EXISTENCE OF PROPER MINIMAL SURFACES OF ARBITRARY TOPOLOGICAL TYPE

LEONOR FERRER, FRANCISCO MARTÍN, AND WILLIAM H. MEEKS, III

ABSTRACT. Consider a domain $\mathcal{D}$ in $\mathbb{R}^3$ which is convex (possibly all $\mathbb{R}^3$) or which is smooth and bounded. Given any open surface M, we prove that there exists a complete, proper minimal immersion $f\colon M \to \mathcal{D}$. Moreover, if $\mathcal{D}$ is smooth and bounded, then we prove that the immersion $f\colon M \to \mathcal{D}$ can be chosen so that the limit sets of distinct ends of M are disjoint connected compact sets in $\partial\mathcal{D}$.

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1. INTRODUCTION

A natural question in the global theory of minimal surfaces, first raised by Calabi in 1965 [2] and later revisited by Yau [25, 26], asks whether or not there exists a complete immersed minimal surface in a bounded domain $\mathcal{D}$ in $\mathbb{R}^3$. In 1996, Nadirashvili [20] provided the first example of a complete, bounded, immersed minimal surface in $\mathbb{R}^3$. However, Nadirashvili's techniques did not provide properness of such a complete minimal immersion in any bounded domain. Under certain restrictions on $\mathcal{D}$ and the topology of an open surface[1] M, Alarcón, Ferrer, Martín, and Morales [1, 10, 11, 12, 19] proved the existence of a complete, proper minimal immersion of M in $\mathcal{D}$.

In this paper we prove that every open surface M can be properly minimally immersed into certain domains $\mathcal{D}$ of $\mathbb{R}^3$ as a complete surface (see Theorem 4). These domains include $\mathbb{R}^3$, all convex domains and all bounded domains with smooth boundary. In contrast to this existence theorem, Martín and Meeks [8] have recently proven that in any Riemannian three-manifold there exist many nonsmooth domains with compact closure which do not admit any complete, properly immersed surfaces with at least one annular end and bounded mean curvature. The above result is a generalization of a previous work for minimal surfaces in $\mathbb{R}^3$ by these authors and Nadirashvili [9]. Thus, some geometric constraint on the boundary of a bounded domain is necessary to insure that it contains complete, properly immersed minimal surfaces of arbitrary topological type.

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[1] We say that a surface is *open* if it is connected, noncompact and without boundary.

When the domain $\mathcal{D}$ is smooth and bounded, we obtain further important control on the limit sets of the ends of M as described in the next theorem; see Definition 2 for the definition of the limit set of an end.

Theorem 1. *If $\mathcal{D}$ is a smooth bounded domain in $\mathbb{R}^3$ and M is an open surface, then there exists a complete, proper minimal immersion of M in $\mathcal{D}$ such that the limit sets of distinct ends of M are disjoint.*

We consider the proof of the above theorem to be the first key point in an approach by the second two authors and Nadirashvili to construct certain complete, properly embedded minimal surfaces M in certain bounded domains of $\mathbb{R}^3$ as described in the next conjecture. The cases described in this conjecture where M is nonorientable appear to be deeper and more interesting than where M is orientable. Our approaches for dealing with the orientable or nonorientable cases in this conjecture are essentially the same by using the theory developed in Section 6; specifically, we refer the reader to Theorem 6 and Propositions 2 and 3, which are closely related to parts 2 and 3 of the next conjecture.

Conjecture 1 (Embedded Calabi-Yau Conjecture, Martín, Meeks, Nadirashvili, Peréz, Ros)**.**

1. *A necessary and sufficient condition for an open surface M to admit complete, proper minimal embeddings in* **every** *smooth bounded domain in $\mathbb{R}^3$ is that M is orientable and every end of M has infinite genus.*
2. *A necessary and sufficient condition for an open surface M to admit a complete, proper minimal embedding in* **some** *smooth bounded domain in $\mathbb{R}^3$ is that every end of M has infinite genus and M has only a finite number of nonorientable ends.*
3. *Let $\mathcal{D}_\infty$ be the bounded domain in $\mathbb{R}^3$ described in Example 3, which is smooth except at one point (see Fig. 10). A necessary and sufficient condition for an open surface M to admit a complete, proper minimal embedding in $\mathcal{D}_\infty$ is that every end of M has infinite genus.*

Embeddedness creates a dichotomy in the Calabi-Yau question. In other words, when the question is asked whether a given domain of $\mathbb{R}^3$ admits a complete, injective minimal immersion of a surface M, the topological possibilities are limited. The first result concerning the embedded Calabi-Yau question was given by Colding and Minicozzi [3]. They proved that complete, embedded minimal surfaces in $\mathbb{R}^3$ with finite topology are proper in $\mathbb{R}^3$. The relevance of their result to the classical theory of complete embedded minimal surfaces is that there are many deep theorems concerning properly embedded minimal surfaces. Recently, Meeks, Pérez and Ros [14] generalized this properness result of Colding and Minicozzi to the larger class of surfaces with finite genus and a countable number of ends.

There are many known topological obstructions for properly minimally embedding certain open surfaces into $\mathbb{R}^3$. For example, the only properly embedded, minimal planar domains in $\mathbb{R}^3$ are the plane and the helicoid which are simply-connected, the catenoid which is 1-connected and the Riemann minimal examples which are planar domains with two limits ends (see [7, 4, 16, 13, 15] for this classification result). Because of these results, the proper minimal immersions described in this paper must fail to be embeddings for certain open surfaces.

The constructive nature of the proper minimal surfaces in our theorems depends on the bridge principle for minimal surfaces and on generalizing to the nonorientable setting the approximation techniques used by Alarcón, Ferrer and Martín in [1]. Also, the construction of the surfaces which we obtain here depend on obtaining certain compact exhaustions for any open surface M; see Section 4 for the case of orientable open surfaces and the proofs of Propositions 2 and 3 in Section 6.3 for the case of nonorientable open surfaces.

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2. Preliminaries and Background

2.1. **Background on convex bodies and Hausdorff distance.** Given E a bounded regular convex domain of $\mathbb{R}^3$ and $p \in \partial E$, we will let $\kappa_2(p) \geq \kappa_1(p) \geq 0$ denote the principal curvatures of ∂E at p (associated to the inward pointing unit normal). Moreover, we write:

$$\kappa_1(\partial E) \stackrel{\text{def}}{=} \min\{\kappa_1(p) \mid p \in \partial E\}, \qquad \kappa_2(\partial E) \stackrel{\text{def}}{=} \max\{\kappa_2(p) \mid p \in \partial E\}.$$

If we consider $\mathcal{N} \colon \partial E \to \mathbb{S}^2$ to be the outward pointing unit normal or Gauss map of ∂E, then there exists a constant $a > 0$ (depending on E) such that $\partial E_t = \{p + t \cdot \mathcal{N}(p) \mid p \in \partial E\}$ is a regular (convex) surface for all $t \in [-a, +\infty[$. Let E_t denote the convex domain bounded by ∂E_t. The normal projection to ∂E is represented as

$$\begin{array}{rccc} \mathcal{P}_E : & \mathbb{R}^3 - E_{-a} & \longrightarrow & \partial E \\ & p + t \cdot \mathcal{N}(p) & \longmapsto & p \, . \end{array}$$

For a subset Υ in $\mathbb{R}^3$ and a real $r > 0$, we define the tubular neighborhood of radius r along Υ in the following way: $T(\Upsilon, r) = \Upsilon + \mathbb{B}(0, r)$, where $\mathbb{B}(0, r) = \{p \in \mathbb{R}^3 \mid \|p\| < r\}$.

A convex set of $\mathbb{R}^n$ with nonempty interior is called *a convex body*. The set $\mathcal{C}^n$ of convex bodies of $\mathbb{R}^n$ can be made into a metric space in several geometrically reasonable ways. The Hausdorff metric is particularly convenient and applicable for defining such a metric space structure. The natural domain for this metric is the set $\mathcal{K}^n$ of the nonempty compact subsets of $\mathbb{R}^n$. For $\mathcal{C}, \mathcal{D} \in \mathcal{K}^n$ the *Hausdorff distance* is defined by:

$$\delta^H(\mathcal{C}, \mathcal{D}) = \min\left\{\lambda \geq 0 \mid \mathcal{C} \subset T(\mathcal{D}, \lambda),\ \mathcal{D} \subset T(\mathcal{C}, \lambda)\right\}.$$

A theorem of H. Minkowski (cf. [18]) states that every convex body C in $\mathbb{R}^n$ can be approximated (in terms of Hausdorff metric) by a sequence C_k of ‘analytic’ convex bodies.

Theorem 2 (Minkowski)**.** *Let $\mathcal{C}$ be a convex body in $\mathbb{R}^n$. Then there exists a sequence $\{\mathcal{C}_k\}$ of convex bodies with the following properties*

1. $\mathcal{C}_k \searrow \mathcal{C}$;
2. $\partial \mathcal{C}_k$ *is an analytic* $(n-1)$*-dimensional manifold;*
3. *The principal curvatures of* $\partial \mathcal{C}_k$ *never vanish.*

A modern proof of this result can be found in [17, §3].

2.2. **Preliminaries on minimal surfaces.** Throughout the paper, whenever we write that M is a *compact minimal surface with boundary*, we will mean that this boundary is regular and M can be extended beyond its boundary. In other words, we will always assume that $M \subset \mathrm{Int}(M')$, where M' is another minimal surface.

For the sake of simplicity of notation and language, we will say that two immersed surfaces in $\mathbb{R}^3$ are *homeomorphic* if and only if their underlying topological surface structures are the same.

The following lemma will be a key point (together with the bridge principle and the existence of simple exhaustions) in the proofs of the main lemmas of this paper. It summarizes all the information contained in Lemma 5, Theorem 3 and Corollary 1 in [1].

Lemma 1 (Alarcón, Ferrer, Martín)**.** *Let $\mathcal{D}'$ be a convex domain (not necessarily bounded or smooth) in $\mathbb{R}^3$. Consider a compact orientable minimal surface M, with nonempty boundary satisfying: $\partial M \subset \mathcal{D} - \overline{\mathcal{D}}_{-d}$, where $\mathcal{D}$ is a bounded convex smooth domain, with $\overline{\mathcal{D}} \subset \mathcal{D}'$, and $d > 0$ is a constant. Let r be a positive constant such that $T(M, r) \subset \mathcal{D}$.*

Then, for any $\varepsilon > 0$, there exists a complete minimal surface M_ε which is properly immersed in $\mathcal{D}'$ and satisfies:

(1) *M_ε has the same topological type as $\mathrm{Int}(M)$;*
(2) *$M_\varepsilon \cap T(M, r)$ contains a connected surface M_ε^r (not a component of $M_\varepsilon \cap T(M, r)$) with the same topological type as $\mathrm{Int}(M)$ and M_ε^r converges in the $\mathcal{C}^\infty$ topology to M, as $\varepsilon \to 0$. Furthermore, the Hausdorff distance $\delta^H(M_\varepsilon^r, M) < \varepsilon$;*
(3) *Each end of $M_\varepsilon - M_\varepsilon^r$ is contained in $\mathbb{R}^3 - \mathcal{D}_{-2d-\varepsilon}$;*
(4) *If $\mathcal{D}$ and $\mathcal{D}'$ are smooth and $\mathcal{D}$ is strictly convex, then $\delta^H(M, M_\varepsilon) < \mathsf{m}(\varepsilon, d, \mathcal{D}, \mathcal{D}')$, where:*

$$\mathsf{m}(\varepsilon, d, \mathcal{D}, \mathcal{D}') \stackrel{\mathrm{def}}{=} \varepsilon + \sqrt{\frac{2(\delta^H(\mathcal{D}, \mathcal{D}') + d + \varepsilon)}{\kappa_1(\partial \mathcal{D})} + (\delta^H(\mathcal{D}, \mathcal{D}') + d + \varepsilon)^2}.$$

2.2.1. *The bridge principle for minimal surfaces.* Let M be a possibly disconnected, compact minimal surface in $\mathbb{R}^3$, and let $P \subset \mathbb{R}^3$ be a thin curved rectangle whose two short sides lie along ∂M and that is otherwise disjoint from M. The *bridge principle* for minimal surfaces states that if M is nondegenerate, then it should be possible to deform $M \cup P$ slightly to make a minimal surface with boundary $\partial(M \cup P)$. The bridge principle is a classical problem that goes back to Paul Lévy in the 1950's. It was involved in the construction of a curve bounding uncountably many minimal disks. The bridge principle is easy to apply to compact minimal surfaces which satisfy the nondegerancy property described in the next definition.

Definition 1. *A compact minimal surface M with boundary is said to be* **nondegenerate** *if there are no nonzero Jacobi fields on M which vanish on ∂M.*

The following version of the bridge principle is the one we need in our constructions.

Theorem 3 (White, [23, 24])**.** *Let M be a compact, smooth, nondegenerate minimal surface with boundary, and let Γ be a smooth arc such that $\Gamma \cap M = \Gamma \cap \partial M = \partial \Gamma$.*

Let P_n be a sequence of bridges on ∂M that shrink nicely to Γ.

Then for sufficiently large n, there exists a minimal surface M_n with boundary $\partial(M \cup P_n)$ and a diffeomorphism $f_n \colon M \cup P_n \to M_n$ such that

(1) area$(M_n) \to$ area(M)*;*

(2) $f_n(x) \equiv x$ *for all* $x \in \partial(M \cup P_n)$*;*
(3) $\|x - f_n(x)\| = O(w_n)$*, where* w_n *is the width of* P_n *and* $O(w_n)/w_n$ *is bounded;*
(4) *The maps* $f_n|_M$ *converge smoothly on compact subsets of* $M - \Gamma$ *to the identity map* $1_M \colon M \to M$*;*
(5) *Each* M_n *is a nondegenerate minimal surface.*

3. Adding handles and ends

In this section we prove two lemmas which represent main tools in our construction procedure. Essentially, they tell to us how we can add a "pair of pants" to a minimal surface with boundary in order to create a new end (Figure 1.(a)) or how to add a handle to increase the genus (Figure 1.(b)).

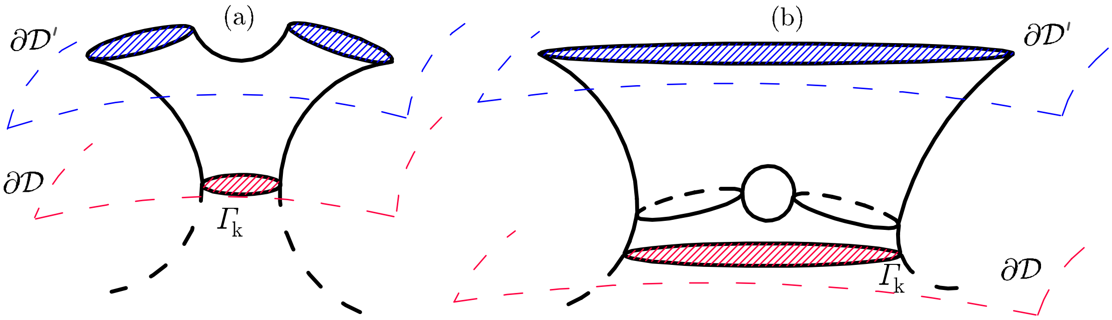

FIGURE 1. We can add a "pair of pants" to a minimal surface with boundary in order to: (a) create a new end, or (b) add a handle.

