Non-simply connected minimal planar domains in $\mathbb{H}^2 \times \mathbb{R}$

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Abstract

We prove that any non-simply connected planar domain can be properly and minimally embedded in $\mathbb{H}^2 \times \mathbb{R}$. The examples that we produce are vertical bi-graphs, and they are obtained from the conjugate surface of a Jenkins-Serrin graph.

1 Introduction

One of the most fruitful methods to obtain minimal surfaces in $\mathbb{H}^2 \times \mathbb{R}$ is by solving the Dirichlet Problem for minimal graphs, with possibly infinite boundary values. This method was originally introduced by H. Jenkins and Serrin [10] for minimal graphs in $\mathbb{R}^3$, and extended to $\mathbb{H}^2 \times \mathbb{R}$ by B. Nelli and H. Rosenberg [16], P. Collin and H. Rosenberg [4], and L. Mazet, H. Rosenberg and the second author [11].

In [16], Nelli and Rosenberg also constructed vertical catenoids and helicoids. L. Hauswirth [7] generalized these examples by studying all minimal surfaces foliated by horizontal constant curvature curves. In this way, he obtained a 2-parameter family of minimal Riemann-type surfaces, which have genus zero and infinitely many ends.

Very recently, J. Pyo [17], F. Morabito and the second author [15] have constructed minimal surfaces of genus zero and finite total curvature. The method of construction in both papers consists of three steps. First, one solves the Jenkins-Serrin problem in a suitable geodesic polygonal domain with vertices $p_1, \ldots, p_{2n}$, satisfying p_{2i-1} in $\mathbb{H}^2$ and p_{2i} in the infinite boundary of $\mathbb{H}^2$ (that we will denote as $\partial_\infty \mathbb{H}^2$.) Secondly, one uses the conjugation introduced by B. Daniel [5] and Hauswirth, R. Sa Earp and E. Toubiana [8] to obtain a minimal graph bounded by n planar geodesics of the surface (not ambient geodesics in $\mathbb{H}^2 \times \mathbb{R}$), all of them at the same height. The complete surface is obtained by doubling the previous graph using Schwarz reflection principle with respect the horizontal slice that contains the horizontal geodesics (see Figure 1).

The main theorem of this paper shows that it is possible to take limits in the method of construction described in the above paragraph. Moreover, we have an important control of this limit surface, in such a way we can prescribe the topology of the resulting minimal surface. This control also allows us to guarantee that the limit set of distinct ends are disjoint. Regarding the conformal structure, the examples can be constructed with *parabolic* conformal type. This is not rare, because in some sense the minimal surfaces that we construct are limits of minimal surfaces with finite total curvature.

So, the main result asserts:

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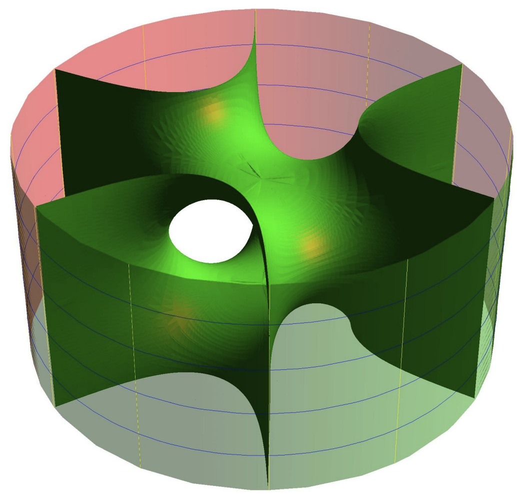

Figure 1: *One of the examples by Rodríguez and Morabito. It has the topology of a sphere minus three points.*

Theorem *Let Σ be a non-simply connected planar domain. Then, there exists a proper minimal embedding $f : \Sigma \to \mathbb{H}^2 \times \mathbb{R}$. Furthermore, f satisfies:*

(1) $f(\Sigma)$ is a vertical bigraph symmetric with respect a horizontal slice.

(2) The annular ends of $f(\Sigma)$ are asymptotic to vertical planes.

(3) The embedding f can be constructed so that for any two distinct ends E_1, E_2 of Σ, the limit sets $L(E_1)$, $L(E_2)$ in $\partial_\infty(\mathbb{H}^2 \times \mathbb{R})$ are disjoint.

(4) $f(\Sigma)$ has parabolic conformal type.

The above theorem, which can be thought as a generalization of the results in [18], gives a partial answer to a more general conjecture proposed to the authors by A. Ros:

Conjecture 1.1 *Let M be an oriented open surface*[1]*, then M can be properly embedded into $\mathbb{H}^2 \times \mathbb{R}$ as a minimal surface.*

Furthermore, the main theorem says to us that we cannot expect classification theorems for properly embedded minimal surfaces in $\mathbb{H}^2 \times \mathbb{R}$ just in terms of their topology, like in $\mathbb{R}^3$. (Meeks, Pérez and Ros recently proved in [14] that the only planar domains properly embedded in $\mathbb{R}^3$ are the plane, the catenoid, the helicoid and Riemann's minimal surfaces.)

2 Preliminaries

We consider the Poincaré disk model for the hyperbolic plane, i.e.

$$\mathbb{H}^2 = \{(x, y) \in \mathbb{R}^2 \mid x^2 + y^2 < 1\}$$

with the hyperbolic metric $g_{-1} = \frac{4}{(1-x^2-y^2)^2} g_0$, where g_0 is the Euclidean metric in $\mathbb{R}^2$, and let $\mathbf{0} = (0, 0)$ be the origin of $\mathbb{H}^2$. In this model, the asymptotic boundary $\partial_\infty \mathbb{H}^2$ of $\mathbb{H}^2$ is identified with the unit circle $\{x^2 + y^2 = 1\}$.

[1]We say that a surface is **open** if it is non-compact and without boundary.

2.1 The existence of simple exhaustions

In this paper we will use that any open orientable surface M has a smooth compact exhaustion $M_1 \subset M_2 \subset \cdots M_n \subset \cdots$, called a *simple exhaustion*, with the following properties:

1. M_1 is a disk.

2. For any $n \in \mathbb{N}$, each component of $M_{n+1} - \text{Int}(M_n)$ has one boundary component in ∂M_n and at least one boundary component in ∂M_{n+1}.

3. For any $n \in \mathbb{N}$, $M_{n+1} - \text{Int}(M_n)$ contains a unique non-annular component which topologically is a pair of pants or an annulus with a handle.

If M has finite topology with genus g and k ends, then we call the compact exhaustion *simple* if properties 1 and 2 hold, property 3 holds for $n \leq g + k$, and when $n > g + k$, all of the components of $M_{n+1} - \text{Int}(M_n)$ are annular.

The reader should note that, for any simple exhaustion of M, each component of $M - \text{Int}(M_n)$ is a smooth, non-compact proper subdomain of M bounded by a simple closed curve and for each $n \in \mathbb{N}$, M_n is connected (see Fig. 2).

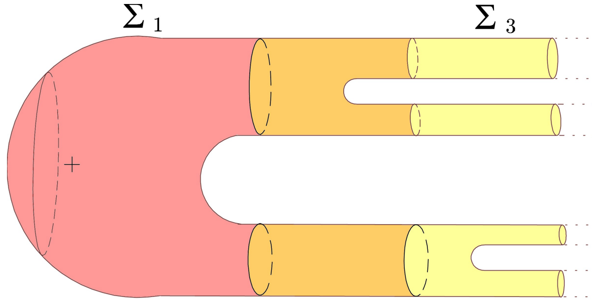

Figure 2: *A topological representation of the terms Σ_1 to Σ_3 in the exhaustion of an open surface M given in Lemma 2.1.*

In [6], Ferrer, Meeks and the first author proved the following result:

Lemma 2.1 ([6]) *Every orientable open surface admits a simple exhaustion.*

A non-simply connected planar domain Σ is a non-compact orientable surface of genus 0. As it has been mentioned in the introduction, our main result is already known for minimal planar domains with finite topology. Hence, we are going to focus on planar domains with infinitely many ends. In this case, Lemma 2.1 gives to us the following:

Corollary 2.2 *Let Σ be a planar domain with an infinite number of ends. Then Σ admits a compact exhaustion $\mathcal{S} = \{\Sigma_1 \subset \Sigma_2 \subset \cdots\}$, satisfying:*

1. *Σ_1 is a sphere minus two disks.*

2. *Each component of $\Sigma_{n+1} - \text{Int}(\Sigma_n)$ has one boundary component in $\partial\Sigma_n$ and at least one boundary component in $\partial\Sigma_{n+1}$.*

3. $\Sigma_{n+1} - Int(\Sigma_n)$ *contains a unique non-annular component which topologically is a* **pair of pants**.

