Analysing theories of meaning in mathematics education from the onto-semiotic approach[1]

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Abstract. The notions of meaning and sense, which are closely related to understanding, play an essential role in educational processes in general and, therefore, in the teaching and learning of mathematics. However, the terms 'meaning' and 'sense' are unevenly used in the theory of language, semiotics, philosophy and psychology. After describing the main characteristics of the referential and pragmatic theories of meaning, in this article, we detail various approaches to the notion of meaning, and present the perspective suggested by the Onto-semiotic Approach (OSA) of mathematical knowledge and instruction. This proposal partially incorporates the semiotic, ontological and epistemological assumptions of three general semiotic theories (by Hjelmslev, Peirce and Wittgenstein), with an holistic approach to the notion of object, sign and meaning, and takes into account some referential and operational theories, as well as other cognitive and socio-cultural theories, about meaning. We also analyse how the meaning of mathematical objects is addressed in three models that have an impact on mathematics education: Frege's logical-realist position, Vergnaud's conceptual triplet and Steinbring's epistemological triangle. In comparing the OSA with these specific semiotic theories, we identify concordances and complementarities between them. We finally suggest some implications for instructional design and teacher education of the OSA holistic approach to the meaning.

Keywords. Meaning, semiotics, epistemology, mathematics education, onto-semiotics.

1. Introduction

The term 'meaning' linked to that of 'understanding' is persistently used in mathematics education research and practice, since it is essential that students acquire the meaning of the mathematical terms, expressions and representations, that is, understand what the mathematical language refers to in its different registers. Balacheff alludes to meaning as a key word in mathematics teaching research: "A problem belongs to a *problématique* of research on mathematics teaching if it is specifically related to the mathematical meaning of pupils' behaviors in the mathematics classroom" (Balacheff, 1990, p. 258).

A fundamental problem for the teaching and learning of mathematics is clarifying the nature, different types and functions of mathematical objects, since it is not possible to approach the teaching and learning of a mathematical content if an ontological position on mathematics is not adopted before. This clarification is closely related to semiotics, since the use of symbols and all kinds of representations is consubstantial with mathematics.

> The significance of semiosis for mathematics education lies in the use of signs; this use is ubiquitous in every branch of mathematics. It could not be otherwise: the objects of mathematics are ideal, general in nature, and to represent them – to others and to oneself – and

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> to work with them, it is necessary to employ sign vehicles, which are not the mathematical objects themselves but stand for them in some way (Presmeg, Radford, Roth and Kadunz, 2018, pp. 2-3).

The terms meaning and sense are used persistently in curricular documents related to understanding mathematics. In the Principles and Standards (NCTM, 2000), the Standard "understanding the meanings of operations and how they relate to each other" is included in all grades from P-K2 to 9-12. It is linked to the meaning of concepts and operations (number, numerals, fraction, equal sign, add, multiply, etc.); meanings and uses of variables, equations, inequations, relationships; meaning of equivalent forms of expressions, similarity, etc. The notion of sense also plays an important role in NCTM (2000), where it is used as a synonymous of meaning, in expressions such as "Develop a sense of whole numbers"; "Making sense of mathematical ideas"; "Mathematics should make sense to students", etc.

From a more general point of view, the main objective of cultural psychology, according to Bruner (1990), is the study of the rules that human beings draw on when creating meanings in cultural contexts. "I argued in favor of a renewal and refreshment of the original revolution, a revolution inspired by the conviction that the central concept of a human psychology is meaning and the processes and transactions involved in the construction of meanings". (Bruner, 1990, p. 33). Dummett (1996) also relates meaning and understanding from a more general perspective:

> "A theory of meaning is a theory of understanding; that is, what a theory of meaning has to give an account of is what it is that someone knows when he knows the language, that is, when he knows the meanings of the expressions and sentences of the language" (Dummett, 1996, p. 3).

Another authoress who considers the idea of meaning fundamental to mathematics education is Sierpinska (1990), who, in turn, relates it intimately to understanding:

> Understanding the concept will then be conceived as the act of grasping this meaning. This act will probably be an act of generalization and synthesis of meanings related to particular elements of the "structure" of the concept (the "structure" being the net of senses of the sentences we have considered). These particular meanings also have to be grasped in acts of understanding (Sierpinska, 1990, p. 27).

Nevertheless, 'meaning' "is one of the most ambiguous and controversial terms in the theory of language" (Ullmann, 1962, p. 62). For example, Speaks (2014, p. 1) suggests that: "The term "theory of meaning" has figured, in one way or another, in a great number of philosophical disputes over the last century. Unfortunately, this term has also been used to mean a great number of different things". In the classic text *The meaning of meaning*, Ogden and Richards (1923) collected no less than seventeen definitions of meaning, to which new uses, implicit or explicit have been added since then, thus increasing its ambiguity. In the case of mathematics education, Pimm (1995) also points to the lack of clarity in the use of the terms understanding and meaning: "What we variously understand by 'understanding' and mean by 'meaning' is far from obvious or clear, despite these being two central terms in any discussion of the learning and teaching of mathematics at whatever level" (Pimm, 1995, p. 3).

The complexity of the linguistic semantic problem increases in the case of mathematics, due to the variety of semiotic registers (ordinary, oral and written language, specific symbols, graphs and tables, material objects, etc.) used in the mathematical practice. Furthermore, we are not only interested in analysing the *meaning* of the mathematical linguistic elements, but also that of the various objects that intervene in the mathematical practices carried out by people when solving problem-situations (languages, concepts, procedures, propositions, arguments). Such objects require competent interpretation and use from teachers and researchers when interested in the teaching and learning processes.

The problem of whether it is possible to develop a theory of meaning specific for mathematics education is still open. This theory should take into account both realistic/referential and pragmatic/operational positions about meaning, and also to serve as a basis to address the epistemological, semiotic, cognitive and sociocultural problems involved in mathematics teaching and learning processes.

The goal of this paper is to systematize and deepen on the characteristics of the theory of meaning proposed by the Onto-Semiotic Approach (OSA) to mathematical knowledge and instruction (Godino & Batanero, 1994; Godino, Batanero, & Font, 2007; 2019). We also describe the general theories of meaning in linguistic, semiotics and philosophy that serve as a basis for OSA, particularly those by Hjelmslev (1943), Peirce (1931-58) and Wittgenstein (1953; 1958), as well as the concordances and complementarities with three semiotic models that have some impact in mathematics education, Frege (1891; 1892), Vergnaud (1982; 1990) and Steinbring (1997; 2006). In this way, we offer a first response to the problem of clarification and comparison of semiotic theories used in mathematics education.

A holistic theory about the meaning of mathematical objects, such as that proposed by the OSA, would allow detailed analysis of mathematical activity, in taking into account the plurality of objects involved in such activity, as well as in articulating the epistemological, semiotic, cognitive and sociocultural dimensions involved in mathematics education research.

In Section 2, we present a synthesis of general theories about meaning, with emphasis on three authors: Hjelmslev, Peirce and Wittgenstein, selected because the onto-semiotic perspective on meaning, presented in Section 3, takes basic notions and assumptions from these authors. In particular, the OSA interprets and assumes Hjelmslev's notion of semiotic function, Peirce's semiotic triad, pragmatic maxim, and Wittgenstein’s notions of meaning as use, language game and form of life. We present in Section 3 the onto-semiotic theory, as a holistic approach to the questions of meaning and sense, as well as to that of object and sign. By grounding on an anthropological (Wittgenstein) and pragmatist (Peirce) conception of mathematics, and adopting the linguistic construct of semiotic function (Hjelmslev), we elaborate an ontology and semiotics that take into account the referential, operational, cognitive and epistemic/cultural theories of meaning. In Section 4 we describe three theories with a strong impact on mathematics education: Frege introduces a key distinction between sense and reference; Vergnaud proposes a cognitive interpretation of meaning, and Steinbring emphasizes an epistemological interpretation of the same. The objective of Section 5 is to begin the study of the concordances and complementarities between the semiotic theories considered, by showing the inclusive nature of the perspective elaborated in the OSA. Finally, in Section 6 we include a synthesis of the article and some implications for mathematics teachers’ education.

2. Theories of meaning

In general terms, two philosophical schools approach the question of meaning from different points of view. Firstly, the realistic or analytical tendency, tries to capture the meaning essence by identifying its main components. The second is the operational or pragmatic tendency, which studies words in action and is less interested in the meaning of words than in how words operate, and how expression and communication tools are used.