Lemma 2 (Adding ends). *Let* $\mathcal{D}$ *and* $\mathcal{D}'$ *be two smooth bounded strictly convex domains in* $\mathbb{R}^3$ *so that* $\vec{0} \in \overline{\mathcal{D}} \subset \mathcal{D}'$*. Consider a compact minimal surface* M *with nonempty boundary and satisfying* $\vec{0} \in \mathrm{Int}(M)$ *and* $\partial M \subset \partial\mathcal{D}$*. Assume that* M *has genus* g *and* k *components at the boundary (*$k \geq 1$*),* $\partial M = \Gamma_1 \cup \ldots \cup \Gamma_k$*. We also assume that* M *intersects* $\partial\mathcal{D}$ *transversally.*

Then for any $\varepsilon > 0$*, there exists a minimal surface* M_ε *satisfying the following properties:*

(1) M_ε *is a smooth, immersed minimal surface with genus* g *and* $k+1$ *boundary components. Moreover,* $\partial M_\varepsilon \subset \partial\mathcal{D}'$*,* ∂M_ε *meets transversally* $\partial\mathcal{D}'$ *and* $\vec{0} \in \mathrm{Int}(M_\varepsilon)$*;*
(2) *The intrinsic distance* $\mathrm{dist}_{M_\varepsilon}(\vec{0}, \partial M_\varepsilon) > \mathrm{dist}_M(\vec{0}, \partial M) + 1$*;*
(3) *The surfaces* $M_\varepsilon \cap \overline{\mathcal{D}}$ *are graphs over* M *and converge in the* C^∞ *topology to* M*, as* $\varepsilon \to 0$*. Furthermore,* $\delta^H(M, M_\varepsilon \cap \overline{\mathcal{D}}) < \varepsilon$*;*
(4) $M_\varepsilon - \mathcal{D}$ *consists of* $k-1$ *annuli, each of whose boundary in* $\partial\mathcal{D}$ *lies in* $T(\Gamma_j, \varepsilon)$*,* $j = 1, \ldots, k-1$*, and a pair of pants, whose boundary in* $\partial\mathcal{D}$ *is a single curve which lies in* $T(\Gamma_k, \varepsilon)$ *(see Figure 1-(a)). Moreover, the two boundary curves of the pair of pants which are contained in* $\partial\mathcal{D}'$ *are disjoint;*
(5) *If* $\mathcal{D}$ *and* $\mathcal{D}'$ *are parallel (boundaries are equidistant), then* $\delta^H(M, M_\varepsilon) < 2\,C(\varepsilon, \mathcal{D}, \mathcal{D}')$*, where:*

$$C(\varepsilon, \mathcal{D}, \mathcal{D}') \stackrel{\text{def}}{=} \varepsilon + \sqrt{\frac{2(\delta^H(\mathcal{D}, \mathcal{D}') + 2\varepsilon)}{\kappa_1(\partial\mathcal{D}')} + (\delta^H(\mathcal{D}, \mathcal{D}') + 2\varepsilon)^2};$$

Proof. Fix $\varepsilon > 0$. The proof of this lemma consists of clever combined applications of the density theorem (Lemma 1) and the bridge principle (Theorem 3). We have divided the proof into three steps.

Step 1. From our assumptions, we know that $M \subset \mathrm{Int}(M')$, where M' is a regular minimal surface. Take $a > 0$ small enough such that $\overline{\mathcal{D}}_a \subset \mathcal{D}'$, and $\delta^H(M, M' \cap \overline{\mathcal{D}}_a) < \varepsilon/4$. Consider $d > 0$ and $\varepsilon_0 > 0$ such that $a > 2\,d + \varepsilon_0$ and $\varepsilon > d + \varepsilon_0$. Let $M'' \subset \mathrm{Int}(M') \cap \overline{\mathcal{D}}_a$ be a compact minimal surface with boundary such that M'' is homeomorphic to M, $\partial M'' \subset \mathcal{D}_a - \overline{\mathcal{D}}_{a-d}$ and

(1) $$\delta^H(M, M'') < \varepsilon/4.$$

Finally, take $r > 0$ such that $T(M'', r) \subset \mathcal{D}_a$. Given $\varepsilon'' \in (0, \min\{\varepsilon_0, \varepsilon/4\}]$, then we apply Lemma 1 to the data: ε'', d, M'', $\mathcal{D}_a$, and $\mathcal{D}'$. So, we obtain a complete, minimal surface $\widetilde{M}$ properly immersed in $\mathcal{D}'$, which satisfies:

- $\widetilde{M}$ has the same topological type as $\mathrm{Int}(M'') \equiv \mathrm{Int}(M)$ and $0 \in \mathrm{Int}(\widetilde{M})$;
- The surface $\widetilde{M} \cap T(M'', r)$ contains a regular compact surface $\widetilde{M}^r$ which is homeomorphic to M'' and these surfaces converge smoothly to M'', as $\varepsilon'' \to 0$. Furthermore, $\delta^H(\widetilde{M}^r, M'') < \varepsilon''$;
- Each end of $\widetilde{M} - \widetilde{M}^r$ is contained in $\mathcal{D}' - \overline{\mathcal{D}}$ (here, we use $a - 2\,d - \varepsilon_0 > 0$);
- $\delta^H(\widetilde{M}, M'') < \varepsilon'' + \sqrt{2\,\frac{\delta^H(\mathcal{D},\mathcal{D}')+d-a+\varepsilon''}{\kappa_1(\partial\mathcal{D}_a)} + (\delta^H(\mathcal{D},\mathcal{D}') + d - a + \varepsilon'')^2}$.

Assume now that $\mathcal{D}$ and $\mathcal{D}'$ are parallel. From our assumptions about d and ε'' and taking into account that $\kappa_1(\mathcal{D}_a) \geq \kappa_1(\mathcal{D}')$, then the last inequality becomes:

$$\delta^H(\widetilde{M}, M'') < \frac{\varepsilon}{4} + \sqrt{2\,\frac{\delta^H(\mathcal{D},\mathcal{D}') + \varepsilon}{\kappa_1(\partial\mathcal{D}')} + (\delta^H(\mathcal{D},\mathcal{D}') + \varepsilon)^2}.$$

Step 2. Consider now $a' > 0$ such that $\overline{\mathcal{D}}_a \subset \mathcal{D}'_{-2a'}$. Let $\widetilde{M}'$ be a compact region of $\widetilde{M}$, with regular boundary, and such that:

(**A**.1) $\partial\widetilde{M}' \subset \mathcal{D}' - \overline{\mathcal{D}'}_{-a'}$;
(**A**.2) $\widetilde{M}^r \subset \widetilde{M}' \subset \widetilde{M}$;
(**A**.3) The origin $\vec{0} \in \mathrm{Int}(\widetilde{M}')$ and $\mathrm{dist}_{\widetilde{M}'}(\vec{0}, \partial\widetilde{M}') > \mathrm{dist}_M(\vec{0}, \partial M) + 1$;
(**A**.4) $\delta^H(\widetilde{M}', M'') < \dfrac{\varepsilon}{4} + \sqrt{2\,\dfrac{\delta^H(\mathcal{D},\mathcal{D}') + \varepsilon}{\kappa_1(\partial\mathcal{D}')} + (\delta^H(\mathcal{D},\mathcal{D}') + \varepsilon)^2}$.

Take $\varepsilon_0' \in (0, \frac{\varepsilon}{4})$ such that $\overline{\mathcal{D}}_a \subset \mathcal{D}'_{-2a'-\varepsilon_0'}$.

At this point, we apply again Lemma 1 to the convex domains $\mathcal{D}'_b$, $\mathcal{D}'$, the constants $d = a'$, $\varepsilon' \in (0, \varepsilon_0']$, $r' > 0$, and the compact minimal surface $\widetilde{M}'$. Thus, we obtain a complete minimal surface $\widehat{M}$ which is properly immersed in $\mathcal{D}'_b$ and satisfies the following conditions:

- $\widehat{M}$ has the same topological type as $\mathrm{Int}(\widetilde{M}')$ (which is homeomorphic to $\mathrm{Int}(M)$), and $\vec{0} \in \mathrm{Int}(\widehat{M})$;
- The surface $\widehat{M} \cap T(\widetilde{M}', r')$ contains a regular compact surface $\widehat{M}^{r'}$ which is homeomorphic to $\widetilde{M}'$ and these surfaces converge smoothly to $\widetilde{M}'$, as $\varepsilon' \to 0$. Furthermore, $\delta^H(\widehat{M}^{r'}, M'') < \varepsilon'$;

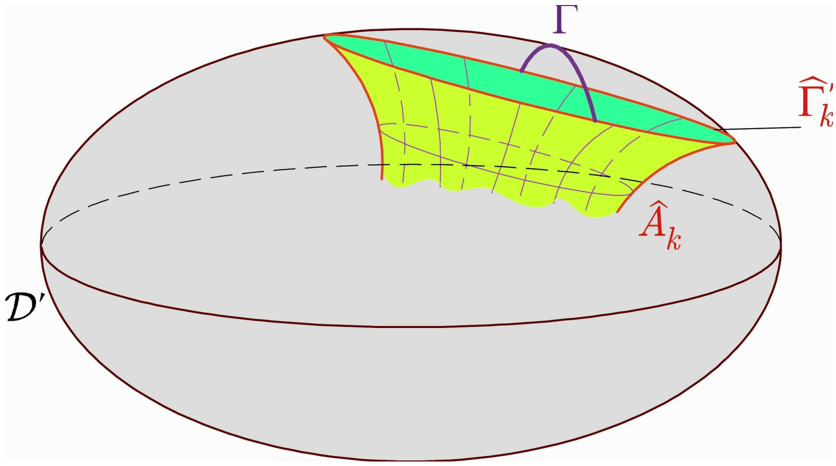

FIGURE 2. Let $\widehat{A}_k \subset \widehat{M}'$ be the (closed) annulus bounded by $\widehat{\Gamma}_k$ in $\overline{\mathcal{D}'} - \mathcal{D}$. Label by $\widehat{\Gamma}_k'$ the boundary component of $\widehat{A}_k$ in $\partial\mathcal{D}'$. Then, we connect two points p and q in $\widehat{\Gamma}_k'$ by a simple smooth arc, $\Gamma \subset \partial\mathcal{D}'$ so that the bridge principle can be applied to the configuration $\widehat{M}' \cup \Gamma$.

- Each end of $\widehat{M} - \widehat{M}^{r'}$ is contained in $\mathcal{D}_b' - \overline{\mathcal{D}_a}$ (here, we use $\overline{\mathcal{D}}_a \subset \mathcal{D}'_{-2a'-\varepsilon_0'}$);
- $\delta^H(\widetilde{M}', \widehat{M}) < \varepsilon' + \sqrt{2\frac{b+a'+\varepsilon'}{\kappa_1(\partial\mathcal{D}')} + (b+a'+\varepsilon')^2}$.

Notice that if b, a' and ε' are taken small enough in terms of $\kappa_1(\mathcal{D}')$, then the last inequality becomes:

$$\delta^H\left(\widetilde{M}', \widehat{M} \cap \overline{\mathcal{D}'}\right) < \varepsilon/4. \tag{2}$$

Step 3. Finally, we consider $\widehat{M}'$ a connected component of $\widehat{M} \cap \overline{\mathcal{D}'}$ with the same topological type as M. Up to an infinitesimal homothety, we can assume that $\widehat{M}'$ meets $\partial\mathcal{D}'$ transversally and that $\widehat{M}'$ is *nondegenerate*. Let $\widehat{\Gamma}_k$ denote the component of $\widehat{M}' \cap \partial\mathcal{D}$ which is contained in the tube $T(\Gamma_k, \frac{\varepsilon}{2})$ and let $\widehat{A}_k \subset \widehat{M}'$ be the (closed) annulus bounded by $\widehat{\Gamma}_k$ in $\overline{\mathcal{D}'} - \mathcal{D}$. Label by $\widehat{\Gamma}_k'$ the boundary component of $\widehat{A}_k$ in $\partial\mathcal{D}'$. Now, we connect two points p and q in $\widehat{\Gamma}_k'$ by a simple smooth arc, $\Gamma \subset \partial\mathcal{D}'$, such that:

- $\Gamma \cap \widehat{M}' = \Gamma \cap \left(\partial\widehat{M}'\right) = \partial\Gamma$, see Figure 2.
- $\delta^H\left(\Gamma \cup \widehat{M}', \widehat{M}'\right) < \varepsilon/4$.

Then we attach a thin bridge B_1 along the arc Γ to the surface $\widehat{M}'$ (see Figure 3). This new minimal surface is called M_ε. Notice that M_ε is nondegenerate (Theorem 3) and, if the bridge B_1 is thin enough, we also have:

$$\delta^H(M_\varepsilon, \widehat{M}') < \varepsilon/4. \tag{3}$$

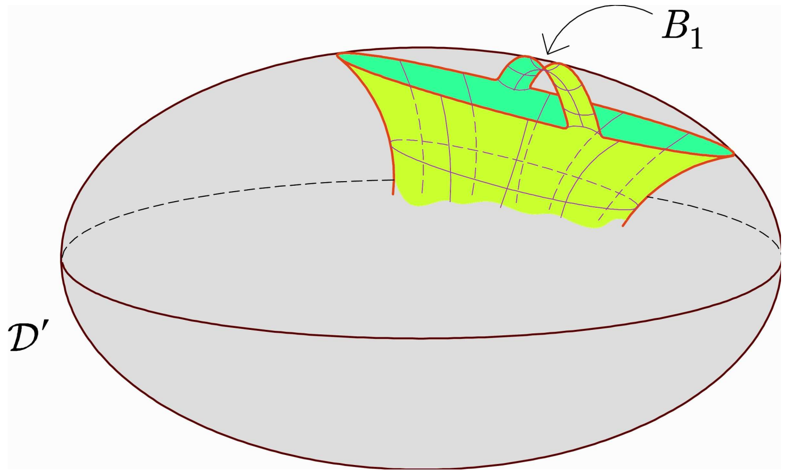

FIGURE 3. We attach a thin bridge B_1 along the arc Γ to the surface $\widehat{M}'$. In this way we obtain the desired surface M_ε.

Moreover, up to an infinitesimal translation and an infinitesimal expansive dilation, we can assume that $\vec{0} \in \mathrm{Int}(M_\varepsilon)$ and that M_ε can be extended beyond its boundary. Taking into account that, outside an open neighborhood of Γ, M_ε converges smoothly to $\widehat{M}'$ as $\varepsilon \longrightarrow 0$ (Theorem 3, item (4)), and the previously described properties satisfied by $\widehat{M}'$ and $\widetilde{M}'$, then it is not hard to see that M_ε satisfies items (1) to (4) in the lemma. Item (5) is a direct consequence of the triangle inequality and the inequalities (1), (A.4), (2), and (3). $\square$

Lemma 3 (Adding handles)**.** *Let $\mathcal{D}$ and $\mathcal{D}'$ be two smooth bounded strictly convex domains in $\mathbb{R}^3$ so that $\vec{0} \in \mathcal{D} \subset \overline{\mathcal{D}} \subset \mathcal{D}'$. Consider a compact minimal surface M with nonempty boundary and satisfying $\vec{0} \in \mathrm{Int}(M)$ and $\partial M \subset \partial \mathcal{D}$. Assume that M has genus g and k boundary components ($k \geq 1$), $\partial M = \Gamma_1 \cup \ldots \cup \Gamma_k$. We also assume that M intersects $\partial\mathcal{D}$ transversally.*

Then for any $\varepsilon > 0$, there exists a minimal surface M_ε satisfying the following properties:

(1) M_ε is a smooth, immersed minimal surface with genus $g+1$ and k boundary components. Moreover, $\partial M_\varepsilon \subset \partial\mathcal{D}'$, ∂M_ε meets transversally $\partial\mathcal{D}'$ and $\vec{0} \in \mathrm{Int}(M_\varepsilon)$;
(2) The intrinsic distance $\mathrm{dist}_{M_\varepsilon}(\vec{0}, \partial M_\varepsilon) > \mathrm{dist}_M(\vec{0}, \partial M) + 1$;
(3) The surfaces $M_\varepsilon \cap \overline{\mathcal{D}}$ are graphs over M and converge in the C^∞ topology to M, as $\varepsilon \longrightarrow 0$. Furthermore, $\delta^H(M, M_\varepsilon \cap \overline{\mathcal{D}}) < \varepsilon$;
(4) $M_\varepsilon - \mathcal{D}$ consists of $k-1$ annuli, whose boundary in $\partial\mathcal{D}$ lies in $T(\Gamma_j, \varepsilon)$, $j = 1, \ldots, k-1$, and an annulus with a handle, whose boundary in $\partial\mathcal{D}$ is a single curve which lie in $T(\Gamma_k, \varepsilon)$ (see Figure 1-(b));
(5) If $\mathcal{D}$ and $\mathcal{D}'$ are parallel, then $\delta^H(M, M_\varepsilon) < 2\, C(\varepsilon, \mathcal{D}, \mathcal{D}')$, where the constant $C(\varepsilon, \mathcal{D}, \mathcal{D}')$ is given in Lemma 2.

