point, the dialectic relation between extension and restriction.”

T: “Is there a natural direction? The path of the exponential function is an expansion.”

I: “How do you present it?”

T: “I work on the extension by continuity.”

I: “Should we go from discrete to continuous or from the continuous to the discrete?”

T: “I go from the discrete to the continuous.”

I: “But here you go in the backward.”

T: “Here I miss something about the real numbers... I tell you how I did it.. I'm going to do it now in my third class. I reconsider the scholastic history of powers. First of all we consider the variable at the exponent, then we recall the first misconception, i.e. power is multiplying a number by itself n times. Then I ask them the properties and to prove them. I focus on 2 raised at 0. What does it mean? A student once asked me: < My teacher explained to me that I multiply it 0 times, then I mustn't write it down, but since we're working with multiplications when we have nothing, it's 1, the neutral element...”

I: “Sometimes a teacher can explain something to herself and then explain it to the students in the same way. Maybe sometimes I do it teaching Physics [I laughs]”

T: “Me too. [T laughs]. I say here two things concerning powers. We have multiplications, the properties and so on. We reach a point in which we have to renounce at this. I miss here which is the exigence to do this, because you finalize it to study the exponential function in the real domain!”

I: “I think that if you divide a number by itself to conserve the formal properties you subtract the exponents and obtain the conventional expression 2 raised at 0.”

T: “I explain to them that to save the properties the mathematicians were forced to abandon the original conception. Paradoxically you define the exponential functions by means of the properties you want it to have. You transform the input value. ”

I: “You build what you want it to be and then you use the properties as a definition.”

T: “Paradoxically this become its definition. The exponential function is.. to the input sums.. that machine that giving sums returns products. Once understood this they don't forget the properties of logarithms.”

I: “I never saw the exponential this way.”

T: “This becomes the new definition because you define this but what is the sense? We do this to conserve that properties. The properties says to us that we can interpret 2 raised at -n as ½ raised at n.”

I: “Are you saying we need a structure?”

T: “Exactly. We renounce to the nursery rhyme: < A power 2 raised at n is the multiplication...> but you have the features we want. Here returns the concept of extending by continuity, even if it's another kind of continuity.”

I: “Maybe the extension by continuity is strange for them because they never extended by continuity what was obtained by means of this process. I take 2 raised at 0, 3, ½, and so on but these are discrete numbers. To trace the line you have to..”

T: “You need the real numbers”

I: “You have to add values that makes it continuous.”

T: “I say <Let's call..>, wait, I'm thinking now. This lead you to 2 raised at x with x rational, but the passage to rational to reals, this scheme cracks.. the powers' properties help you until x rational.”

I: “It's analogous to what Dedekind said about the fact we have to invent numbers. I put here a real number, I have 2 raised at x, but if x is unknown, you invent something that makes continuous the functions.”

T: “I like so much this idea of inventing since its trace remains in the moment in which we have to write that numbers, because we use symbols. To express the irrational numbers you have to use symbols I can't calculate them in their entirety. For instance root square of 2, Pi, 2 raised at Pi.”

I: “In this moment a teacher could read Dedekind, when he wrote that the construction is a free act of the thought. Also a mathematician talking about irrational ...”

T: “What is not obvious is making them aware that here is a hole! The density of Q makes it not obvious. Overall when you want to represent an infinitesimal visually … it's not easy. I travel again through the powers' properties, they drive me until here, but I say. I know that also 2 raised at root square of 2 exists, so how can we give sense to this?”

I: “The know that root square of 2 is not rational?”

T: “Yes, I prove it many times.”

I: “You say: < Make it full>. You put there 2, and then how do you extend it by continuity? If you ask them to it associating values. You could ask if in your mind this is complete. If they answer yes, we can conjecture that the choice of representing number on the line by means of collection of points is a big mistake. They should take something for every point. Or they have a misconception concerning Q that fill the line R you have gratis the problem to solve constructing real numbers.”

T: “The root square of 2 is known, they are able to construct.”

I:. “You could ask them to complete a function in an interval rather than in a point. They could understand the need of adding points to complete by continuity but also associate a value to a limit point. If you are on the rational... you create a critical situation in which to have the definition of continuity as limit is necessary.”

T: “Also if you complete it by continuity you're taking for granted it's continuous innately. What does it mean to complete by continuity? It's a chaos. We have to rethink the previous paths because you arrive..”

I: “You have a function presented like this and ask them to complete it by continuity.”

T: “It would also be nice.. Wait, now I tell you which my exigence is. First of all to investigate their image of continuity, so this could be an exercise: I give three images and you say if it' continuous and why, because the mathematical continuity is also this one [two pieces defined o intervals]. Do you bet they think that this is continuous? [one pieces]. In the assessment of the next week in the fifth class next week I'll put one question of this kind. Here the function is defined. F is continuous on R. What of these graphics are acceptable? In my mind this [cuspid] is not associated to continuity because they associate it to be smooth, with a unique equation... In fact they take for granted that.. they have a concept of similarity added to not to interrupt the trace."

I: "Maybe in their mind the function stops here and restart there ..."

T: "The word continuity is used very soon, also in the previous years. We have to investigate where you name continuity."

I: "Maybe the line is the first critical point. You draw I thanks to an axiom. Then the hyperbolic curve is the first critical function. In physics they only see a part with the inverse proportionality. When do we transform first the line in a function at school?"