2.1. Realistic or analytical theories of meaning

According to Kutschera (1975), there are two categories, realistic and pragmatic, of theories of meaning: Realistic (or referential) theories conceive meaning as a conventional relationship between signs and concrete or ideal entities, which exist independently of linguistic signs; consequently, they involve a conceptual realism. "According to this conception, the meaning of a linguistic expression does not depend on its use in specific situations, but rather it is governed by its meaning, being possible a sharp division between semantics and pragmatics" (Kutschera, 1975, p. 34).

A word becomes meaningful for someone when he/she assigns that word an object (concept or proposition) as its meaning. In this way, some, not necessarily concrete, although always objectively given entities are the meanings of the words.

- The simplest form of realistic semantic appears when attributing to linguistic expressions only a semantic function, consisting in designating (by virtue of some conventions) certain entities; for example: The meaning of a proper name is the object or person that is designated by that name (for example, *Granada*).
- Predicates (for example, *this is red*; *A is bigger than B*) denote properties, relations or general attributes.
- Simple sentences (subject - predicate - object) designate facts (for example, *Madrid is a city).*

Therefore, in realistic theories (such as those advocated by Frege, Carnap, or in Wittgenstein's *Tractatus*), linguistic expressions have a relationship of attribution with certain entities (objects, attributes, facts).

2.2. Operational or pragmatic theories of meaning

The two basic ideas of the operational or pragmatic theories of meaning are the following:

- The meaning of linguistic expressions depends on the context in which they are used.
- A scientific, empirical and intersubjective observation of abstract entities - as concepts or proposition- which is implicitly admitted by realistic theories is not possible. Only the linguistic use of language is accessible to observation in a scientific investigation of language and it is from such use how we infer the meaning of abstract objects.

The worth of this operational approach is defining the meaning in contextual, purely empirical terms, without the need to resort to vague, intangible and subjective mental states or processes.

Wittgenstein (1953), in his *Philosophical Investigations,* openly defends a pragmatic or operational conception of meaning. In his formulation, a word is made meaningful by the fact of playing a certain function in a linguistic game, alongside of being used in this game in a certain way and for a specific purpose. A word can become meaningful, although there is nothing that is the meaning of that word, according to realistic theories.

For some authors the realistic and the operational views on meaning are irreconcilable. However, Ullman (1962) suggests that pragmatic theories (which he calls operational or contextual) are valid and necessary complement to realistic theories (which he calls referential):

> The researcher should begin by gathering an adequate sample of contexts and then deal them with an open spirit, by allowing meaning or meanings to emerge from the contexts themselves. Once this phase has been completed, he can safely move into the "referential" phase and attempt to formulate the meaning or meanings thus identified (Ullmann, 1962, pp. 76-77).

This observation by Ullmann is fundamental, and serves as a support for the model of meaning proposed by the OSA (described in section 3), where meaning is conceived pragmatically, by relating to problem-solving practices and contexts of use. Nevertheless, in addition, such practices involve words, symbols, and various kinds of representations that refer to other objects and systems; that is, a referential-type meaning is brought into play.

2.3. Semiotics and philosophy of language

Because mathematical objects cannot be apprehended directly by the senses, their ontological status requires the use of signs, such as symbols or diagrams. Consequently, *semiotics*, understood as the systematic study of the signs nature, properties and types, is receiving increasing attention in mathematics education research. "Semiotics has been a fruitful theoretical lens used by researchers investigating diverse issues in mathematics education in recent decades" (Presmeg, 2014, p. 539). However, what semiotics can offer to mathematics education is a complex issue since there are different semiotic approaches. Some of them may offer interesting avenues for the problems which we deal with in mathematics education, while others may be less useful (Radford, 2001).

In our case, we have been particularly interested in the Danish linguist Hjelmslev (1943) language theory, which can be useful to describe mathematical activity and the cognitive processes involved, both in the production and in communication of mathematical knowledge.

To describe and analyse the mathematical instruction processes, the linguistic manifestations of participants and the events that take place in didactical interactions should be transcribed in textual form. To carry out their work, didactical researchers collect the written planning of the instructional process, the lessons transcripts, interviews and written answers to assessment tests, etc. In short, their analyses are mainly applied to texts recording the mathematical activity developed by the participants.

Starting from the text as data, Hjelmslev's linguistic theory suggests the way that leads to a self-consistent and exhaustive description of texts, on account of its analysis, whose basic principle is:

> both the object under examination and its parts have existence only by virtue of these dependences; the whole of the object under examination can be defined only by their sum total; and each of its parts can be defined only by the dependences joining it to other coordinated parts, to the whole, and to its parts of the next degree, and by the sum of the dependences that these parts of the next degree contract with each other (Hjelmslev 1943, p. 23).

A key notion in Hjelmslev's theory of language is that of *function*, which is conceived as the dependence between the text and its components and between these components themselves. The terminals of a function that is, any object that has function with others, are called its *functives*. This notion of function is midway between the logical-mathematical and the etymological meaning of the term, closer to the first in the formal sense, but not identical to it.

> We shall be able to say that an entity within the text (or within the system) has certain functions, and thereby think, first of all with approximation to the logical-mathematical meaning, that the entity has dependences with other entities, such that certain entities premise others – and secondly, with approximation to the etymological meaning, that the entity functions in a definite way, fulfils a definite role, assumes a definite "position" in the chain (Hjelmslev, 1943, p. 34).

The sign function

For Hjelmslev, language is a system of signs, and a sign (or sign expression) is characterized, firstly and foremost, by being a sign of something else, in such manner that operates as a function. "A "sign" functions, designates, denotes; a "sign", in contradistinction to a non-sign, is the bearer of a meaning" (Hjelmslev, 1943, p. 43). "Any entity, and thus also any sign, is defined relatively, not absolutely, and only by its place in the context" (Hjelmslev, 1943, p. 45).

Among the possible types of dependence that can be identified between parts of a text those in which one part designates or denotes another thing stand out; the first part (expression plane) works as or represents the second term (content plane), that is, the first part points towards a content that is outside the expression. This function was designated as a *sign function* by Hjelmslev and presented by Eco (1991, p. 83) as *semiotic function*[2].

2.4. Peirce's pragmatism and semiotics

Charles Sanders Peirce (1839-1914) wrote a large number of works, which are receiving special attention in recent years in various fields on various topics related to philosophy, mathematics and semiotics, among other disciplines. In this section we summarize some ideas of special interest, which are used as foundations in various investigations in mathematics education (Campos, 2010, Otte, 2006; Presmeg, Radford, Roth, & Kadunz, 2018; Sáen-Ludlow, & Kadunz, 2016).

Pragmatism

Pragmatism is a philosophical current that emerged at the end of the 19th century in the United States. William James and Charles S. Peirce were the main promoters of this doctrine, which is characterized by the search for the practical consequences of thought. Pragmatism places the criterion of truth in the effectiveness and value of thought for life. In this trend of thought, understanding the practical use of a concept is more important than its conceptual definition. For pragmatists, the relevance of data arises from the interaction between human being and their environment, which leads to the rejection of invariable meanings and absolute truths: ideas, for pragmatism, are only provisional and can change from future research. By establishing the meaning of things from their consequences, pragmatism is often associated with practicality and utility according to the context.

Peirce oriented his pragmatism to investigate the way in which a generic individual uses signs to form new ideas and concepts and to reach the truth, instead of centering on what signs mean in the heart of social life. In his work, *How to make your ideas clear?* Peirce supported his pragmaticist idea of how to understand concepts clearly. His pragmatic maxim is a logical statement proposed as a normative recommendation or regulatory principle to 'achieve clarity in apprehension' in an optimal way. Peirce enunciated this pragmatic maxim over the years in various ways, one of which is easier to understand:

> 402. It appears, then, that the rule for attaining the third grade of clearness of apprehension is as follows: Consider what effects, that might conceivably have practical bearings, we conceive the object of our conception to have. Then, our conception of these effects is the whole of our conception of the object (Peirce, 1931-58, CP 5.402).

According to Burch (2014, p. 8) in suggesting that the full meaning of a clear conception is the complete set of its practical effects, Peirce has in mind that a meaningful conception should have some sort of 'effective experiential value'; it must, in some way, be related to some collection of possible empirical observations under specifiable conditions.

The notion of sign

The development of semiotics, or theory of signs, was fundamental in Peirce's intellectual life. According to Atkin (2010), we can distinguish three stages in Peirce's semiotics from 1860 to 1910, in which progressively the notion of sign and its different types are enriching, although the basic structure of signs and the process of signification are largely maintained.

