Teaching real numbers and continuum in secondary school: an onto-semiotic approach to the investigation of teachers' choices

Enseñanza de los números reales y el continuo en la escuela secundaria: una aproximación ontosemiótica a la investigación de las elecciones de los profesores

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Resumen

En este trabajo se presenta un estudio de caso sobre enseñanza y aprendizaje de los números reales y el continuo en la escuela secundaria italiana. Se aplicó un cuestionario a una muestra de profesores de educación secundaria en servicio diseñado para investigar los enfoques, orientaciones y recursos que caracterizan las elecciones didácticas de los profesores para la enseñanza de los números reales y el continuo. Los datos han sido analizados usando herramientas teóricas del Enfoque Ontosemiótico (EOS). Dos objetivos específicos de la investigación fueron: investigar en qué medida los recursos didácticos, fines y orientaciones afectan al proceso de toma de decisiones, y buscar criterios de idoneidad didáctica usados por los profesores para argumentar a favor de sus elecciones didácticas declaradas. Después de discutir la complejidad epistemológica de estos objetos matemáticos presentamos un estudio de caso, con el fin de mostrar, entre otros resultados, el papel del EOS en esta investigación.

Palabras clave: Números reales, continuo, escuela secundaria, profesores, idoneidad didáctica

Abstract

In this paper, we present a case study concerning teaching and learning of real numbers and continuum in the Italian high school. In the research, secondary school in-service teachers, after answering a questionnaire designed to investigate their resources, orientations and goals, have been asked to declare and motivate their didactical choices concerning the teaching and learning of real numbers and continuum. Data have been analysed within a framework including OSA. In particular, two goals were: to investigate how far teachers' resources, goals and orientations affect the process of decision-making and to look for criteria of didactical suitability used by the teacher to argument in favour of their declared choices. After discussing the epistemological complexity of these mathematical objects, we present a *case study*, in order to make emerge, among other results, the role of OSA in such a research.

Keywords: Real numbers, continuum, secondary school, teachers, didactical suitability

1. Introduction

The investigation of teaching and learning of continuity and real numbers are classical topics in Mathematics education. The relation between the intuition of continuity of the line and the formal constructions of the field(s) of real numbers is very complex from an epistemological point of view (Giusti, 2012). In Italy, according to the national curricula, an introduction to Calculus is proposed to students in the end of secondary school (Italy is K13) while the transition from Calculus to Analysis (Bergé, 2008) is a

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challenge that university lecturers in the first year should face.

According to Bergé (ibid.) what characterize more this transition is the transformation of intuitive model of the line into formal concepts and a formal construction of R is important. Usually in secondary school teachers are asked to introduce: limits, continuous functions and related theorems (Bolzano and Weierstrass theorems), derivatives, Rolle and Lagrange theorems, examples of computations of limits and derivatives, Riemann integral, examples of integration based on Torricelli-Barrow theorem and, maybe, very simple examples of differential equations. Italian textbooks are usually based on formal definitions and intuitive examples that are not suitably connected each other. Secondary school teachers who have been taught as secondary school students in the most traditional way have in their mind examples of such a not problematized way of teaching Calculus, based on attempts of making intuitive and simplifying formal definitions, that are considered undeniable.

As Lindgren (1996) stressed, experiences as students affect in a very significant way future teachers' choices: teachers often teach the way they have been taught. Furthermore, in Italy mathematics secondary school teachers may not have a strong background in *advanced mathematics*. Combining a very complex topic from the epistemological point of view with a lack of good teaching materials and the habit to reproduce traditional choices rather than re-think teaching and learning practices, the results in terms of quality of the teachers' didactical proposal may be very weak.

A couple of paradigmatic examples in the literature in Mathematics education encouraged us to investigate teachers' choices in this case. Real numbers and the formal definitions of limits and continuity have often been considered intrinsically counterintuitive, but researches in which suitably complex activities were proposed to students showed that irrational numbers and formal continuity should be made somehow intuitive (Tall & Vinner, 1981; Fischbein, Jehiam & Cohen, 1995). The researchers, independently, came to the same conclusion: oversimplification of complex topics doesn't help students but, on the contrary, is responsible for a lot of the most common ingenuous misconceptions.

In this work, that is part of a PhD thesis (Branchetti, 2016), we discuss a *case study* to show that, according to the previous considerations, it is relevant to understand how teachers decide to deal with such a complex didactical issue and what criteria they use to make their choices.