We are also interested in the asymptotic behavior of the minimal surfaces we are going to construct. So, we need some background about the limit set of an end. In what follows, we will use the *ideal boundary* of $\mathbb{H}^2 \times \mathbb{R}$; $\partial_\infty(\mathbb{H}^2 \times \mathbb{R}) = \left(\partial_\infty \mathbb{H}^2 \times \mathbb{R}\right) \cup \left(\mathbb{H}^2 \times \{\pm\infty\}\right)$.

Definition 2.3 *Let $f\colon M \to \mathbb{H}^2 \times \mathbb{R}$ be a proper embedding of a surface M with possibly non-empty boundary. The* **limit set** *of M is*

$$L(M) = \bigcap_{\alpha \in I} (\overline{f(M) - f(C_\alpha)}),$$

where $\{C_\alpha\}_{\alpha \in I}$ is the collection of compact subdomains of M and the closure $\overline{f(M) - f(C_\alpha)}$ is taken in $\partial_\infty(\mathbb{H}^2 \times \mathbb{R})$. The **limit set** $L(E)$ **of an end** E **of** M *is defined to be the intersection of the limit sets of all properly embedded subdomains of M with compact boundary which represent E. Notice that $L(M)$ and $L(E)$ are closed sets of $\partial_\infty(\mathbb{H}^2 \times \mathbb{R})$.*

2.2 Minimal graphs

Given an open domain $\Omega \subset \mathbb{H}^2$ and a smooth function $u : \Omega \to \mathbb{R}$, the graph surface of u is minimal in $\mathbb{H}^2 \times \mathbb{R}$ when

$$\operatorname{div}\left(\frac{\nabla u}{\sqrt{1 + |\nabla u|^2}}\right) = 0, \tag{1}$$

where all terms are calculated with respect to the metric of $\mathbb{H}^2$.

Definition 2.4 *We say that a domain $\Omega \subset \mathbb{H}^2$ is polygonal when it is bounded by geodesic arcs. A polygonal domain $\Omega \subset \mathbb{H}^2$ with a finite number of vertices (possibly at the infinite boundary $\partial_\infty \mathbb{H}^2$ of $\mathbb{H}^2$) is said to be semi-ideal when no two consecutive vertices are ideal (i.e. they are at $\partial_\infty \mathbb{H}^2$) nor interior (i.e. they lie in $\mathbb{H}^2$).*

Let Ω be a semi-ideal domain. In particular, Ω has an even number of vertices $p_1, \ldots, p_{2k}$ (cyclically ordered), with $p_{2i-1} \in \partial_\infty \mathbb{H}^2$ and $p_{2i} \in \mathbb{H}^2$, for any $i = 1, \ldots, k$. We call A_i (resp. B_i) the geodesic arc joining p_{2i-1}, p_{2i} (resp. p_{2i}, p_{2i+1}); i.e.

$$A_i = (p_{2i-1}, p_{2i})_{\mathbb{H}^2}, \qquad B_i = (p_{2i}, p_{2i+1})_{\mathbb{H}^2}.$$

We consider a horocycle H_{2i-1} at each ideal vertex p_{2i-1}. Assume $H_{2i-1} \cap H_{2j-1} = \emptyset$ for any $i \neq j$. Given a polygonal domain $\mathcal{P}$ inscribed in Ω (i.e. a polygonal domain $\mathcal{P} \subset \Omega$ whose vertices are vertices of Ω, possibly at $\partial_\infty \mathbb{H}^2$), we denote by $\Gamma(\mathcal{P})$ the part of $\partial \mathcal{P}$ outside the horocycles. (Observe that $\Gamma(\mathcal{P}) = \partial \mathcal{P}$ in the case all the vertices of $\mathcal{P}$ are in $\mathbb{H}^2$.) Also let us call

$$\alpha(\mathcal{P}) = \sum_{i=1}^{k} |A_i \cap \Gamma(\mathcal{P})| \qquad \text{and} \qquad \beta(\mathcal{P}) = \sum_{i=1}^{k} |B_i \cap \Gamma(\mathcal{P})|,$$

where $|\bullet| = \text{length}^2_{\mathbb{H}}(\bullet)$.

Definition 2.5 A domain $\Omega \subset \mathbb{H}^2$ is called *admissible* when:

1. It is a convex semi-ideal polygonal domain with vertices $p_1, \ldots, p_{2k}$, with $p_{2i-1} \in \partial_\infty \mathbb{H}^2$ and $p_{2i} \in \mathbb{H}^2$.

2. There exists a choice of disjoint horocycles H_{2i-1} at the ideal vertices p_{2i-1} such that:

   (i) $\mathrm{dist}_{\mathbb{H}^2}(p_{2i-2}, H_{2i-1}) = \mathrm{dist}_{\mathbb{H}^2}(p_{2i}, H_{2i-1})$.

   (ii) $2\alpha(\mathcal{P}) < |\Gamma(\mathcal{P})|$ and $2\beta(\mathcal{P}) < |\Gamma(\mathcal{P})|$, for every polygonal domain $\mathcal{P}$ inscribed in Ω, $\mathcal{P} \neq \Omega$.

Up to an isometry of $\mathbb{H}^2$, we can assume that the origin $\mathbf{0} = (0,0)$ is contained in Ω. We say that (Ω, u) is an *admissible pair* if Ω is an admissible domain and $u : \Omega \to \mathbb{R}$ is a solution to the minimal graph equation (1) with $u(\mathbf{0}) = 0$ and whose boundary values are $+\infty$ on each edge A_i and $-\infty$ on each B_i.

We remark that condition (i) in the above definition does not depend on the choice of horocycles; and if the inequalities of condition (ii) are satisfied for some choice of horocycles, then they continue to hold for "smaller" horocycles (see the argument given by Collin and Rosenberg in [4]).

The following lemma is very useful to know when a domain satisfying conditions 1 and 2-(i) in the above definition is admissible. We will use this characterization in the proof of Lemma 3.1.

Lemma 2.6 ([18]) *Let Ω be a convex semi-ideal polygonal domain with vertices $p_1, \ldots, p_{2k}$, with $p_{2i-1} \in \partial_\infty \mathbb{H}^2$ and $p_{2i} \in \mathbb{H}^2$. Suppose there exists a choice of disjoint horocycles H_{2i-1} at the ideal vertices p_{2i-1} such that $dist_{\mathbb{H}^2}(p_{2i-2}, H_{2i-1}) = dist_{\mathbb{H}^2}(p_{2i}, H_{2i-1})$. Then Ω is admissible if, and only if, $p_{2j} \in \mathbb{H}^2 - \overline{D_{2i-1}}$ for any $i \neq j, j+1$, where D_{2i-1} is the horodisk at p_{2i-1} passing through p_{2i-2} and p_{2i}.*

The following theorem says that, given an admissible domain, it exists a unique solution $u : \Omega \to \mathbb{R}$ to the minimal graph equation (1) on Ω such that (Ω, u) is an admissible pair.

Theorem 2.7 ([4, 11, 15]) *Let Ω be an admissible domain with edges $A_1, B_1, \ldots, A_k, B_k$ (cyclically ordered). Then there exists a solution u for the minimal graph equation (1) in Ω with boundary values*

$$u|_{A_i} = +\infty \quad \textit{and} \quad u|_{B_i} = -\infty, \quad \textit{for any} \;\; i = 1, \ldots, k.$$

This solution is unique up to an additive constant.