For Peirce, the world of appearances is entirely composed of signs, which refer to qualities, relationships, events, states, regularities, habits, laws, etc., and which have meanings or

[2] A sign is always constituted by one (or more) elements of an EXPRESSION PLANE conventionally placed in correlation with one (or more) elements of a CONTENT PLANE [...]. A semiotic function is performed when two functives (expression and content) enter into mutual correlation. (Eco, 1991, pp. 83-84).

interpretations. A sign is one of the terms of a triad that are indissolubly connected to one another by an essential triadic relation, which Peirce calls the *sign relation*. In the definition given by Peirce: "A sign, or ***representamen***, is something which stands to somebody for something in some respect or capacity" (CP 2.228) the three basic elements: sign, object and interpretant are included.

The *sign* (also called *representamen*) is the term that is usually said to represent or mean something. The *object* is what is ordinarily understood as the *thing* signified or represented by the sign, that thing for which the sign is a sign o*f*. The *interpretant*[3] is the understanding that we reach of some relation between the sign and the object, such as the translation or development of the original sign (Atkin, 2010, p. 4). Under Peirce's definition of the sign relation, the interpretant must itself be a sign and a sign of the same object that is (or was) represented by the (original) sign. That is, the interpretant is a second signifier of the object, but one that now has openly a mental status. However, this second sign must itself have an interpretant, which in turn is a new, third sign of the original object, and again it is one with an openly mental status. And so on. In this way, if there is a sign of any object, then there is a sequence of signs of the same object. Therefore, since anything in the world of appearances is a sign, there is an infinite sequence of mental interpretants of an object.

Peirce's pragmatic maxim is interpreted and adopted by the OSA (Section 3) when this framework proposes to conceive the meaning of a mathematical concept in terms of the systems of operative and discursive practices performed by a person (or institution) to respond to a type of problem-situations. The OSA also interprets the notion of semiotic function taken from Hjelmslev, and articulates this idea with the Peirce's semiotic triad and unlimited semiosis process.

2.5. Wittgenstein's language games and forms of life

The realistic conception of word meaning is based on considering each signifier as a name, and this idea informs most reflection in the philosophy of mathematics and psychology. Mathematical expressions such as '0', '-2'; '$\sqrt{-1}$'; or even '+', 'x^4' and 'e^x', are taken as names of entities, and the question *What do they mean*?, is reduced to *Instead of what they are*? (Baker & Hacker, 1985).

Meanwhile, Wittgenstein (1953; 1956) argued that we should consider words as tools and clarify their uses in our *language games*. For example, the numerals are instruments for counting, ordering and measuring, and the mastery of the series of natural numbers is based on counting training. The language games and forms of life are main concepts in Wittgenstein's philosophy, which, together with considering mathematical concepts and propositions as grammatical rules of the languages we use to describe our worlds, characterize an anthropological approach to mathematics (Bloor, 1983). Since the meaning of words is conceived as its use made in different contexts, the meaning of "language game" should be found in Wittgenstein's use of this expression. Thus, for example, the communicative interaction that a builder A established with his assistant B when asking him for materials is considered as a language game. The communicative processes through which children learn their mother tongue are other examples. In the paragraph 23 of *Philosophical Investigations W*ittgenstein develops this idea with new examples:

> "23. ... Here the term "language-game" is meant to bring into prominence the fact that the speaking of language is part of an activity, or of a form of life.
>
> Review the multiplicity of language-games in the following examples, and in others:

[3] Sáenz- Ludlow and Kadunz (2016, page 3) represent the triadic sign with the word SIGN (in capital), to distinguish it from the representamen or vehicle-sign component. They indicate that understanding the process of meaning construction involves understanding the active role of the interpreting Person in the reconstruction of the real Object of a SIGN from the keys and indications provided by the sign-vehicles, which only indicate certain aspects of the real Object.

Giving orders, and obeying them—

Describing the appearance of an object, or giving its measurements-

Constructing an object from a description (a drawing)—

Reporting an event—

Speculating about an event—

Forming and testing a hypothesis—

Presenting the results of an experiment in tables and diagrams— …"

According to Marrades (2014), the expression *form of life* always appears in connection with language and, more specifically, with particular language games. In addition, in most examples, the form of life is characterized as a way of acting that is at the base of the use of language. This author indicates that the recourse to this notion occurs in a field of problems that concern the conceptual conditions of language comprehension. Understanding the meaning of an expression requires, not only to appeal to the rules governing its use, but also to see that use by reference to a broader existential structure, of which the language game is part:

> More specifically, a way of life designates, for Wittgenstein, a factual framework of relations between linguistic behavior, non-linguistic behavior and situations in the world, within which a language game develops. [...] Life forms are always social forms of life, social practices (Marrades, 2014, p. 146).

The constructs form of life and language game are incorporated into the OSA notion of institution (Section 3). An institution or community of practices shares some types of problems, specific ways of addressing these problems, as well as habits, norms, material and linguistic resources, which is equivalent to say that the members of the institution share life forms and language games. This is a basic postulate accepted by any sociocultural approach to knowledge in general and to mathematical knowledge in particular.

In the next section, we analyse how the OSA relies on the notions of sign function (Hjelmslev) and mathematical practice, which makes operational the anthropological view of Wittgenstein's mathematics and its relativity regarding language games and forms of life. In addition, the OSA interprets Peirce's semiotic triad in terms of function or correspondence between two terms, antecedent and consequent, connected by a correspondence criterion or rule. Peirce's pragmatic maxim is also translated in the OSA in terms of systems of operative and discursive practices, which are used to propose a conceptualization of the pragmatic meaning of mathematical objects, as opposed to the mentalist or idealistic views on mathematical concepts.

3. Meaning in the onto-semiotic approach

Under the OSA framework (Godino and Batanero, 1994; Godino, et al., 2007; Godino, et al, 2019) the notion of meaning and its relation to the notions of practice and object plays a central role. A practice is "any action or manifestation (linguistic or otherwise) carried out by somebody to solve mathematical problems, to communicate the solution to other people, so as to validate and generalize that solution to other contexts and problems" (Godino & Batanero, 1998, p. 182).

> In our conception, it is the fact that certain types of practices are carried out within certain institutions, which determines the progressive emergence of "mathematical objects" and that the "meaning" of these objects is intimately linked to the problems and activity carried out for its resolution, not being able to reduce this meaning of the object to its mere mathematical definition (Godino & Batanero, 1994, p. 331).

Although the OSA initial aim was to develop a theoretical model that would analyse the meaning of mathematical concepts, in successive developments this goal has been expanded and the framework is now applied to any type of object involved in mathematical practices; in addition, a categorization for these objects is proposed. The assumption is that the solution of mathematics education epistemological, cognitive and instructional problems previously requires to deal with the ontological problem, that is, to clarify the nature and types of mathematical objects whose teaching and learning is intended.

In the OSA framework, *meaning,* in a first approximation, is what is referred by a word, a symbol or any other means of expression, issued by a person in a communicative act, taking place in a specific context with another person or with himself. This approach gives an account of the referential meaning of the means of expression. However, words and symbols do not only mention or represent things but through them things are also *done*, that is, they intervene in operative practices. One operates and calculates with the words and symbols so that new objects are produced. For example, with the numerical symbols 2, 3 and the word "sum," by following certain agreed rules, the result 5 is produced, as well as a new mathematical object, the proposition that 2 + 3 is equal to 5, which is accepted to be true when deduced from the agreed rules.

Therefore, what role, besides the representational aspect, does this word, symbol or expression play in this operative practice? This central problem has to be addressed by a holistic theory about meaning, which takes into account both the referential and operational uses, and serves to model the meaning of expressions that refer to concepts (ideal, abstract objects), or to any other type of object.

In this section, we describe the conception of meaning in the OSA, and its relationship with the notions of mathematical practice and object. We contextualize the explanation with the example included in Figure 1 of a possible demonstration of the elementary arithmetic statement 2+3=5. We accept that practices 1) to 7) are performed by an epistemic subject who shares the language and way of life of people who know and are competent in Peano's axiomatic.

Proposition: 2 + 3 = 5

Demonstration:

1) The symbols, 2, 3 and 5 represent natural numbers.