T: "With the conic."

Appendix C: Teachers' answers to the written questionnaire (in Italian)

Teacher 1 (4)

Formazione (Laurea in..... e/o TFA, SSIS, altro): Laurea in Matematica- Dottorato in Matematica-TFA49
Anni di insegnamento: (0-5, 6-10, 11 o più): 0-5

D2: Ho studiato i numeri reali:
all'università di Matematica in un corso di Analisi
a scuola

D3: Le proprietà fondamentali dell'insieme dei numeri reali sono:
continuo
completo
ordinato
archimedeo

D4: In ogni estensione numerica (N → Z, Z → Q) c'è una operazione "critica" sugli elementi dell'insieme di cui si effettua l'estensione, ad esempio sottrazione, divisione, che porta alla definizione di un nuovo insieme. Come si può effettuare la costruzione di R a partire da Q?
per chiusura rispetto al concetto di "limite" di successioni di numeri razionali: partendo dall'operazione inversa della potenza si fa notare che "nascono" nuovi numeri che hanno la caratteristica di poter essere stimati dal basso e dall'alto in maniera sempre più precisa (cioè con differenza che sempre più piccola) dai numeri razionali. A questo punto si fa la "chiusura" di Q rispetto a questa operazione cioè si considerano tutti i numeri che si possono ottenere in questo modo.

D5: Secondo Lei si può definire un punto di accumulazione nell'insieme Q oppure è necessario usare
i numeri reali?
si ad esempio 0 è un punto di accumulazione di 1/n

D6: Nelle "Indicazioni nazionali per il sistema dei Licei" …

Le situazioni descritte sono sufficienti per
comprendere il "problema della formalizzazione dei
numeri reali"
3
La conoscenza dei numeri reali è fondamentale in
matematica
4
La conoscenza dei numeri reali è necessaria per
risolvere equazioni e disequazioni di secondo grado
1

D7: La padronanza delle proprietà dell'insieme dei
numeri reali è indispensabile per introdurre (può
selezionare più risposte):
calcolo integrale, calcolo differenziale

D8: Osservi il primo minuto del video al seguente link:http://www.youtube.com/watch?v=jk08WkwqTQCosa pensa dei supporti materiali utilizzati (cartoncino, disegni, etc.)? Scelga una o due opzioni

per l'apprendimento di questi contenuti non sono necessari supporti di questo tipo
,aiutano gli studenti a crearsi una immagine dei numeri reali che potrà essere utile in seguito

D9: Ha dei suggerimenti (inerenti il linguaggio, i supporti utilizzati, i fini dichiarati, ...) per migliorare il primo minuto di questo video?

L'incipit è molto buono (problema del cartoncino), e anche la parte che riporta il problema sulla retta reale; non mi piace la frase "dalla geometria sappiamo che": per me la definizione di radice quadrata è il lato del quadrato che ha area pari al radicando, quindi non è per "una regola geometrica"; non mi piace neanche il riferimento alla calcolatrice: questo problema è nato molto prima della calcolatrice e la calcolatrice non lo risolve.

D10: Osservi il video al seguente link: http://www.youtube.com/watch?v=kuKTyp_b8WIIl video può aiutare uno studente a comprendere la corrispondenza biunivoca tra numeri reali e punti della retta?

No, perché lo slider sale necessariamente con una scansione (cioè ha un passo) visibile, quindi non dà l'idea che ad esempio tra due numeri qualsiasi e "vicinissimi" ce ne stanno infiniti. In effetti lo slider mostra solo numeri razionali. Piuttosto userei una scala fissa sulla retta e il cursore che scorre lungo la retta. Oppure un' effetto zoom che va sempre più in piccolo facendo vedere che per quanto piccolo vai resta sempre un segmento lunghissimo se zoommi.

D11: Osservi il video seguente, in particolare dal minuto 10:20 al minuto 12:10http://www.youtube.com/watch?v=UEBK5DfPxvk Lei cambierebbe qualcosa nella spiegazione?

Probabilmente non distinguerei tra soluzione algebrica e grafica, ma direi semplicemente che visualizziamo meglio l'insieme delle soluzioni se le rappresentiamo sulla retta

D12: Crede che sia opportuna la distinzione tra soluzione algebrica e grafica di una disequazione?

No, perché non sono due "soluzioni" sono due modi diversi di scrivere una soluzione

D13: Quali tra queste metterebbe come soluzione dell'esercizio nel libro di testo? Le n°
2,5

D14: Cosa pensa delle soluzioni fornite dagli studenti?

le n° .... non sono da accettare come soluzioni
perché

la n° 4 non è accettabile perché 2 e 4 non sono soluzioni del sistema; la 1 è corretta quindi accettabile, ma credo che per uno studente sia molto difficile interpretarlo come intervallo centrato in 3 di raggio 1; inoltre non mi sembra (ma forse sono io che non lo vedo) che ci si arrivi naturalmente, mi sembra molto forzata

D15: Esprima i suoi giudizi sulle risposte fornite:
Quali soluzioni sono più adeguate al problema?
Perché?
2 e 5: sono quelle che meglio rendono in maniera più immediata l'idea intervallo di numeri reali,
Altro mi piacerebbe molto anche la soluzione]2,4[

D16: Quali tra queste metterebbe come soluzione dell'esercizio nel libro di testo e perché? Le n° ....
nessuna, ma la migliore è la 5; le 1,2,3,4,mi fanno pensare ad un intervallo continuo di soluzioni e dato che la
disequazione è in Q non è così (in effetti credo che non abbia senso risolvere disequazioni in Q, eventualmente ha un senso farlo in Z)

D17: Cosa pensa delle soluzioni proposte dagli studenti?
le n° .... non sono da accettare come soluzioni
perché
la 1 la eviterei perché è troppo esplicito il riferimento ad R; nella 2, 3, 4 aggiungerei esplicitamente l'intersezione con Q

D18: Esprima i suoi giudizi sulle risposte fornite
Quali soluzioni sono più adeguate perché saranno necessarie per introdurre altri concetti? Perché?

