A note on the uniqueness of the periodic Callahan-Hoffman-Meeks surfaces in terms of their symmetries

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Abstract

We give a new characterization of the periodic Callahan-Hoffman-Meeks surfaces in terms of their symmetries.

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1 Introduction and preliminaries

The first examples of periodic minimal surfaces with more than one end, and different from Riemann's examples, were discovered by Callahan, Hoffman, and Meeks in [2]. They found examples of complete embedded minimal surfaces, with an infinite number of annular ends. For each odd integer $k > 0$, they constructed a surface M_k which is invariant under a cyclic group of vertical translations, $\mathcal{T}$. The quotient $M_k/\mathcal{T}$ has: genus k, two planar ends and $4(k+1)$ symmetries.

D. Rodriguez and the author studied the surfaces M_k in [5], and obtained the following classification theorem:

Theorem 1 (—, Rodríguez) *Let $\widetilde{M}$ be a properly embedded minimal surface in $\mathbb{R}^3$ satisfying the following:*

1. *$\widetilde{M}$ has an infinite number of annular ends;*
2. *$\widetilde{M}$ is invariant under a cyclic group of translations $\mathcal{T}$;*
3. *$\widetilde{M}/\mathcal{T}$ has genus k, $k \geq 2$, and 2 ends;*
4. *$Iso(\widetilde{M}/\mathcal{T})$ has at least $4(k+1)$ elements.*

Then k is odd and $\widetilde{M}$ is, up to scaling and rigid motion, the Callahan-Hoffman-Meeks example of genus k.

In this paper we prove that the above theorem is true if we change the translation by a non-trivial *screw motion*. More specifically, we have obtained the following theorem.

Theorem 2 *Let $\widetilde{M}$ be a properly embedded minimal surface in $\mathbb{R}^3$ satisfying:*

1. *$\widetilde{M}$ is invariant under a screw motion $S_\theta = \widetilde{R}_\theta \circ T$, where $\widetilde{R}_\theta$ is a rotation about the x_3-axis and T is a vertical translation,*
2. *$\widetilde{M}$ has an infinite number of annular ends,*
3. *$M = \widetilde{M}/\langle S_\theta \rangle$ has genus k, $k \geq 2$, and two ends,*
4. *$|Iso(M)| \geq 4(k+1)$.*

Then, k is odd, $\theta \in \frac{4\pi}{1+k}\mathbb{Z}$, the rigid motions $\widetilde{R}_\theta$, T belong to $Sym(\widetilde{M})$ and $\widetilde{M}$ is the Callahan-Hoffman-Meeks surface of genus k.

Observe that we obtain a non-existence result for properly embedded minimal surfaces in $\mathbb{R}^3$ verifying hypotheses *1,2, 3,* and *4*, for angle $\theta \notin \frac{4\pi}{1+k} \cdot \mathbb{Z}$. A non-existence theorem similar in style was obtained by Callahan, Hoffman, and Karcher [1] for $\theta = \frac{2\pi}{1+k}$.

Firstly, we need to briefly summarize the principal results for complete, embedded, singly periodic minimal surfaces.

Let $\widetilde{x} : \widetilde{M} \longrightarrow \mathbb{R}^3$ be a proper minimal embedding of a surface $\widetilde{M}$ in three dimensional Euclidean space, invariant under a screw motion, S. Using isothermal parameters, $\widetilde{M}$ has a natural conformal structure and we label $(\widetilde{g}, \widetilde{\eta})$ as the Weierstrass data of $\widetilde{x}$. Remember that the stereographic projection of the Gauss map $\widetilde{g}$ is a meromorphic function on $\widetilde{M}$, and $\widetilde{\eta}$ is a holomorphic one-form on $\widetilde{M}$.

The total curvature of M is denoted by $\mathcal{C}(M)$. In the remainder of this section we assume that $\mathcal{C}(M) > -\infty$.

A surprising relationship exists between the topology of a properly embedded periodic minimal surface in $\mathbb{R}^3$ and the behavior of its ends. This was studied by Callahan, Hoffman and Meeks who proved, in [3], the following

Theorem 3 (Callahan, Hoffman, Meeks) *Suppose $\widetilde{M}$ is a properly embedded minimal surface in $\mathbb{R}^3$, with more than one end, whose symmetry group is infinite. Then either $\widetilde{M}$ is a catenoid or:*

1. *$\widetilde{M}$ has an infinite number of ends;*
2. *$\widetilde{M}$ is invariant under a screw motion $\mathcal{S}$;*
3. *all annular ends of $\widetilde{M}$ are flat ends;*

4. the total curvature of $M = \widetilde{M}/\mathcal{S}$ is finite if and only if M has finite topology, in which case

$$\mathcal{C}(M) = 2\pi(\chi(M) - r(M)),$$

where $r(M)$ is the number of ends of M.