2) Natural numbers are a set of symbols that satisfy the Peano's axioms, in particular, there is a first element, 1, and a following (successor), injective function $s: \mathbb{N} \to \mathbb{N}$, is defined. In this set, the sum, +, is defined recursively as:

$$n + 1 = s(n)\ ;\ n + s(m) = s(n + m)$$

3) In the sequence, 2 is the successor of 1, $2 = s(1) = 1 + 1$; 3 is the successor of 2, $3 = s(2) = 2 + 1$; and 5 is the successor of 4 which is next of 3, $5 = s(4) = s(s(3))$.

4) The sign = indicates the equivalence of two expressions.

5) The expression 2 + 3 represents the sum of the natural numbers 2 and 3.

6) Taking into account the definition of the sum of natural numbers and successor

$$2 + 3 = 2 + s(2) = s(2 + 2) = s(2 + s(1)) = s(s(2 + 1)) = s(s(3)) = s(4) = 5.$$

7) Therefore, the expressions 2 + 3 and 5 are equivalents.

Figure 1. Demonstration of an elementary arithmetic proposition (2 + 3 = 5)

3.1. Practices, objects and meanings

In the proposition, 2 + 3 = 5, the symbols 2, 3 and 5 refer to the natural numbers 2, 3 and 5, + refers to the addition arithmetic operation, and the symbol = indicates that the result of add 2 and 3 is the number 5.

In making these interpretations of the symbols, we are following rules agreed in the mathematical culture, so that if we understand the numbers, the addition and equality symbol, we must necessarily accept that "two plus three equals five".

From a conceptualist - idealist perspective of mathematics it is assumed that in the expression 2 + 3 = 5, in addition to visible material signs, other non-visible immaterial objects, usually considered as concepts, intervene; in this example the concepts of number 2, 3, 5, sum, and equality. To understand the justification of the proposition 2 + 3 = 5 truth it is necessary to explain what is meant by natural number, in particular what are the concepts of 2, 3, 5, sum and equality, or what is equivalent, what meanings must be attributed to these concepts.

To avoid falling into the idealistic Platonism trap, denounced by Wittgenstein, the OSA, assumes a pragmatist interpretation of concepts when speaking of its meaning. Thus, Godino and Batanero (1998, pp. 186-187) introduced the following definitions of meaning, where *problem field* refers to the types of mathematical problem-situations in whose solution the object plays a decisive role:

> DEFINITION 8: The meaning of an institutional object O_I is the system of institutional practices linked to the problem field from which O_I emerges at a given time.
>
> DEFINITION 9: The meaning of a personal object O_p is the system of personal practices that a person p carries out to solve the problem field from which the object O_p emerges at any given time.

The example of proposition 2 + 3 = 5 helps to clarify the scope of definitions 8 and 9, and therefore, of the OSA pragmatist assumptions. The statement O: 2 + 3 = 5 is a propositional mathematical object that requires a justification within the Peano's axiomatic specific language game. In response to the problem-situation consisting of proving O, the epistemic/institutional subject solving the problem replies what does O mean? Under OSA, the pragmatic meaning of O is the system of practices 1) to 7). The problem can be posed to a student, who will surely give a different answer. In any case, the pragmatic meaning of O for the student will be the system of operative and discursive practices that he/she uses to proof the veracity of the proposition.

The practices 1) to 7) (Figure 1) together constitute the argument that justifies the proposition 2 + 3 = 5, in which besides linguistic and conceptual entities a procedural entity intervenes: the technique of applying recursively the definition of sum of natural numbers. Through discursive and operative practices, the rules that fix the meaning of the concepts and procedures are evoked, in concluding with the discursive–normative practice 7): *Therefore, the expressions 2 + 3 and 5 are equivalent*.

Concepts, propositions and procedures can participate in mathematical activity, as *unitary* entities, which are described by a definition or a statement that fixes the rule of use of such objects: for example, the definition of natural number given in practice 2) of the demonstration (Figure 1). Nevertheless, we know that it is possible to find other definitions of natural number using different axiomatic systems, or depending on different contexts or institutional frameworks in which numbers are used. Each of these definitions brings into play different operative and discursive practices involving other objects as well, and therefore they imply different pragmatic meanings.

In the OSA ontology, the term 'object' is used in a broad meaning to refer to any entity involved some way in mathematical practice and that can be identified as a unit. The use of object is metaphoric, since a mathematical concept, is usually conceived as an ideal or abstract entity, and not something tangible, such as a rock, a drawing, or a manipulative. This general idea of object, consistent with the proposal in symbolic interactionism (Blumer, 1969; Cobb & Bauersfeld, 1995), becomes useful when considering a typology of mathematical objects, by taking into account their different roles and nature in the mathematical activity.

Symbols, external material representations and manipulatives, are involved in school and professional mathematical activity and, consequently, they are considered mathematical objects, because they intervene in mathematical practices. The concepts of number, fraction, derivative, etc., are mathematical objects of different nature and function than ostensive representations; they are non-ostensive, mental objects (when they intervene in personal, or individual practices), or institutional objects (when they intervene in shared social-cultural practices). In both cases, they are objects that regulate the mathematical activity, while their ostensive representations support or facilitate the performance of the said activity.

Each type of object can be considered from different dual points of view, as indicated in Figure 2. In particular, an object can be considered from a personal (individual subject) or institutional (social, shared) point of view, thus having, a double nature, mental/cognitive, and cultural/epistemic. The personal-institutional duality applies to practices, objects and meanings, which allows us to describe the semiosis processes (expression-content duality) from the cognitive and cultural points of view.

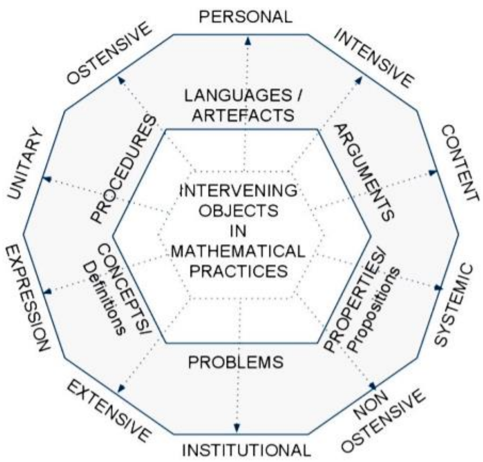

Figure 2. The OSA ontology (Font et al., 2013, p. 117)

There are neither objects without practices, nor practices without objects. Concepts, propositions, procedures, in their unitary version (as a whole or unity), are understood, -as proposed by Wittgenstein- as grammatical rules of the languages used in the operative and discursive practices made to describe our world and solve the situations-problems we face. In the OSA, mathematical objects are also viewed from a systemic perspective (as a system of components), where the diverse partial meanings of the same object are identified and articulated[4]. In addition, various *personal meanings* can be identified in the semiotic analysis of individual subject's practices when solving problems involving particular objects (numbers, probability, etc.).

3.2. Use and intentionality of practices

The operational theories of meaning assume that words, symbols or expressions do not refer or be in place of other things, but are used to *do* something with them. For example, numerals are instruments for counting, ordering and measuring, and statements about numbers describe rules for the use of such words. According to this assumption, 2 + 3 = 5 is not a property that establishes a relation

[4] Godino, Font, Wilhelmi and Lurduy (2011) describe different meanings of natural numbers from the institutional point of view. Batanero & Diaz (2007) identifies various meanings of probability.

between conceptual entities, as it happen with the expression "lions are carnivores". On the contrary, it is a rule about how the symbols 2, 3, 5, +, =, should be used: it states that, whenever you have the expression 2 + 3, you can replace it with 5, and vice versa.

The justification of mathematical propositions is done by a sequence of operative and discursive practices (as shown in Figure 1) having a given purpose. Each practice carried out to solve a problem, -which can be intra-mathematical, such as demonstrating that 2 + 3 = 5, or involving an extra mathematical context- plays a role in the resolution process. Table 1 summarizes the use or *operational/pragmatist meaning* of the practices required in the proof of proposition 2 + 3 = 5 (Figure 1).

