Bounded domains which are universal for minimal surfaces

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Abstract

We construct open domains in $\mathbb{R}^3$ which do not admit complete properly immersed minimal surfaces with an annular end. These domains can not be smooth by a recent result of Martín and Morales [5].

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1 Introduction

The main goal of this paper is to construct bounded open domains in $\mathbb{R}^3$ which do not contain any complete properly immersed minimal

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surfaces with at least one annular end. It is our belief that these open domains are in fact universal according to the following definition: A connected region of space which is open or the closure of an open set is *universal for minimal surfaces*, if every complete properly immersed minimal surface in the region is recurrent for Brownian motions. In particular, a bounded domain is universal if and only if it contains no complete properly immersed minimal surfaces.

Theorem 1. *Let $\mathcal{D}$ be any bounded open domain in $\mathbb{R}^3$. Then there exists a proper countable collection $\mathcal{F}$ of pairwise disjoint horizontal simple closed curves in $\mathcal{D}$ such that the complementary domain $\widetilde{\mathcal{D}} = \mathcal{D} - \mathcal{F}$ is universal for minimal surfaces with at least one annular end. In particular, any complete immersed minimal surface of finite genus in $\widetilde{\mathcal{D}}$ must have an uncountable number of ends.*

The construction of the domains $\widetilde{\mathcal{D}}$ that appear in the above theorem are motivated by a related unpublished example of the third author. We will explain a variant of his original example at the end of Section 2.

Interest in results like Theorem 1 dates back to an earlier question by Calabi. Calabi asked whether or not it is possible for a complete minimal surface in $\mathbb{R}^3$ to be contained in the ball $B = \{x \in \mathbb{R}^3 \mid \|x\| < 1\}$. In [10], Nadirashvili constructed a complete minimal surface in B. After Nadirashvili negative solution to Calabi's question, Martín and Morales [7] proved that there exist complete properly immersed minimal disks in B. Recently [6], they improved on their original techniques and were able to show that every bounded domain with $C^{2,\alpha}$-boundary admits a complete properly immersed minimal disk whose boundary limit set is close to a prescribed simple closed curve on the boundary of the domain. In contrast to these existence results for complete properly immersed minimal disks in bounded domains, Colding and Minicozzi [2] recently proved that any complete embedded minimal surface in $\mathbb{R}^3$ is properly embedded in $\mathbb{R}^3$. By results of Meeks and Rosenberg, [9, 8], any properly embedded minimal surface of finite topology in $\mathbb{R}^3$ is recurrent for Brownian motion. Hence, every domain in $\mathbb{R}^3$ is universal for embedded minimal surfaces of finite topology. Finally, we remark that Collin, Kusner, Meeks and Rosenberg [3] proved that any properly immersed minimal surface with boundary in a closed convex domain in $\mathbb{R}^3$ has full harmonic measure on its boundary.

At the end of Section 2, we give an estimate for the growth of the absolute curvature function $|K_M|$ for any complete properly immersed minimal surface M in a smooth bounded domain $\mathcal{D} \subset \mathbb{R}^3$ in terms of the distance function $d_{\partial\mathcal{D}}$ of M to $\partial\mathcal{D}$. This estimate implies the function $|K_M|\, d^2_{\partial\mathcal{D}}$ is never bounded.

2 Proof of Theorem 1

Let $\mathcal{D}$ be an open connected bounded set of $\mathbb{R}^3$ and let $\overline{\mathcal{D}}$. Without loss of generality we may assume that $\overline{\mathcal{D}}$ is contained in the closed slab

$$\left\{(x_1, x_2, x_3) \in \mathbb{R}^3 \mid 0 \leq x_3 \leq 1\right\}$$

and $\overline{\mathcal{D}}$ contains points at heights 0 and 1.

For $t \in (0, 1)$, let P_t denote the horizontal plane at height t. Let $C_t = \mathcal{D} \cap P_t$, which consists of a collection $\{C_{t,i}\}_{i \in I_t}$ of connected components, for some countable indexing set I_t. For each t and for each $i \in I_t$, choose an exhaustion of $C_{t,i}$ by smooth compact domains $C_{t,i,k}$, $k \in \mathbb{N}$, and where $C_{t,i,k} \subset C_{t,i,k+1}, \quad \forall k \in \mathbb{N}$. Finally, let $C_t(k) \stackrel{\text{def}}{=} \bigcup_{i \in I_t} C_{t,i,k}$.

Now consider the following sequence of ordered rational numbers:

$$Q = \left\{\frac{1}{2}, \frac{1}{3}, \frac{2}{3}, \frac{1}{4}, \frac{2}{4}, \frac{3}{4}, \ldots, \frac{1}{n}, \frac{2}{n}, \ldots, \frac{n-1}{n}, \ldots\right\}.$$

Let t_k the k-th rational number in Q. Define $\mathcal{F}$ to be the collection of boundary curves to all of the domains $\bigcup_{k \in \mathbb{N}} C_{t_k}(k)$, and define $\widetilde{\mathcal{D}} \stackrel{\text{def}}{=} \mathcal{D} - \mathcal{F}$.