La 3 è interessante perché è l'unica"scrittura" in cui è rilevante il fatto che la disequazione sia in Q

Teacher 3 (115)

Formazione (Laurea in..... e/o TFA, SSIS, altro): Laurea in Matematica
Anni di insegnamento: (0-5, 6-10, 11 o più):18

D2: Ho studiato i numeri reali:
a scuola

---

all'università di Matematica in un corso di Analisi
sui libri divulgativi e sui testi originali da autodidatta

D3: Le proprietà fondamentali dell'insieme dei numeri reali sono:
assiomi delle operazioni, di ordinamento, di completezza

D4: In ogni estensione numerica (N → Z, Z → Q) c'è una operazione "critica" sugli elementi dell'insieme di cui si effettua l'estensione, ad esempio sottrazione, divisione, che porta alla definizione di un nuovo insieme. Come si può effettuare la costruzione di R a partire da Q?

storicamente iniziò con l'introduzione degli irrazionali algebrici (radicali) partendo da considerazioni geometriche - chiedere ad Ippaso da Metaponto quanto vale il rapporto tra diagonale e lato del quadrato; equivalentemente, si può partire ad es. dalla risoluzione di equazioni algebriche di grado almeno 2 andando più in là, verso la trascendenza, a me fa impazzire il teorema di Lindemann-Weierstrass; tra l'altro, la dimostrazione platonica (molto... reale in verità) dell'irrazionalità di radice di 2 è didatticamente ancora un ottimo strumento deduttivo

D5: Secondo Lei si può definire un punto di accumulazione nell'insieme Q oppure è necessario usare i numeri reali?
direi che occorre R

D6: Nelle "Indicazioni nazionali per il sistema dei Licei" sono presenti queste frasi:(per il primo biennio)Lo studente acquisirà una conoscenza intuitiva dei numeri reali, con particolare riferimento alla loro rappresentazione geometrica su una retta. La dimostrazione dell'irrazionalità di radice di 2 e di altri numeri sara un'importante occasione di approfondimento concettuale. Lo studio dei numeri irrazionali e delle espressioni in cui essi compaiono fornirà un esempio significativo di applicazione del calcolo algebrico e un'occasione per affrontare il tema dell'approssimazione.(per il secondo biennio)Lo studio della circonferenza e del cerchio, del numero π, e di contesti in cui compaiono crescite esponenziali con il numero e, permetteranno di approfondire la conoscenza dei numeri reali, con riguardo alla tematica dei numeri trascendenti. Attraverso una prima conoscenza del problema della formalizzazione dei numeri reali lo studente si introdurrà alla problematica dell'infinito matematico e delle sue connessioni con il pensiero filosofico.Per ognuna delle seguenti affermazioni indichi digitando il numero corrispondente alla scelta è: per niente d'accordo (1) solo parzialmente d'accordo (2)abbastanza d'accordo (3)completamente d'accordo (4)
Le situazioni descritte sono sufficienti per comprendere il "problema della formalizzazione dei numeri reali" 1

La conoscenza dei numeri reali è fondamentale in matematica4
La conoscenza dei numeri reali è necessaria per risolvere equazioni e disequazioni di secondo grado4

D7: La padronanza delle proprietà dell'insieme dei numeri reali è indispensabile per introdurre (può selezionare più risposte):

calcolo differenziale

calcolo integrale
intervalli di R
limiti
sistemi di equazioni
Altro (specificare) numeri complessi - NB Le proprietà di R sono ovviamente indispensabili per TRATTARE TUTTI gli argomenti proposti, ma quelli che non ho indicato possono essere INTRODOTTI anche senza R (es. exp e log con una tastiera)

D8: Osservi il primo minuto del video al seguente link:http://www.youtube.com/watch?v=jk08WkwqT-QCosa pensa dei supporti materiali utilizzati (cartoncino, disegni, etc.)? Scelga una o due opzioni

aiutano gli studenti a comprendere i numeri reali

aiutano gli studenti a crearsi una immagine dei numeri reali che potrà essere utile in seguito

D9: Ha dei suggerimenti (inerenti il linguaggio, i supporti utilizzati, i fini dichiarati, ...) per migliorare il primo minuto di questo video?

farei vedere un gomitolo di spago

D10: Osservi il video al seguente link: http://www.youtube.com/watch?v=kuKTyp_b8WIIl video può aiutare uno studente a comprendere la corrispondenza biunivoca tra numeri reali e punti della retta?

Sì, perchéfino ad un certo punto, visto che lo step è di 0.1, praticamente "discreto"

No, perché forse collega il concetto più alla "distanza" che alla "ascissa"

D11: Osservi il video seguente, in particolare dal minuto 10:20 al minuto 12:10 http://www.youtube.com/watch?v=UEBK5DfPxvk Lei cambierebbe qualcosa nella spiegazione?

soprattutto NON direi che si tratta di "due tipi di soluzione" !!!