Moreover, if we denote by Σ^* the conjugate surface, then Σ^* is a graph of a function u^* over an ideal domain Ω^* with

$$\partial\Omega^* = \gamma_1^* \cup \delta_1^* \cup \ldots \cup \gamma_k^* \cup \delta_k^*, \quad \textit{(cyclically ordered)},$$

where:

1. *$\delta_1^*, \ldots, \delta_k^*$ are concave curves, with respect to Ω^*,*

2. *$u^*|_{\delta_i^*} = 0$, for $i = 1, \ldots, k$,*

3. *$\gamma_1^*, \ldots, \gamma_k^*$ are geodesics and $u^*|_{\gamma_i^*} = +\infty$, for any $i = 1, \ldots, k$,*

4. *δ_i^* is a horizontal geodesic curvature line of symmetry of Σ^*, for $i = 1, \ldots, k$,*

5. *δ_i^* and γ_i^* (resp. δ_i^* and γ_{i+1}^*) are asymptotic at their common endpoint at $\partial_\infty \mathbb{H}^2$.*

In the following subsections we present some useful tools used in the proof of Theorem 2.7, which will also been used along the present paper.

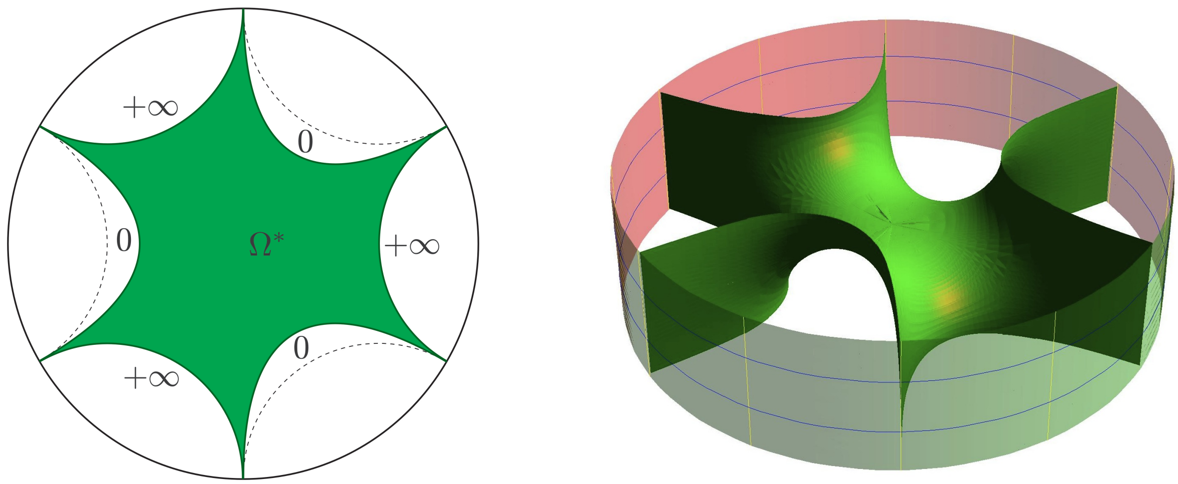

Figure 3: *Left: The domain Ω^*. Right: The conjugate graph Σ^*.*

2.2.1 Flux of a minimal graph along a curve

Let u be a minimal graph defined on a domain $\Omega \subset \mathbb{H}^2$. Assume $\partial\Omega$ is piecewise smooth and u extends continuously to $\overline{\Omega}$ (possibly with infinite values). We define the *flux* of u along a curve $\Gamma \subset \partial\Omega$ as

$$F_u(\Gamma) = \int_\Gamma \left\langle \frac{\nabla u}{\sqrt{1+|\nabla u|^2}}, \eta \right\rangle ds,$$

where η is the outer normal to $\partial\Omega$ in $\mathbb{H}^2$ and ds is the arc-length of $\partial\Omega$.

In the case $\Gamma \subset \Omega$, we can see Γ in the boundary of different subdomains of Ω, with two possible induced orientations. The flux $F_u(\Gamma)$ of u along Γ is then well-defined up to sign, and $|F_u(\Gamma)|$ is well-defined.

Lemma 2.8 ([16]) *Let u be a minimal graph on a domain $\Omega \subset \mathbb{H}^2$.*

(i) For every subdomain $\Omega' \subset \Omega$ such that $\overline{\Omega'}$ is compact, we have $F_u(\partial\Omega') = 0$.

(ii) Let Γ be a piecewise smooth curve contained in the interior of Ω, or a convex curve in $\partial\Omega$ where u extends continuously and takes finite values. Then $|F_u(\Gamma)| < |\Gamma|$.

(iii) If $T \subset \partial\Omega$ is a geodesic arc such that u diverges to $+\infty$ (resp. $-\infty$) as one approaches T within Ω, then $F_u(T) = |T|$ (resp. $F_u(T) = -|T|$).

Lemma 2.9 ([11]) *Let u be a minimal graph on a domain $\Omega \subset \mathbb{H}^2$, and $T \subset \partial\Omega$ such that $|F_u(T)| = |T|$ (resp. $|F_u(T)| = -|T|$). Then u goes to $+\infty$ (resp. $-\infty$) as we approach T within Ω.*

2.2.2 Divergence lines

Let $\Omega \subset \mathbb{H}^2$ be a domain and $\{u_k\}_k$ a sequence of minimal graphs on Ω. We define the *convergence domain* of $\{u_k\}_k$ as

$$\mathcal{B} = \{p \in \Omega \mid \{|\nabla u_k(p)|\}_k \text{ is bounded}\},$$

and the *divergence set* of $\{u_k\}_k$ as

$$\mathcal{D} = \Omega - \mathcal{B}.$$

The following proposition describes the convergence domain and the divergence set of a sequence of minimal graphs.

Proposition 2.10 ([11]) *Let $\Omega \subset \mathbb{H}^2$ be a domain and $\{u_k\}_k$ be a sequence of minimal graphs on Ω. Then:*

1. *$\mathcal{D}$ is composed of geodesic arcs contained in Ω (called divergence lines), each one joining two points of $\partial\Omega$ (including the vertices of Ω).*

2. *Let $L \subset \mathcal{D}$ be a divergence line. Passing to a subsequence, $|F_{u_k}(T)| \to |T|$ as $k \to +\infty$, for any geodesic arc $T \subset L$.*

3. *If $\mathcal{D} = \emptyset$, then a subsequence of $\{u_k - u_k(p)\}_k$ converges uniformly on compact subsets of Ω to a minimal graph, for any $p \in \Omega$.*

2.3 Conjugate minimal surfaces

Let Σ be a simply connected Riemann surface and $X = (\varphi, h) : \Sigma \to \mathbb{H}^2 \times \mathbb{R}$ be a conformal minimal immersion. It is known that h is a real harmonic function and $\varphi = \pi \circ X$ is a harmonic map from Σ to $\mathbb{H}^2$. Daniel [5] and Hauswirth, Sa Earp and Toubiana [8] proved that there exists a minimal immersion $X^* = (\varphi^*, h^*) : \Sigma \to \mathbb{H}^2 \times \mathbb{R}$, called *conjugate minimal immersion of X*, whose induced metric on Σ coincides with the one induced by X, and such that h^* is the real harmonic conjugate function of h and the Hopf differential of φ^* is $-Q_\varphi$, being Q_φ be the Hopf differential of φ. X^* is well-defined up to an isometry of $\mathbb{H}^2 \times \mathbb{R}$.

If N (resp. N^*) denotes the unit normal to X (resp. X^*), then $\langle N, \partial_t \rangle = \langle N^*, \partial_t \rangle$ (i.e. their angle maps coincide). Moreover, the correspondence $X \leftrightarrow X^*$ maps:

- Vertical geodesics of $\mathbb{H}^2 \times \mathbb{R}$ to horizontal geodesic curvature lines along which the normal vector field of the surface is horizontal.

- Horizontal geodesics of $\mathbb{H}^2 \times \mathbb{R}$ to geodesic curvature lines contained in vertical geodesic planes along which the normal vector field is tangent to the plane.

We will consider the conjugate surfaces of minimal graphs defined on convex domains. The surfaces obtained in this way are also minimal graphs (and consequently embedded), as ensured by the following Krust-type theorem given by Hauswirth, Toubiana and Sa Earp.

Theorem 2.11 ([8]) *If Σ is a minimal graph over a convex domain Ω of $\mathbb{H}^2$, then Σ^* is also a minimal graph over a (non-necessarily convex) domain of $\mathbb{H}^2$.*

3 Main Theorem

Recall that the purpose of this paper is to show that any domain in the plane which is not simply connected, can be properly embedded into $\mathbb{H}^2 \times \mathbb{R}$ as a minimal bi-graph. Since this fact is known in the case of finite topology [15, 17], then we will focus throughout this section in the construction of examples with infinite topology. The case of surfaces with an uncountable number of ends will be particularly interesting.