Suppose that $f : M \to \widetilde{\mathcal{D}}$ is a complete properly immersed minimal surface with an annular end E and we will obtain a contradiction. First, note that the limit set $L(E)$ of E is a connected set contained in $\partial\widetilde{\mathcal{D}}$, where

$$\partial\widetilde{\mathcal{D}} = \overline{\widetilde{\mathcal{D}}} - \widetilde{\mathcal{D}} = \partial\mathcal{D} \cup \mathcal{F}.$$

Our initial goal is to prove that $x_3|_{L(E)}$ is constant, from which we will easily obtain a contradiction.

If $L(E)$ intersects one of the horizontal curves C in $\mathcal{F}$, then $L(E) \subset C$ (recall that $L(E)$ is connected) and we have proved that $x_3|_{L(E)}$

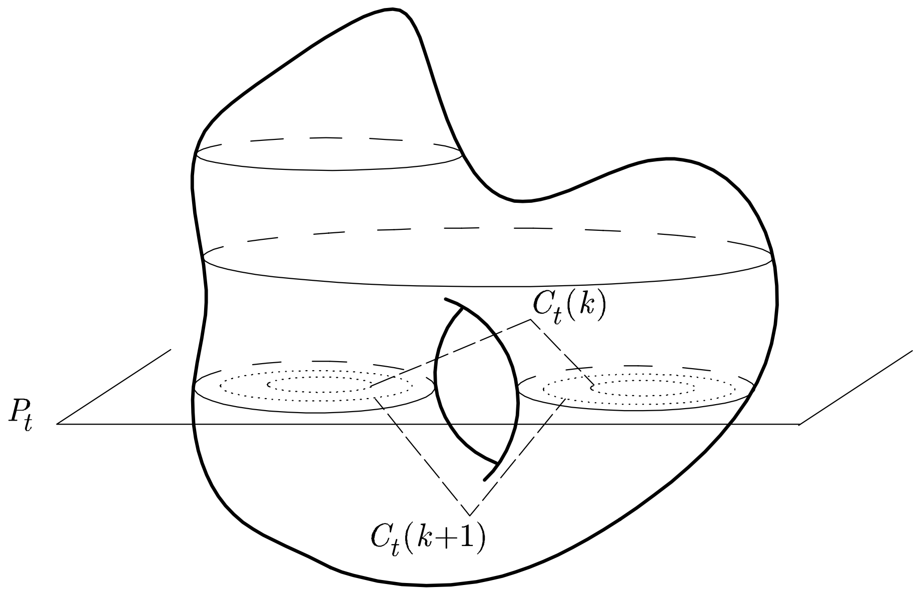

Figure 1: The domain $\mathcal{D}$ and the sets $C_t(k)$.

is constant. So, suppose that $p \in L(E) \subset \partial\mathcal{D}$. If $x_3|_{L(E)}$ is not constant, then there exists a point $q \in L(E)$ with $x_3(p) \neq x_3(q)$. Choose a positive rational number t which lies between $x_3(p)$ and $x_3(q)$. Notice that t can be represented by an infinite subsequence $\{t_{k_1}, t_{k_2}, \ldots, t_{k_n}, \ldots\} \subset Q$. Since the plane P_t separates p and q, for every subend $E' \subset E$, $P_t \cap E'$ is nonempty. On the other hand, the subdomains $C_t(k_n)$ give a compact exhaustion to $P_t \cap \mathcal{D}$ with boundaries disjoint from E. Therefore, every component of $P_t \cap E$ is compact. Since $P_t \cap E$ is noncompact, then there exist a pair of disjoint simple closed curves in $P_t \cap E \subset E$ which bound a compact domain in E, since E is an annulus. But then the harmonic function x_3 restricted to this domain has an interior maximum or minimum which is impossible. This contradiction proves that $x_3|_{L(E)}$ *is constant.* Let a denote this constant.