D12: Crede che sia opportuna la distinzione tra soluzione algebrica e grafica di una disequazione?

No, perché non sono "due tipi di soluzione" ma due forme diverse della stessa soluzione (=sostanza), che vanno "visualizzate" assieme

D13: Quali tra queste metterebbe come soluzione dell'esercizio nel libro di testo? Le n°

2 e 5, ma anche la 4 correggendola in]2,4[

D14: Cosa pensa delle soluzioni fornite dagli studenti?

a. sono tutte ugualmente accettabili forse equivalenti ma non accettabili come forma "finale", sintetizzata

b. le n° .... non sono da accettare come soluzioni perché1 ha ancora la forma di una disequazione, 4 comprende gli estremi, 3 è equivalente ma non compattata

c. è meglio se gli studenti usano tutti la stessa sempre, cioè la n°..........., perché omologazione? mai!

D15: Esprima i suoi giudizi sulle risposte fornite:

Quali soluzioni sono più adeguate al problema? Perché?la soluzione è la stessa, le sue forme che preferisco quelle da me prima indicate

Quali soluzioni sono più adeguate perché saranno necessarie per introdurre altri concetti? Perché?2 >> condizioni di esistenza (domini di funzioni); 3 e 4 >> domini come sottoinsiemi di R

D16: Quali tra queste metterebbe come soluzione dell'esercizio nel libro di testo e perché? Le n° ....

tutte tranne la 3, che comprende gli estremi; ma specificherei in tutte che x deve stare in Q...

D17: Cosa pensa delle soluzioni proposte dagli studenti?

b. le n° .... non sono da accettare come soluzioni perché3: comprende gli estremi

d. nessuna è accettabile perché occorre specificare che x sta in Q

D18: Esprima i suoi giudizi sulle risposte fornite

Quali soluzioni sono più adeguate al problema? Perché?1, 2 e 4

Quali soluzioni sono più adeguate perché saranno necessarie per introdurre altri concetti? Perché?

la 5 è la forma standard di un intorno circolare (di 0), utile per i limiti

Teacher 4

Formazione (Laurea in..... e/o TFA, SSIS, altro):laurea, dottorato in matematica e SSIS
Anni di insegnamento: (0-5, 6-10, 11 o più):0-5

D2: Ho studiato i numeri reali:
all'università di Matematica in un corso di Analisi

D3: Le proprietà fondamentali dell'insieme dei numeri reali sono:
Due operazioni che rendono i reali un campo (di caratteristica zero). Ordinamento totale, compatibile con le operazioni, completo.

D4: In ogni estensione numerica (N → Z, Z → Q) c'è una operazione "critica" sugli elementi dell'insieme di cui si effettua l'estensione, ad esempio sottrazione, divisione, che porta alla definizione di un nuovo insieme. Come si può effettuare la costruzione di R a partire da Q?
Due costruzioni equivalenti: 1) il metodo delle sezioni di Dedekind (elementi separatori di due insiemi di unione Q, che però non hanno minimo o massimo in Q, esempio: {x in Q| x^2<2} e {x in Q|x^2>2}), 2) insieme delle successioni di Cauchy, modulo le successioni convergenti.

D5: Secondo Lei si può definire un punto di accumulazione nell'insieme Q oppure è necessario usare i numeri reali?
Si può definire anche in Q. Per esempio, l'insieme {x in Q| x>0} ha 0 come punto di accumulazione.

D6: Nelle "Indicazioni nazionali per il sistema dei Licei" …

Le situazioni descritte sono sufficienti per comprendere il "problema della formalizzazione dei numeri reali" 2

La conoscenza dei numeri reali è fondamentale in matematica2
La conoscenza dei numeri reali è necessaria per risolvere equazioni e disequazioni di secondo grado1
D7: La padronanza delle proprietà dell'insieme dei numeri reali è indispensabile per introdurre (può selezionare più risposte):

calcolo differenziale

calcolo integrale
successioni e serie

D8: Osservi il primo minuto del video al seguente link:http://www.youtube.com/watch?v=jk08WkwqT-QCosa pensa dei supporti materiali utilizzati (cartoncino, disegni, etc.)? Scelga una o due opzioni

aiutano gli studenti a crearsi una immagine dei numeri reali che potrà essere utile in seguito

D9: Ha dei suggerimenti (inerenti il linguaggio, i supporti utilizzati, i fini dichiarati, ...) per migliorare il primo minuto di questo video?
Mi lasciano perplesse due cose: 1) di solito si conosce (si misura) il lato di un quadrato e si calcola l'area, non viceversa. Forse si potrebbe enunciare il problema come un problema di "raddoppiamento"... 2) c'è un po' di confusione tra il problema di arrotondare un numero fino ad avere abbastanza cifre decimali per motivi pratici, e quello d stabilire che un numero abbia infinite cifre decimali aperiodiche. Questo secondo problema è astratto e indipendente dal primo. Potrebbe essere che radice di 2 sia periodico di periodo 1000000. Gli esempi dati non permetterebbero di distinguerlo da un numero decimale aperiodico...

D10: Osservi il video al seguente link: http://www.youtube.com/watch?v=kuKTyp_b8WIIl video può aiutare uno studente a comprendere la corrispondenza biunivoca tra numeri reali e punti della retta?