The main tool in all this construction is Lemma 3.1, which gives us the approximation of an admissible pair by other admissible pair with an extra ideal vertex. Its proof follows from the ideas of Lemma 3.2 in [18]. Roughly speaking, this means that we are able to increase the topology of the conjugate graph by using surfaces which are close enough on compact regions. This kind of ideas has been extensively used in the study of the Calabi-Yau problem for minimal surfaces in $\mathbb{R}^3$.

Given an admissible pair (Ω, u), we call $\mathcal{V}_i(\Omega)$ the set of interior vertices of Ω, and $\mathcal{V}_\infty(\Omega)$ the set of its ideal vertices. We will finally call $\mathcal{V}(\Omega)$ the set of vertices of Ω, i.e.

$$\mathcal{V}(\Omega) = \mathcal{V}_i(\Omega) \cup \mathcal{V}_\infty(\Omega).$$

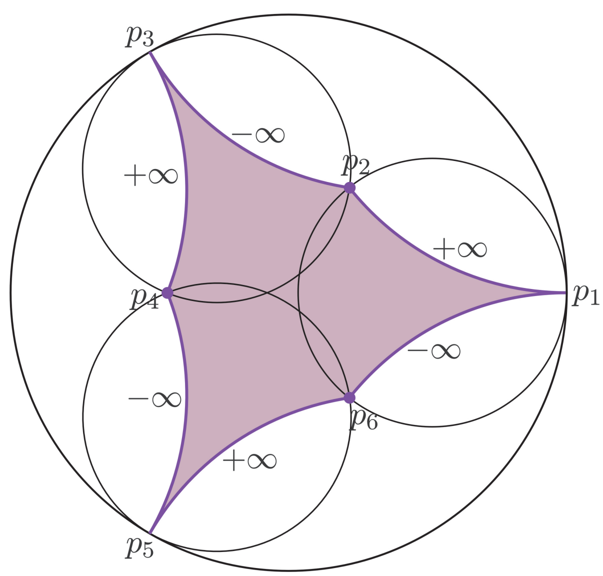

Figure 4:

Lemma 3.1 *Let ε, δ be positive numbers, and (Ω, u) an admissible pair. For any ideal vertex P of Ω and any $R > 0$ such that the hyperbolic disk $B(R)$ centered at $(0,0)$ of radius R contains all the interior vertices of Ω, there exists an admissible pair $(\widetilde{\Omega}, \widetilde{u})$ verifying:*

1. *Each boundary edge of Ω that does not have P as an endpoint, is contained in the boundary of $\widetilde{\Omega}$. In particular, $\mathcal{V}(\Omega) - \{P\} \subset \mathcal{V}(\widetilde{\Omega})$.*

2. *$\widetilde{\Omega}$ only contains two ideal vertices and an interior vertex which are not vertices of Ω; this is, $\mathcal{V}_\infty(\widetilde{\Omega}) - \mathcal{V}_\infty(\Omega) = \{P_1, P_2\}$ and $\mathcal{V}_i(\widetilde{\Omega}) - \mathcal{V}_i(\Omega) = \{P_0\}$.*

3. *$\Omega \cap B(R) \subset \widetilde{\Omega} \cap B(R)$. In particular, $P_0 \in \mathbb{H}^2 - B(R)$.*

4. *$\|\widetilde{u} - u\|_n < \varepsilon$ in $\Omega_\delta \cap B(R)$, for any $n \in \mathbb{N}$, where $\Omega_\delta = \{p \in \Omega \mid dist_{\mathbb{H}^2}(p, \partial\Omega) \geq \delta\}$.*

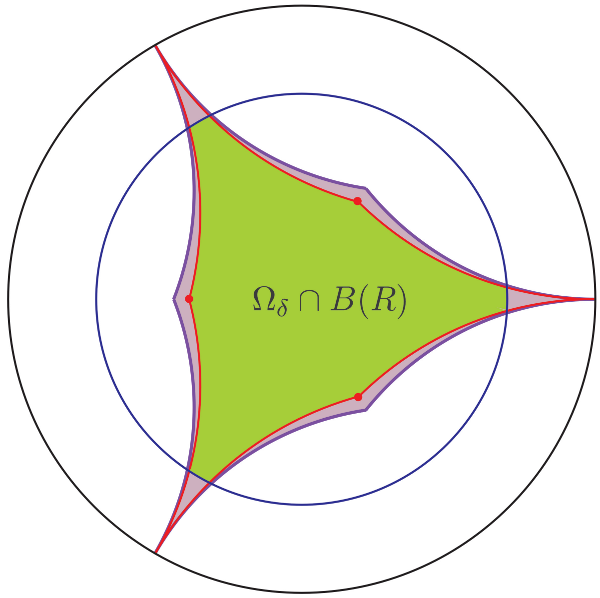

Figure 5:

Proof. Up to an isometry of $\mathbb{H}^2$, we can assume $P = (1, 0)$. We call $p_1, p_2, \cdots, p_{2k}$ the vertices of Ω, cyclically ordered, so that $p_1 = P$. We consider $P_n^+ = e^{i/n}$, $P_n^- = e^{-i/n}$, for any $n \in \mathbb{N}$. It is clear that $P_n^\pm \to P$ as $n \to +\infty$. We call C_n^+ (resp. C_n^-) the horocycle at P_n^+ (resp. P_n^-) passing through p_2 (resp. p_{2k}). For n big enough, $C_n^+ \cap C_n^- \neq \emptyset$. We call P_n^0 the intersection point in $C_n^+ \cap C_n^-$ which is closer to P (in the sense that the horodisk at P passing throught P_n^0 is contained in the horodisk at P passing throught the other point in $C_n^+ \cap C_n^-$). We take n big enough to assure $P_n^0 \in \mathbb{H}^2 - B(R)$.

We call $p_1(n) = P_n^+$, $p_2(n) = p_2, \cdots, p_{2k}(n) = p_{2k}$, $p_{2k+1}(n) = P_n^-$, $p_{2k+2}(n) = P_n^0$, and Ω_n the polygonal domain with vertices $p_1(n), p_2(n), \cdots, p_{2k+2}(n)$. From the fact that Ω is an admissible domain and using that all the interior vertices of Ω_n remain fixed except for $p_{2k+2}(n)$, we can deduce that Ω_n is an admissible domain for n large (here we use Lemma 2.6). Let $u_n : \Omega \to \mathbb{R}$ be the solution to the minimal graph equation (1) on Ω_n such that (Ω_n, u_n) is an admissible pair (it exists by Theorem 2.7). It is clear that $\Omega_n \to \Omega$ as $n \to +\infty$. Let us prove that $u_n \to u$ uniformly on compact sets of Ω. By Proposition 2.10, it suffices to prove that the sequence $\{u_n\}$ does not have any divergence line.

Suppose by contradiction that $L \subset \Omega$ is a divergence line for $\{u_n\}$. We call L_n the intersection of Ω_n with the complete geodesic of $\mathbb{H}^2$ containing L. Since Ω_n is convex (by the choice of P_n^0), we get that L_n is connected. Let $\mathcal{P}_n$ be a component of $\Omega_n - L_n$.