Our next step consists of proving that if $x_3|_{L(E)}$ is constant, then the minimal immersion $f : M \to \widetilde{\mathcal{D}}$ is incomplete, which is contrary to our assumptions. Indeed, consider a conformal parameterization of the end E by the annulus $A = \{z \in \mathbb{C} \mid r \leq |z| < 1\} \subset \mathbb{C}$, for some

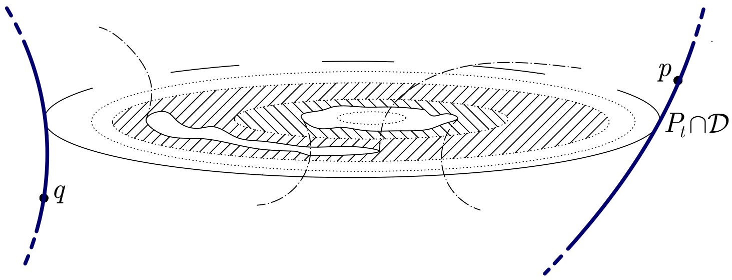

Figure 2: The subdomains $C_t(k_n)$ give a compact exhaustion to $P_t \cap \mathcal{D}$ with boundaries disjoint from E. Therefore, every component of $P_t \cap E$ is compact. Since $P_t \cap E$ is noncompact, then there exist a pair of disjoint simple closed curves in $P_t \cap E \subset E$ which bound a compact domain in E

$0 < r < 1$. Since x_3 is a *bounded* harmonic function defined on A, then by Fatou's theorem x_3 has radial limit a.e. in $\mathbb{S}^1 = \{z \in \mathbb{C} \mid |z| = 1\}$. Furthermore, the function x_3 is determined by the Poisson integral of its radial limits (see for instance [4].) Since the limit $\lim_{\rho \to 1} x_3(\rho\theta) = a$, at almost every point θ in $\mathbb{S}^1$, then x_3 admits a regular extension to $\overline{A}$. In particular, $\|\nabla x_3\|$ is bounded in A. On the other hand, as x_2 is also a bounded harmonic function, then a result by Bourgain [1, Theorem 2] asserts that the set

$$\mathcal{S} = \left\{ \theta \in \mathbb{S}^1 \mid \int_r^1 \|\nabla x_2(\rho\,\theta)\|\, d\rho < +\infty \right\}$$

has Hausdorff dimension 1, in particular $\mathcal{S}$ is nonempty. Moreover, for a conformal minimal immersion it is well known [11] that $\|\nabla x_1\| \leq \|\nabla x_2\| + \|\nabla x_3\|$.

Hence, as a consequence of all these facts, if θ is a point in $\mathcal{S}$ then

$$\int_r^1 \sqrt{\|\nabla x_1(\rho\,\theta)\|^2 + \|\nabla x_2(\rho\,\theta)\|^2 + \|\nabla x_3(\rho\,\theta)\|^2}\, d\rho < \infty,$$

which means that the divergent curve $f(\rho\,\theta)$, $\rho \in (r, 1)$, has finite length, and so f is not complete. This contradiction proves the theorem.

We now explain a modification of the original unpublished example of Nadirashvili which motivated our construction of the domains $\widetilde{\mathcal{D}}$ given in Theorem 1.

Let $\mathcal{D}$ be the open cube:

$$\mathcal{D} = (-1,1) \times (-1,1) \times (-1,1).$$

Let $F_1 = \{1\} \times [-1,1] \times [-1,1]$, $F_2 = [-1,1] \times \{1\} \times [-1,1]$ and $F_3 = [-1,1] \times [-1,1] \times \{1\}$ be the three coordinate faces of $\mathcal{D}$. Let $S_i \overset{\text{def}}{=} \partial F_i$ be the related boundary square curves. As in the construction of the domains in Theorem 1, we need to define a countable proper collection $\mathcal{F}$ of planar simple closed curves in the cube $\mathcal{D}$, so that $\mathcal{D} - \mathcal{F}$ admits no complete properly immersed minimal surfaces with an annular end.

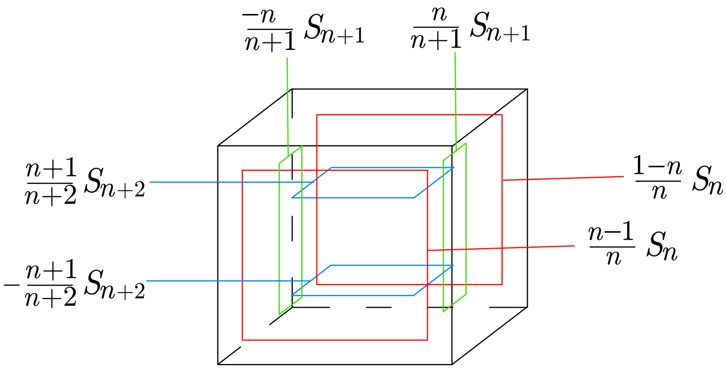

Figure 3: The cube $\mathcal{D}$.