Proof. The proof of this lemma is identical to the one of Lemma 2, except for Step 3 which is slightly different. We construct the surface M_ε, like in the third step of the previous lemma. But this time we add a second bridge B_2 along a curve γ joining two opposite points in ∂B_1 (see

Figure 4). Notice that, in this way, the old annular component $\widehat{A}_k$ becomes an annulus with a handle.

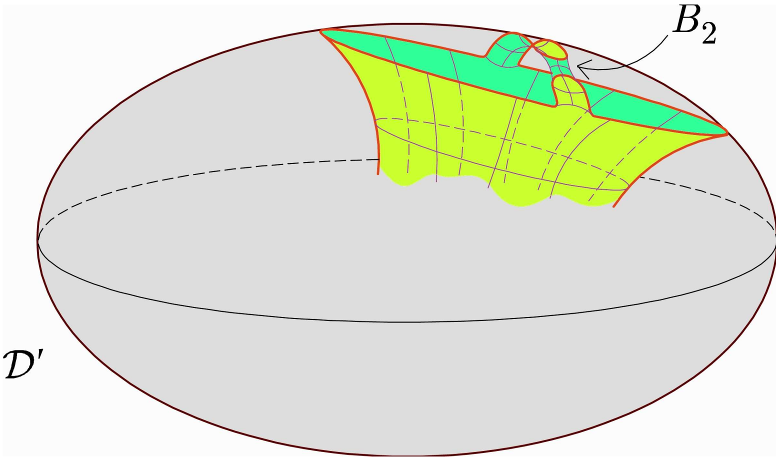

FIGURE 4. This time, we construct the surface M_ε, like in the third step of Lemma 2. But this time we add a second bridge B_2 along a curve γ joining two opposite points in ∂B_1

□

4. THE EXISTENCE OF SIMPLE EXHAUSTIONS

In this section we prove that any open orientable surface M of infinite topology has a smooth compact exhaustion $M_1 \subset M_2 \subset \cdots M_n \subset \cdots$, called a *simple exhaustion*. The defining properties for this exhaustion to be simple when M is orientable are:

1. M_1 be a disk.

For all $n \in \mathbb{N}$:

2. Each component of $M_{n+1} - \text{Int}(M_n)$ has one boundary component in ∂M_n and at least one boundary component in ∂M_{n+1}.

3. $M_{n+1} - \text{Int}(M_n)$ contains a unique nonannular component which topologically is a pair of pants or an annulus with a handle.

If M has finite topology with genus g and k ends, then we call the compact exhaustion *simple* if properties 1 and 2 hold, property 3 holds for $n \leq g+k$, and when $n > g+k$, all of the components of $M_{n+1} - \text{Int}(M_n)$ are annular.

The reader should note that for any simple exhaustion of M, each component of $M - \text{Int}(M_n)$ is a smooth, noncompact proper subdomain of M bounded by a simple closed curve and for each $n \in \mathbb{N}$, M_n is connected (see Fig. 5).

The following elementary lemma plays an essential role in the proofs of Theorems 1 and 2.

Lemma 4. *Every orientable open surface admits a simple exhaustion.*

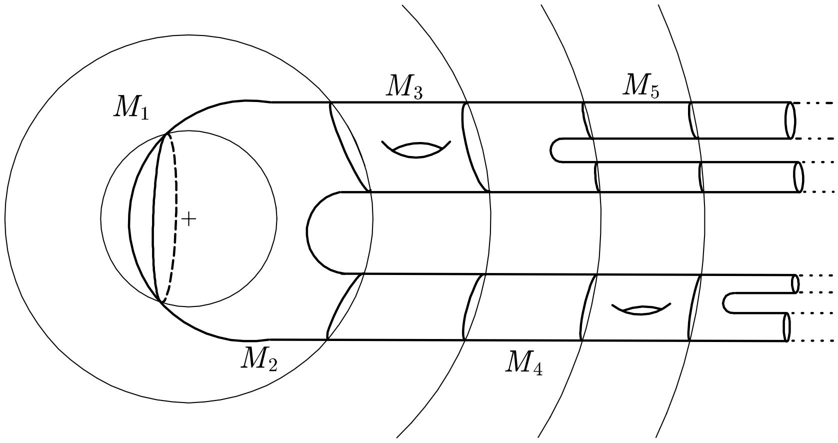

FIGURE 5. A topological representation of the terms M_1 to M_5 in the exhaustion of the open surface M given in Lemma 4.

Proof. If M has finite topology, the proof of the existence of a simple exhaustion is a straightforward consequence of the arguments we are going to use in the infinite topology situation. Assume now that M has infinite topology.

Consider a smooth compact exhaustion $W_1 \subset \cdots \subset W_n \subset \cdots$ of M such that W_1 is a disk. We first show that:

Assertion 4.1. *The exhaustion can be modified so that for every $j \in \mathbb{N}$, W_j is connected.*

If this assertion fails to hold for the given exhaustion, there exists a smallest $n > 1$ such that W_n consists of a finite collection of components $W_n(1), \ldots, W_n(m)$ with $m > 1$ and where $W_1 \subset W_n(1)$. For each $j \in \{2, \ldots, m\}$, choose a smooth embedded arc $\alpha_j \subset M - \mathrm{Int}(W_n)$ joining a point in the boundary of $W_n(j)$ to a point in the boundary of $W_n(1)$ and so that these arcs form a pairwise disjoint collection. Let W_n' be the union of W_n together with a closed regular neighborhood in M of the union of these arcs; W_n' is connected since W_{n-1} is connected. Suppose $W_n' \subset W_{n+k}$ for some k. Consider the new exhaustion $W_1 \subset \cdots \subset W_{n-1} \subset W_n' \subset W_{n+k} \subset \cdots$ for M. Repeating this argument inductively, one obtains a new compact exhaustion satisfying the connectedness condition stated in the assertion.

Assume now that the exhaustion fulfills the above assertion.

Assertion 4.2. *The exhaustion can be modified so that for all $j \in \mathbb{N}$, W_j is connected and there are no compact components in $M - \mathrm{Int}(W_j)$.*

If assertion were to fail, then for some smallest n, $M - \mathrm{Int}(W_n)$ contains a maximal (possibly disconnected) compact domain F. For some $k > 0$, the connected compact domain $W_n \cup F$ is a subset of W_{n+k} and so, we obtain a new exhaustion

$$W_1 \subset \cdots \subset W_{n-1} \subset W_n \cup F \subset W_{n+k} \subset \cdots .$$

Repeating this argument inductively, we obtain a new compact exhaustion satisfying the conclusions of Assertion 4.2.

Assume now that the exhaustion satisfies Assertion 4.2.

Assertion 4.3. *The exhaustion can be modified so that, for every $j \in \mathbb{N}$, each boundary curve of W_j separates M, each W_j is connected and there are no compact components in $M - \mathrm{Int}(W_j)$.*

If this new condition fails to hold for our given exhaustion, there exists a smallest $n > 1$ such that some boundary curve α in ∂W_n does not separate M and ∂W_n contains at least one other component different from α. In this case, there exists a simple closed curve β which intersects α transversally in a single point and is transverse to ∂W_n. Let W_n' be the union of W_n and a closed regular neighborhood of the embedded arc in $\beta \cap (M - \mathrm{Int}(W_n))$ whose ends points are contained in α and in a second boundary component of ∂W_n. The surface W_n' is connected and $M - \mathrm{Int}(W_n')$ has no compact components because $M - \mathrm{Int}(W_n)$ has none. Since W_n' contains one less boundary component than W_n, after a finite number of modifications of this type to W_n, we obtain a new connected surface W_n'' such that each boundary component of this surface separates M and $M - \mathrm{Int}(W_n'')$ has no compact components. The surface W_n'' is a subset of some W_{n+k}. Consider the new exhaustion $W_1 \subset \cdots \subset W_{n-1} \subset W_n'' \subset W_{n+k} \subset \dots$. Repeating this argument inductively, one obtains a new compact exhaustion $W_1 \subset \cdots \subset W_n \subset \dots$ with the desired properties.

Assertion 4.4. *The exhaustion can be modified to satisfy property 3 in the definition of simple exhaustion, and so that the exhaustion continues to satisfy the conclusions of Assertion 4.3*

Suppose that for $k \leq n-1$, $W_{k+1} - \mathrm{Int}(W_k)$ satisfies property 3 in the definition of simple exhaustion but $W_{n+1} - \mathrm{Int}(W_n)$ fails to satisfy this property. One way that $W_{n+1} - \mathrm{Int}(W_n)$ can fail to satisfy this property is that $W_{n+1} - \mathrm{Int}(W_n)$ consists entirely of annuli. Since M has infinite topology, there is a smallest $m > n$ such that $W_m - \mathrm{Int}(W_n)$ has a connected component F which is not an annulus. Thus, after removing the indexed domains W_j, $n < j < m$, from the exhaustion and reindexing, we may assume that $W_{n+1} - \mathrm{Int}(W_n)$ contains a compact component Δ that is not an annulus and which satisfies:

- Δ has exactly one boundary component δ_1 in ∂W_n; the existence of δ_1 is a consequence of Assertion 4.3.
- Δ has at least one boundary component in ∂W_{n+1}.

After the above modification, if $W_{n+1} - \mathrm{Int}(W_n)$ fails to satisfy property 3, then $|\chi(W_{n+1})| > 1$, where $\chi(\cdot)$ denotes the Euler characteristic. Let $\{\delta_1, \delta_2, \dots, \delta_\alpha\}$ be the components of ∂W_n and let A_i, $i = 1, \dots, \alpha$, be a small annular neighborhood of δ_i contained in $\mathrm{Int}(W_{n+1})$. If the genus of Δ is positive, then there exists a compact annulus with a handle $\Delta' \subset \mathrm{Int}(\Delta)$ with $\delta_1 \subset \partial \Delta'$ and $A_1 \subset \Delta'$. If the genus of Δ is zero, there exits a pair of pants $\Delta' \subset \mathrm{Int}(\Delta)$ with $\delta_1 \subset \partial \Delta'$ such that each of the other two boundary curves of Δ' separates M into two noncompact domains, and $A_1 \subset \Delta'$. In either case, define

$$W_{n+1}'' = W_n \cup \Delta' \cup \left(\bigcup_{i=1}^{\alpha} A_i \right).$$

Observe that $0 \leq |\chi(W_{n+1}'')| < |\chi(W_{n+1})|$. Also note that the compact exhaustion

$$W_1 \subset \cdots \subset W_n \subset W_{n+1}'' \subset W_{n+1} \subset W_{n+2} \subset \cdots$$

satisfies Assertion 4.3 and property 3 in the definition of simple exhaustion for levels $k \leq n$. After a smallest positive integer $j \leq |\chi(W_{n+1} - \mathrm{Int}(W_n))|$ of modifications of this sort, we arrive at the refined exhaustion:

$$W_1 \subset \cdots \subset W_n \subset W''_{n+1} \subset W''_{n+2} \subset \cdots \subset W''_{n+j} \subset W_{n+1} \subset \cdots,$$

such that $W_{n+1} - \mathrm{Int}(W''_{n+j})$ consists of annuli. It is straightforward to check that the new refined exhaustion

$$W_1 \subset \cdots \subset W_n \subset W''_{n+1} \subset W''_{n+2} \subset \cdots \subset W''_{n+j-1} \subset W_{n+1} \subset \cdots,$$

fulfills property 3 of a simple exhaustion through the domain W_{n+1} and such that Assertion 4.3 also holds. Repeating these arguments inductively, we obtain an exhaustion which satisfies property 3 in the definition of a simple exhaustion.

An exhaustion which satisfies Assertion 4.4 is a simple exhaustion and the lemma now follows. □

5. Proof of the main theorems

In this section we prove Theorem 1 in the case of open orientable surfaces. First, we need the following definition.

Definition 2. *Let $f\colon M \to \mathcal{D}$ be a proper immersion of an open surface M into a domain $\mathcal{D}$ in $\mathbb{R}^3$. We define the* **limit set** *of an end e of M as*

$$L(e) = \bigcap_{\alpha\in I} (\overline{f(E_\alpha)} - f(E_\alpha)),$$

where $\{E_\alpha\}_{\alpha\in I}$ is the collection of proper subdomains of M with compact boundary which represent e. Notice that $L(e)$ is a compact connected set of $\partial\mathcal{D}$.

Theorem 4. *Let M be an open orientable surface and let $\mathcal{D}$ be a domain in $\mathbb{R}^3$ which is either convex (possibly all $\mathbb{R}^3$) or bounded and smooth. Then, there exists a complete, proper minimal immersion $f\colon M \to \mathcal{D}$. Moreover, we have:*

(1) There exists a smooth exhaustion $\{\mathcal{D}_n \mid n \in \mathbb{N}\}$ of the domain $\mathcal{D}$ such that $\{M_n = f^{-1}(\overline{\mathcal{D}_n}) \mid n \in \mathbb{N}\}$ is simple exhaustion of M;

(2) If $\mathcal{D}$ is convex, then for any simple exhaustion $\{M_n \mid n \in \mathbb{N}\}$ of M and for any smooth exhaustion $\{\mathcal{D}_n \mid n \in \mathbb{N}\}$, where $\mathcal{D}_n$, $n \in \mathbb{N}$, are bounded and strictly convex[2], the immersion f can be constructed in such a way that $f(M_n) = f(M) \cap \overline{\mathcal{D}_n}$;

(3) Suppose $\mathcal{D}$ is smooth and bounded, and fix some open subset $U \subsetneq \partial\mathcal{D}$ such that U has positive mean and positive Gaussian curvature, with respect to the inward pointing normal to $\partial\mathcal{D}$. Then the minimal immersion $f: M \to \mathcal{D}$ can be constructed in such a way that the limit set of different ends of M are disjoint subsets of U.

Proof. In the proof of this theorem, we will distinguish three cases, depending on the nature of the domain $\mathcal{D}$.

Case 1. *$\mathcal{D}$ is a general convex domain, not necessarily bounded or smooth.*

Let M be an open surface and $\mathcal{M} = \{M_1 \subset M_2 \subset \cdots \subset M_n \subset \cdots\}$ be a simple exhaustion of M. Consider $\{\mathcal{D}_n,\ n \in \mathbb{N}\}$ a smooth exhaustion of $\mathcal{D}$, where $\mathcal{D}_n$ is bounded and strictly

[2]Any convex domain admits such a exhaustion by a classical result of Minkowski.

convex, for all n. The existence of such an exhaustion is guaranteed by a classical result of Minkowski (see [18, §2.8]).