For any $i = 1, \cdots, k+1$, we call $D_{2i-1}(n)$ the open horodisk at $p_{2i-1}(n)$ passing through $p_{2i-2}(n), p_{2i}(n)$, and we consider a sequence of nested horocycles $H_{2i-1}(n, m)$ at $p_{2i-1}(n)$ contained in $D_{2i-1}(n)$ such that $\mathrm{dist}_{\mathbb{H}^2}(H_{2i-1}(n, m), \partial D_{2i-1}(n)) = m$, for any m. In particular, for m large we have $H_{2i-1}(n, m) \cap H_{2j-1}(n, m) = \emptyset$, if $i \neq j$. Let $\mathcal{P}_n(m)$ be the polygonal domain bounded by the part of $\partial \mathcal{P}_n$ outside the horocycles $H_{2i-1}(n, m)$, together with geodesic arcs joining the corresponding

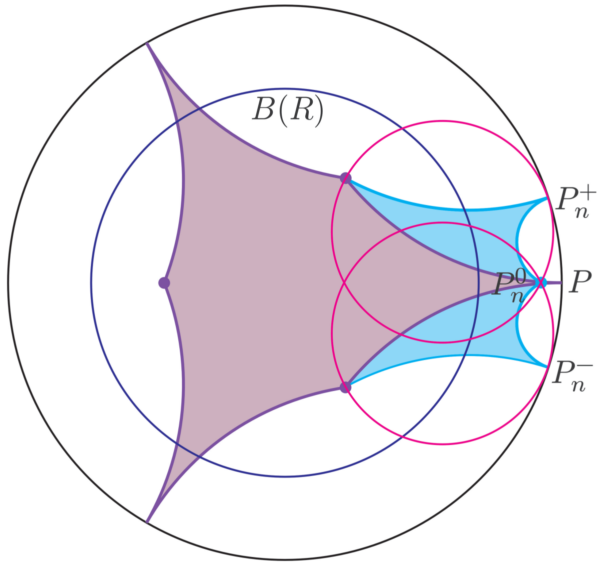

Figure 6:

points in $\partial\mathcal{P}_n \cap (\cup_i H_{2i-1}(n,m))$. We also denote

$$\alpha_n(m) = \sum_{i=1}^{k+1} |A_i^n \cap \partial\mathcal{P}_n(m)|, \qquad \beta_n(m) = \sum_{i=1}^{k+1} |B_i^n \cap \partial\mathcal{P}_n(m)|,$$

$$f_n(m) = F_{u_n}(\partial\mathcal{P}_n(m) - \partial\mathcal{P}_n),$$

where $A_i^n = (p_{2i-1}(n), p_{2i}(n))_{\mathbb{H}^2}$ and $B_i^n = (p_{2i}(n), p_{2i+1}(n))_{\mathbb{H}^2}$. We observe that, for any fixed n, $|f_n(m)| < |\partial\mathcal{P}_n(m) - \partial\mathcal{P}_n| \to 0$ as $m \to +\infty$. We can choose $\mathcal{P}_n$ to have

$$\beta_n(m) \geq \alpha_n(m).$$

We consider similar definitions associated to Ω: For any $i = 1, \cdots, k$, let D_{2i-1} be the open horodisk at p_{2i-1} passing through p_{2i-2}, p_{2i}, and we consider a sequence of nested horocycles $H_{2i-1}(m)$ at p_{2i-1} contained in D_{2i-1} such that $\mathrm{dist}(H_{2i-1}(m), \partial D_{2i-1}) = m$, for any m.

We denote by $L(m)$ (resp. $L_n(m)$) the geodesic arc in L (resp. L_n) outside the horocycles $H_{2i-1}(m)$ (resp. $H_{2i-1}(n,m)$). By Lemma 2.8,

$$F_{u_n}(L_n(m)) = \beta_n(m) - \alpha_n(m) - f_n(m).$$

We observe that $F_{u_n}(L_n(m)) \geq 0$ for m large.

- Suppose L has finite length. Then L joins a point $q_1 \in [p_{2i}, p_{2i+1})_{\mathbb{H}^2} \cup (p_{2i+1}, p_{2i+2})_{\mathbb{H}^2}$ to a point $q_2 \in [p_{2j}, p_{2j+1})_{\mathbb{H}^2} \cup (p_{2j+1}, p_{2j+2})_{\mathbb{H}^2}$, with $0 \neq i \neq j$ (see Figure 7). We consider m large enough so that $L(m) = L$ and $L_n(m) = L_n$. The endpoints of L_n are q_1 and another point that we are going to call $q_2(n)$ (notice that $q_2(n) = q_2$ when $j \neq 0$). For n large, one has $L \subset L_n$ and $|L_n| = |L| + \delta_n < +\infty$, where $\delta_n \geq 0$ converges to zero as $n \to +\infty$ ($\delta_n = 0$ in the case $j \neq 0$).

In this case, $c_n = \beta_n(m) - \alpha_n(m)$ does not depend on m (it is also constant on n when $j \neq 0$). Taking limits when m goes to $+\infty$, we get $F_{u_n}(L_n) = c_n$. On the other hand, $|F_{u_n}(L_n)| \to |L|$ as $n \to +\infty$. Then $c_n \to |L|$. Let us see this is not possible. We call C_1 (resp. C_2, $C_2(n)$) the horocycle at p_{2i+1} (resp. $p_{2j+1}, p_{2j+1}(n)$) passing through q_1 (resp. q_2, $q_2(n)$), and

$$d_1 = \mathrm{dist}(C_1, p_{2i}), \quad d_2 = \mathrm{dist}(C_2, p_{2j}), \quad d_2(n) = \mathrm{dist}(C_2(n), p_{2j}(n)).$$

We have that $|d_1 - d_2(n)| = c_n$. Suppose $d_1 > d_2$ (the case $d_2 > d_1$ follows analogously). Thus, $d_1 > d_2(n)$ for n large enough. Taking limits as $n \to \infty$ we have $d_1 = |L| + d_2$. That implies that p_{2j} (if $q_2 \in [p_{2j}, p_{2j+1})_{\mathbb{H}^2}$) or p_{2j+2} (if $q_2 \in (p_{2j+1}, p_{2j+2})_{\mathbb{H}^2}$) lies on D_{2i+1}, a contradiction with the fact that Ω is admissible (see Lemma 2.6).

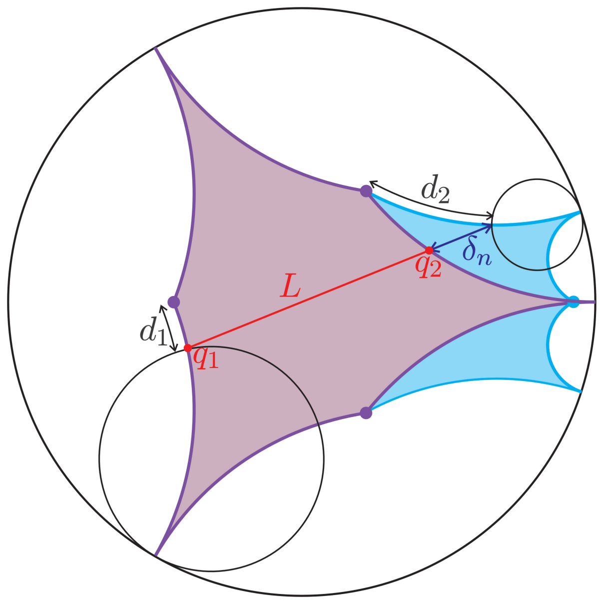

Figure 7:

- Now we suppose that L joins an ideal vertex p_{2i+1}, $i \neq 0$, to $q \in [p_{2j}, p_{2j+1})_{\mathbb{H}^2} \cup (p_{2j+1}, p_{2j+2})_{\mathbb{H}^2}$, with $j \neq i$. It holds $|L(m)| = m + d$, for some constant $d \in \mathbb{R}$. And for n large, $L \subset L_n$ and $|L_n(m)| = |L(m)| + \delta_n$, with $\delta_n \geq 0$ converging to zero.
  On the other hand, for m large we have that $c_n = m + \alpha_n(m) - \beta_n(m) \geq 0$ is constant on m ($c_n = 0$ when $q = p_{2j}$). Then
  $$|L(m)| - |F_{u_n}(L(m))| = d + c_n + f_n(m) + \lambda_n,$$
  where $\lambda_n = F_{u_n}(L_n(m) - L(m))$ converges to zero as $n \to +\infty$. Since $|L(m)| - |F_{u_n}(L(m))| \to 0$ as $n \to +\infty$, we conclude that $c_n \to -d$. That implies that $p_{2j} \in D_{2i+1}$, if $q \in [p_{2j}, p_{2j+1})_{\mathbb{H}^2}$, or $p_{2j+2} \in D_{2i+1}$, if $q \in (p_{2j+1}, p_{2j+2})_{\mathbb{H}^2}$, a contradiction.

- We consider now that L joins two ideal vertices p_{2i+1}, p_{2j+1}, with $i \neq j$ both different from zero. Then we have $\alpha_n(m) = \beta_n(m)$ because of the choice of horocycles above. For any compact geodesic arc $T \subset L_n$ and m large, we have $|F_{u_n}(T)| \leq |F_{u_n}(L_n(m))| = |f_n(m)|$. Taking $m \to +\infty$, we get $F_{u_n}(T) = 0$. But this contradicts that $|F_{u_n}(T)| \to |T|$ as $n \to +\infty$.