For a real number λ, let $\lambda\, S_i = \{\lambda\,(x_1, x_2, x_3) \mid (x_1, x_2, x_3) \in S_i\}$, for $i = 1, 2, 3$. Let $\mathcal{F}$ be the collection of curves

$$\left\{ \tfrac{n-1}{n}\, S_{n(\text{mod}3)}\,,\ \tfrac{1-n}{n}\, S_{n(\text{mod}3)} \right\}_{n\in\mathbb{N}}.$$

Then, a small modification of the arguments given in the proof of Theorem 1 implies that one of the coordinate functions (x_1, x_2, x_3) restricted to the limit set of an annular end of a complete immersed minimal surface in $\widetilde{\mathcal{D}} \overset{\text{def}}{=} \mathcal{D} - \mathcal{F}$ is constant. As in the proof of Theorem 1, we obtain a contradiction.

Finally, we explain that if M is a complete properly immersed minimal surface in a convex or smooth bounded domain, then the function $|K_M|\, d_{\partial\mathcal{D}}{}^2$ is not bounded.

We proceed by contradiction. Assume there exists a constant $C > 0$ so that $|K_M|\, d_{\partial D} \leq C$. Since the convex hull of a nonflat complete minimal surface with bounded curvature in $\mathbb{R}^3$ is all of $\mathbb{R}^3$ [12], the curvature function is not bounded in M. Thus, take an arbitrary sequence of points $q_n \in M$ such that $|K_M(q_n)| \geq n^2$. Let $p'_n \in M \cap B_M(q_n, 1)$ be a maximum of

$$h_n = |K_M|\, d_M(\cdot, \partial B_M(q_n, 1))^2,$$

where d_M denotes the intrinsic distance of M and $B_M(q_n, 1)$ means the intrinsic ball centered at q_n with radius 1.

We label $\lambda'_n = \sqrt{|K_M(p'_n)|}$. Notice that:

$$\lambda'_n \geq \lambda'_n\, d_M(p'_n, \partial B_M(q_n, 1)) = \sqrt{h_n(p'_n)} \geq \sqrt{h_n(q_n)} = \sqrt{|K_M(q_n)|} = n.$$

Fix $t > 0$. Notice that the sequence of extrinsic balls

$$\left\{ \lambda'_n \mathbb{B}\left(p'_n, \frac{t}{\lambda'_n}\right) \right\}_{n\in\mathbb{N}}$$

converges to the ball $\mathbb{B}(t)$, where we have indentified p'_n with $\vec{0}$. Similarly, we can consider $\{\lambda'_n B_M(p'_n, t/\lambda'_n)\}_{n\in\mathbb{N}}$ as a sequence of minimal surfaces with boundary, passing through $\vec{0}$ with curvature -1 at the origin. From our assumption, we know that $D_n(t) = \partial\mathcal{D} \cap \mathbb{B}\left(p'_n, \frac{t}{\lambda'_n}\right)$ is nonempty, for any $t > C$.

We assert that the curvature of these minimal surfaces with boundary is uniformly bounded. Indeed, pick a point z in $B_M(p'_n, t/\lambda'_n)$. Then we have

$$\frac{\sqrt{|K_M(z)|}}{\lambda'_n} = \frac{\sqrt{h_n(z)}}{\lambda'_n\, d_M(z, \partial B_M(q_n, 1))} \leq \frac{d_M(p'_n, \partial B_M(q_n, 1))}{d_M(z, \partial B_M(q_n, 1))} \qquad (1)$$

By the triangle inequality, one has

$$d_M(p'_n, \partial B_M(q_n, 1)) \leq \frac{t}{\lambda'_n} + d_M(z, \partial B_M(q_n, 1)),$$

and so

$$\frac{d_M(p'_n, \partial B_M(q_n,1))}{d_M(z, \partial B_M(q_n,1))} \leq 1 + \frac{t}{\lambda'_n\, d_M(z, \partial B_M(q_n,1))} \leq$$
$$1 + \frac{t}{\lambda'_n \left(d_M(p'_n, \partial B_M(q_n,1)) - \frac{t}{\lambda'_n} \right)} \leq 1 + \frac{t}{n-t},$$

which tends to 1 as $n \to \infty$.

After extracting a subsequence, it follows that $\lambda'_n\, B_M(p'_n, \frac{t}{\lambda'_n})$ converge smoothly to a minimal surface $M_\infty(t)$ contained in $\mathbb{B}(t)$. Since $\lim_{n\to\infty} \lambda'_n = +\infty$, then $\lambda'_n\, D_n(t)$ converges either to a plane in the case that $\mathcal{D}$ is a regular domain or to the boundary of a convex body if $\mathcal{D}$ is a convex domain. In any case, $M_\infty(t)$ is contained in one of the halfspaces determined by the plane, or in the interior of the convex body. Note that $M_\infty = \cup_{t\geq C} M_\infty(t)$ is a complete nonflat minimal surface. By construction, M_∞ has bounded curvature and is contained in a convex domain which is not $\mathbb{R}^3$. But this is contrary to the aforementioned result by Xavier. This contradiction proves that $|K_M|\, {d_{\partial\mathcal{D}}}^2$ is not bounded.

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