2. Intuitive and formal aspects of continuum

Lots of historical and epistemological analyses of the evolution of the conceptions of continuum have been published (see Giusti, 2012 and Bell, 2014 for a complete overview) and the theme has been of interest also for many important modern mathematicians like Weierstrass, Bolzano, Cantor, Dedekind - who were directly involved in the program of arithmetization of Analysis - or like Hilbert - who defined the field of real numbers in an axiomatic way - but also like Russell, Brouwer, Weyl, Poincaré and many other mathematicians who felt the necessity to clarify to themselves what was the deep relation between intuition and formalization of continuity (Bell, 2014). The topic is indeed related with important issues that are relevant for mathematicians: what does a theory founded on continuum needs in terms of rigor and possibility of proving theorems within a formalized theory? At what degreee real numbers represent continuum and what properties of continuum are really necessary and must be included in real numbers? Weyl (1987, p.108, in Bell, 2014) stated in a provoking way: "[…] the conceptual world of mathematics is so foreign to what the intuitive continuum presents to us that the demand for coincidence between the two must be dismissed as absurd". The dialectic between intuitive and formal aspects didn't end up in a unique solution nor historically nor from a didactical point of view. In some occasions intuitive models of continuum are used also in *advanced mathematics* and sometimes formal aspects are dismissed as unuseful and disconnected from the intuitive models (Tall & Vinner, 1981). Lakoff and Nunez (2000) explored also in depth the relations between a natural conception of continuity and the formal one - named Cauchy-Weierstrass continuity by the authors - showing the intrinsic metaphorical nature of such a relation. Of this metaphorical relation was already aware Bolzano, who described this way in 1817 the need for a bend in mathematics concerning continuity: "No one will deny that the concepts of time and motion are just as foreign to general mathematics as the concept of space. We strictly require only this: that examples never be put forward instead of proofs and that the essence of a deduction never be based on the merely metaphorical use of phrases or on their related ideas, so that the deduction itself would become void as soon as these were changed". To decide how to balance these partially conflicting tendencies that characterize the relation between real numbers and continuum, between formalization and intuition, would thus be, implicitly or not, an important choice for a teacher.

3. Research problem

Many factors may affect the teachers' decision making processes. Teachers' background in mathematics is of course a very important one, in particular if we think about complex topics as real numbers are. In particular, if we think about teachers' experiences as mathematics students a much longer experience also as researchers in mathematics could make teachers' personal knowledge wider and deeper. We decided thus to involve also people with a PhD in Mathematics who are, at the same time, mathematics secondary school teachers in order to understand what factors beyond the experience in the discipline could be taken in account by a teacher asked to design teaching and learning activities concerning continuum and real numbers.

4. Theoretical framework

First of all, we clarify what we mean with the term *mathematical object*, consistently with the perspective we rely on. The definition of mathematical object in the Onto-semiotic Approach (OSA) is based on the notion of practice (P). Godino and Batanero (1998, p. 182) consider "mathematical practice any action or manifestation (linguistic or otherwise) carried out by somebody to solve mathematical problems, to communicate the solution to other people, so as to validate and generalize that solution to other contexts and problems". According to Godino and Batanero (1998):

- *mathematical object O* indicating any entity or thing to which we refer, or talk about it, be it real or imaginary and that intervenes in some way in mathematical activity;
- *meaning of the object O* indicating the system of practices related to the object O that a person carries on (personal meaning), or that are shared within an institution (institutional meaning).

There are six types of primary mathematical objects involved in practices defined in Godino, Batanero and Font (2007): language (terms, expressions, notations, graphics); situations (problems, applications, exercises, ...); concepts, given by their definitions or descriptions (number, point, straight line, mean, function etc.); propositions, properties or attributes; procedures (operations, algorithms, techniques, ...); arguments used to validate and explain the propositions and procedures (deductive, inductive etc.). New objects can emerge from the system of practices, that we call configuration of objects (CO), i.e. network of objects characterized by relationships established between them. The COs emerge by mean of sequences of practices, called *processes (*communication; problem posing; definition; enunciation; procedures; argumentation). To explain the nature of the relationship between objects, the definition of semiotic function is crucial. *Semiotic function* is "the correspondences (relations of dependence or function) between an antecedent (expression, signifier) and a consequent (content, signified or meaning), established by a subject (person or institution) according to certain criteria or a corresponding code." (Godino, Batanero and Font, 2007, p. 4).

All the six different types of objects can be expression or content of the semiotic functions. We call a *representation* an object that is put in place to another through a representational semiotic function. Objects can have different facets, that will be coupled in pairs since they are complementary according to Godino, Batanero & Font (2007). In particular, we are interested in *personal – institutional:* personal objects emerge from practices carried on by a person; institutional objects emerge from systems of practices shared within an institution. In our perspective, this is very useful because we assume the mathematical object to be personal and a very complex result of many elaborations in different practices over years.