Our purpose is to construct a sequence of minimal surfaces $\{\Sigma_n \mid n \in \mathbb{N}\}$ with nonempty boundary satisfying:

(1_n) $\vec{0} \in \Sigma_n$ and $\partial\Sigma_n \subset \partial\mathcal{D}_n$;

(2_n) For $i = 1, \ldots, n-1$, $\Sigma_n \cap \overline{\mathcal{D}}_i$ is a normal graph over its projection $\Sigma_{i,n} \subset \Sigma_i'$, where Σ_i' is a larger compact minimal surface containing Σ_i in its interior. Furthermore, if we write $\Sigma_n = \{p + f_{n,i}(p) \cdot N_i(p) \mid p \in \Sigma_{i,n}\}$, where N_i is the Gauss map of $\Sigma_{i,n}$, then:

(2_n- a) $|\nabla f_{n,i}| \leq \displaystyle\sum_{k=i+1}^{n} \varepsilon_k$, and

(2_n- b) $\delta^H(\Sigma_n \cap \overline{\mathcal{D}}_i, \Sigma_i) \leq \displaystyle\sum_{k=i+1}^{n} \varepsilon_k, \quad$ for $i = 1, \ldots, n-1$.

where $\varepsilon_k > 0$, for all k, and $\displaystyle\sum_{k=1}^{\infty} \varepsilon_k < 1$.

(3_n) $\operatorname{dist}_{\Sigma_n}(\vec{0}, \partial\Sigma_n) \geq \operatorname{dist}_{\Sigma_1}(\vec{0}, \partial\Sigma_1) + n - 1$;

The sequence $\{\Sigma_n \mid n \in \mathbb{N}\}$ is obtained by recurrence.

In order to define the first element of the family, we consider an analytic Jordan curve Γ_1 in $\partial\mathcal{D}_1$ and we solve the classical Plateau problem associated to this curve. The minimal disk obtained in this way is smooth and embedded [17] and it is the first term of the sequence Σ_1. Up to a suitable translation in $\mathbb{R}^3$, we can assume that $\vec{0} \in \operatorname{Int}(\Sigma_1) \subset \mathcal{D}_1$. It is obvious that Σ_1 satisfies Properties (1_1) and (4_1) (notice that the other two properties do not make sense for $n = 1$.)

Assume now we have defined Σ_n, satisfying items from (1_n) to (4_n). We are going to construct the minimal surface Σ_{n+1}. As the exhaustion $\mathcal{M}$ is simple, then we know that $M_{n+1} - \operatorname{Int}(M_n)$ contains a unique nonannular component N which topologically is a pair of pants or an annulus with a handle. Label γ as the connected component of ∂N that is contained in ∂M_n. We label the connected components of $\partial\Sigma_n$, $\Gamma_1, \ldots, \Gamma_k$, in such a way that γ maps to Γ_k by the homeomorphism which maps M_n into Σ_n. Then, we apply Lemma 2 or 3 (depending on the topology of N) to the data

$$\mathcal{D} = \mathcal{D}_n, \quad \mathcal{D}' = \mathcal{D}_{n+1}, \quad M = \Sigma_n.$$

Then, we obtain a family of minimal surfaces with boundary, Σ_ε, satisfying:

(i) $\partial\Sigma_\varepsilon \subset \partial\mathcal{D}_{n+1}$ and $\vec{0} \in \operatorname{Int}(\Sigma_\varepsilon)$;

(ii) $\operatorname{dist}_{\Sigma_\varepsilon}(\vec{0}, \partial\Sigma_\varepsilon) > \operatorname{dist}_{\Sigma_n}(\vec{0}, \partial\Sigma_n) + 1 \geq \operatorname{dist}_{\Sigma_1}(\vec{0}, \partial\Sigma_1) + n$ (notice that Σ_n satisfies property (3_n));

(iii) The surfaces $\Sigma_\varepsilon \cap \overline{\mathcal{D}_n}$ are diffeomorphic to Σ_n and converge in the C^∞ topology to Σ_n, as $\varepsilon \to 0$. Furthermore, $\delta^H(\Sigma_n, \Sigma_\varepsilon \cap \overline{\mathcal{D}_n}) < \varepsilon$;

(iv) $\Sigma_\varepsilon - \mathcal{D}_n$ consists of $k-1$ annuli whose boundary in $\partial\mathcal{D}_n$ lies in $T(\Gamma_j, \varepsilon)$, $j = 1, \ldots, k-1$, and a nonannular piece which is homeomorphic to N whose boundary in $\partial\mathcal{D}_n$ is a single curve which lies in $T(\Gamma_k, \varepsilon)$;

Item (iii) and property (2_n) imply that $\Sigma_\varepsilon \cap \overline{\mathcal{D}}_i$ can be expressed as a normal graph over its projection $\Sigma_{i,\varepsilon} \subset \Sigma_i'$, $i = 1, \ldots, n$; $\Sigma_\varepsilon \cap \overline{\mathcal{D}}_i = \{p + f_{\varepsilon,i}(p)\, N_i(p) \mid p \in \Sigma_{i,\varepsilon}\}$. Since as $\varepsilon \to 0$

Σ_ε converges smoothly to Σ_n in $\mathcal{D}_n$ and Σ_n satisfies (2_n-a), then we have:

$$|\nabla f_{\varepsilon,i}| < \sum_{k=i+1}^{n+1} \varepsilon_k. \tag{4}$$

Moreover, if we take $\varepsilon < \varepsilon_{n+1}$, then item (iii) and property (2_n-2) implies that

$$\delta^H \left(\Sigma_\varepsilon \cap \overline{\mathcal{D}}_i, \Sigma_i\right) < \sum_{k=i+1}^{n+1} \varepsilon_k; \tag{5}$$

here we have also used the triangle inequality for δ^H.

Then, we define $\Sigma_{n+1} \stackrel{\text{def}}{=} \Sigma_\varepsilon$, where ε is chosen small enough in order to satisfy (4) and (5). It is clear that Σ_{n+1} so defined fulfills (1_{n+1}), (2_{n+1}) and (3_{n+1}).

Now, we have constructed our sequence of minimal surfaces $\{\Sigma_n\}_{n\in\mathbb{N}}$. Taking into account properties (2_n), for $n \in \mathbb{N}$, and using Ascoli-Arzela's theorem, we deduce that the sequence of surfaces $\{\Sigma_n\}_{n\in\mathbb{N}}$ converges to an open immersed minimal surface Σ in the C^m topology, for all $m \in \mathbb{N}$. Moreover, $\Sigma \cap \overline{\mathcal{D}}_i$ is a normal graph over its projection $\Sigma_{i,\infty} \subset \Sigma_i'$, for all i, and the norm of the gradient of the graphing functions its at most 1 (see properties (2_n-a)).

Finally, we check that Σ satisfies all the statements in the theorem.

• Σ *is properly immersed in* $\mathcal{D}$. To see this, we consider $K \subset \mathcal{D}$ a compact subset. We have to prove that $\Sigma \cap K$ is compact. As $\{\mathcal{D}_n \ : \ n \in \mathbb{N}\}$ is an exhaustion of $\mathcal{D}$, then we know that there exists $n_0 \in \mathbb{N}$ such that $K \subset \mathcal{D}_{n_0}$. We also know that $\Sigma \cap \overline{\mathcal{D}}_{n_0}$ is a graph over Σ_{n_0} which is compact. Therefore $\Sigma \cap \overline{\mathcal{D}}_{n_0}$ is compact and $\Sigma \cap K$ is a closed subset compact set, consequently $\Sigma \cap K$ is compact.

• Σ *is complete.* Consider the compact exhaustion $\Sigma \cap \mathcal{D}_n$ of Σ and note that $\Sigma \cap \mathcal{D}_n$ is quasi isometric to $\Sigma_{n,\infty}$ with respect to constants that are independent of n. Then properties (3_n), $n \in \mathbb{N}$, trivially imply that Σ is complete.

• Σ *is homeomorphic to* M. If we consider the exhaustions $\{\Sigma \cap \overline{\mathcal{D}}_n \mid n \in \mathbb{N}\}$ of Σ and $\{M_n \mid n \in \mathbb{N}\}$ of M, then we know (from the way in which we have constructed Σ) that $\Sigma \cap \overline{\mathcal{D}}_n$ is homeomorphic to M_n. Label this homeomorphism as $f_n \colon \Sigma \cap \overline{\mathcal{D}}_n \to M_n$.

$$\begin{array}{ccc} \Sigma \cap \overline{\mathcal{D}}_i & \xrightarrow{\ \mathrm{i}\ } & \Sigma \cap \overline{\mathcal{D}}_n \\ \Big\downarrow {\scriptstyle f_n|_{\Sigma\cap\overline{\mathcal{D}}_i}} & & \Big\downarrow {\scriptstyle f_n} \\ M_i & \xrightarrow{\ \mathrm{i}\ } & M_n \end{array}$$

Moreover, we have that $f_n|_{\Sigma\cap\overline{\mathcal{D}}_i}$ is also a homeomorphism between $\Sigma \cap \overline{\mathcal{D}}_i$ and M_i which coincides with the corresponding homeomorphism f_i. Then, after taking the limit as $n \to \infty$, we conclude that Σ and M are homeomorphic.

Case 2. *$\mathcal{D}$ is a smooth strictly convex domain.*

First of all, we can assume, up to a suitable shrinking of $\mathcal{D}$, that $\kappa_1(\partial\mathcal{D}) = 1$. This time the proof is slightly different from the previous case. Our aim is to create a sequence:

$$\Theta_n = \{t_n, \varepsilon_n, \delta_n, \mathcal{D}_n, \Sigma_n\}_{n\in\mathbb{N}},$$

where:

- $\{t_n\}_{n\in\mathbb{N}}$, $\{\varepsilon_n\}_{n\in\mathbb{N}}$, $\{\delta_n\}_{n\in\mathbb{N}}$, are sequences of real numbers decreasing to 0. Moreover, $\displaystyle\sum_{n=i+1}^{\infty} \varepsilon_n < \delta_i$ for any $i \in \mathbb{N}$.
- $\mathcal{D}_n \stackrel{\text{def}}{=} \mathcal{D}_{-t_n}$ is the convex domain parallel to $\mathcal{D}$ at distance t_n.
- Σ_n is a compact, connected, minimal surface with nonempty boundary.

This sequence can be constructed in such a way so that it satisfies:

(1_n) $\vec{0} \in \Sigma_n$ and $\partial\Sigma_n \subset \partial\mathcal{D}$;

(2_n) For $i = 1, \dots, n-1$, $\Sigma_n \cap \overline{\mathcal{D}}_i$ is a normal graph over its projection $\Sigma_{i,n} \subset \Sigma_i'$, where Σ_i' is a larger compact minimal surface containing Σ_i in its interior. Furthermore, if we write $\Sigma_n = \{p + f_{n,i}(p) \cdot N_i(p) \mid p \in \Sigma_{i,n}\}$, where N_i is the Gauss map of $\Sigma_{i,n}$, then:

(2_n- a) $\displaystyle |\nabla f_{n,i}| \leq \sum_{k=i+1}^{n} \varepsilon_k$, and

(2_n- b) $\displaystyle \delta^H(\Sigma_n, \Sigma_i) \leq \sum_{k=i+1}^{n} \varepsilon_k, \quad$ for $i = 1, \dots, n-1$.

where $\varepsilon_k > 0$, for all k, and $\displaystyle\sum_{k=1}^{\infty} \varepsilon_k < 1$.

(3_n) $\operatorname{dist}_{\Sigma_n}(\vec{0}, \partial\Sigma_n) \geq \operatorname{dist}_{\Sigma_1}(\vec{0}, \partial\Sigma_1) + n - 1$;

(4_n) Let $2\,\delta_i \stackrel{\text{def}}{=} \min_{j\neq k} \operatorname{dist}_{\mathbb{R}^3}(C_j, C_k)$, where C_j are the connected components of $\Sigma_i \cap (\overline{\mathcal{D}} - \mathcal{D}_i)$. If there is only one component in $\Sigma_i \cap (\overline{\mathcal{D}} - \mathcal{D}_i)$, then we define $\delta_i \stackrel{\text{def}}{=} 1/2$. If C and C' are two different connected components of $\Sigma_n \cap \big(\overline{\mathcal{D}} - \mathcal{D}_i\big)$, then the distance $\operatorname{dist}_{\mathbb{R}^3}(C, C') > \delta_i$.

The sequence $\{\Theta_n\}_{n\in\mathbb{N}}$ is obtained in a recurrent way. In order to define Σ_1, we consider an analytic Jordan curve Γ_1 in $\partial\mathcal{D}$. We solve the Plateau problem for this curve and let Σ_1 be the solution minimal disk. Up to a translation in $\mathbb{R}^3$, we can assume that $\vec{0} \in \operatorname{Int}(\Sigma_1) \subset \mathcal{D}$.

Suppose that we have constructed the term Θ_n in the sequence. The idea is to apply Lemma 2 or Lemma 3 (depending on the topology of $M_{n+1} - \operatorname{Int}(M_n)$) to produce the next minimal surface Σ_{n+1}, like in the proof of Case 1. However, this time we have to be more careful. First, we take $t_{n+1} \in (0, t_n)$ small enough so that:

- Σ_n intersects $\partial\mathcal{D}_{-t_{n+1}}$ transversally and $\Sigma_n \cap \overline{\mathcal{D}}_{-t_{n+1}}$ contains a connected component $\widehat{\Sigma}_n$ with the same topological type than Σ_n and satisfies
$$\operatorname{dist}_{\widehat{\Sigma}_n}(\vec{0}, \partial\widehat{\Sigma}_n) \geq \operatorname{dist}_{\Sigma_1}(\vec{0}, \partial\Sigma_1) + n - 1.$$
- The constant $C(\varepsilon', \mathcal{D}_{-t_{n+1}}, \mathcal{D}) = \varepsilon' + \sqrt{(t_{n+1} + 2\varepsilon' + 1)^2 - 1} < \varepsilon_{n+1}$ for ε' sufficiently small.

Then apply one of the lemmas to the data Σ_n, $\mathcal{D}_{n+1}$ and $\mathcal{D}$. In this way, we obtain the new immersion Σ_{n+1} satisfying properties (1_{n+1}) to (4_{n+1}). Let us check (4_{n+1}). Take C and C' two components of $\Sigma_{n+1} \cap (\overline{\mathcal{D}} - \mathcal{D}_i)$. Then C and C' lie in tubular neighborhoods of radius $\sum_{k=i+1}^{n+1} \varepsilon_k$ of some components of $\Sigma_i \cap (\overline{\mathcal{D}} - \mathcal{D}_i)$, that we label $\widetilde{C}$ and $\widetilde{C}'$, respectively. Then one has

$$\operatorname{dist}_{\mathbb{R}^3}(C, C') \geq \operatorname{dist}_{\mathbb{R}^3}(\widetilde{C}, \widetilde{C}') - \sum_{k=i+1}^{n+1} \varepsilon_k > 2\delta_i - \sum_{k=i+1}^{n+1} \varepsilon_k > \delta_i.$$

If $\sum_{k=n+2}^{\infty} \varepsilon_k \geq \delta_{n+1}$, then we modify the sequence $\{\varepsilon_n\}_{n\in\mathbb{N}}$ as follows:

- $\varepsilon'_k = \varepsilon_k$, for $k = 1, \ldots, n+1$;
- $\varepsilon'_k = \delta_{n+1}\, \varepsilon_k$, for $k > n+1$.

At this point in the proof, we have obtained a sequence of compact minimal surfaces $\{\Sigma_n\}_{n\in\mathbb{N}}$ with regular boundary in $\partial\mathcal{D}$, whose interiors converge smoothly on compact sets of $\mathcal{D}$ to a complete minimal surface Σ, properly immersed in $\mathcal{D}$. As in the previous step, we have that $\Sigma \cap \overline{\mathcal{D}_i}$ is homeomorphic to M_i, for all $i \in \mathbb{N}$, and for each $i \in \mathbb{N}$, $\Sigma \cap \overline{\mathcal{D}_i}$ is a small graph over Σ_i. Furthermore, properties (4_n), $n \in \mathbb{N}$, imply that the distances between any two components of $\Sigma \cap (\mathcal{D} - \mathcal{D}_i)$ are larger than δ_i. Note that two different ends e_1, e_2 of Σ can be represented by distinct components C_1, C_2 of $\Sigma - \mathcal{D}_j$, for some j sufficiently large. By Definition 2, the distance between $L(e_1)$ and $L(e_2)$ is at least equal to the distance between C_1 and C_2 which is greater than δ_j. This completes the proof of Case 2.