We have written as $Iso(\widetilde{M})$ the isometry group of $\widetilde{M}$. The subgroup of $Iso(\widetilde{M})$, which is denoted by $Sym(\widetilde{M})$, consists of isometries which are the restriction of a rigid motion in $\mathbb{R}^3$ leaving $\widetilde{x}(\widetilde{M})$ invariant. H. I. Choi, W. H. Meeks and B. White [4] proved the following:

Theorem 4 (Choi, Meeks, White) *A properly embedded minimal surface $\widetilde{M}$ in $\mathbb{R}^3$ which has more than one end is minimally rigid. In particular $Sym(\widetilde{M}) = Iso(\widetilde{M})$.*

Finally, we consider the behavior of the Gauss map at an end and at a point on the x_3-axis in the presence of a rotational symmetry $\widetilde{R}$ of $\widetilde{M}$.

Remark 1 *A planar end must correspond to a zero or a pole of $\widetilde{g}$ with order $jm + 1$, where j is a positive integer and m is the order of $\widetilde{R}$. On the other hand, if $\widetilde{Q} \in \widetilde{M}$ is a point of the x_3-axis, then the order of $\widetilde{Q}$ as zero or pole of g is $jm - 1$ (for details see [2]).*

2 The main results

Consider $\widetilde{x} : \widetilde{M} \to \mathbb{R}^3$ a proper minimal embedding satisfying:

1. $\widetilde{x}(\widetilde{M})$ is invariant under a screw motion S_θ,
2. $\widetilde{x}(\widetilde{M})$ has an infinite number of annular ends,
3. $M = \widetilde{M}/\langle S_\theta \rangle$ has genus k, $k \geq 2$, and two ends,
4. $|Iso(M)| \geq 4(k+1)$.

From Theorem 3 we deduce that the ends of $\widetilde{M}$ are flat and the total curvature of $M = \widetilde{M}/\langle S_\theta \rangle$ is $-4\pi(k+1)$. As we have mentioned in Section 1, M has the conformal type of a compact Riemann surface, $\overline{M}$, punctured in two points, P_1 and P_2, which correspond to the ends. Meeks and Rosenberg [6] proved that the Weierstrass data $(\widetilde{g}, \widetilde{\eta})$ extend meromorphically to the ends of $\widetilde{M}$. Since $\widetilde{x}$ is an embedding the ends are parallel and we can suppose that, up to a rigid motion of $\mathbb{R}^3$, the limit normal vectors are $\pm(0,0,1)$. Furthermore, the limit normal vector is upwards at one end and downwards at the next alternately.

Let $\widetilde{A} \in Sym(\widetilde{M})$, then $\widetilde{A}(x) = \mathcal{A} \cdot x + \vec{a}$, where $\vec{a} \in \mathbb{R}^3$ is vertical, and $\mathcal{A} \in \mathcal{O}(3, \mathbb{R})$ preserves the x_3-axes. If we write $S_\theta(x) = \mathcal{R} \cdot x + \vec{v}$, then it is straightforward to check that:

$$\left(\widetilde{A} \circ S_\theta \circ \widetilde{A}^{-1}\right)(x) = \mathcal{R}^{\pm 1} \cdot x \pm \vec{v}. \tag{1}$$

Claim 1 *The screw motion $S'_\theta(x) = \mathcal{R} \cdot x - \vec{v}$ is also a symmetry of $\widetilde{M}$.*

Proof: We proceed by contradiction. Assume $S'_\theta \notin \mathrm{Sym}(\widetilde{M})$. From (1), the previous assumption means that $\langle S_\theta \rangle$ is normal in $\mathrm{Sym}(\widetilde{M}) = \mathrm{Iso}(\widetilde{M})$.

In this case, if we write $\Delta = \mathrm{Iso}(\widetilde{M})/\langle S_\theta\rangle$, then $\Delta = \mathrm{Iso}\ (M)$. Since the subgroup of holomorphic transformations has index one or two in Δ, then Hurwitz's theorem implies that Δ is finite.

We label $H = \{F \in \Delta\ : F \text{ is holomorphic }\}$ and $G = \{F \in H\ :\ F(P_1) = P_1\}$. It is obvious that:

$$[\Delta : H] \leq 2, \quad [H : G] \leq 2. \tag{2}$$

Up to a translation, we suppose that P_1 is asymptotic to the plane $x_3 = 0$.

If we take D, a conformal disk centered at P_1, invariant under G, then we can see G as a finite group of biholomorphisms of the disk fixing the origin, and so G is cyclic. Let J be a generator of G.