No, perché non è chiara la corrispondenza; è fissato l'origine ma non una scala (il punto corrispondente a 1). Difficile giustificare la posizione dei numeri negativi

D11: Osservi il video seguente, in particolare dal minuto 10:20 al minuto 12:10http://www.youtube.com/watch?v=UEBK5DfPxvk Lei cambierebbe qualcosa nella spiegazione?

Almeno dire "rappresentazione" della soluzione: la soluzione è l'insieme dei numeri che soddisfano la disequazione: essa è rappresentabile in modi differenti, tra cui quello "algebrico" e quello "grafico"
D12: Crede che sia opportuna la distinzione tra soluzione algebrica e grafica di una disequazione?
No, perché No, perché la soluzione è un ente astratto: ciò che cambia è la sua rappresentazione

D13: Quali tra queste metterebbe come soluzione dell'esercizio nel libro di testo? Le n°
2 e 5
D14: Cosa pensa delle soluzioni fornite dagli studenti?
b. le n° .... non sono da accettare come soluzioni perché la 4 è sbagliata (gli estremi non sono inclusi)
D15: Esprima i suoi giudizi sulle risposte fornite:
Quali soluzioni sono più adeguate al problema? Perché? la 2 e la 5 indicano in maniera più chiara l'insieme delle soluzioni

D16: Quali tra queste metterebbe come soluzione dell'esercizio nel libro di testo e perché? Le n° ....
nessuna
D17: Cosa pensa delle soluzioni proposte dagli studenti?
d. nessuna è accettabile perché solitamente le espressioni usate indicano intervalli reali, non razionali
D18: Esprima i suoi giudizi sulle risposte fornite
Quali soluzioni sono più adeguate al problema? Perché? nessuna, vedi sopra

Teacher 8

Formazione (Laurea in..... e/o TFA, SSIS, altro):*laurea in matematica*
Anni di insegnamento: (0-5, 6-10, 11 o più):*più di 20*

D2: Ho studiato i numeri reali:
a scuola
all'università di Matematica in un corso di Analisi

D3: Le proprietà fondamentali dell'insieme dei numeri reali sono:
insieme denso, corpo

D4: In ogni estensione numerica (N → Z, Z → Q) c'è una operazione "critica" sugli elementi dell'insieme di cui si effettua l'estensione, ad esempio sottrazione, divisione, che porta alla definizione di un nuovo insieme. Come si può effettuare la costruzione di R a partire da Q?
rapporto tra la misura del lato quadrato e diagonale

D5: Secondo Lei si può definire un punto di accumulazione nell'insieme Q oppure è necessario usare i numeri reali?
è necessario usare R

D6: Nelle "Indicazioni nazionali per il sistema dei Licei"
Le situazioni descritte sono sufficienti per comprendere il "problema della formalizzazione dei numeri reali"*3*
La conoscenza dei numeri reali è fondamentale in matematica*4*
La conoscenza dei numeri reali è necessaria per risolvere equazioni e disequazioni di secondo grado*3*
D7: La padronanza delle proprietà dell'insieme dei numeri reali è indispensabile per introdurre (può selezionare più risposte):
funzione esponenziale
funzione logaritmica
calcolo differenziale
calcolo integrale
intervalli di R
limiti

D8: Osservi il primo minuto del video al seguente link: http://www.youtube.com/watch?v=jk08WkwqT-Q Cosa pensa dei supporti materiali utilizzati (cartoncino, disegni, etc.)? Scelga una o due opzioni
aiutano gli studenti a comprendere i numeri reali
aiutano gli studenti a crearsi una immagine dei numeri reali che potrà essere utile in seguito
D9: Ha dei suggerimenti (inerenti il linguaggio, i supporti utilizzati, i fini dichiarati, ...) per migliorare il primo minuto di questo video?
no

D10: Osservi il video al seguente link: http://www.youtube.com/watch?v=kuKTyp_b8WIIl video può aiutare uno studente a comprendere la corrispondenza biunivoca tra numeri reali e punti della retta?
No, perché *i due slider mostrano la stessa cosa, non aggiunge niente*

D11: Osservi il video seguente, in particolare dal minuto 10:20 al minuto 12:10http://www.youtube.com/watch?v=UEBK5DfPxvk Lei cambierebbe qualcosa nella spiegazione?

Quella non è la soluzione grafica, che è il confronto tra i grafici i due rette

D12: Crede che sia opportuna la distinzione tra soluzione algebrica e grafica di una disequazione?
Sì, perché *si perché educa alla flessibilità*

D13: Quali tra queste metterebbe come soluzione dell'esercizio nel libro di testo? Le n°
2 o la 4
D14: Cosa pensa delle soluzioni fornite dagli studenti?
b. le n° .... non sono da accettare come soluzioni perché *5 perchè non è espressa con linguaggio formale*
D15: Esprima i suoi giudizi sulle risposte fornite:
Quali soluzioni sono più adeguate perché saranno necessarie per introdurre altri concetti? Perché? *3 introduce alla logica*

D16: Quali tra queste metterebbe come soluzione dell'esercizio nel libro di testo e perché? Le n° ....
la 2 con aggiunta x appartiene. a Q
D17: Cosa pensa delle soluzioni proposte dagli studenti?
d. nessuna è accettabile perché *non si distingue tra Q ed R*
D18: Esprima i suoi giudizi sulle risposte fornite
Quali soluzioni sono più adeguate al problema? Perché?*2*