- If L joins p_1 to another ideal vertex p_{2i+1}, $i \neq 0$, then $L_n \subset L$ for any n. We have $\beta_n(m) - \alpha_n(m) = m - c_n$, with $c_n \geq 0$ independent of m ($c_n = 0$ when L_n finishes at $p_{2k+2}(n)$), and $|L_n(m)| = m + \delta_n$, where $\delta_n \in \mathbb{R}$. Then,
$$|L_n(m)| - |F_{u_n}(L_n(m))| = \delta_n + c_n + f_n(m) \to \delta_n + c_n, \quad \text{as } m \to +\infty.$$
Since $|L_n(m)| - |F_{u_n}(L_n(m))| \to 0$ as $n \to +\infty$, we conclude that $\delta_n + c_n \to 0$. That implies that, for n big enough, $p_{2k+2}(n) \in D_{2i+1}$, a contradiction, as Ω_n is admissible.

- Finally, let us consider that L joins p_1 to a point $q \in [p_{2j}, p_{2j+1})_{\mathbb{H}^2} \cup (p_{2j+1}, p_{2j+2})_{\mathbb{H}^2}$, with $j \neq 0$ (excluding the case $q = p_2, p_{2k}$). In this case we have $L_n \subset L$, $|L_n| < +\infty$ and $|L_n| = |L_n(m)|$ for big m. When $n \to +\infty$, $|L_n(m)| - |F_{u_n}(L_n(m))| \to 0$, for any m. On the other hand, $|L_n(m)| - |F_{u_n}(L_n(m))| = |L_n| - |F_{u_n}(L_n)| \to |L_n| - c_n$ as $m \to +\infty$, where $c_n = \beta_n(m) - \alpha_n(m)$ for any m. The only possibility is $|L_n| - c_n \to 0$ as $n \to +\infty$. That contradicts the fact that $|L_n| \to +\infty$ when $n \to +\infty$ while c_n remains bounded.

Then we get that $\{|\nabla u_n|\}_n$ is uniformly bounded on compact sets of Ω. Then Lemma 3.1 holds for $(\widetilde{\Omega}, \widetilde{u}) = (\Omega_{n_0}, u_{n_0})$ with some n_0 big enough, taking $P_0 = P^0_{n_0}$, $P_1 = P^+_{n_0}$ and $P_2 = P^-_{n_0}$. □

Using Lemma 3.1 we are able to prove the main result of this paper.

Theorem 3.2 *Let Σ be a non-simply connected planar domain. Then, there exists a proper minimal embedding $f : \Sigma \to \mathbb{H}^2 \times \mathbb{R}$. Furthermore, f satisfies:*

(1) $f(\Sigma)$ is a vertical bigraph, symmetric with respect a horizontal slice.

(2) The annular ends of $f(\Sigma)$ are asymptotic to vertical planes.

(3) The embedding f can be constructed so that for any two distinct ends E_1, E_2 of Σ, the limit sets[2] $L(E_1)$, $L(E_2)$ in $\partial_\infty(\mathbb{H}^2 \times \mathbb{R})$ are disjoint.

Proof. In what follows, we are going to assume that Σ has an infinite number of ends. Otherwise, we refer to [15] . From Corollary 2.2, the domain Σ admits a simple exhaustion $\{\Sigma_1 \subset \Sigma_2 \subset \cdots \subset \Sigma_n \subset \cdots\}$. We are going to give a labeling of the boundary components of the simple exhaustion that will give us a description of the set of ends of Σ.

The boundary components of Σ_1 will be denoted by ∂_0 and ∂_1. The difference $\Sigma_2 \setminus \Sigma_1$ consists of a pair of pants P_2 and a cylinder C_2. If the cylinder has ∂_i as a common boundary with Σ_1 then we denote as $\partial_{i,0}$ to the other boundary component of C_2. On the other hand, if ∂_j is the boundary component of P_2 that touches Σ_1, then we label $\partial_{j,0}$ and $\partial_{j,1}$ as the other two boundary components of P_2.

Now, assume we have already labeled the boundary components of Σ_n. We are going to label the connected components of $\partial\Sigma_{n+1}$. We know that $\Sigma_{n+1} \setminus \Sigma_n$ consists of cylinders $C^1_{n+1}, \ldots, C^k_{n+1}$ and just one pair of pants P_{n+1}. For a cylinder C^i_{n+1}, if the boundary component of C^i_{n+1} which touches Σ_n is labeled as $\partial_{i_1,\ldots,i_n}$, then we represent by $\partial_{i_1,\ldots,i_n,0}$ the other boundary component. In the case of the pair of pants P_{n+1}, if the boundary component of P_{n+1} which touches Σ_n is labeled as $\partial_{j_1,\ldots,j_n}$, then we denote by $\partial_{j_1,\ldots,j_n,0}$ and $\partial_{j_1,\ldots,j_n,1}$ the other two connected components of ∂P_{n+1}.

At this point, we are going to construct a sequence of admissible pairs (Ω_n, u_n), where Ω_n is an admissible domain with $2(n+1)$ edges, and a sequence of radius $\{R_n\}_{n\geq 2}$ and positive constants $\{\varepsilon_n\}_{n\geq 2}$, $\{\delta_n\}_{n\geq 2}$, satisfying:

[2] See Definition 2.3 for the definition of the limit set of an end of a surface in a three-manifold. Recall that $\partial_\infty(\mathbb{H}^2 \times \mathbb{R}) = (\partial_\infty \mathbb{H}^2 \times \mathbb{R}) \cup (\mathbb{H}^2 \times \{\pm\infty\})$.

(a) $\varepsilon_n, \delta_n \in (0, 1/2^n)$. In particular, $\sum_{n\geq 2} \varepsilon_n < +\infty$, and $\sum_{n\geq 2} \delta_n < +\infty$.

(b) Ω_{n+1} contains all the vertices of Ω_n, except for and ideal vertex p.

(c) Ω_{n+1} only contains two ideal vertices and an interior vertex which are not vertices of Ω_n. In particular, each boundary edge of Ω_n that does not contain p, is contained in $\partial\Omega_{n+1}$.

(d) $\Omega_n \cap B(R_{n+1}) \subset \Omega_{n+1} \cap B(R_{n+1})$.

(e) For any $k \in \mathbb{N}$, we have $\|u_{n+1} - u_n\|_k < \varepsilon_{n+1}$ in the domain $\Delta_n \stackrel{\text{def}}{=} \Omega_n(\delta_{n+1}) \cap B(R_{n+1})$. We recall that $\Omega_n(\delta_{n+1}) = \{p \in \Omega_n \mid \text{dist}_{\mathbb{H}^2}(p, \partial\Omega_n) > \delta_{n+1}\}$.

(f) If G_n denotes the graph of u_n, then the surface S_n obtained by doubling the conjugate graph G_n^* has the same topological type as Σ_n.

(g) If $q_{i_1,\dots,i_n}$ is an interior vertex of Ω_n and $x_{i_1,\dots,i_n}$ is a point in $\partial\Omega_n(\delta_{n+1})$ with

$$\text{dist}_{\mathbb{H}^2}(q_{i_1,\dots,i_n}, x_{i_1,\dots,i_n}) = \delta_{n+1},$$

then the third coordinate of $(x_{i_1,\dots,i_n}, u(x_{i_1,\dots,i_n}))^*$ is less than $1/n$, where $(x_{i_1,\dots,i_n}, u(x_{i_1,\dots,i_n}))^*$ means the conjugate point in the conjugate graph G_n^* corresponding to $(x_{i_1,\dots,i_n}, u(x_{i_1,\dots,i_n}))$.