Case 3. *$\mathcal{D}$ is a smooth bounded domain.*

In this case we take U to be an open disk in $\partial\mathcal{D}$ so that the principal curvatures with respect to the inner pointing normals are positive and bounded away from zero. Then, it is possible to find a smooth convex domain $\mathcal{D}_U \subset \mathcal{D}$ with $U \subset \partial\mathcal{D}_U$. Then we consider the curve $\Gamma_1 \subset U$ as in the previous case, and we solve the classical Plateau problem to obtain a compact minimal disk Σ_1. We take the series $\sum_{k=1}^{\infty} \varepsilon_k$ to satisfy:

$$\sum_{k=1}^{\infty} \varepsilon_k < \frac{1}{2} \operatorname{dist}_{\mathbb{R}^3}(\Sigma_1, \partial\mathcal{D} - U).$$

Thus, we apply Case 2 to obtain a complete minimal surface Σ satisfying the conclusions of the theorem for the domain $\mathcal{D}'$ and the limit set of Σ is contained in U. Then the surface Σ is also properly immersed in $\mathcal{D}$. This concludes the proof of the theorem. □

Suppose M is a proper minimally immersed open surface in $\mathbb{R}^3$ and passes through the origin. After a small translation of M assume that M is transverse to the boundary sphere of the balls $\mathbb{B}(n)$ of radius n, $n \in \mathbb{N}$. Then the maximum principle implies that the exhaustion $\{M_n = M \cap \mathbb{B}(n)\}$ of M is a smooth compact exhaustion where for all

$$n \in \mathbb{N}, \quad M - \operatorname{Int}(M_n) \quad \text{has no compact components.}$$

We will call a smooth compact exhaustion $M_1 \subset M_2 \subset \cdots \subset M_n \subset \cdots$ *admissible* if it satisfies the above property. The next result is an immediate corollary of Theorem 4.

Theorem 5. *Let M be an open orientable surface with an admissible exhaustion $M_1 \subset M_2 \subset \cdots M_n \subset \cdots$. There exists a proper minimal immersion $f\colon M \longrightarrow \mathbb{R}^3$ satisfying $f(M_n) = f(M) \cap \mathbb{B}(n)$.*

The question concerning the existence of complete proper minimal surfaces in the unit ball $\mathbb{B}(1)$ such that the limit sets are the entire unit sphere $\mathbb{S}^2(1)$ was proposed to the second author by Nadirashvili in 2004. The techniques used to prove Theorem 4 allow us to give a positive answer to this former question.

Proposition 1. *Let M be an open orientable surface and $\mathcal{D}$ a convex open domain. Then there exists a complete proper minimal immersion $f : M \to \mathcal{D}$ such that the limit set of $f(M)$ is $\partial\mathcal{D}$.*

The proof of the above proposition consists of a suitable use of the bridge principle in the proof of Lemmas 2 and 3. In this case the curve Γ used in Step 3 in both lemmas is substituted by a smooth arc in $\partial\mathcal{D}'$ which is ε close to every point of $\partial\mathcal{D}'$. With these new versions of the lemmas we can modify the proof of Case 1 (when $\partial\mathcal{D}$ is convex) as follows: we construct the sequence $\{\Sigma_n\}_{n\in\mathbb{N}}$ in such a way that $\partial\Sigma_n$ is $\frac{1}{n}$ close to every point in $\partial\mathcal{D}_n$. So, the limit immersion Σ would satisfy that its limit set $L(\Sigma)$ is $\partial\mathcal{D}$.

As a consequence of Proposition 1, we obtain the following result.

Corollary 1. *Any convex domain of $\mathbb{R}^3$ is the convex hull of some complete minimal surface.*

6. Nonorientable minimal surfaces

The main goal of this section is to develop the necessary theory for dealing with complete, properly immersed or embedded, nonorientable minimal surfaces in domains in $\mathbb{R}^3$. First we explain how to modify arguments in the proof of the Density Theorem in [1] to the case of nonorientable surfaces, i.e., given a compact nonorientable surface M, we describe how to approximate it by a complete, nonorientable hyperbolic surface $\widetilde{M}$ which is homeomorphic to the interior of M. Once this generalization of the Density Theorem is seen to hold, we apply it to prove that Theorem 4 holds for nonorientable surfaces, which then completes the proof of Theorem 1 stated in the Introduction.

Since one of the goals in the Embedded Calabi-Yau Conjecture is to construct nonorientable, properly embedded minimal surfaces in bounded domains of $\mathbb{R}^3$, we construct in Section 6.3 complete, proper minimal immersions of any open surface M with a finite number of nonorientable ends into some certain smooth nonsimply connected domain such that distinct ends of M have disjoint limit sets and such that the immersed surface is properly isotopic to a proper (incomplete) minimal embedding of M in the domain. In Example 3, we construct a bounded domain $\mathcal{D}_\infty$ in $\mathbb{R}^3$ which is smooth except at one point p_∞ and has the property that every open surface M admits a complete, proper minimal immersion $f : M \to \mathcal{D}_\infty$ which can be closely approximated in the Hausdorff distance by a proper, noncomplete, minimal embedding of M in $\mathcal{D}_\infty$.

6.1. **Density theorems for nonorientable minimal surfaces.** The results contained in [1] remain true when the minimal surfaces involved in the construction are nonorientable. In order to obtain a result similar to Lemma 1 in the nonorientable setting, we work with the orientable double covering. But then all the machinery must be adapted in order to be compatible with the antiholomorphic involution of the change of sheet in the orientable covering. In verifying this construction, there are three points that are nontrivial and they are explained in paragraphs 6.1.1, 6.1.2, and 6.1.3 below.

First, we need some notation. Let M' denote a connected compact Riemann surface of genus $\sigma \in \mathbb{N} \cup \{0\}$. Let $I : M' \to M'$ be an antiholomorphic involution without fixed points. Then, the surface $\widetilde{M}' \stackrel{\text{def}}{=} M'/\langle I \rangle$ is a compact connected nonorientable surface.

For $\text{E} \in \mathbb{N}$, consider $\mathbb{D}_1, \ldots, \mathbb{D}_\text{E} \subset M'$ open disks so that $\{\gamma_i \stackrel{\text{def}}{=} \partial\mathbb{D}_i, \quad i = 1, \ldots, \text{E}\}$ are piecewise smooth Jordan curves and $\overline{\mathbb{D}}_i \cap \overline{\mathbb{D}}_j = \emptyset$ for all $i \neq j$.

Definition 3. *Each curve γ_i will be called a cycle on M' and the family $\mathcal{J} = \{\gamma_1, \ldots, \gamma_{\text{E}}\}$ will be called a* **multicycle** *on M'. We denote by $\text{Int}(\gamma_i)$ the disk $\mathbb{D}_i$, for $i = 1, \ldots, \text{E}$. We also define $M(\mathcal{J}) = M' - \cup_{i=1}^{\text{E}} \overline{\text{Int}(\gamma_i)}$. Notice that $M(\mathcal{J})$ is always connected.*

We will say that $\mathcal{J}$ is **invariant under** *I iff for any disk $\mathbb{D}_i$ there exist another disk in the family $\mathbb{D}_j$ such that $I(\mathbb{D}_i) = \mathbb{D}_j$. Observe that $i \neq j$ and so the number of cycles in $\mathcal{J}$ is even in this case.*

Given $\mathcal{J} = \{\gamma_1, \ldots, \gamma_{\text{E}}\}$ and $\mathcal{J}' = \{\gamma_1', \ldots, \gamma_{\text{E}}'\}$ two multicycles in M', we write $\mathcal{J}' < \mathcal{J}$ if $\overline{\text{Int}(\gamma_i)} \subset \text{Int}(\gamma_i')$ for $i = 1, \ldots, \text{E}$. Observe that $\mathcal{J}' < \mathcal{J}$ implies $\overline{M(\mathcal{J}')} \subset M(\mathcal{J})$.

6.1.1. *Runge functions on nonorientable minimal surfaces.* Runge-type theorems are crucial in obtaining the theorems for orientable surfaces obtained previously in [1]. So, the first step in the proof of Lemma 1 in the nonorientable case consists of proving a suitable Runge theorem for nonorientable minimal surfaces. To be more precise, we need the following.

Lemma 5. *Let $\mathcal{J}$ be a multicycle in M'which is invariant under I and let $F : \overline{M(\mathcal{J})} \to \mathbb{R}^3$ be a nonorientable minimal immersion with Weierstrass data (g, Φ_3)*[3]. *Consider K_1 and K_2 two disjoint compact sets in $M(\mathcal{J})$ and $\Delta \subset M'$ satisfying:*

(a) There exists a basis of the homology of $M(J)$ contained in K_2 and $I(K_2) = K_2$;
(b) $\overline{\Delta} \subset M' - (K_1 \cup I(K_1) \cup K_2)$ and $I(\Delta) = \Delta$;
(c) Δ has a point in each connected component of $M' - (K_1 \cup I(K_1) \cup K_2)$.

Then, for any $m \in \mathbb{N}$ and any $t > 0$, there exists a holomorphic function without zeros $H : M(\mathcal{J}) - \Delta \to \mathbb{C}$ such that:

(1) $H \circ I = 1/\overline{H}$;
(2) $|H - t| < 1/m$ in K_1;
(3) $|H - 1| < 1/m$ in K_2;
(4) The nonorientable minimal immersion given by the Weierstrass data $\widetilde{g} \stackrel{\text{def}}{=} g/H$ and $\widetilde{\Phi}_3 := \Phi_3$ is well-defined (has no real periods.)

Proof. If σ represents the genus of M' and 2E is the number of cycles in $\mathcal{J}$, notice that the dimension of $H_1(M(\mathcal{J}), \mathbb{R})$ is $2\sigma + 2\text{E} - 1$.

Assertion 6.1. *There exists a basis for the first real homology group of $M(\mathcal{J})$*

$$B = \{\gamma_1, \ldots, \gamma_{\sigma+\text{E}}, \Gamma_1, \ldots, \Gamma_{\sigma+\text{E}-1}\},$$

which is contained in K_2 and satisfies:

- *$I_*(\gamma_j) = \gamma_j$, for $j = 1, \ldots, \sigma + \text{E}$,*
- *$I_*(\Gamma_j) = -\Gamma_j$, for $j = 1, \ldots, \sigma + \text{E} - 1$.*

The proof of this assertion is a standard topological argument that can be found in [6], for instance.

Assertion 6.2. *If τ is a holomorphic differential in $M(\mathcal{J})$ satisfying $I^*(\tau) = \overline{\tau}$, then $\text{Re}\left(\int_\gamma \tau\right) = 0$, for all γ in $H_1(M(\mathcal{J}), \mathbb{R})$ if and only if $\int_{\gamma_j} \tau = 0$, for all $j = 1, \ldots, \sigma + \text{E}$.*

In addition, if τ is holomorphic on M', then $\tau = 0$ if and only if $\int_{\gamma_j} \tau = 0$, for all $j = 1, \ldots, \sigma + \text{E}$.

[3]Recall that $g \circ I = -1/\overline{g}$, $I^*\Phi_3 = \overline{\Phi_3}$.

Proof. The proof of the first part of this claim is straightforward. For the second part, take into account that a holomorphic differential on a compact Riemann surface is zero if and only if it has imaginary periods. □

Assertion 6.3. *Consider* $(b_1, \ldots, b_{\sigma+\mathrm{E}}) \in \mathbb{R}^{\sigma+\mathrm{E}} - \{\vec{0}\}$ *and* $c = \sum_{j=1}^{\sigma+\mathrm{E}} b_j \cdot \gamma_j$, *then there exists a holomorphic differential on* $\overline{M(\mathcal{J})}$ *satisfying* $I^*\tau = -\overline{\tau}$ *and* $\int_c \tau \neq 0$.

Furthermore, given L *an integral divisor in* M', *invariant under* I *and with* $\operatorname{supp}(L) \subset \overline{M(\mathcal{J})}$, *then* τ *can be chosen in such a way that* $(\tau)_0 \geq L$, *where* $(\cdot)_0$ *means the divisor of zeros.*

Proof. The first holomorphic De Rham cohomology group, $H^1_{\mathrm{hol}}(\overline{M(\mathcal{J})})$ is a complex vector space of dimension ϱ. If we define $F\colon H^1_{\mathrm{hol}}(\overline{M(\mathcal{J})}) \longrightarrow H^1_{\mathrm{hol}}(\overline{M(\mathcal{J})})$

$$F([\omega]) \stackrel{\mathrm{def}}{=} \left[\overline{I^*(\omega)}\right],$$

then F is a (real) linear involution of $H^1_{\mathrm{hol}}(\overline{M(\mathcal{J})})$. Hence, $H^1_{\mathrm{hol}}(\overline{M(\mathcal{J})}) = V^+ \oplus V^-$, where $V^+ = \{[\omega] \mid F([\omega]) = [\omega]\}$ and $V^- = \{[\omega] \mid F([\omega]) = -[\omega]\}$. Moreover, the linear map $[\omega] \mapsto [\mathrm{i}\,\omega]$ establishes an isomorphism between V^+ and V^-. Then, we have that the real dimension $\dim_{\mathbb{R}} V^+ = \varrho$. So, the linear map:

$$T : V^- \longrightarrow \left(\mathrm{i} \cdot \mathbb{R}^{\sigma+\mathrm{E}}\right) \times \mathbb{R}^{\sigma+\mathrm{E}-1}$$

$$T([\psi]) = \left(\int_{\gamma_1} \psi, \ldots, \int_{\gamma_{\sigma+\mathrm{E}}} \psi, \int_{\Gamma_1} \psi, \ldots, \int_{\Gamma_{\sigma+\mathrm{E}-1}} \psi\right),$$

is an isomorphism where $\mathrm{i} = \sqrt{-1}$. In particular, there exists $[\psi]$ in V^- such that

$$T([\psi]) \notin \left\{(z_1, \ldots, z_{\sigma+\mathrm{E}}, w_1, \ldots, w_{\sigma+\mathrm{E}-1}) \in \left(\mathrm{i} \cdot \mathbb{R}^{\sigma+\mathrm{E}}\right) \times \mathbb{R}^{\sigma+\mathrm{E}-1} \mid \sum_{j=1}^{\sigma+\mathrm{E}} b_j z_j = 0\right\}.$$

Hence $\operatorname{Im}\left(\int_c \psi\right) \neq 0$. Now, using Claim 3.2 in [1], we can prove the existence of a holomorphic differential on $\overline{M(\mathcal{J})}$, $\widetilde{\psi}$, with the same periods as ψ and such that $(\widetilde{\psi})_0 \geq L$. Then, we define the 1-form $\tau \stackrel{\mathrm{def}}{=} \frac{1}{2}\left(\widetilde{\psi} - \overline{I^*(\widetilde{\psi})}\right)$. From the definition, it is clear that $I^*(\tau) = -\overline{\tau}$ and $(\tau)_0 \geq L$. Moreover, as ψ and $\widetilde{\psi}$ have the same periods, one has:

$$\int_c \tau = \frac{1}{2}\left(\int_c \widetilde{\psi} - \overline{\int_c \widetilde{\psi}}\right) = \mathrm{i}\operatorname{Im}\left(\int_c \widetilde{\psi}\right) = \mathrm{i}\operatorname{Im}\left(\int_c \psi\right) \neq 0.$$

□

From this point on in the proof, we can follow the proof of Lemma 1 in [6] to obtain the existence of the function H satisfying all the assertions in the lemma. For completeness, we include a sketch of this proof.