Teacher 9

Formazione (Laurea in..... e/o TFA, SSIS, altro):*laurea in scienze statistiche ed economiche*
Anni di insegnamento: (0-5, 6-10, 11 o più):*27*

D2: Ho studiato i numeri reali:
a scuola

D3: Le proprietà fondamentali dell'insieme dei numeri reali sono:
in una classe seconda si introducono i numeri reali come elemento separatore di due classi contigue ,dopo aver presentato l'esistenza di numeri irrazionali con un approccio storico grazie al teorema di Pitagora

D4: In ogni estensione numerica (N → Z, Z → Q) c'è una operazione "critica" sugli elementi dell'insieme di cui si effettua l'estensione, ad esempio sottrazione, divisione, che porta alla definizione di un nuovo insieme. Come si può effettuare la costruzione di R a partire da Q?
il problema nasce dal verificare che i numeri razionali non permettono di esaurire gli insiemi numerici e quindi se arrivo a Q utilizzando ad esempio la media aritmetica di 2 numeri successivi, per arrivare all'insieme degli irrazionali presento il teorema di Pitagora che mi porta ad un numero che non è razionale ,necessariamente

D5: Secondo Lei si può definire un punto di accumulazione nell'insieme Q oppure è necessario usare i numeri reali?
é necessario usare i numeri reali

D6: Nelle "Indicazioni nazionali per il sistema dei Licei" …

Le situazioni descritte sono sufficienti per comprendere il "problema della formalizzazione dei numeri reali" *3*

La conoscenza dei numeri reali è fondamentale in matematica *4*

La conoscenza dei numeri reali è necessaria per risolvere equazioni e disequazioni di secondo grado *4*

D7: La padronanza delle proprietà dell'insieme dei numeri reali è indispensabile per introdurre (può selezionare più risposte):

funzione esponenziale

funzione logaritmica
calcolo differenziale
calcolo integrale
intervalli di R
limiti
sistemi di equazioni

D8: Osservi il primo minuto del video al seguente link: http://www.youtube.com/watch?v=jk08WkwqT-QCosa pensa dei supporti materiali utilizzati (cartoncino, disegni, etc.)? Scelga una o due opzioni
aiutano gli studenti a comprendere i numeri reali

queste immagini saranno molto utili al momento di imparare a risolvere le disequazioni di secondo grado

D9: Ha dei suggerimenti (inerenti il linguaggio, i supporti utilizzati, i fini dichiarati, ...) per migliorare il primo minuto di questo video?
no

D10: Osservi il video al seguente link: http://www.youtube.com/watch?v=kuKTyp_b8WIIl video può aiutare uno studente a comprendere la corrispondenza biunivoca tra numeri reali e punti della retta?
No, perché *no in quanto inutile*

D11: Osservi il video seguente, in particolare dal minuto 10:20 al minuto 12:10http://www.youtube.com/watch?v=UEBK5DfPxvk Lei cambierebbe qualcosa nella spiegazione?
Non la chiamerei soluzione bensì rappresentazione della soluzione in via grafica

D12: Crede che sia opportuna la distinzione tra soluzione algebrica e grafica di una disequazione?
No, perché *no la soluzione viene presentata per via grafica*

D13: Quali tra queste metterebbe come soluzione dell'esercizio nel libro di testo? Le n°
in un biennio va bene la 2, in un triennio è meglio]2,4[

D14: Cosa pensa delle soluzioni fornite dagli studenti?
a. sono tutte ugualmente accettabili *no*

b. le n° .... non sono da accettare come soluzioni perché*4 è errata*
c. è meglio se gli studenti usano tutti la stessa sempre, cioè la n°.........., perché*5 perché è corretta*
d. nessuna è accettabile perché *no*

D15: Esprima i suoi giudizi sulle risposte fornite:
Quali soluzioni sono più adeguate al problema? Perché? *la sol 5 mostra l'intervallo*

D16: Quali tra queste metterebbe come soluzione dell'esercizio nel libro di testo e perché? Le n° ....
4

D17: Cosa pensa delle soluzioni proposte dagli studenti?
a. sono tutte ugualmente accettabili *no*

b. le n° .... non sono da accettare come soluzioni perché*3 è errata*
c. è meglio se gli studenti usano tutti la stessa sempre, cioè la n°.........., perché *2 è corretta perché siamo in Q*
d. nessuna è accettabile perché*n0*
D18: Esprima i suoi giudizi sulle risposte fornite
Quali soluzioni sono più adeguate al problema? Perché? *solo la 2*

Teacher 10

Formazione (Laurea in..... e/o TFA, SSIS, altro):*Laurea in matematica e SSIS in matematica e Fisica*
Anni di insegnamento: (0-5, 6-10, 11 o più):*15*

D2: Ho studiato i numeri reali:
a scuola
all'università di Matematica in un corso di Analisi
in un corso di formazione per insegnanti

D3: Le proprietà fondamentali dell'insieme dei numeri reali sono:
I numeri reali ammettono inverso moltiplicativo e additivo, hanno l'elemento neutro di addizione e moltiplicazione

D4: In ogni estensione numerica (N → Z, Z → Q) c'è una operazione "critica" sugli elementi dell'insieme di cui si effettua l'estensione, ad esempio sottrazione, divisione, che porta alla definizione di un nuovo insieme. Come si può effettuare la costruzione di R a partire da Q?
In classe presenterei i numeri reali a partire da costruzioni geometriche di segmenti non commensurabili come la diagonale di un pentagono costruita a partire dalla sezione aurea del lato e la costruzione del rettangolo aureo.