Under these conditions, up to a composition with a suitable power of S_θ, we can assume that there exists a lift $\widetilde{J}$ of J to $\widetilde{M}$ which preserves the x_3-axis and fixes the origin. Taking (2) and hypothesis *4* into account, we have $\mathrm{ord}(\widetilde{J}) \geq k+1 > 2$. Then we can assume that either $\widetilde{J}$ is a counterclockwise rotation around the x_3-axis by angle $2\pi/\mathrm{ord}(\widetilde{J})$ or $\mathrm{ord}(\widetilde{J})$ is even and $\widetilde{J}$ is a counterclockwise rotation around the x_3-axis by angle $2\pi/\mathrm{ord}(\widetilde{J})$ followed by a reflection in the (x_1, x_2)-plane.

But observe that $\widetilde{J}$ cannot be a rotation about the x_3-axis followed by a reflection in the (x_1, x_2)-plane, because, in that case $\widetilde{J} \circ S_\theta \circ \widetilde{J}^{-1} = S'_\theta$, which is contrary to our assumption.

Thus $\widetilde{J}$ is a rotation about the x_3-axis. Therefore, as we mentioned in Remark 1, each end E of $\widetilde{M}$ must correspond to a zero or a pole of $\widetilde{g}$ with order greater than or equal to $\mathrm{ord}(\widetilde{J})+1 \geq k+2$. Taking into account that the ends are flat, the 1-form $\widetilde{\Phi}_3$ is holomorphic on $\widetilde{M}$. Furthermore, if E is an end of $\widetilde{M}$, then $\widetilde{\Phi}_3$ has a zero of order $|\mu| - 2$ at E, where μ is the order of the zero or pole of g at E. Let Φ_3 the corresponding 1-form on $\overline{M}$ induced by $\widetilde{\Phi}_3$ (recall that $S_\theta^* \widetilde{\Phi}_3 = \widetilde{\Phi}_3$). Thus, Φ_3 has two zeroes of order greater or equal to k at P_1 and P_2. Denote $Z(\Phi_3)$ and $P(\Phi_3)$ as the number of zeroes and poles (counting multiplicities) of Φ_3, respectively. A well-known formula for meromorphic 1-forms on compact Riemann surfaces implies that $2(k-1) = -\chi(\overline{M}) = Z(\Phi_3) - P(\Phi_3) \geq 2k$, which is absurd. This contradiction completes the proof. □

Claim 1 implies that $\widetilde{M}$ is invariant under the translation $T(x) \stackrel{\mathrm{def}}{=} (S_\theta \circ S'^{-1}_\theta)(x) = x + 2\vec{v}$. Consider the quotient $N = \widetilde{M}/\langle T\rangle$, and write (g, η) as its Weierstrass data. Reasoning as above, we infer that N has the conformal type of a compact Riemann surface $\overline{N}$ punctured in four points: Q_1, Q_2, Q_3 and Q_4.

Claim 2 *Let $\widetilde{M}$ be a properly embedded minimal surface in $\mathbb{R}^3$. Suppose that $\widetilde{M}$ is invariant under a rotation about the x_3-axis by angle $\frac{2\pi}{k}$, $\widetilde{R}$, and under a vertical translation T. If we label $S = \widetilde{R} \circ T$, then:*

1. $|Iso(\widetilde{M}/\langle T\rangle)| \geq |Iso(\widetilde{M}/\langle S\rangle)|$,
2. $\mathcal{C}(\widetilde{M}/\langle T\rangle) = \mathcal{C}(\widetilde{M}/\langle S\rangle)$.

Proof: Consider $\mathcal{M} = \widetilde{M}/\langle T, \widetilde{R}\rangle$. Both $\widetilde{M}/\langle T\rangle$ and $\widetilde{M}/\langle S\rangle$ are k-sheeted conformal branched coverings of $\mathcal{M}$. Then, Statement 2 holds easily. To obtain 1, observe that

$$\mathrm{ord}(\widetilde{R}) \cdot \left|\mathrm{Iso}\left(\widetilde{M}/\langle T\rangle\right)\right| = \left|\mathrm{Iso}\left(\widetilde{M}/\langle \widetilde{R}, S\rangle\right)\right| \geq \mathrm{ord}(\widetilde{R}) \cdot \left|\mathrm{Iso}\left(\widetilde{M}/\langle S\rangle\right)\right|,$$

which concludes the proof. $\square$

Applying the above claim to the screw S_θ^2 and the translation T, we have

$$|\mathrm{Iso}(N)| \geq 2\,|\mathrm{Iso}(M)| \geq 8(k+1), \tag{3}$$
$$\mathcal{C}(N) = 2\,\mathcal{C}(M) = -8\pi(k+1). \tag{4}$$

Notice that $\langle T\rangle$ is a normal subgroup of $\mathrm{Sym}(\widetilde{M})$. If we write $\Delta = Sym(\widetilde{M})/\langle T\rangle$, then as a consequence of the rigidity of $\widetilde{M}$ (Theorem 4) we obtain $Iso(N) = \Delta$. If we label H' as the subgroup of the holomorphic transformations of Δ and $G' = \{A \in H' \;/\; A(Q_1) = Q_1\}$, it is obvious that:

$$[\Delta : H'] \leq 2, \quad [H' : G'] \leq 4. \tag{5}$$

Up to a translation, we suppose that Q_1 is asymptotic to the plane $x_3 = 0$.