The existence of such a sequence is obtained by using Lemma 3.1 in a recursive way: First, we take (Ω_1, u_1) as an admissible pair, where Ω_1 is an admissible geodesic quadrilateral. We call q_0, p_0, q_1, p_1 the vertices of Ω_1, with $p_0, p_1 \in \partial_\infty\mathbb{H}^2$. For the sake of clarity, we are going to construct the admissible domain Ω_2. Take $R_2 > 0$ such that $B(R_2)$ contains q_0, q_1, and $\varepsilon_2 \in (0, 1/4)$. We choose $\delta_2 \in (0, 1/4)$ small enough so that $\Omega_1(\delta_2) \cap \partial B(R_2)$ has two components. According to the notation we have introduced for the exhaustion Σ_n, $n \in \mathbb{N}$, we should add an interior vertex and two new ideal vertices around p_j: We apply Lemma 3.1 to $\Omega_1, \varepsilon_2, \delta_2, R_2$ and p_j. We call them $q_{j,1}$ and $p_{j,0}, p_{j,1}$, respectively. The remain vertices q_i, p_i, q_j of Ω_1 remains fixed, and we call them $q_{i,0}, p_{i,0}, q_{j,0}$. The vertices of Ω_2 are then $q_{i,0}, p_{i,0}, q_{j,0}, p_{j,0}, q_{j,1}, p_{j,1}$, consecutively ordered. Note that this action has *the topological effect of adding a pair of pants* to the surface obtained by doubling the conjugate graph. In order to see this, we call $\Gamma_{q_i} \stackrel{\text{def}}{=} \{q_i\} \times \mathbb{R}$, $i = 0, 1$, the vertical lines contained in the graph of u_1, denoted by G_1. Let $\Gamma_{q_0}^*$ and $\Gamma_{q_1}^*$ be the conjugate curves in G_1^*. By Theorem 2.7, $\Gamma_{q_0}^*$ and $\Gamma_{q_1}^*$ are horizontal lines of symmetry placed at height zero. Since Ω_1 is convex, then we know by Theorem 2.11 that G_1^* is a vertical graph over a domain that we call Ω_1^*. Similarly, we denote by $\gamma_{p_i}^*$, $i = 0, 1$, the geodesics in $\partial\Omega_1^*$ given by Theorem 2.7, where $u_1^*|_{\gamma_{p_i}^*} = +\infty$. When we reflect G_1* with respect to the slice $\{t = 0\}$ and obtain a properly embedded minimal surface S_1 with genus zero and two ends. The ends are asymptotic to the vertical geodesic planes $\gamma_{p_i}^* \times \mathbb{R}$. In this sense, we could say that there exists a natural correspondence between the ends of S_1 and the ideal vertices of Ω_1, p_0 and p_1. After the application of Lemma 3.1, we are substituting the end associated to p_j by two new ends; the ones associated to $p_{j,0}$ and $p_{j,1}$, respectively. These two new ends are linked by the horizontal curve of symmetry $\Gamma_{q_{j,1}}^*$ (see Figure 8.)

Now, assume we have (Ω_n, u_n) satisfying conditions above, and let us construct (Ω_{n+1}, u_{n+1}). We fix $R_{n+1} > 0$ such that $B(R_{n+1})$ contains all the interior vertices of Ω_n. We choose $\delta_{n+1} \in (0, 1/2^{n+1})$ small enough so that $\Omega_n(\delta_{n+1}) \cap \partial B(R_{n+1})$ has $n+1$ components. We also take $\varepsilon_{n+1} \in (0, 1/2^{n+1})$. As above, the effect of adding a pair of pants to the boundary $\partial_{j_1,\dots,j_n}$ of Σ_n means that

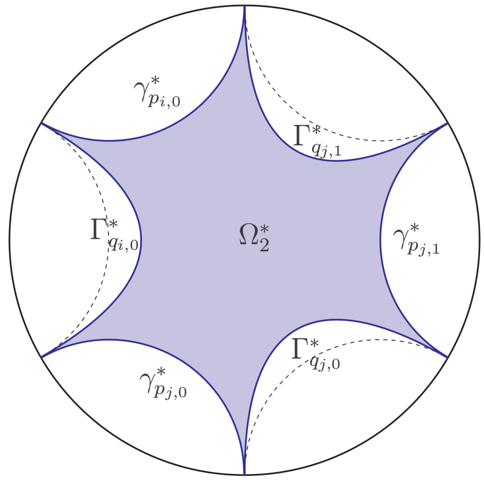

Figure 8: The domain Ω_2^*.

we have to substitute the ideal vertex $p_{j_1,\ldots,j_n}$ by two new ideal vertices, that we will call $p_{j_1,\ldots,j_n,0}$ and $p_{j_1,\ldots,j_n,1}$. To do this we apply, as before, Lemma 3.1 to: $\Omega_n, \varepsilon_{n+1}, \delta_{n+1}, R_{n+1}$ and $p_{j_1,\ldots,j_n}$. A new interior vertex also appears, we call it $q_{j_1,\ldots,j_n,1}$. Finally, we relabel the other vertices just by adding a 0 in the subindex.

Let us define $\Omega \stackrel{\text{def}}{=} \bigcup_{n=1}^{\infty} \Delta_n$. It is not hard to prove that $\Omega = \bigcup_{n=1}^{\infty} (\Omega_n \cap B(R_{n+1}))$ and Ω is *convex*.

Taking into account that the sequence $\{u_n\}_{n\in\mathbb{N}}$ satisfies item (e) and that $\sum_n \varepsilon_n$ converges, then we obtain that $\{u_n\}_{n\in\mathbb{N}}$ is a Cauchy sequence, with respect to the smooth convergence on compact sets in Ω. Ascoli-Arcela's theorem implies that $\{u_n\}_{n\in\mathbb{N}}$ converges to a smooth function u which is also a solution of (1) on Ω. Label as G the graph surface of u. As Ω is convex, then Theorem 2.11 says us that G^* is a also a graph over a domain that we call Ω^*. In particular, G^* is embedded.

Claim 3.3 *The limit graph G contains vertical straight lines placed over the interior vertices of Ω_n, for all $n \in \mathbb{N}$.*

In order to prove this claim, we fix $n_0 \in \mathbb{N}$ and let q be a (fixed) interior vertex of Ω_{n_0}. Two geodesics in $\partial\Omega_{n_0}$ arrive at this point, denoted by $\gamma_{n_0}^+$ and $\gamma_{n_0}^-$, with the properties that $u_{n_0|\gamma_{n_0}^\pm} = \pm\infty$. Recall that q is an interior vertex of Ω_n, for all $n \geq n_0$. Consider the corresponding boundary geodesics γ_n^+, γ_n^- in $\partial\Omega_n$ with $u_{n|\gamma_n^\pm} = \pm\infty$.

First, we focus on the sequence $\{\gamma_n^+\}_{n\in\mathbb{N}}$. Notice that, from the way in which we have obtained our sequence $\{\Omega_n\}_{n\in\mathbb{N}}$, the initial conditions of the geodesic γ_n^+ are given by $\gamma_n^+(0) = q$, $(\gamma_n^+)'(0) = \mathrm{e}^{\mathrm{i}\theta_n}$, where the sequence of arguments $\{\theta_n\}_{n\in\mathbb{N}}$ is monotone and bounded. So, $\{\theta_n\}_{n\in\mathbb{N}}$ converges to a real number θ. Let γ^+ be the geodesic starting at q with $(\gamma^+)'(0) = \mathrm{e}^{\mathrm{i}\theta}$. By construction, $\{\gamma_n^+\}_{n\in\mathbb{N}}$ smoothly converges to γ^+. The geodesic γ^+ joins q with a point $p^+ \in \partial_\infty\mathbb{H}^2$. Moreover, γ^+ is part of $\partial\Omega$. Let ρ^+ be the radial geodesic arriving at p^+. Taking our method of construction into account, we can guarantee that there are no interior vertices of Ω_n, $n \geq n_0$, in the triangle R^+ whose sides consists of γ^+, a bounded piece of $\gamma_{n_0}^-$ starting at q that we call σ and a convex curve α (convex with respect to R^+) which is asymptotic to ρ^+ at p^+ (see Figure 9.) Let v be the solution to the Dirichlet problem associated to equation (1) on R^+ with boundary data $+\infty$ on γ^+, $-\infty$ on σ and $\inf_{n\geq n_0} u_n$ on α.

Notice that $\inf_{n\geq n_0} u_n$ is continuous over α and then solution v exists by Theorem 4.9 in [11]. Then, the generalized maximum principle given by Collin and Rosenberg in [4, Theorem 2] (see also [11, Theorems 4.13 and 4.16]) gives us that $v \leq u_n$ in $\Omega_n \cap R^+$, for all $n \geq n_0$. This fact implies that $u_{|\gamma^+} = +\infty$.