Assertion 6.4. *Let* $\mathcal{H}^-\left(\overline{M(\mathcal{J})}\right)$ *be the* real *vector space of the holomorphic functions* $t : \overline{M(\mathcal{J})} \to \mathbb{C}$*, satisfying* $t \circ I = -\overline{t}$*. Then the linear map* $F : \mathcal{H}^-\left(\overline{M(\mathcal{J})}\right) \to \mathbb{R}^{2(\sigma+\mathrm{E})}$*, given by:*

$$F(t) = \left(\int_{\gamma_j} t\ \Phi_3\left(\frac{1}{g}+g\right), -i\int_{\gamma_j} t\ \Phi_3\left(\frac{1}{g}-g\right)\right)_{j=1,\dots,\sigma+\mathrm{E}}$$

is surjective.

Proof. We proceed by contradiction. Assume F is not onto. Then, there exist

$$(\vartheta_1,\dots,\vartheta_{\sigma+\mathrm{E}},\mu_1,\dots,\mu_{\sigma+\mathrm{E}}) \in \mathbb{R}^{2(\sigma+\mathrm{E})} - \{(0,\dots,0)\},$$

such that:

$$\sum_{j=1}^{\sigma+\mathrm{E}}\left[\vartheta_j\int_{\gamma_j} t\ \Phi_3\left(\frac{1}{g}+g\right) - i\,\mu_j\int_{\gamma_j} t\ \Phi_3\left(\frac{1}{g}-g\right)\right] = 0 \quad \forall t \in \mathcal{H}^-\left(\overline{M(\mathcal{J})}\right). \tag{6}$$

Assertion 6.3 guarantees the existence of a differential τ satisfying

(i) $(\tau)_0 \geq \left(\left(\left(\frac{1}{g}+g\right)\Phi_3\right)_{|\overline{M(\mathcal{J})}}\right)_0^2 \left(\left(d\left(\frac{1-g^2}{1+g^2}\right)\right)_{|\overline{M(\mathcal{J})}}\right)_0,$

(ii) $-i\sum_{j=1}^{\sigma+\mathrm{E}}\mu_j\int_{\gamma_j}\tau \neq 0,$

(iii) $I^*\tau = -\overline{\tau}.$

Let us define $y \stackrel{\text{def}}{=} \dfrac{\tau}{d\left(\frac{1-g^2}{1+g^2}\right)}$, and $t \stackrel{\text{def}}{=} \dfrac{d(y)}{\left(\frac{1}{g}+g\right)\Phi_3}$. Taking the choice of τ into account, the function t belongs to $\mathcal{H}^-\left(\overline{M(\mathcal{J})}\right)$. In this case and after integrating by parts, (6) becomes $-i\sum_{j=1}^{\sigma+\mathrm{E}}\mu_j\int_{\gamma_j}\tau = 0$, which is absurd. This contradiction proves the claim. □

Using the previous claim we infer the existence of $\{t_1,\dots,t_{2(\sigma+\mathrm{E})}\} \subset \mathcal{H}^-\left(\overline{M(\mathcal{J})}\right)$ such that $\det(F(t_1),\dots,F(t_{2(\sigma+\mathrm{E})})) \neq 0$. Up to changing $t_i \leftrightarrow t_i/x$, $x>0$ large enough, we can assume that

$$\left|\exp\left(\sum_{i=1}^{2(\sigma+\mathrm{E})} x_i t_i(p)\right) - 1\right| < 1/(2m), \tag{7}$$

$\forall (x_1,\dots,x_{2(\sigma+\mathrm{E})}) \in \mathbb{R}^{2(\sigma+\mathrm{E})},\ |x_i|<1,\ i=1,\dots,2(\sigma+\mathrm{E}), \quad \forall p \in \overline{M(\mathcal{J})}.$

Assertion 6.5. *For each* $n \in \mathbb{N}$*, there is* $t_0^n \in H^-(\overline{M(\mathcal{J})})$ *such that:*

(i) $|t_0^n - n| < 1/n$ *in* K_1 *(and so* $|t_0^n + n| < 1/n$ *in* $I(K_1)$*),*

(ii) $|t_0^n| < 1/n$ *in* K_2.

Proof. Given $n \in \mathbb{N}$, we apply a Runge-type theorem on $\overline{M}$, see [22, Theorem 10], and obtain a holomorphic function $T_0^n : \overline{M(\mathcal{J})} \to \mathbb{C}$ satisfying

- $|T_0^n - n| < 1/n$ in K_1,

- $|T_0^n + n| < 1/n$ in $I(K_1)$,
- $|T_0^n| < 1/n$ in K_2.

We take $t_0^n = \frac{1}{2}(T_0^n - \overline{T_0^n \circ I})$. From this, it is trivial to check Properties *(i)* and *(ii)*. □

For $\Theta = (\lambda_0, \ldots, \lambda_{2(\sigma+\text{E})}) \in \mathbb{R}^{2(\sigma+\text{E})+1}$, we define

$$h^{\Theta,n}(p) \stackrel{\text{def}}{=} \exp\left[\lambda_0\, t_0^n(p) + \sum_{j=1}^{2(\sigma+\text{E})} \lambda_j\, t_j(p)\right], \quad \forall p \in \overline{M(\mathcal{J})}.$$

Label $g^{\Theta,n} = g/h^{\Theta,n}$ and $\Phi_3^{\Theta,n} = \Phi_3$. As $\left\{t^n_{0|_{K_2}}\right\}_{n\in\mathbb{N}}$ is uniformly bounded, then, up to a subsequence, we have $\left\{t^n_{0|_{K_2}}\right\} \to t_0^\infty \equiv 0$, uniformly on K_2. We also define on K_2 the Weierstrass data $g^{\Theta,\infty} = g/h^{\Theta,\infty}$, $\Phi_3^{\Theta,\infty} = \Phi_3$, where

$$h^{\Theta,\infty}(p) \stackrel{\text{def}}{=} \exp\left[\sum_{j=1}^{2(\sigma+\text{E})} \lambda_j\, t_j(p)\right], \quad \forall p \in K_2.$$

Observe that third Weierstrass differential of the aforementioned holomorphic data has no real periods. Therefore, we must only consider the period problem associated to $\Phi_j^{\Theta,n}$, $j = 1, 2$. To do this, we define the period map $\mathcal{P}_n : \mathbb{R}^{2(\sigma+\text{E})+1} \to \mathbb{R}^{2(\sigma+\text{E})}$, $n \in \mathbb{N} \cup \{\infty\}$;

$$\mathcal{P}_n(\Theta) = \left(\int_{\gamma_j} \Phi_1^{\Theta,n}, \int_{\gamma_j} \Phi_2^{\Theta,n}\right)_{j=1,\ldots,\sigma+\text{E}}.$$

Since the initial immersion X is well-defined, then one has $\mathcal{P}_n(0) = 0$, $\forall n \in \mathbb{N} \cup \{\infty\}$. Moreover, it is not hard to check that

$$\text{Jac}_{\lambda_1,\ldots,\lambda_{2(\sigma+k)}}(\mathcal{P}_n)(0) = \det(F(t_1), \ldots, F(t_{2(\sigma+\text{E})})) \neq 0, \quad \forall n \in \mathbb{N} \cup \{\infty\}.$$

Applying the Implicit Function Theorem to the map $\mathcal{P}_n$ at $0 \in [-\epsilon, \epsilon] \times \overline{B}(0, r)$, we get an smooth function $L_n : I_n \to \mathbb{R}^{2(\sigma+\text{E})}$ satisfying $\mathcal{P}_n(\lambda_0, L_n(\lambda_0)) = 0$, $\forall \lambda_0 \in I_n$, where I_n is a maximal open interval containing 0 (here, maximal means that L_n can not be regularly extended beyond I_n).

We next check that the supremum ϵ_n of the connected component of $L_n^{-1}(\overline{B}(0, r)) \cap [0, \epsilon]$ containing $\lambda_0 = 0$ belongs to I_n. Indeed, take a sequence $\{\lambda_0^k\}_{k\in\mathbb{N}} \nearrow \epsilon_n$. As $\{L_n(\lambda_0^k)\} \subset \overline{B}(0, r)$, then, up to a subsequence, $\{L_n(\lambda_0^k)\}_{k\in\mathbb{N}} \to \Lambda_n \in \overline{B}(0, r)$. Taking into account that $\text{Jac}_{\lambda_1,\ldots,\lambda_{2(\sigma+k)}}(\mathcal{P}_n)(\epsilon_n, \Lambda_n) \neq 0$, the local unicity of the curve $(\lambda_0, L_n(\lambda_0))$ around the point (ϵ_n, Λ_n), and the maximality of I_n, we infer that $\epsilon_n \in I_n$. Therefore, either $\epsilon_n = \epsilon$, or $L_n(\epsilon_n) = \Lambda_n \in \partial(B(0, r))$.

We will now see that $\epsilon_0 \stackrel{\text{def}}{=} \liminf\{\epsilon_n\} > 0$. Otherwise, there would be a subsequence $\{\epsilon_n\} \to 0$. Without loss of generality, $\epsilon_n < \epsilon$, $\forall n \in \mathbb{N}$, and so $\Lambda_n \in \partial(B(0, r))$, $\forall n \in \mathbb{N}$. Up to a subsequence, $\{\Lambda_n\} \to \Lambda_\infty \in \partial(B(0, r))$. The fact $\mathcal{P}_\infty(0, 0) = \mathcal{P}_\infty(0, \Lambda_\infty) = 0$ would contradict the injectivity of $\mathcal{P}_\infty(0, \cdot)$ in $\overline{B}(0, r)$. Hence the function $L_n : [0, \epsilon_0] \to \overline{B}(0, r)$ is well-defined, $\forall n \geq n_0$, n_0 large enough.

Label $(\lambda_1^n, \ldots, \lambda_{2(\sigma+\text{E})}^n) = L_n(\epsilon_0)$. From (7) we have $|\exp[\sum_{j=1}^{2(\sigma+\text{E})} \lambda_j^n t_j] - 1| < 1/(2m)$ on $D(\overline{\mathfrak{p}})$. Hence, if n $(\geq n_0)$ is large enough, the function:

$$H(z) \stackrel{\text{def}}{=} \exp\left[\epsilon_0\, t_0^n(z) + \sum_{j=1}^{2(\sigma+\text{E})} \lambda_j^n\, t_j(z)\right]$$

satisfies items *1*, *2* and *3* in Lemma 5. Since the period function $\mathcal{P}_n$ vanishes at $\Theta_n = (\epsilon_0, \lambda_1^n, \ldots, \lambda_{2(\sigma+\text{E})}^n)$, then the minimal immersion $\widetilde{F}$ associated to the Weierstrass data $g^{\Theta_n,n}$, $\Phi_3^{\Theta_n,n} = \Phi_3$ is well-defined. This proves item *4* in the lemma.

□

6.1.2. *The existence of a holomorphic differential without zeros.* In the paper [1], the existence of a holomorphic 1-form without zeros ω on $M(\mathcal{J}_0)$ is used over and over again, for a given multicycle $\mathcal{J}_0$. In our new setting, we need the following related result:

Lemma 6. *Given $\mathcal{J}_0$ a multicycle in M', which is invariant under I, there exists a holomorphic 1-form ω' in $M(\mathcal{J}_0)$, without zeros, and satisfying $I^*(\omega') = \overline{\omega'}$.*

Proof. Let $\pi : M' \to \widetilde{M}'$ be the projection and let $\widetilde{h}_1, \ldots, \widetilde{h}_\sigma$ be a basis of the harmonic 1-forms on $\widetilde{M}'$. Since I is an orientation reversing isometry of the orientable surface M', then I leaves invariant the harmonic 1-forms $h_i \stackrel{\text{def}}{=} \pi^*(\widetilde{h}_i)$ and $I^*(\star h_i) = -\star h_i$, where $\star$ denotes the Hodge operator. Hence, $I^*(\omega_i) = \overline{\omega}_i$, where $\omega_i \stackrel{\text{def}}{=} h_i + \mathrm{i} \star h_i$. A simple Euler characteristic calculation shows that $\omega_1, \ldots, \omega_\sigma$ is a basis for the holomorphic differentials of M'. Let $W = (\omega_1, \ldots, \omega_\sigma)$, then the Abel-Jacobi map $f : M' \to \mathbb{C}^\sigma/\Lambda$ satisfies:

$$f(I(p)) = \left[\int_{p_0}^{I(p)} W\right] = \left[\int_{p_0}^{I(p_0)} W + \int_{I(p_0)}^{I(p)} W\right] = v_0 + \left[\int_{p_0}^{p} I^*(W)\right] = v_0 + \left[\int_{p_0}^{p} \overline{W}\right] = v_0 + c \circ f(p), \tag{8}$$

where c is the map on $\mathbb{C}^\sigma/\Lambda$ induced by the complex conjugation in $\mathbb{C}^\sigma$ and $p_0 \in M'$ is a base point.

Let $U \subseteq M'$ be an open region and let $\mathrm{Div}(U)$ denote the set of divisors in M' whose support is contained in U. Then the map f can be extend linearly to $\mathrm{Div}(U)$ as follows:

$$f\left(\sum_{j=1}^{k} n_j \cdot p_j\right) = \sum_{j=1}^{k} n_j \cdot f(p_j).$$

Assertion 6.6. *Let* $\mathrm{Div}_{\sigma-1}(U)$ *denote the subset of divisors in* $\mathrm{Div}(U)$ *of degree* $\sigma - 1$. *Then* $f\colon \mathrm{Div}_{\sigma-1}(U) \to \mathbb{C}^\sigma/\Lambda$ *is onto.*

Let n in $\mathbb{N}$ and consider S_n the group of permutations of $(1, \ldots, n)$. S_n acts on the cartesian product $(M')^n$; the quotient $S^n(M')$ is called the n^{th} *symmetric power of* M'. $S^n(M')$ is a complex manifold of dimension n whose points can be identified with divisors of the form $D = \sum_{j=1}^{n} P_j$. It is well-known [21, Chap. 15] that the set of $D \in S^\sigma$, such that the rank at D

of the differential of $f\colon S^\sigma(M')\to\mathbb{C}^\sigma/\Lambda$ is maximal, $=\sigma$, is open and dense in $S^\sigma(M')$. In particular, $f(S^\sigma(U))$ contains an open subset of $\mathbb{C}^\sigma/\Lambda$. So, if we consider

$$f : S^{n\,\sigma-1}(U)\times S^{(n-1)\sigma}(U)\to\mathbb{C}^\sigma/\Lambda$$
$$f(D,E)=f(D)-f(E),$$

then the image $f\left(S^{n\,\sigma-1}(U)\times S^{(n-1)\sigma}(U)\right)\subseteq f(\mathrm{Div}_{\sigma-1}(U))$ contains an open subset whose diameter diverges, in terms of n. This completes the proof of Assertion 6.6.