Let D' be a conformal disk centered at Q_1, invariant under G'. We can see G' as a finite group of biholomorphisms of the disk fixing the origin, and so G' is cyclic. Let F be a generator of G'.

Up to a composition with a suitable power of T, we can assume that there exists a lift $\widetilde{F}$ of F to $\widetilde{M}$ which fixes the origin. Moreover, recall that $\widetilde{F}$ preserves the x_3-axis. Taking (3) and (5) into account, we have $\mathrm{ord}(\widetilde{F}) \geq k+1 \geq 3$ ($k \geq 2$). Then we can assume that either $\widetilde{F}$ is a counterclockwise rotation around the x_3-axis by angle $2\pi/\mathrm{ord}(\widetilde{F})$ or $\mathrm{ord}(\widetilde{F})$ is even and $\widetilde{F}$ is a counterclockwise rotation around the x_3-axis by angle $2\pi/\mathrm{ord}(\widetilde{F})$ followed by a reflection in the (x_1, x_2)-plane.

Claim 3 *The map $\widetilde{F}$ is a rotation about the x_3-axis followed by a reflection in the (x_1, x_2)-plane.*

Proof: If $\widetilde{F}$ were a rotation, then, using Remark 1, each end corresponds to a zero or a pole of $\tilde{g}$ of order at least $k+2$. In the quotient N there are two ends with the same character (as zero or pole) then $\deg(g) \geq 2k+4$. So, $\mathcal{C}(N) < -8\pi(k+1)$ which is contrary to (4). $\square$

Claim 4 *$\widetilde{M}$ is invariant under the translation $x \mapsto x+\vec{v}$. So, using Claim 2 and Theorem 1, k is odd and $\widetilde{M}$ is the Callahan-Hoffman-Meeks example of genus k.*

Proof: Observe that $T^{-1} \circ S_\theta^2$ is a rotation about the x_3-axis by angle 2θ. Then, $T^{-1} \circ S_\theta^2 = \widetilde{F}^{2m}$, $m \in \mathbb{Z}$.

Assume m is odd. Then

$$\widetilde{F}^m \circ S_\theta^{-1}(x_1, x_2, x_3) = (x_1, x_2, b - x_3),$$

which corresponds to a reflection in the plane $\Pi = \{(x_1, x_2, x_3) \in \mathbb{R}^3 \ / \ x_3 = b/2\}$. If this plane were not asymptotic to one end then there would be more than two ends in slab $0 \leq x_3 < b$, which is absurd. On the other hand, if Π were asymptotic to one end then $\widetilde{x}(\widetilde{M}) \cap \Pi$ would be a nonempty, 1-dimensional variety formed by double points, which is also absurd.

Hence, m is even, and so $(\widetilde{F}^{-m} \circ S_\theta)$ is the translation that we are looking for. This completes the proof. $\square$

Claim 4 completes the proof of Theorem 2.

References

[1] M. Callahan, D. Hoffman, H. Karcher. *A family of singly periodic minimal surfaces invariant under a screw motion.* Experimental Mathematics, **2**, No. 3, 157-182 (1993).

[2] M. Callahan, D. Hoffman, W. H. Meeks III. *Embedded minimal surfaces with an infinite number of ends.* Invent. math., **96** (1989), 459-505.

[3] M. Callahan, D. Hoffman, W. H. Meeks III. *The structure of singly-periodic minimal surfaces.* Invent. math., **99** (1990), 455-481.

[4] H. I. Choi, W. H. Meeks III, B. White. *A rigidity theorem for properly embedded minimal surfaces in $\mathbf{R}^3$.* J. Differential Geometry, **32** (1990), 65-76.

[5] F. Martín, D. Rodríguez. *A characterization of the periodic Callahan-Hoffman-Meeks surfaces in terms of their symmetries.* Duke Math. J., **89**, No. 3 (1997), 445-463.

[6] W. H. Meeks III, H. Rosenberg. *The geometry of periodic minimal surfaces.* Comment. Math. Helvetici, **68** (1993), 538-578.

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