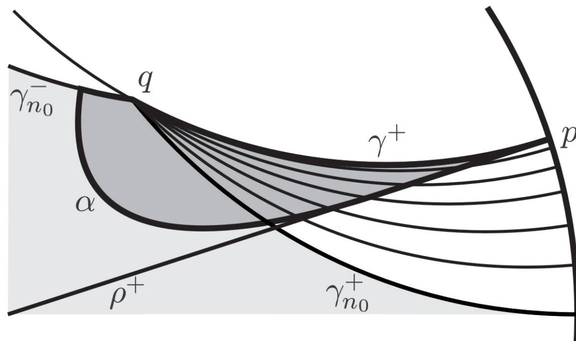

Figure 9:

A similar argument gives us that $u_{|\gamma^-} = -\infty$, where γ^- is the limit of the sequence $\{\gamma_n^-\}$. So, the graph of u extends to a vertical line over the point q. This concludes the proof of Claim 3.3.

Let q be an interior vertex of Ω and $\Gamma_q \stackrel{\text{def}}{=} \{q\} \times \mathbb{R}$ the vertical line contained in the graph of u, called M. Then, the conjugate curve $\Gamma_q^* \subset M^*$ is a horizontal curvature line of symmetry (see Subsection 2.3.)

Claim 3.4 *For any interior vertex q in Ω, Γ_q^* is contained in the plane $\{t = 0\}$. In particular, we can see Γ_q^* as a part of $\partial\Omega^*$. In this sense, Γ_q^* is concave with respect to Ω^*. Moreover, the endpoints of Γ_q^* in $\partial_\infty\mathbb{H}^2$ are distinct.*

In order to prove this claim, we assume that q is an interior vertex of Ω_n, for $n \geq k$. As a vertex of Ω_n, q appears represented as $q_{i_0,\dots,i_n}$, with $i_j \in \{0,1\}$, $j = 1,\dots,n$. Let $x_{i_0,\dots,i_n}$ be the corresponding point given by item (g). By construction, the sequence $\{x_{i_0,\dots,i_n}\}_{n\in\mathbb{N}}$ converges to q. So, $\{(x_{i_0,\dots,i_n}, u(x_{i_0,\dots,i_n}))^*\}_{n\in\mathbb{N}}$ is a sequence of points in $\mathbb{H}^2 \times \mathbb{R}$ accumulating to Γ_q^*. Taking item (g) into account (and using that the intrinsic distance between two vertical geodesics in the boundary of the graphs G_n remains uniformly bounded), this means that Γ_q^* is contained in the slice $\{t = 0\}$, for any q. The concavity of Γ_q^* with respect to Ω^* is a simple consequence of the maximum principle for minimal surfaces, using that Γ_q^* is a curve of symmetry.

Now, we are going to see that the endpoints of Γ_q^* are distinct. We proceed by contradiction. We suppose that both branches of Γ_q^* arrive to the same ideal point $d \in \partial_\infty\mathbb{H}^2$, and let σ_ε be the geodesic in $\mathbb{H}^2$ whose endpoints $d_\varepsilon^\pm$ are disposed symmetically in $\partial_\infty\mathbb{H}^2$ with respect to d and such that $\mathrm{dist}_{\mathbb{R}^2}(d, d_\varepsilon^\pm) = \varepsilon$. We consider the bounded convex region $\mathcal{D}$ in $\mathbb{H}^2$ bounded by Γ_q^* and σ_ε. If we apply Gauss-Bonnet formular for ε small enough, we obtain that

$$\mathrm{Area}(\mathcal{D}) \leq \int_{\Gamma_q^*} k_g - \pi, \tag{2}$$

where k_g is the geodesic curvature of Γ_q^* in $\mathbb{H}^2$. Since the normal vector field of M rotates less than π along Γ_q, we get $\int_{\Gamma_q^*} k_g \leq \pi$, which contradicts (2).

We consider the closed set $D_n = \overline{\Omega'_n} \cap \overline{B(R_{n+1})}$, where Ω'_n is a domain in Ω_n with the same vertices than Ω_n joined by arcs which are contained in $\Omega_n \setminus \Omega_n(\delta_n)$. Denote M_n as the graph of u over D_n. M_n is a minimal surface whose boundary contains vertical segments over the interior vertices of Ω_n. Then the conjugate surface M_n^* can be reflected with respect to the horizontal slice $\mathbb{H}^2 \times \{0\}$, and we obtain a surface S_n which is homeomorphic to Σ_n. Furthermore, if we label $f_n : \Sigma_n \to S_n$ to this homeomorphism, we have for all $i \leq n$ that $f_n|_{\Sigma_i}$ coincides with the corresponding homeomorphism $f_i : \Sigma_i \to S_i$, since $D_i \subset D_n$.

Let S be complete surface obtained by gluing together both G^* and its reflection with respect to $\mathbb{H}^2 \times \{0\}$. We have that S_n is a simple exhaustion of S and the sequence of homeomorphisms $\{f_n\}_{n \in \mathbb{N}}$ has a limit $f : \Sigma \to S$.

In order to prove item (3) in the statement of the theorem, we consider E_1 and E_2 two different ends of $f(\Sigma)$. Then there is a first natural $n \in \mathbb{N}$ so that E_1 and E_2 are represented by two different components of $\Sigma - (\cup_{i=1}^n \Sigma_i)$. This is $\partial_{i_1,\ldots,i_n}$ is the boundary of a component representing both ends E_1 and E_2, but $\partial_{i_1,\ldots,i_n,0}$ represents E_1 and $\partial_{i_1,\ldots,i_n,1}$ represents E_2. Consider the points $q_1 = q_{i_1,\ldots,i_n,0}$ and $q_2 = q_{i_1,\ldots,i_n,1}$ which are interior vertices of Ω. From Claim 3.4 we know that $\Gamma^*_{q_1}$ and $\Gamma^*_{q_2}$ are curves in $\partial\Omega^*$ with distinct endpoints. Moreover, these two curves cannot be asymptotic. Let η_1 and η_2 be the geodesics in $\mathbb{H}^2$ joining an end point of $\Gamma^*_{q_1}$ to an endpoint of $\Gamma^*_{q_2}$ in such a way that $\eta_1 \cup \Gamma^*_{q_1} \cup \eta_2 \cup \Gamma^*_{q_2}$ bounds an open ideal quadrilateral $\mathcal{Q}$. Hence, the limit sets $L(E_1)$ and $L(E_2)$ lie in different components of $\partial_\infty((\mathbb{H}^2 - \mathcal{Q}) \times \mathbb{R})$. □

Finally, we would like to discuss about the underlying conformal structure of the minimal surfaces we have just constructed. A good reference for the notation and results we are going to use is [1, §6 and §15].

As we have mentioned before, it is important to note that if Σ has a finite number of ends, then the examples provided in the above theorem are those already constructed by Morabito and the second author. These examples have total curvature $-4\pi(k-1)$, where k represents the number of ends. Thus, using a classical result by Huber [9], Morabito-Rodríguez's surfaces are conformally equivalent to a sphere minus k points. In particular, they are parabolic (see definition below). The examples with infinite topology given by Theorem 3.2 no longer have finite total curvature. However, we would like to point out that they can be constructed with parabolic conformal type, as explained in Remark 3.6.

Definition 3.5 *An open Riemann surface W is said to be parabolic if there are no non-constant negative subharmonic functions on W.*

Among other important characterizations of parabolicity, we know that W is parabolic if and only if one of the following conditions is fulfilled:

- the maximum principle for harmonic maps is valid on W;
- the harmonic measure of the ideal boundary of W vanishes;
- there is no Green's function defined on W.

Remark 3.6 *The embedding $f : \Sigma \to \mathbb{H}^2 \times \mathbb{R}$ in Theorem 3.2 can be constructed in such a way that $f(\Sigma)$ is parabolic. To do this, we consider the simple exhaustion*

$$S_1 \subset S_2 \subset \cdots \subset S_n \subset \cdots$$

given in the proof of the theorem. We denote by λ_n *the* extremal length *between* ∂S_1 *and* ∂S_n *and by* μ_n *the harmonic modulus* $\mu_n \stackrel{\mathrm{def}}{=} \mathrm{e}^{\lambda_n}$*. Notice that the surface obtained by doubling the graph* G_n^* *is parabolic (it has finite total curvature). So, using Lemma 3.1 in a suitable way, we could guarantee in our inductive process that* $\mu_n \geq n-1$*. This fact implies that* $S = f(\Sigma)$ *is parabolic.*

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