Table 1. Use and intentionality of the practices to demonstrate that 2 + 3 = 5

Sequence of elementary practices	Use /intentionality
1) The symbols, 2, 3 and 5 represent natural numbers.	Attributing meaning to the symbols 2, 3, 5 as natural numbers
2) The natural numbers are a set of symbols that satisfy the Peano's axioms, in particular, there is a first element, 1, and a following (successor), injective function s:N→N is defined. In this set, the sum, +, is defined recursively as: n+1=s(n) ; n+s(m)=s(n+m)	Evoking the rules that define natural numbers and their sum, within the framework of a specific axiomatic theory.
3) In the sequence, 2 is the successor of 1, 2=s(1)=1+1; 3 is the successor of 2, 3=s(2)=2+1, and 5 is the successor of 4 which is next of 3, 5=s(4)=s(s(3)).	Interpreting the meaning of symbols 2, 3, 5 in Peano's axiomatic theory of natural numbers.
4) The sign = indicates the equivalence of two expressions.	Evoking the meaning of the equality of natural numbers as equivalence of two expressions.
5) The expression 2 + 3 represents the sum of the natural numbers 2 and 3.	Interpreting the meaning of + as the sum of natural numbers.
6) Taking into account the definition of the sum of natural numbers and successor 2+3=2+s(2)=s(2+2)=s(2+s(1))=s(s(2+1))=s(s(3))=s(4)=5.	Applying the rules that define the following function (successor) and addition of natural numbers.
7) Therefore, the expressions 2 + 3 and 5 are equivalent.	Fixing the new rule of use of the numerical symbols (declare the truth of the proposition).

3.3. Meaning and semiotic function

Between the symbol 2 and the concept of number 2, as well as between the concept of natural number and the system of practices from which such mathematical object emerges, there is relationship termed in the OSA *semiotic function* (taking the term from Hjelmslev, 1943; and Eco, 1991). The semiotic function is the correspondence between an *antecedent* object (expression/signifier) and another *consequen*t (content/meaning) established by a subject (person or institution) according to a criterion or rule of correspondence. This notion is intended to include any use that is given to meaning: meaning is the content of a semiotic function.

Each practice that intervenes in the demonstration of proposition 2 + 3 = 5 (Figure 1) has a function or role in the argumentative process, and therefore, this role is the *operational meaning* of the practice (Table 1). Nevertheless, in each practice, and in the set of them, a network of objects intervenes

(Table 2) whose identification is necessary to understand and manage the teaching and learning processes.

Table 2. Objects that intervene in the practices needed to demonstrate 2 + 3 = 5

Sequence of elementary practices	Objects involved
1) The symbols, 2, 3 and 5 represent natural numbers.	Languages: symbolic; natural. Concepts: natural number.
2) The natural numbers are a set of symbols that satisfy the Peano's axioms, in particular, there is a first element, 1, and a following (successor), injective function s:N→N is defined. In this set, the sum, +, is defined recursively as: n+1=s(n) ; n+s(m)=s(n+m)	Language: natural, symbolic. Concepts: natural number; set (of symbols); successor, function; first element; sum. Propositions: Peano's axioms.
3) In the sequence, 2 is the successor of 1, 2=s(1)=1+1; 3 is the successor of 2, 3=s(2)=2+1; and 5 is the successor of 4 which is next of 3, 5=s(4)=s(s(3)).	Languages: natural; symbolic. Concepts: sequence; successor, sum. Proposition: 2 is the successor of 1, 3 is the successor of 2, and 5 is the successor of the successor of 3. Arguments: convention based on the properties of the successor function.
4) The sign = indicates the equivalence of two expressions.	Languages: symbolic; naturally. Concepts: equivalence of expressions; equality.
5) The expression 2 + 3 represents the sum of the natural numbers 2 and 3.	Languages: natural and symbolic. Concepts: sum of natural numbers.
6) Taking into account the definition of the sum of natural numbers and successor 2+3=2+s(2)=s(2+2)=s(2+s(1))=s(s(2+1))=s(s(3))= s(4)=5.	Languages: natural and symbolic. Proposition: 2 + 3 = 5. Procedure: addition and successor operations. Argument: deductive, based on the definitions of natural numbers, sum and the successor function.
7) Therefore, the expressions 2 + 3 and 5 are equivalent.	Languages: natural and symbolic Proposition: statement of practice 7). Argument: deductive sequence of practices 1) to 6).

We understand the *semiotic function* as an interpretation of the Peircean sign.

> A representation is that character of a thing by virtue of which, for the production of a certain mental effect, it may stand in place of another thing. The thing having this character I term a ***representamen***, the mental effect, or thought, its ***interpretant***, the thing for which it stands, its ***object*** (Peirce, 1931-58, CP 1.564).

In the OSA, the Peircean's interpretant is conceived as the rule (habit, norm) of correspondence between the representation and the object, established by a person, or within an institution, in the corresponding interpretative act (personal or institutional meanings). When, for example, in practice 1) it is stated that 2 refers to the "concept of natural number two" (Figure 1), we are following a rule (habit, agreement) that is learned in the community of school mathematics practices. That is, the interpretant between the sign 2 and the concept *two* is a cultural agreement followed by the subject that makes the interpretation.

In addition, the OSA assumes that any entity taking part in a semiosis process, interpretation, or language game, is an object, and thus, it can play the role of expression (signifier), content (meaning) or interpretant (rule that relates expression and content). The systems of operative and discursive practices themselves are objects and can be components of the semiotic function. In this way, any use given to the word meaning is modelled.

In summary, the pragmatist/anthropological semiotics assumed by the OSA consider that the functives of semiotic functions are not only the ostensive linguistic objects (words, symbols, expressions, diagrams, etc.), but also the concepts, propositions, procedures, arguments; even the situations-problems, can also be functives of semiotic functions. This assumption makes sense, because it is necessary to question about both for the meaning of the concept of number, and the meaning of the propositions, procedures, arguments, situations and representations that intervene in the numerical practices. The functives in the semiotic function can furthermore be unitary or systemic entities, particular or general, material or immaterial, personal or institutional. A variety of types of meaning is generated in this way, which guides and supports the onto-semiotic analysis of mathematical activity at the macro and micro levels, both from the epistemic (institutional) and cognitive (personal) points of view (Font, Godino & Gallardo, 2013) .

3.4. Relativity of practices, objects and meanings

Since mathematical practices are carried out in an ecological background (material, biological and social), which determines an institutional, personal and contextual relativity of the practices, objects and meanings, the OSA accepts a relativity with respect to the *language games* and *form of life* (Wittgenstein, 1953). It is assumed, therefore, a sociocultural perspective of semiosis, in which the social, cultural, and historical dimension of the signs is emphasized. "In these perspectives signs are understood not as artifacts to which an individual resorts to represent or present knowledge, but as artifacts of communication and signification" (Presmeg, et al, 2018, p. 4).

In the example presented in Figure 1, the context of modular arithmetic changes the meaning of 2 + 3, just as the meaning of the natural number concept is different if the axiomatic is changed, or the sets construction of the numbers is used. The meaning of the numbers is different in the different communities of practices formed by diverse cultural groups or in different historical moments.

Consequently, the didactic-mathematical analysis must intend to characterize the different meanings of the objects and their interrelationships, thus constructing a *global meaning* that serves as a reference in the analysis of mathematical instruction processes. This first level of onto-semiotic analysis of the mathematical activity would help become aware of the mathematical objects plurality and relativity of meanings. In this first level of analysis the types of situations - problems where the object in question intervenes, as well as the mathematical practices which answer these problems should be identified, classified and described.

The existence of social, material and biological contexts (ecological background), sustaining and conditioning the mathematical activity, implies the relativity of practices and objects, which can be viewed from different polarities (personal-institutional, unitary-systemic, etc.). Moreover, such practices are related to the different institutional frameworks and contexts of use of a particular object.

For education in general and for mathematics education in particular, a holistic theory of meaning that includes the personal-institutional duality for meanings is necessary. Both a cognitive semiotics and an epistemic/cultural semiotics are required: the meanings are established between individual persons, in discursive and operative practices; but also between a person and the cultural knowledge whose learning is intended. In the mathematical culture the terms, symbols, concepts, etc., have a crystallized, socially shared meaning, formed in a historical-cultural process, which is the result of multiple discursive and operative practices between individual subjects, mediated by the use of different languages and artifacts. This approach is consistent with the cultural semiotics proposed by Radford (2006) for the meaning of mathematical concepts: "mathematical objects are conceptual forms of historically, socially, and culturally embodied, reflective, mediated activity" (Radford, 2006, p. 59).

From the point of view of education, meanings should neither be reduced to mental objects, nor to cultural objects; it is necessary to attribute a double personal and institutional nature, in order to account for the dialectical relationship established between them in the teaching and learning processes.