Consider ω a nonzero holomorphic 1-form satisfying $I^*\omega=\overline{\omega}$, then the divisor of ω has this form $(\omega)=\sum_{j=1}^{\sigma-1}p_j+\sum_{j=1}^{\sigma-1}I(p_j)$. If we label $\mathcal{K}=\sum_{j=1}^{\sigma-1}f(p_j)$, then (8) implies that $f((\omega))=2\Re(\mathcal{K})+(\sigma-1)\,v_0$, where $\Re$ is the map induced by the real projection **Re**: $\mathbb{C}^\sigma\to\mathbb{R}^\sigma$. If we consider one of the disks $\mathbb{D}_i$ in the complement of $\overline{M(\mathcal{J}_0)}$, then Assertion 6.6 gives the existence of $D\in\mathrm{Div}(\mathbb{D}_i)$ so that $\deg(D)=\sigma-1$ and $f(D)=\mathcal{K}$. So, one has that $\deg(D+I(D))=2\sigma-2$ and

$$f(D+I(D))=\mathcal{K}+c(\mathcal{K})+\deg(D)\,v_0=2\Re(\mathcal{K})+(\sigma-1)\,v_0=f((\omega)).$$

Abel's theorem gives the existence of a meromorphic function h on M' such that $(h)=(w)-D-I(D)$. In other words, the meromorphic 1-form $\tau\stackrel{\text{def}}{=}\omega/h$ satisfies:

$$(\tau)=\left(\overline{I^*(\tau)}\right)=D+I(D).$$

Therefore, $\tau=a\,\overline{I^*(\tau)}$, for some complex constant $a\in\mathbb{C}^*$. Since I is an involution, then we deduce that $|a|=1$.

If $a=-1$, then $\omega'\stackrel{\text{def}}{=}\mathrm{i}\tau$ is the 1-form that we are looking for. If not, we define $\omega'\stackrel{\text{def}}{=}\frac{1+\overline{a}}{2}\tau$ and it satisfies the assertions of this lemma. □

6.1.3. *López-Ros parameters adapted to nonorientable minimal surfaces.* In order to obtain that the examples constructed in [1] were proper, we used special types of functions with simple poles at some points near the boundary of the surface and which were approximated by 1 in almost the entire surface. To do the same thing in the nonorientable case, we need to modify the proof of Lemma 2 in [1] according to the following explanation.

The holomorphic function $\zeta_{i,k}\,:\,M(\mathcal{J}_0)-\{p_i^k\}\longrightarrow\mathbb{C}$ having a simple pole at p_i^k [1, subsection 4.1.1, p. 14] must be replaced by a holomorphic function on $M(\mathcal{J}_0)-\{p_i^k\}$ having a simple pole at p_i^k and a zero (not necessarily simple) at $I(p_i^k)$. The existence of such a function is guaranteed by Noether's gap theorem (see [5].)

Now, for $\Theta=\left(\lambda_0,\lambda_1,\ldots,\lambda_{2(\sigma+\mathrm{E})}\right)\in\mathbb{R}^{2(\sigma+\mathrm{E})+1}$, we consider the function h^Θ (compare with [1, subsection 4.1.1, equation (3.10)]):

$$h^\Theta=\frac{\lambda_0\,\theta_i^k\,\zeta_{i,j}+\exp\left(\sum_{j=1}^{2(\sigma+\mathrm{E})}\lambda_j\,\varphi_j\right)}{\overline{\lambda_0\,\theta_i^k\,(\zeta_{i,j}\circ I)+\exp\left(-\sum_{j=1}^{2(\sigma+\mathrm{E})}\lambda_j\,\varphi_j\right)}}. \tag{9}$$

Then, the function h^Θ in subsection 4.1.1 of [1] must be replaced by this new one and then all the arguments work in the same way. The reason for changing h^Θ is because we need that

$$h^\Theta\circ I=\frac{1}{\overline{h^\Theta}},$$

in order to use this function as a López-Ros parameter for nonorientable minimal surfaces.

This concludes our discussion on how to adapt the proof of Theorem 4 to the nonorientable case, which completes the proof of Theorem 1.

6.2. **A nonexistence theorem for nonorientable minimal surfaces properly immersed in smooth bounded domains.** In this section, we describe a topological obstruction to the existence of certain proper immersions of open nonorientable surfaces into a given smooth bounded domain. For this description we need the following definition.

Definition 4. *Let $\mathcal{D}$ be a smooth bounded domain. We say that a proper immersion $f : M \to \mathcal{D}$ of an open surface M is properly isotopic to a properly embedded surface in $\mathcal{D}$ if there exists a proper continuous map $F : M \times [0,1] \to \mathcal{D}$ such that for each $t \in [0,1]$, $F_t = F|_{M\times\{t\}}$ is a proper immersion into $\mathcal{D}$, F_0 corresponds to f and F_1 is a proper embedding.*

Theorem 6. *Suppose $\mathcal{D}$ is a smooth bounded domain in $\mathbb{R}^3$ with boundary being a possibly disconnected surface of genus g and M is a properly immersed surface in $\mathcal{D}$. If M is properly isotopic to a properly embedded surface in $\mathcal{D}$, then M has at most g nonorientable ends*[4].

Proof. Since M is properly isotopic to an embedded surface M' in $\mathcal{D}$, then M is homeomorphic to M'. In particular, the number of nonorientable ends of M' and M is the same. Hence, it suffices to prove the theorem in the special case that M is properly embedded, a property that we now assume holds.

Arguing by contradiction, suppose M has at least $g+1$ nonorientable ends $e_1, e_2, \dots e_{g+1}$. Since $\mathcal{D}$ is smooth, then for some small $\varepsilon > 0$, $\overline{\mathcal{D}(\varepsilon)} = \{x \in \overline{\mathcal{D}} \mid \operatorname{dist}_{\mathbb{R}^3}(x, \partial\mathcal{D}) \leq \varepsilon\}$ is a smooth domain which is diffeomorphic to $\partial\mathcal{D}\times[0,1]$, where $\partial\mathcal{D}$ is a smooth compact surface of genus g. For some ε sufficiently small, $\overline{\mathcal{D}(\varepsilon)} \cap M$ contains a collection $\{E_1, E_2, \dots, E_{g+1}\}$ of pairwise disjoint, proper subdomains of M with compact boundary and such that E_i represents the end e_i for $i \in \{1, 2, \dots, g+1\}$. In this case, after reindexing, we may assume that there is a component ∂ of $\partial\mathcal{D}$ of genus k such that the limit sets $L(E_1), \dots, L(E_{k+1})$ are contained in ∂.

For some small positive δ with $\delta < \varepsilon$, the surfaces $\partial_\varepsilon, \partial_\delta$ in $\mathcal{D}$ parallel to ∂ of distance ε, δ, respectively, are embedded and the closed region $R(\varepsilon,\delta) \subset \mathcal{D}$ bounded by $\partial_\varepsilon \cup \partial_\delta$ is topologically $\partial \times [0,1]$. Since each E_i is nonorientable, for δ sufficiently small, $R(\varepsilon,\delta) \cap E_i$ contains a connected, smooth, compact nonorientable domain F_j with $\partial F_j \subset \partial R(\varepsilon,\delta) = \partial_\varepsilon \cup \partial_\delta$ for $j \in \{1, 2, \dots, k+1\}$.

Since for each $j \in \{1, 2, \dots, k+1\}$, F_j is nonorientable and $R(\varepsilon,\delta)$ is orientable, there is a simple closed curve $\gamma_j \subset F_j$ such that $\overline{F}_j \cap \overline{\gamma}_j = 1 \in H_0(R(\varepsilon,\delta), \mathbb{Z}_2)$, where $\overline{\gamma}_j \cap \overline{F}_j$ is the homological intersection number mod 2 of γ_j and F_j relative to $\partial R(\varepsilon,\delta)$. Since the domains $F_1, \dots, F_{k+1}$ are pairwise disjoint, we conclude that $\overline{F}_i \cap \overline{\gamma}_j = \delta_{i,j}$ for $i, j \in \{1, 2, \dots, k+1\}$.

Now let α_j be a closed curve in ∂_ε which is homologous in $R(\varepsilon,\delta)$ to γ_j. Since $\overline{F}_i \cap \overline{\gamma}_j = \delta_{i,j}$, then $\overline{\partial F_i \cap \partial_\varepsilon} \cap \overline{\alpha}_j = \delta_{i,j}$, where we consider $\partial F_i \cap \partial_\varepsilon$ to represent an element in $H_1(\partial_\varepsilon, \mathbb{Z}_2\}$. In particular, the collection of **pairwise disjoint** simple closed curves that make up $\bigcup_{i=1}^{k+1} \partial F_i \cap \partial_\varepsilon$ represent at least $k+1$ independent homology classes in $\mathbb{H}_1(\partial_\varepsilon, \mathbb{Z}_2)$, which

[4]An end of a surface M is said to be *nonorientable* if every proper subdomain with compact boundary which represents the end is nonorientable.

is impossible since ∂_ε is a compact orientable surface of genus k. This contradiction completes the proof of the theorem. □

The following corollary is an immediate consequence of Theorem 6.

Corollary 2. *If M is an open surface with an infinite number of nonorientable ends, then there does not exist a proper immersion of M into any smooth bounded domain, such that the immersion is properly isotopic to a properly embedded surface in the domain.*

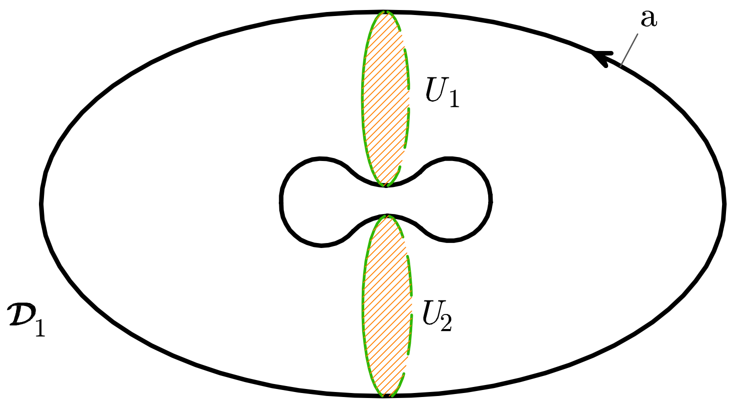

FIGURE 6. The domain $\mathcal{D}_1$, the curve $\mathbf{a}$ and the disks U_1 and U_2

6.3. **The description of the universal domains of Conjecture 1.** The main goal of this section is to describe bounded domains of $\mathbb{R}^3$ which are candidates for solving parts (2) and (3) of the embedded Calabi-Yau conjecture. From the previous theorem, we know that some restrictions are necessary in order to properly embed a nonorientable surface in a smooth bounded domain. That condition is that the number n of nonorientable ends can not be greater than the genus of the boundary of the domain. We will actually construct a sequence of domains $\{\mathcal{D}_n\}_{n\in\mathbb{N}}$ which are solid n-holed donuts and which contain certain properly embedded nonorientable minimal surfaces. We conjecture that:

(1) If M is a nonorientable open surface with no nonorientable ends, then it can be properly minimally embedded in $\mathcal{D}_1$ with a complete metric.
(2) If $n \geq 1$ and M has n nonorientable ends, then it can be properly and minimally embedded in $\mathcal{D}_n$ with a complete metric.

Example 1. *Consider a smooth compact solid torus $\overline{\mathcal{D}_1}$ satisfying the following properties (see Figure 6):*

(1) $\overline{\mathcal{D}_1}$ is invariant under reflections in the coordinate planes P_{xy}, P_{xz} and P_{yz}.
(2) The intersection of P_{yz} with $\overline{\mathcal{D}_1}$ consists of two compact convex disks, U_1 and U_2.
(3) The intersection of P_{xy} with $\partial\mathcal{D}_1$ consists of two curves, and the exterior one $\mathbf{a}$ is convex.
(4) There exists an open neighborhood N of $\partial U_1 \cup \mathbf{a} \cup \partial U_2$ in $\partial\mathcal{D}_1$ with $\kappa_1(N) > 1$.

Example 2. *For $n > 1$, consider now a smooth compact solid n-holed torus $\overline{\mathcal{D}_n}$ satisfying the following properties (see Figure 8 for the case of $\overline{\mathcal{D}_3}$):*

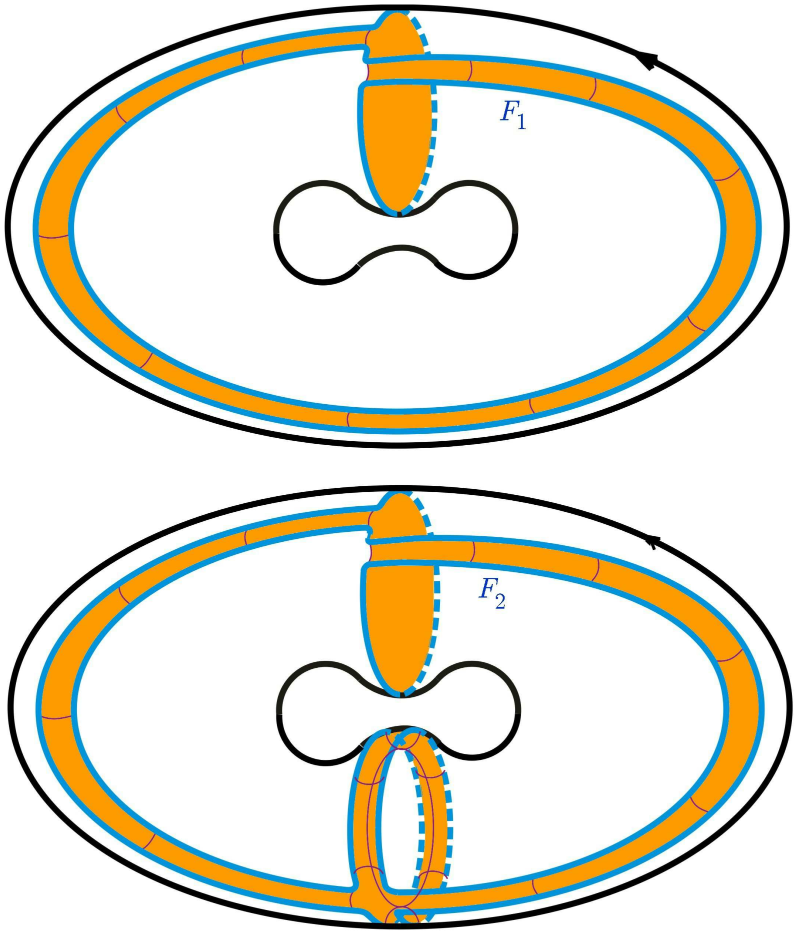

FIGURE 7. The minimal surface F_1 has the topology of a Möbius strip and F_2 is topologically a minimal Klein bottle minus a disk.

(1) $\overline{\mathcal{D}_n}$ is invariant under reflections in the coordinate planes P_{xy}, P_{xz} and P_{yz}.
(2) For each integer k in $[-n+1, n-1]$, one of the components in the intersection of the plane $P_k = \{x = k\}$ and $\overline{\mathcal{D}_n}$ is a compact convex disk, U_k with positive y-coordinate.
(3) The intersection of P_{xy} with $\partial\mathcal{D}_n$ consists of $n+1$ curves, and the exterior one $\mathbf{a}$ is convex.
(4) There exists an open neighborhood N of $\mathbf{a} \cup \left(\bigcup_{k=-n+1}^{n-1} \partial U_k\right)$ in $\partial\mathcal{D}_n$ with $\kappa_1(N) \geq \varepsilon_n > 0$.

Finally, we described the domain $\mathcal{D}_\infty$.