Another important notion is *didactical suitability*. The components of didactical suitability concerning our analysis are some of those "objective criteria that serve to improve the teaching and learning and guide the evaluation of the teaching/learning processes. [...] Epistemic suitability measures the extent to which the implemented meaning represents adequately the intended meaning (the curricular guidelines for this course or classroom). Cognitive suitability is the degree to which [...] the implemented meaning is included in students' zone of proximal development, and whether the students' learning (personal meaning achieved) is close to the intended meaning. Ecological suitability is the extent to which the teaching process is in agreement with the school and society educational goals, and takes into account other possible social and cultural factors." (Godino, Ortiz, Roa & Wilhemi, 2011). To choose criteria a teacher could use to evaluate the didactical suitability of some possible choices we refer to Breda & Lima (2016), who specified some criteria to analyse the teachers reports of didactical sequences and, in particular, the reasons why the teachers thinks their proposals are good. We report here some of the markers proposed in the paper we quoted before that we took in account to analyse data.

Epistemic suitability

EPS1: *Errors:* no practices are wrong from a mathematical point of view.

EPS2: *Ambiguity:* there are no ambiguities that can confuse the students: wrong or not clear definitions, not adequate definitions or explications for the students' level.

EPS3: *Processes richness:* the sequence includes relevant processes from the mathematical point of view (modelling, argumentation, problems solutions).

EPS4: *Representativeness:* the presented meaning are representative examples of the complexity of the mathematical objects that the teachers is going to teach and is included in the national curricula; for every partial meaning problems are presented; for every partial meaning different ways to represent objects and to connect them each other are presented.

Cognitive suitability

CS1*:* *Previous knowledge*: the students own the necessary previous knowledge to study the new topic; the expected meanings to reach are possible to achieve in all their components.

CS2: *High cognitive request:* Relevant cognitive and metacognitive processes are activated .

Ecological suitability

ECS1: *Adaptation to the national curricula:* the contents, their implementation and evaluation correspond to the curricular indications.

ECS2: *Didactical innovation:* Innovation based on researches and reflections is taken in account.

As theoretical framework for teachers' choice we chose the goal-oriented decision-making theory by Schoenfeld (2010). Drawing on this theoretical framework we will consider teachers as decision-makers, whose choices are determined by "their resources (their knowledge, in the context of available material and other resources), goals (the conscious or unconscious aims they are trying to achieve), and orientations (their beliefs, values, biases, dispositions, etc.) [...] at both macro and micro levels." (Schoenfeld, 2010, p. 14). This framework concerns in particular choices of the teachers in real-time. In Schoenfeld's model the teacher is analysed as a decision-maker who has to solve continuously didactical problems. However, investigating a hypothetical design we provide significant information in order to foresee potential teachers' behaviours in the classroom and to design a future long-term research based on the comparison of a priori hypotheses about teachers' real-time choices and the observed teachers' reactions when they really implemented their planned activities.

5. Research methodology

Our research questions are the following:

RQ1: What systems of practices concerning real numbers does the teacher declare to prefer and to propose to students?

RQ2: What criteria of didactical suitability do the teacher use to make her choices?

RQ3: What are the relations between teachers' resources, orientations and goals and the criteria she used to make her choices?

We carried out a mixed-methodological study – qualitative and quantitative (Onwuegbuzie and Leech, 2004). Our methodology was inspired first of all by two methodologies of qualitative research presented by Neuman and Pirie in the monograph on qualitative research in mathematics education edited by Teppo (1998), even if the subject involved in their researches were students. As Neuman we opted for a phenomenological observation technique and for the construction of descriptive categories. While in this framework the results depend strictly on the model, in our research analysing data other questions emerged that we didn't consider in the beginning. This is contemplated by Pirie's methodology, that allow to let emerge new questions from data, going on with cyclic analyses and making more and more precise the categorization. Then we inspired our investigation to methodologies used before in researches framed in the OSA, in particular those referred to epistemic analysis of mathematical objects' complexity (that are not reported here), teachers' cognitive configurations and didactical suitability. Teachers were asked to answer an online questionnaire designed to investigate:

I. teachers' background (master degree, training courses attended)

II. teachers' personal object (configurations of objects they associated to R set)

III. practices chosen by teachers involving elements or subsets of R or objects traditionally used in the constructions of real numbers like inequalities, Q etc.