D5: Secondo Lei si può definire un punto di accumulazione nell'insieme Q oppure è necessario usare i numeri reali?
Si si può definire

D6: Nelle "Indicazioni nazionali per il sistema dei Licei" …
Le situazioni descritte sono sufficienti per comprendere il "problema della formalizzazione dei numeri reali"*3*
La conoscenza dei numeri reali è fondamentale in matematica*4*
La conoscenza dei numeri reali è necessaria per risolvere equazioni e disequazioni di secondo grado*3*
D7: La padronanza delle proprietà dell'insieme dei numeri reali è indispensabile per introdurre (può selezionare più risposte):
funzione esponenziale
funzione logaritmica
calcolo differenziale
calcolo integrale
intervalli di R
limiti
sistemi di equazioni

D8: Osservi il primo minuto del video al seguente link: http://www.youtube.com/watch?v=jk08WkwqT-Q Cosa pensa dei supporti materiali utilizzati (cartoncino, disegni, etc.)? Scelga una o due opzioni
aiutano gli studenti a comprendere i numeri reali

aiutano gli studenti a crearsi una immagine dei numeri reali che potrà essere utile in seguito
D9: Ha dei suggerimenti (inerenti il linguaggio, i supporti utilizzati, i fini dichiarati, ...) per migliorare il primo minuto di questo video?
no

D10: Osservi il video al seguente link: http://www.youtube.com/watch?v=kuKTyp_b8WIIl video può aiutare uno studente a comprendere la corrispondenza biunivoca tra numeri reali e punti della retta?
No, perché *No, mi sembra inutile*

D11: Osservi il video seguente, in particolare dal minuto 10:20 al minuto 12:10http://www.youtube.com/watch?v=UEBK5DfPxvk Lei cambierebbe qualcosa nella spiegazione?
Non la chiamerei soluzione, ma al limite rappresentazione grafica della soluzione
D12: Crede che sia opportuna la distinzione tra soluzione algebrica e grafica di una disequazione?
No, perché *No, la soluzione è unica ed è un sottoinsieme di elementi dell'insieme considerato per le quali la proposizione è vera. Possiamo scegliere di rappresentarla in vari modi, ma è sempre la soluzione*

D13: Quali tra queste metterebbe come soluzione dell'esercizio nel libro di testo? Le n°
La due è la disequazione equivalente che comparirebbe nel libro di testo
D14: Cosa pensa delle soluzioni fornite dagli studenti?
a. sono tutte ugualmente accettabili *La risposta 5 è l'unica che rappresenta la soluzione anche se per via grafica, le altre sono disequazioni equivalenti*

b. le n° .... non sono da accettare come soluzioni perché*4*
c. è meglio se gli studenti usano tutti la stessa sempre, cioè la n°.........., perché *5*
d. nessuna è accettabile perché *no*
D15: Esprima i suoi giudizi sulle risposte fornite:
Quali soluzioni sono più adeguate al problema? Perché?*2*

Quali soluzioni sono più adeguate perché saranno necessarie per introdurre altri concetti? Perché?*2,4*

D16: Quali tra queste metterebbe come soluzione dell'esercizio nel libro di testo e perché? Le n° ....
se le soluzioni devono essere date in Q la soluzione 2 potrebbe essere la scelta migliore se si specifica che x appartiene a Q
D17: Cosa pensa delle soluzioni proposte dagli studenti?
a. sono tutte ugualmente accettabili *no*

b. le n° .... non sono da accettare come soluzioni perché*3-1-4*
c. è meglio se gli studenti usano tutti la stessa sempre, cioè la n°.........., perché*2 se si specifica x in Q*
d. nessuna è accettabile perché *no*
D18: Esprima i suoi giudizi sulle risposte fornite
Quali soluzioni sono più adeguate al problema? Perché?*2*

Quali soluzioni sono più adeguate perché saranno necessarie per introdurre altri concetti? Perché?*2*

Teacher 11

Formazione (Laurea in..... e/o TFA, SSIS, altro): *Laurea in Matematica, Dottorato in Analisi, SSIS, ...*
Anni di insegnamento: (0-5, 6-10, 11 o più):*10*

D2: ***Ho studiato i numeri reali:***
all'università di Matematica in un corso di Analisi
all'università in altri corsi

D3: ***Le proprietà fondamentali dell'insieme dei numeri reali sono:***
La completezza rispetto all'ordinemento (Dedekind). La completezza per successione (alla Cauchy) R è archimedeo (altrimenti le completezze sopra citate non sarebbero equivalenti) La densità di Q in R R non è numerabile

D4: ***In ogni estensione numerica (N → Z, Z → Q) c'è una operazione "critica" sugli elementi dell'insieme di cui si effettua l'estensione, ad esempio sottrazione, divisione, che porta alla definizione di un nuovo insieme. Come si può effettuare la costruzione di R a partire da Q?***
Ci sono diversi modi equivalenti per costruire R - Estendendo Q ad un campo ordinato e completo (completa nel senso che ogni sottoinsieme non vuoto e superiormente limitato ha estremo superiore oppure, equivalentemente, completo nel senso di Dedekind, ossia richiedendo che ogni coppia di classi contigue abbia elemento separatore). - Estendo Q ad un insieme in cui ogni successione di Cauchy sia convergente, ossia ad un insieme sequenzialmente completo. NOTA Essendo R archimedeo la completezza rispetto all'ordine è equivalente a quella per successioni) Altre costruzioni possibili: - Sarebbe anche possibile costruire R tramite allineamenti decimali, ma le difficoltà tecniche sarebbe maggiori - Sarebbe possibile anche costruire R cercando di estendere Q ad un insieme in cui si possa costruire una soddisfacente teoria della misura...