3.5. Some examples

The construction of the set of natural numbers N and its arithmetic is at the base of the mathematics that every educator should know. The formal construction of N based on the theory of one-to-one correspondence of sets defines the arithmetic operations in different way from the definitions when N is constructed from the Peano's axioms. These are two possible partial meanings of the natural numbers and its arithmetic, but they are not the only ones. Each partial meaning of the natural numbers is a semiotic system, characterized by a concrete configuration of operative and discursive practices. The global meaning of numbers (Figure 3) is made up of the articulation of the different subsystems determining each partial meaning.

> "In terms of learning, however, meanings are relative, not absolute. There are degrees of meanings; degrees of what may be termed extent, exactness, depth, complexity; and growth in meanings may take place in any of these dimensions. For relatively few aspects of life, for relatively few aspects of the school's curriculum (including arithmetic), do we seek to carry meanings to anything like their fullest development. Moreover, whatever the degree of meaning we want children to have, we cannot engender it all at once. Instead, we stop at different levels with different concepts; we aim now at this level of meaning, later at a higher level, and so on" (Brownell, 1947, p. 257)

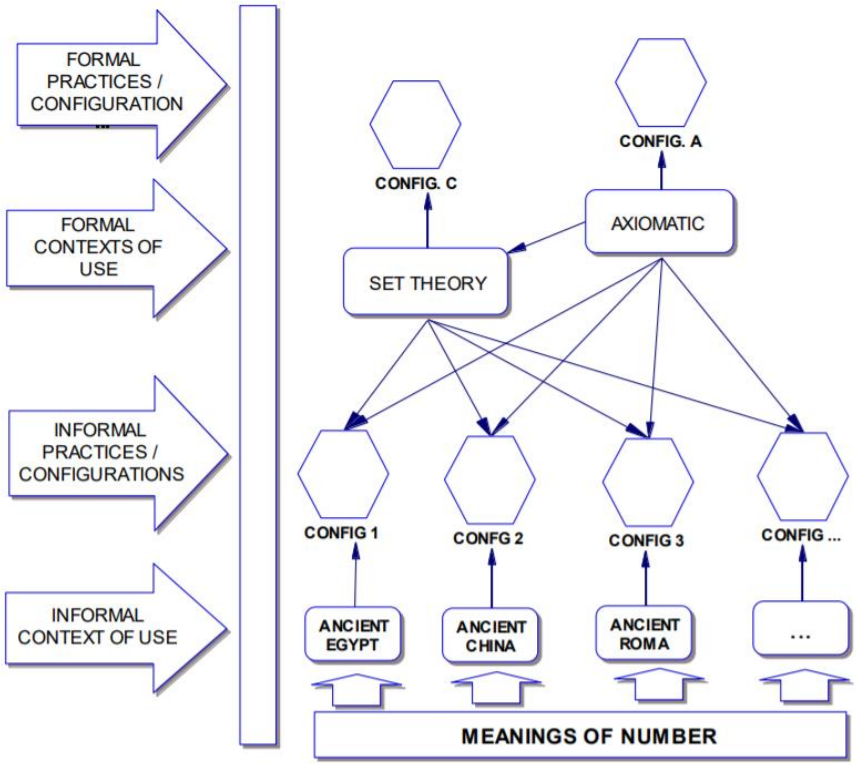

Figure 3. Plurality of meanings for number (Godino et al, 2011, p. 252)

The identification of the various partial meanings of a mathematical object and its articulation must therefore be a phase of the onto-semiotic analysis of mathematical activity. This analysis helps to formulate hypotheses about critical points in the interaction between the various agents in which there may be gaps of meaning or disparity of interpretations that require processes of negotiation of meanings and changes in the instruction process.

Batanero and Díaz (2007) apply the OSA theoretical notions to analyse the historical emergence of probability and its different current meanings (intuitive, classical, frequentist, propensity, logical, subjective and axiomatic). They, furthermore, describe mathematical activity as a chain of semiotic functions and use the idea of semiotic conflict to give an alternative explanation to some widespread probabilistic misconception.

Font and Contreras (2008) apply the notion of semiotic function and the OSA mathematical ontology to analyse the processes of generalization and particularization in the teaching and learning of mathematics. Using the analysis of the definition of derivative of a function in a high school textbook as a reflection context these authors address the following problems:

- The delimitation of particularization and generalization processes with respect to materialization and idealization processes;
- The elaboration of a typology of generalization processes;
- The role that the generic element plays in the relation particular–general;
- The relation of generalization processes with other mathematical processes.

Montiel, Wilhelmi, Vidakovic and Elstak (2009) apply OSA to analyse the mathematical notion of different coordinate systems, as well as some situations and university students' actions related to these coordinate systems in the context of multivariate calculus. The authors identify the objects emerging from the mathematical activity and make a first intent to describe an epistemic network for this activity. In other paper, Montiel, Wilhelmi, Vidakovic and Elstak (2012) approaches different coordinate systems through the process of change of basis, as developed in the context of linear algebra, as well as the similarity relationship between the matrices that represent the same linear transformation with respect to different bases.

4. Theories of meaning in mathematics education

The clarification of the notions of meaning and sense is a topic of interest for mathematical education and is approached from different perspectives. In this section, we synthetically describe three semiotic theories specifically oriented to mathematical knowledge: Frege's logical-semantic theory, Vergnaud's cognitive perspective and Steinbring's epistemological approach. Frege is a classic author who raises the distinction between sense and reference, which is a starting point for the Steinbring epistemological triangle, a model developed from a mathematics education explicit position. Vergnaud is representative of the theories of meaning from the constructivist psychological perspective. In these three theories, there is an interest in relating the question of the meaning of terms and expressions to the ontological problem on the nature of mathematical concepts, which is a central issue in the OSA. In Section 5, we analyse some concordances and complementarities between these semiotic theories and the OSA.

The concern for the meaning of mathematical terms and concepts leads directly to the inquiry into the nature of mathematical objects, to the ontological and epistemological reflection on the personal and cultural genesis and their mutual interdependence of mathematical knowledge. Reciprocally, behind any theory on the formation of concepts, or more generally, of any learning theory, there are ontological assumptions about the nature of concepts, and therefore, a more or less explicit theory about their meaning.

4.1. Sense and reference in Frege

Different triangular models have been proposed to analyse the relationships between symbols and meanings. One of them is introduced by Frege (1892) in the work *On meaning and reference*.

> It is natural, now, to think of there being connected with a sign (name, combination of words, letter), besides that to which the sign refers, which may be called the referent of the sign, also what I would like to call the sense of the sign, wherein the mode of presentation is contained (Frege, 1892, p. 210).

For example, let *a, b, c* be the segments joining the vertices of a triangle with the midpoints of the opposite sides. The point of intersection of *a* and *b* is also the intersection point of *b* and c (centroid). We, therefore, have different designations for the same point, and these names ("point of intersection of *a* and *b*" and "point of intersection of *b* and *c*") indicate at the same time the mode of presentation, and that is why the proposition contains effective knowledge. Both expressions have the same reference but different senses.

To each sign corresponds a given sense and to this sense, in turn, a specific reference, while a reference (to an object) neither is linked to only one sign, nor receives a single sense. The same sense has different expressions in different languages, and it may happen that an expression makes sense but has not a reference. For example, the expression "the series that converges more slowly" possess a meaning but has no reference, since for each convergent series, another series that converges more slowly can be found. Therefore, by grasping a sense, one is not sure there is a reference (Frege, 1892, p. 87).

Frege provides advice to distinguish between the reference and the sense of a sign, with regards to the representation associated with them, since the representation is something internal to subjects. If the reference of a sign is a perceptible object, then someone's representation is an image originating in the person's memories of sensorial impressions, and of internal and external activities. Even for the same person, not always, the same representation is linked with the same sense. Representation is subjective: the one's representation is not that from another.

> The sense of an expression was supposed to consist in the way in which we determined its reference: but now it appears that, often, there is no one favored way to determine the reference of an expression, but that different people may determine it in different ways, and even that what is taken at one time as an acceptable means of determining it may later be dropped as not agreeing with the others. If so, then what is objective about the employment of an expression, what is shared by all the speakers of the language, is after all its reference (Dummett, 1973, p. 102).

Initially, the theory of sense and reference was developed for proper names: "The referent of a proper name is the object itself which we designate by its means; the conception, which we thereby have, is wholly subjective; in between lies the sense, which is indeed no longer subjective like the conception, but is yet not the object itself" (Frege, 1892, p. 213).