IV. representations of intervals of rational and real numbers they considered best to address a goal.

In the first part teachers were asked to answer open questions knowledge and closed questions about goals, while in the second part the questions aimed at making emerge preferences and orientations in an indirect way. Using Schoenfeld’s categories (2010) the questions have been planned in order to collect information about resources, the other questions concerned the goals and the orientations. Teachers have then asked to comment videos and teaching materials. After answering the questionnaire teachers were interviewed in focus groups (3 or 4 members) and then individually; we guided a discussion on questionnaire answers in order to permit to make explicit their personal choices and the motivations of their choices and to investigate their general orientations concerning the teaching and learning of real numbers and students’ perceived difficulties. We defined some *a priori* categories standing on the literature review we presented before. Only significant categories are reported. In this paper, we present just a case study: a secondary school teacher with PhD in Mathematics and experience in teaching Calculus in secondary school.

6. Data analysis

Background: PhD in Mathematics, National qualification as a Mathematics and Physics teacher in the high school. Years of experience as a teacher: 5.

Studied real numbers: at the University in a course of Calculus and at school.

The fundamental properties of real numbers are: continuous, complete, ordered, Archimedean.

In every numerical set enlargement (N → Z, Z → Q) there is a “critical operation” involving the elements of the set to enlarge, like subtraction and division, that lead to a new set. How is it possible to construct R starting from Q?:

> By closure in respect of the concept of “limit” of sequences of rational numbers: starting from the power inverse operation you show that new numbers “arise” that have the following feature: they can be estimated from the bottom and from the top in a more and more precise (i.e. with a decreasing difference) using rational numbers. At this point you can make the “closure” of Q in respect of this operation, i.e. you consider all the numbers that you can obtain this way.

Is it possible to define a limit point in Q or is it necessary to use real numbers?

> Yes, for example,.., 0 is a limit point of 1/n.

Asked what R properties are necessary for, selecting from a list of topic and practices traditionally introduced within the R field, she just selected only differential and integral calculus. In the next questions the teacher had been asked to comment on teaching materials or audio video-taped lessons concerning: a concrete in-context approach to the introduction of irrationals numbers as parts of the reality; the correspondence between points of a line and real numbers "showed" (as something evident and non to be theoretically discussed) by means of the flow of a sliding point; representations of intervals as solutions of inequalities (algebraic and graphic). According to the teacher "The first video may help the students to create an image of real numbers that could be useful in the future" even if "The supports (concrete objects) used is not necessary". After she watched the second video and said that the visualization of a flow could not help students to have intuitions about real numbers properties because the slider has a pace necessarily and doesn't represent the density of the number set; she would have preferred to zoom in more and more. In the third part, she stated that the graphic representation allows to visualize better the solutions and that there are not two solutions but only two representations of the solution. In further questions concerning different ways of representing R intervals but also "Q intervals" (proposed in order to check how far the teacher considered problematic such a concept) the teacher's position is: some representations are clear and must not be mediated; the graphic representation is better/more intuitive, synthetic; a drawing of a segment can represent the infinite real solutions of an inequality. Usual representations of real intervals can't represent a subset of Q since make her thinking about a continuous set of solutions. Here are some excerpts from the interview:

1. I: ”In which situations do you introduce real numbers? How do you explain the students why you introduce real numbers and how do you represent them?”

> 2. T: ”I would try ... It's hard ... anyway quite soon. Not from the first years but very soon. I think I would introduce them, very easily, showing that not all the numbers are rational. ” [...]

4. I: ”Would you use in the last year [grade 13] some construction of R to introduce the Calculus? Is there something that needs the properties of Q and R?”

> 5. T: ”The mathematization of a primitive intuition. In my classroom, this year, in the Calculus...”

6. I: ”Didn't you recall R? Did you find difficulties?”

> 7. T: ”No, because it's quite natural. I did many examples... for instance…, that emerges a bit… when we talked about limit points, the book didn't propose a definition .... instead I proposed a definition ... we talked for some minutes and since they had studied the sequences I tried to make them reflect about the fact that, while when you calculate the limit of a sequence only to the infinite, when you study a function on the real numbers you calculate limits also in finite points, be only limit point for N is the infinite, while for R... But, in the reality, it's a thing that only a few students gather ... the straight line topology is so banal... to be honest no... but it's transformed by the books in a such banal thing that all [the teachers? the books?] always present all the theorems on the neighbourhoods, on the intervals… because the interval at any rate works... so you don't gather intuitively... Calculus is so complex for them that, being asked to learn new tools, that a discourse about these things may be very good for a few students ... In fact, I presented a lesson about the topology of the straight line, but I had not time enough. They had talked about contiguous classes, they see the continuum [represents simultaneously a limit point on the line]. They see all but that ... the problem of discontinuity. For them there is all. This is the reason why I say that I'm sure that for them these are real numbers [traces a segment] and that, however I take out stuff, there other stuff remains here, close,

this is sure. They didn't understand the sense of talking about limit points. [...]