D5: ***Secondo Lei si può definire un punto di accumulazione nell'insieme Q oppure è necessario usare i numeri reali?***
In linea generale sì: Q è uno spazio metrico e quindi si può definire il concetto di punto di accumulazione. La non completezza di Q lo rende poco fecondo come concetto...

D6: ***Nelle "Indicazioni nazionali per il sistema dei Licei" …***

Le situazioni descritte sono sufficienti per comprendere il "problema della formalizzazione dei numeri reali"*2*
La conoscenza dei numeri reali è fondamentale in matematica*4*
La conoscenza dei numeri reali è necessaria per risolvere equazioni e disequazioni di secondo grado*2*
D7: ***La padronanza delle proprietà dell'insieme dei numeri reali è indispensabile per introdurre (può selezionare più risposte):***
funzione esponenziale
funzione logaritmica
calcolo differenziale
calcolo integrale
successioni e serie
intervalli di R
limiti

D8: ***Osservi il primo minuto del video al seguente link:http://www.youtube.com/watch?v=jk08WkwqT-QCosa pensa dei supporti materiali utilizzati (cartoncino, disegni, etc.)? Scelga una o due opzioni***
confondono gli studenti

Altro (specificare)*Aiutano a mostrare che la costruzione di R nasce anche da esigente metriche... ma i materiali utilizzati non aiutano certo l'astrazione necessaria per capire tale costruzione! Sono perplessa...*

D9: ***Ha dei suggerimenti (inerenti il linguaggio, i supporti utilizzati, i fini dichiarati, ...) per migliorare il primo minuto di questo video?***
Eviterei di lasciar intendere che se avessimo a disposizione una calcolatrice potremmo calcolare il valore esatto!

D10: ***Osservi il video al seguente link: http://www.youtube.com/watch?v=kuKTyp_b8WIIl video può aiutare uno studente a comprendere la corrispondenza biunivoca tra numeri reali e punti della retta?***
No, perché *Ovviamente si sta confondendo il concetto di densità con quello di completezza. Terrificante! I numeri visualizzati nello slide sono solo razionali... se il puntino che scorre fosse adimensionale farebbe dei salti...*

D11: ***Osservi il video seguente, in particolare dal minuto 10:20 al minuto 12:10http://www.youtube.com/watch?v=UEBK5DfPxvk Lei cambierebbe qualcosa nella spiegazione?***
Preferirei un linguaggio più preciso e meno ambiguo: la soluzione ---> l'insieme delle soluzioni la soluzione grafica e algebrica ---> l'insieme delle soluzioni si può rappresentare in modi diversi... riga continua---> dipende da quello che i ragazzi sanno già sui reali

D12: ***Crede che sia opportuna la distinzione tra soluzione algebrica e grafica di una disequazione?***
No, perché *la soluzione è unica (inteso come sottoinsieme dell'insieme in cui si è chiesto di risolvere la dissertazione), diversi sono i modi di rappresentarlo*

D13: ***Quali tra queste metterebbe come soluzione dell'esercizio nel libro di testo? Le n°***
Prima di tutto inserire la consegna specificando cosa si chiede di fare (risolvere il sistema? Scrivi un'equazione equivalente. Oppure Interpretarlo graficamente?...) Se la consegna fosse: risolvi in R il seguente sistema sceglierei la 2. Se la consegna fosse: scrivi un'equazione equivalente: sceglierei 1
D14: ***Cosa pensa delle soluzioni fornite dagli studenti?***
b. le n° .... non sono da accettare come soluzioni perché *No. La 4 è proprio errata. Mentre 1 e 3 non propongono il risultato nella maniere più efficace, pur mostrando delle abilità*
D15: ***Esprima i suoi giudizi sulle risposte fornite:***
Quali soluzioni sono più adeguate al problema? Perché?*2 e 5 perché comunicano con più immediatezza quali numeri sono soluzione.*

Quali soluzioni sono più adeguate perché saranno necessarie per introdurre altri concetti? Perché? *Non ho capito... Tutto dipende dalla consegna dell'esercizio...*

D16: ***Quali tra queste metterebbe come soluzione dell'esercizio nel libro di testo e perché? Le n° ....***
2 o 3 specificando con x in Q, e utilizzando una corretta notazione insiemistica. Senza modifiche reputerei tutti i risultati non idonei

***D17:* Cosa pensa delle soluzioni proposte dagli studenti?**
a. sono tutte ugualmente accettabili *purché si intersechi con Q. Infatti tutte queste scritture identificano gli stessi numeri razionali.*
***D18:* Esprima i suoi giudizi sulle risposte fornite**
Quali soluzioni sono più adeguate al problema? Perché? *Nessuna. Preferirei (x in Q| meno radice 5< x < radice 5)*