Next, Frege expanded his theory of meaning and reference for declarative sentences, that is, statements that affirm a judgment to be true or false, and for common names or concepts. "Every declarative sentence concerned with the referents of its words is therefore to be regarded as a proper name, and its referent, if it exists, is either the true or the false" (Frege, 1892, p. 216).

Frege distinguishes between object and concept, which in logic is closely related to that of function, which Frege defines as: "A function of x was taken to be a mathematical expression containing x, a formula containing the letter x" (Frege, 1891, p. 138). For Frege: "a concept is a function whose value is always a truth-value" (Frege, 1891, p. 146); the values given to the argument of the function are the objects that fall under the concept. In logic, "we can designate as an extension the value-range of a function whose value for every argument is a truth-value" (Frege, 1891, p. 146).

In reference to the idea of object, Frege affirms: "An object is anything that is not a function, so that an expression for it does not contain any empty place" (Frege, 1891, p. 147). Value-range of functions are objects, whereas functions themselves are not. Likewise, extensions of concepts are also objects, although concepts themselves are not.

Frege's logical-semantic model distinguishes whether a sign refers to an object or to a concept, under a certain modality or meaning (sign, sense, reference). This is a first step in accepting that a concept admits a plurality of possible interpretations, uses or partial meanings. There is only an object/concept, but this object can be seen from different perspectives: for example, the centroid can be linked to the medians a, b of a triangle or to b and c.

Although Frege's philosophy of mathematics is undoubtedly realistic - Platonist, when assuming that a mathematical object has its own independent existence, his theory of sense and reference of signs, words and expressions, opens a window to the relativism of the psychological and anthropological positions. One word designates or refers to an object or a concept, but it is always accompanied by a thought, one sense or specific way of seeing the object or concept in the context in which communication takes place. Such senses are considered in an intersubjective way, and consequently, the problem of identifying and characterizing the possible universe of senses attributable to the object can be raised.

4.2. Vergnaud's conceptual triplet

A scientific challenge for Vergnaud (1982) is promoting the study of mathematics learning and teaching as a well-defined research field, not reducible to mathematics, psychology, linguistics, sociology or other sciences. This goal requires the analysis of the different mathematical contents, in their specificity, and the empirical study of their teaching and learning, in order to take into account both the long-term knowledge acquired by children and adolescents, and the short term change of conceptions when solving new situations. Consequently, Vergnaud produced the theory of conceptual fields in which he proposes a definition of concept useful to study the evolutionary development of mathematical knowledge. "Therefore, from a developmental point of view, a concept is altogether: a set of situations, a set of operational invariants (contained in schemes), and a set of linguistic and symbolic representations" (Vergnaud, 2009, p.94).

Vergnaud suggest that a concept cannot be reduced to its definition when we are interested in its learning and teaching (Vergnaud, 1990, p. 133). A concept acquires meaning for the child through situations and problems that should be solved. Vergnaud's study of a concept development and functioning, in the course of learning or during its use, lead him distinguish three planes or components, the triplet (S, I, G), as constituents of a concept C, where,

S: set of situations that give meaning to the concept (the reference).

I: set of invariants on which the operation of the schemes rests (the meaning)

G: set of linguistic and non-linguistic forms that allow symbolically represent the concept, its properties, situations and treatment procedures (the signifier).

There is neither general bijection between signifier and meaning, nor between invariant and situation. It is neither possible to reduce the meaning to the signifier, nor to the situation. The notion of meaning is understood as a relation of the subject to situations and the signifier. "More precisely, the schemes evoked by the individual subject in a situation or by a signifier are what constitute the subject's meaning of this situation or signifier" (Vergnaud, 1990, p. 158). For example, the meaning of addition for a subject is the set of schemes that can be used to deal with the situations that face the subject which imply the idea of addition. It also includes the set of schemes involved when operating on the symbols (numerical, algebraic, graphic or linguistic) which represent addition.

Vergnaud (1982; 1990) goes one-step further than Frege in problematizing the mathematical concept when addressing the problem of learning and teaching: the concept itself is a complex, systemic entity formed by the interaction between three types of objects: representation systems, problem situations and operative invariants.

4.3. The epistemological triangle

Steinbring (1997; 2006) interprets Frege's triangle and that proposed by Ogden and Richards (1923), in adopting an epistemological perspective that helps to understand the interpretation, communication and construction of meanings processes that take place in the mathematics classroom.

His epistemological triangle includes three elements (Figure 4): the sign or symbol, the object or reference context, and the concept, which is understood as an ideal or abstract mathematical concept.

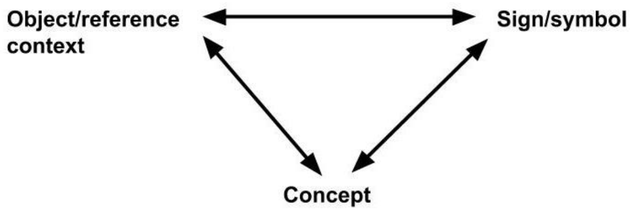

Figure 4. Epistemological triangle (Steinbring, 2006, p. 135)

Through the epistemological triangle, a semiotic (representational) mediation is modeled. “The links between the corners of the epistemological triangle are not defined explicitly and invariably, they rather form a mutually supported, balanced system. In the course of further developing knowledge, the interpretations of sign systems and their accompanying reference contexts will be modified” (Steinbring, 1997, p. 52).

Steinbring attributes two functions to mathematical signs:

> (1) A semiotic function: the role of the mathematical sign as “something which stands for something else”.
>
> (2) An epistemological function: the role of the mathematical sign in the frame of the epistemological constitution of mathematical knowledge (Steinbring, 2006, p. 134).

To understand the Steinbring’s semiotic-epistemological model of mathematical knowledge, the nature of the triangle vertices should be clarified. The author assumes that "The true mathematical object, that is the mathematical concept, may not be identified with its representations" (Steinbring, 2006, p. 137). Then, *what are for him the mathematical concepts? What are the objects/reference contexts*?

Steimbring’s application of his epistemological triangle to the concept of probability (Figure 5) helps us to understand the features of this theoretical model of mathematical knowledge.

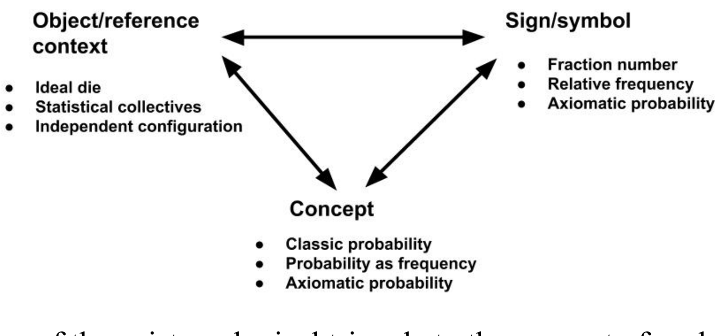

Figure 5. Application of the epistemological triangle to the concept of probability (Steinbring, 1997, p. 53)

In this figure, diverse expressions are included (fractional number, relative frequency, axioms) within the category of sign/symbol. Within the category of *objects /reference context* the model includes problem-situations where probability is applied, such as determining whether a given die is biased or not (ideal die), the statistical collectives to determine the probability, and the calculation of probabilities of independent event configurations. Within the category *concept,* it includes different meanings of the probability concept: classical, frequency and axiomatic probability meanings.

Thus, unless explicitly mentioned, it is assumed that the concept of probability has different meanings, depending on the contexts or situations-problems where it intervenes, and that such situations and meanings involve different systems of representation, although the reference to the axioms is very vague. Possibly, the author refers to the symbolic expression of the axioms, since the axioms themselves are not representations but properties of probability that link it with other mathematical objects, such as the union or intersection of events.

> The epistemological triangle is a model for making the invisible mathematical knowledge accessible with regard to its structural character, for describing its particularities and also for analyzing interactive processes of constructing mathematical knowledge – thus invisible relations that are embodied in exemplary contexts and activities (Steinbring, 2006, p. 144).

The Steinbring's epistemological triangle implicitly suggests, that the concept, reference signs/symbols and objects/reference setting, include a variety of general structures (various constructions of probability, natural number, etc.). The reciprocal relationships of conceptual structures with the representation systems and the different contexts and situations of use should be taken into account to organize and explain the generation of mathematical knowledge, that is, the concept epistemology. In this model, a systemic perspective is adopted, both for the structure of concepts and for the symbol systems and contexts. Frege attributes various senses to the mathematical concept, while for Steinbring these senses are reciprocally related to various symbolic systems and contexts.