8. " The problem is that they don't have the concept of approximation. [...] which is the essential difference between the real numbers and their practical uses. My students in the end of the 5 years [9-13 grades, nba] don't know it. I would expect them to know this at least: that they have understood the difference between a real number and one of its approximations. A lot of them... it's something subtle for them, but it shouldn't be a subtle thing! This is what should emerge as the strongest thing, i.e. the difference between a real number and its approximations...that there is an infinity of approximations… They don't know the great difference between continuum and discrete, interpreted precisely as the number and the approximation.

7. Results and conclusions

1. *Intuitive and formal meanings are identified.* The teacher is PhD in Mathematics and knew the problem of the definition of real numbers starting from Q. She said half times it's hard to talk about real numbers and half times that's innate, intuitive, so we wondered: why does she expect that for the students R is innate? How does this belief affect her choices? What criteria does she use to choose practices and definitions, since she shows this inconsistency? Looking at the way she to talk about R and continuum, she identified completely different "partial meanings" with the complex meaning, as if the first were good representations of the second. Maybe her knowledge became so much part of her way of thinking that she considers it innate and considers all the meanings resumed in every representation of real numbers.

2. Representativeness without cognitive suitability. The teacher showed to take care in general of representativeness (to define limit points, to show R has not the power of N and Q, to consider crucial the difference between real numbers and approximations) (EPS4), but not to deepen into the complexity of the relation between R and the line, causing a lack of high cognitive request (no CS2); she explained that this is not considered useful, because the students got more confused rather than understand better. A lot of her choices are due to high attention to students' intuitions. She had already reflected on her own on the students' difficulties but she had considered them unavoidable ("this is hard for the most of them"; "they see the segment full"). She declared that they had already studied real numbers and she expected them to know R (no CS1), so she had decided to take R for granted, but the fact that the teacher didn't take in account the complexity of R made highly probably her responsible for the failure of their attempts to teach concepts like limit point, that are not intuitive in the graphic representation (Bagni, 2000). She had tried to simplify, limiting everything to the intuitions, but, at the same time, she hadn't abandon the goal to be introduce formal concepts and had not considered their personal objects (no CS1): she had created thus an "intuitive world" in which it was no more possible to present the right problems to students to understand what she was trying to explain.

3. Connections between teacher's personal meaning, orientations and choices. The teacher has a strong background in Mathematics and attended teachers' training courses, but the practices she declared to propose are not rich enough (no EPS3) to represent the complexity of the relation between intuitive and formal aspects. She takes care to introduce significant configurations (EPS4), also present in the national curricula (ECS1), but not presenting the systems of practices and the connections (no EPS3) and in the end she doesn't succeed in presenting significantly the complexity of real numbers, "jumping" from very poor meanings to very general expected meanings, identifying the configurations, in strength of the orientation that complex meanings may often be transformed easily in something innate and intuitive. She declared to put together, without explaining their relations: the comparison between limits in the discrete and in the dense case; comparison between limits of sequences and functions; infinite as limit point in N; the "topology of the line"; intervals of real numbers as segments obtained by tracing; the difference and the relation between approximation and real numbers. She presents complex configurations avoiding to enter their complexities and trusting students' intuitions and innate predisposition to "think continuously", without introducing suitable problems and connections. She said, for instance, "students perceive the numbers in a continuous way"; "in the students' minds there is already a spontaneous conception of real number as a sequence of numbers"; "students see the continuum, the contiguous classes"; "it's easy to show that Q is equipotent to N". In the interview, she appeared to be aware that students' conceptions are not suitable for grasping some concepts like the limit points and so on, but she attributes the difficulties to the fact that real numbers are not introduced as soon as possible distinguishing immediately between the numbers and their approximations, rather than to the lack of suitable complex practices (no CS1), so she doesn't feel she could change things in the last year of secondary school. We think that this analysis adds complexity to cognitive suitability: this teacher takes care of students' intuitions, but she considers them somehow independent from the system of practices from which their personal configurations of R and continuum emerge. This aspect convinced the teacher to avoid richness in processes (no EPS3) but to aim at the same time at representativeness (EPS4 and ECS1), not taking care of ambiguity (no EPS2) and basing all the choices on students' intuitions. More attention to innovation and researches in mathematics education (no ECS2), in particular results from OSA researches on complexity of mathematical objects, would be useful for this teacher to interpret better students' difficulties and to be more effective.

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