5. Some concordances and complementarities between semiotic theories

The question of how the terms meaning and sense are used by various authors and disciplines is intriguingly and unclearly linked to the notion of object, and in the case of mathematics, to the nature of abstract objects. Therefore, semiotics is essentially linked with ontology, the various types of objects referred to by signs, and the various modalities in which objects can participate in communication and interpretation. The answers to the question of meaning of Frege, Vergnaud and Steinbring differ substantially in the nature of the objects referred to or represented by the signs, although the three models are triadic. In Frege assumes a platonic, transcendentalist position on the reference (the object referred to). The centroid, for example, is unique although it can be represented in different ways and each of them provides a different meaning. In a way, the models of Vergnaud and Steinbring respond similarly to the question of what represents, for example, the word 'number': it represents the (ideal, abstract) concept of number; but for the question of what number means, or what it is number the response is different: a heterogeneous system formed by three components (triplet): situations, invariants, representations (Vergnaud); the triplet sign, object, concept (Steinbring).

In the OSA, we find relevant differences in the response given to the question of meaning of a mathematical concept, in considering that these objects cannot be disentangled from the mathematical practices, and in assuming the anthropological perspective for mathematics, that is, conceiving mathematics as a human activity. In addition, objects and practices can adopt an institutional (social or shared practices), or personal (idiosyncratic practices of a subject) perspective and a systemic or unitary perspective.

When considering objects in a unitary way, their meaning would be one of the possible definitions (rules that intensively define the object). When viewing the objects systemically, the meaning is the system of operative and discursive practices in which said object intervenes in a critical manner, thus including one of the possible definitions, together with the situations, languages, properties and argumentations involved (partial meaning). The epistemological and didactic analysis of a mathematical object should also take into account the diversity of partial meanings of the object and their articulation in a *global meaning*, as can be seen in Batanero and Diaz (2007) for the probability or in Wilhelmi, Godino and Lacasta (2007) for the equality of real numbers.

Progress is made in the onto-semiotic framework towards an increasing refinement of the idea of mathematical concept by firstly connecting it with human activity, mediated by the linguistic and material artifacts put into play in the resolution of specific problem-situations. Secondly, each sense, or partial meaning, is linked to a specific rule (concept-definition) of the use of linguistic elements to solve a class of situations (contexts, phenomena) and other procedural, propositional and argumentative objects. Finally, the different partial meanings - senses - are organized in a global meaning formed by the network of senses and related objects.

The OSA also takes into account the functional or operational interpretations of meaning, that is, as the use of objects in the various practices. Thus, for example, numerical symbols do not only refer to the corresponding concepts, but they are also instruments for counting, numbering, ordering, etc.

6. Synthesis and implications

Along the paper, we have presented a synthesis of the theory of meaning that has been elaborated from the perspective of the OSA, which is supporting a new field of reflection on what could be called an *onto-semiotic* perspective, in which the study of signs should be linked to the analysis of the objects referred to by the signs. We have described the sources on which this onto-semiotics is based and the attempt to combine realistic and operational theories of meaning, since the problem is approached from the educational context, that is the context of construction, communication and learning of mathematical knowledge.

Although the problem of meaning interests to various disciplines (philosophy, linguistics, psychology, semiotics, etc.), the field of education, and, in particular, mathematics education, provides a rich perspective to address this problem. The OSA proposes not to separate the problem of signs and their interpretation from the ontological problem. This is understood in terms of inquiry about the nature and types of entities referred to by the signs, as well as the instrumental role played by them in the knowledge construction, communication and learning. The solution of the onto-semiotic problem, in turn, implies new ways of approaching the epistemological problem related to the origin and evolution of knowledge, which is undoubtedly essential to address the educational-instructional problem (Godino, et al., 2019).

In the OSA there are no objects without practices, which are made by people, and objects (concepts, propositions, representations, etc.) have a double "reality", personal (mental, cognitive), and institutional (cultural, shared). In this way, the OSA tries to articulate the cognitive with the epistemological problem (understood from an anthropological and therefore, historical-cultural perspective) in the mathematics learning.

In the curricular documents the terms meaning and sense are used informally, without really specifying what is meant by meaning or sense. In some cases, a cognitive mental vision of meaning is emphasized, while in others an epistemic/cultural vision is reinforced, when the design and analysis of mathematical instruction processes require articulating both perspectives. Consequently, we believe that the elaboration of a holistic theory of meaning that also takes into account the notions of sense and understanding is very useful for educational research and practice. This theory is applicable in the analysis of mathematical objects both at the macro level (such as when talking about developing in students the "numerical sense"), and at the micro level (in statements of the type, "students do not understand the meaning of the fraction concept, Pythagoras' theorem, the pie chart, etc.)". In the macroscopic analysis of any mathematical object it is necessary to take into account the various partial meanings and plan the learning of each one of them, as well as its progressive articulation. In addition, when solving problem-situations, in the processes of demonstration, representation, generalization, etc., teachers should consider the configuration of objects and meanings that students should know, interpret and understand.

Instructional design research or didactic engineering (Artigue, 1989; 2009) establish the phases of preliminary study, design, implementation and retrospective analysis in the teaching and learning processes. An epistemological analysis of the teaching contents is performed in the first stage, in order to have well-founded criteria to design the learning tasks in the second phase. The OSA theoretical tools, in particular the notion of partial meaning and its articulation to the global meaning, together with the categories of objects and processes involved in the practices (Figure 2) can support these phases of instructional design. As regards the implementation and retrospective analysis phases, the notion of didactic suitability offers a guideline to support the reflection on the teacher practice (Esqué & Breda, 2020; Morales-Maure, Durán-González, Pérez-Maya, & Bustamante, 2019).

The OSA to the meaning of mathematical objects has implications for teachers' education, since it highlights the complexity of knowledge and, therefore, the need to investigate pedagogical interventions, as well as making teachers be aware of this complexity.

> Teachers surely need mathematical content knowledge and pedagogical knowledge; and within the domain of pedagogical content knowledge, they also need epistemological knowledge so they are able to assess the epistemological constraints of mathematical knowledge in different social settings of teaching, learning, and communicating mathematics (Steinbring, 1998, p. 160).

In this sense, mathematics teacher should know the different meanings of mathematical objects, as well as the network of objects and processes involved in the mathematical practices, in order to be able to plan the teaching, manage the interactions in the classroom, understand the difficulties and assess the students' learning.

The notion of *didactic suitability* has been developed within the OSA framework (Godino et al, 2007; Breda, Font, & Pino-Fan, 2018) to provide criteria that mathematical instruction processes should fulfil to optimise learning, when taking into account the context constraints. With this aim, it is necessary to consider the knowledge provided by didactic research on the epistemic, cognitive, affective, interactional, mediational and ecological facets involved in the instructional processes. Moreover, the OSA pragmatic view of institutional and personal meanings provides an essential element for assessing the epistemic and cognitive suitability of mathematical instruction processes. On the one hand, to achieve high epistemic suitability the planned or implemented meanings should represent the global meaning of the mathematical object studied. On the other hand, an adequate level of cognitive suitability requires that the personal meanings constructed by the students agree with the institutional planned or implemented meanings (Pino-Fan, Font, Gordillo, Larios, & Breda, 2018), and the students are able to establish connections between the different meanings and objects involved.

This pragmatic approach to meaning can serve to analyse the representativeness of the meanings intended for a mathematical object given by the curriculum of a particular educational level in a given country (Pino-Fan, Parra-Urrea, & Castro, 2019; Burgos & Godino, 2020) or in textbooks (Burgos, Castillo, Beltrán-Pellicer, & Godino, 2020). In addition, introducing the students to a representative sample of the partial meanings of mathematical objects allows them to develop their problem solving competence, according to the different contexts in which these objects are used.

The notion of didactic suitability is being widely used as a tool to analyse the didactic sequences designed and implemented by teachers, in order to achieve an adequate teaching of mathematics (Breda, 2020; Morales & Font, 2019; Sousa, Silva Gusmão, Font, & Lando, 2020). It also is employed to organise training programmes focused on the reflection on teaching practice (Esqué & Breda, 2020; Morales-Maure, Durán-González, Pérez-Maya, & Bustamante, 2019). The didactic suitability construct facilitates the systematic reflection of teachers on the complexity of the mathematical objects they teach and on the factors involved in their study.

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