Example 3. *We consider an infinitely many holed solid donut $\mathcal{D}_\infty$ with a single nonsmooth point p_∞ on its boundary which is accumulation point of the holes of $\mathcal{D}_\infty$. This domain satisfies the following properties (see Figure 10):*

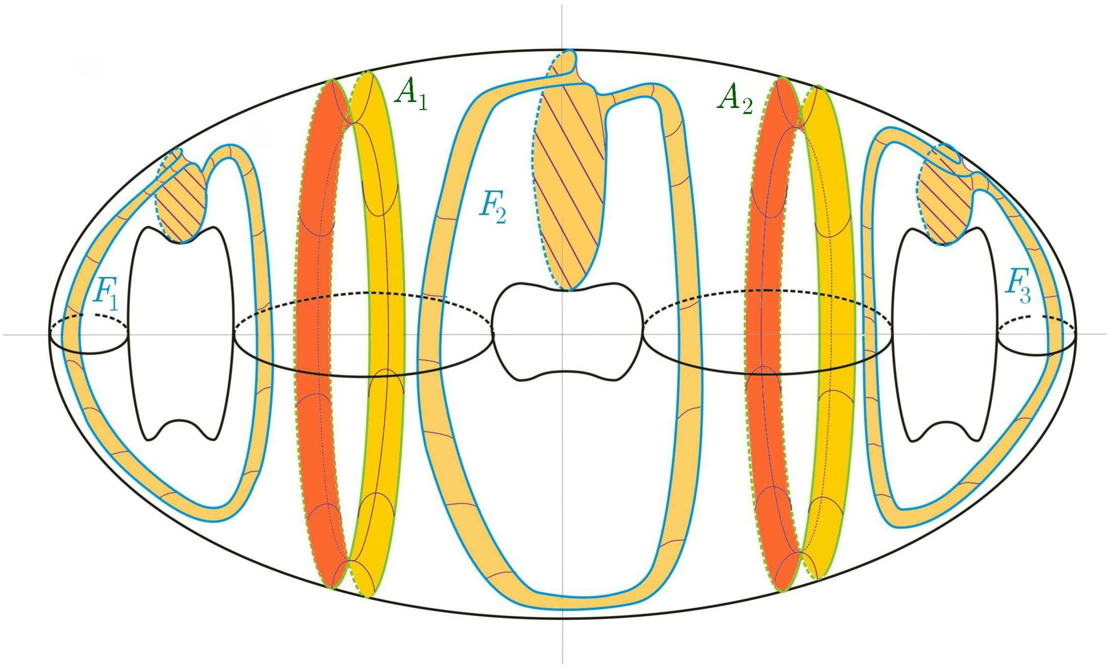

FIGURE 8. The domain $\mathcal{D}_3$

(1) *The domain $\mathcal{D}_\infty$ is contained in the slab $\{0 \leq x \leq 1\}$ and $p_\infty = (1, 0, 0)$.*
(2) *$\overline{\mathcal{D}_\infty}$ is invariant under reflections in the coordinate planes P_{xy} and P_{xz}.*
(3) *There exists positive real numbers r_n, s_n, $n \in \mathbb{N}$, such that:*
  (a) *$r_1 < s_1 < r_2 < s_2 < r_3 < \cdots < r_n < s_n < r_{n+1} < \cdots$ and $\lim_n r_n = 1$,*
  (b) *the planes $\{x = r_n\}$ intersect $\mathcal{D}_\infty$ in two convex disks, one of them contained in the half space $\{y > 0\}$ that we call $U(r_n)$,*
  (c) *the planes $\{x = s_n\}$ intersect $\mathcal{D}_\infty$ in one convex disk, which we call $V(s_n)$,*
(4) *The intersection of P_{xy} with $\partial\mathcal{D}_\infty$ contains a unique exterior curve $\mathbf{a}$ which is convex and smooth.*
(5) *There exists an open neighborhood N of $\mathbf{a} \cup \left(\bigcup_{k=1}^{\infty} \partial U(r_k) \cup \partial V(s_k) \right)$ in $\partial\mathcal{D}_\infty - \{p_\infty\}$ with $\kappa_1(N) \geq \varepsilon_\infty > 0$, for some positive ε_∞.*

Using the bridge principle, the classification of noncompact surfaces and a suitable choice of a compact exhaustion, we next prove the following proposition.

Proposition 2. *For every $n \in \mathbb{N}$, the smooth domain $\mathcal{D}_n$ satisfies:*

(1) *For any nonorientable open surface M with no nonorientable ends, there exists a proper, stable, minimal noncomplete embedding $f\colon M \to \mathcal{D}_1$.*
(2) *For any open surface M with n nonorientable ends, there exists a proper, stable, minimal noncomplete embedding $f\colon M \to \mathcal{D}_n$.*

Furthermore, the embedding f satisfies that the limit sets of distinct ends of $f(M)$ are disjoint.

Proof. We are going to divide the proof into the case where M has orientable ends and the case where M has n nonorientable ends.

Case 1. M is nonorientable and it has orientable ends. By the classification of compact nonorientable surfaces, there exists a compact exhaustion of M, $\mathcal{M} = \{M_k \mid k \in \mathbb{N}\}$, such that:

- M_1 is either a Möbius strip or a Klein bottle with a disk removed, and $M - M_1$ is orientable.
- Consider the surface M' formed by attaching a disk D along the boundary of $M - M_1$ and the associated exhaustion $\mathcal{M}' = \{M'_1 = D, M'_k = M_k \mid k \geq 2\}$. Then the new exhaustion $\mathcal{M}'$ is a simple exhaustion of M'.

Recall from the description in Example 1 that N is an open neighborhood of $\partial U_1 \cup \mathbf{a} \cup \partial U_2$. Consider a simple arc Γ in N with distinct end points on ∂U_1 and which is almost parallel to $\mathbf{a}$. Let F_1 be the compact embedded minimal Möbius strip obtained by adding a thin bridge to U_1 along Γ as described in Figure 7. Notice that we can guarantee that $\partial F_1 \subset N$ by choosing the bridge thin enough. Let F_2 be the embedded compact Klein bottle minus a disk obtained by adding a thin bridge along ∂U_2 to the surface F_1 in such a way that $\partial F_2 \subset N$ as in Figure 7.

We now describe how to construct the desired proper minimal immersion. If M_1 is a Möbius strip, then we choose Σ_1 to be F_1. Since $M - M_1$ is an orientable surface with a "simple exhaustion" and $\kappa_1(N) > 1$, then we can follow the proof of Case 2 in Theorem 4 in order to construct a proper minimal embedding $f : M \to \mathcal{D}_1$ such that the limit set of different ends of M are disjoint. Of course, this construction is now much easier since we do not have to deal with the density theorem; one just uses the bridge principle to construct compact embedded minimal surfaces $\Sigma_n \subset \mathcal{D}_n$. If M_1 is a Klein bottle with a disk removed, then we take $\Sigma_1 = F_2$ and repeat the same argument to construct the desired immersion.

Case 2. M is nonorientable and it has n nonorientable ends.

Using again the classification of compact nonorientable surfaces and arguments similar to those in the proof of Lemma 4, there exists a compact exhaustion of M, $\mathcal{M} = \{M_k \mid k \in \mathbb{N}\}$, such that:

- M_1 is the compact nonorientable surface with n boundary components and Euler characteristic $\chi(M_1) = -2n + 1$.
- Every boundary curve of each M_k separates M into two components.
- For each $k \in \mathbb{N}$, $M_{k+1} - \mathrm{Int}(M_k)$ contains exactly one nonannular component Δ_{k+1} which is either a Möbius strip minus a disk, a pair of pants, or an annulus with a handle.
- If Δ_{k+1} is an annulus with a handle, then the component of $M - \mathrm{Int}(M_k)$ which contains Δ_{k+1} is orientable.
- If Δ_{k+1} is a pair of pants, then at most one of the two components of $M - \mathrm{Int}(M_{k+1})$ which intersects $\partial\Delta_{k+1}$ is nonorientable.

For the following construction of the domain $\mathcal{D}_3$, see Figure 8. The planes P_{-n+2}, P_{-n+4}, ..., P_{n-2}, separate $\partial\mathcal{D}_n$ into n open regions that we call $R_1, R_2, \ldots, R_n$ and which are ordered by their relative x-coordinates. Let $A_1, A_2, \ldots, A_{n-1}$ be compact stable minimal annuli in $\mathcal{D}_n$ with $\partial A_i \subset \partial\mathcal{D}_n$, ordered by their relative x-coordinates, with boundaries close and parallel to the boundaries of the regions $R_1, R_2, \ldots, R_n$, respectively. Let $F_1, F_2, \ldots, F_n$ be the compact stable minimal Möbius strips with $F_i \subset R_i$, $i = 1, \ldots, n$, constructed by attaching bridges to the disks U_{-n+1}, U_{-n+3}, ..., U_{n-1} in a manner similar to the construction of F_1 in Case

1. Furthermore, we can assume that the boundary curves of these annuli and Möbius strips are contained in the neighborhood N. We obtain our surface Σ_1 by connecting the annuli and Möbius strips by thin minimal bridges in N close to the intersection of P_{xz} and $\partial\mathcal{D}_n$ and where $z > 0$. Finally, we can also assume that $\partial\Sigma_1 \subset N$ and Σ_1 has n boundary curves $b_i \subset R_i$, $i = 1, \ldots, n$, where we fix an orientation of each boundary curve.

We now describe how to finish the construction of the desired proper minimal immersion. In Case 1, the changes in the topology of Σ_m, $m \in \mathbb{N}$, occur near one (prescribed) point in the boundary of Σ_1. In our case, we prescribe n points, $p_i \in b_i$, $i = 1, \ldots, n$, where p_i lies on the boundary of the bridge used to make F_i. The process of adding a pair of pants or an annulus with a handle to Σ_m is the same as in the orientable case; one attaches a very thin bridge B near a point of the boundary of Σ_n or one attaches B and then a second bridge B' in the center of B in order to attach an annulus with a handle (see Figures 4.)

The process to add a Möbius strip to Σ_m is by attaching a very thin bridge B along a short oriented simple arc in $N - \partial\Sigma_m$ with end points on an oriented component $\gamma \subset \partial\Sigma_m$ and which has the same intersection numbers with γ at each of its end points. For example, suppose that Δ_2 is a Möbius strip attached to ∂M_1 along a boundary component corresponding to $b_i \subset \Sigma_1$. In this case, we choose a short arc Γ connecting p_i to its opposite point $\widehat{p_i}$ in the corresponding bridge used to produce the Möbius strip F_i (see Figure 9). Note that the intersection number of Γ with b_i at p_i is opposite to the intersection number at $\widehat{p_i}$. In this case we add a bridge B_1 to Σ_1 along Γ like in Figure 9 to make Σ_2. Since the component of $M - M_1$ containing Δ_2 has exactly one nonorientable end, then there exists a smallest $k > 2$ such that Δ_k is a Möbius strip minus a disk contained in this component. So, in the construction of Σ_k, we will again attach a bridge, this time inside B_1 (see Figure 9.) Combining all the arguments described in the last two paragraphs, we obtain a limit surface Σ contained in $\mathcal{D}_n$ and satisfying all of the statements of the proposition except stability. By choosing the bridges in the construction of Σ sufficiently thin, then Σ is also stable. □

Proposition 3. *Every open surface M admits a proper stable minimal embedding in $\mathcal{D}_\infty$.*

Proof. Without loss of generality we can assume that M is not simply-connected, since $U(r_1)$ is simply-connected and properly embedded in $\mathcal{D}_\infty$.

Let $X_1 = \mathcal{D}_\infty \cap \{x < s_1\}$ and for $n > 1$, define $X_n = \mathcal{D}_\infty \cap \{s_{3n} < x < s_{3n+1}\}$. For each $n \in N$, define $Y_n = \mathcal{D}_\infty \cap \{s_{3n-2} < x < s_{3n-1}\}$, and $Z_n = \mathcal{D}_\infty \cap \{s_{3n-1} < x < s_{3n}\}$. In each region X_n we construct a compact stable embedded minimal Möbius strip F_n by attaching a thin bridge to the disk $U(r_{3n-2})$ like in Proposition 2. Similarly, in each Y_n let A_n be a compact stable embedded minimal annulus close to the boundary of $U(r_{3n-1})$. Finally, in each region Z_n we construct a stable compact embedded minimal disk with a handle H_n by attaching a bridge to a stable compact minimal annulus near the boundary of $U(r_{3n})$. Note that the collection $\{X_n, Y_n, Z_n\}_{n\in\mathbb{N}}$ is a pairwise disjoint family of compact domains whose union is a properly embedded surface with boundary in $\mathcal{D}_\infty - \{p_\infty\}$, where $p_\infty = (1, 0, 0)$. We can assume that the curve $\mathbf{a}$ intersects all of these compact stable surfaces, F_n, A_n, H_n, $n \in \mathbb{N}$ and $\bigcup_{n=1}^{\infty}(\partial F_n \cup \partial A_n \cup \partial H_n) \subset N$ (see Figure 10).

The case where M has finite topology is easily obtained by connecting a finite number of the components F_n, A_n and H_n by bridges. Hence, from now on we assume that M has infinite topology.

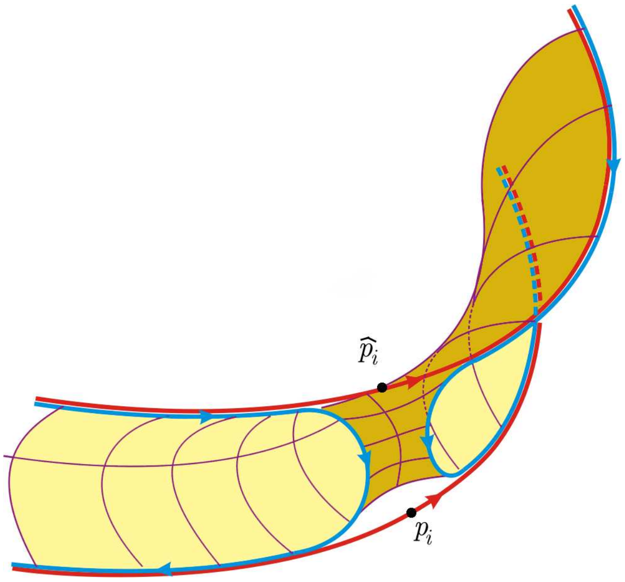

FIGURE 9. We choose a short arc Γ connecting p_i to its opposite point $\widehat{p_i}$ in the corresponding bridge used to produce the Möbius strip F_i. Note that the intersection number of Γ with b_i at p_i is opposite to the intersection number at $\widehat{p_i}$. In this case we add a bridge B_1 to Σ_1 along Γ.

Following similar ideas to those in the proof of Case 2 in the previous proposition, we can choose a compact exhaustion of M such that:

- M_1 is a Möbius strip, an annulus or a disk with a handle.
- Every boundary curve of each M_k separates M into two components, one of them containing M_1.
- For each $k \in \mathbb{N}$, $M_{k+1} - \mathrm{Int}(M_k)$ contains exactly one nonannular component Δ_{k+1} which is either a Möbius strip minus a disk, a pair of pants, or an annulus with a handle.

Once again we construct the surface inductively. To do this we only need to explain how to apply the bridge principle to add a pair of pants, an annulus with a handle or a Möbius strip to a given Σ_m. To guarantee the stability of Σ_m, we choose bridges sufficiently narrow.

Let Σ_1 be either F_1, A_1 or H_1 depending on the topology of M_1. Then we connect Σ_1 to the compact minimal surface W_2 in $\{F_2, A_2, H_2\}$, which is homeomorphic to $\Delta_2 \subset M_2 - \mathrm{Int}(M_1)$ with a disk added to its boundary, by a thin bridge contained in N to make the compact embedded minimal surface Σ_2. We can do this connection along an arc that travels from a point in $\partial\Sigma_1 \cap \mathbf{a}$ to a point in $\partial W_2 \cap \mathbf{a}$.

The surface Σ_m is obtained from Σ_{m-1} by first finding a connection curve $\gamma(m)$ joining a component ∂_{m-1} of $\partial\Sigma_{m-1}$ to the boundary of one of the surfaces $W_m \in \{F_m, A_m, H_m\}$, where W_m depends on the topology of Δ_m. For the construction to work well, it is helpful that $\gamma(m)$ be chosen to be contained in a particular domain $C^m_\infty \subset N$ which is defined inductively as follows. For $\gamma(k)$, $1 \leq k \leq m-1$, there exists a small regular neighborhood

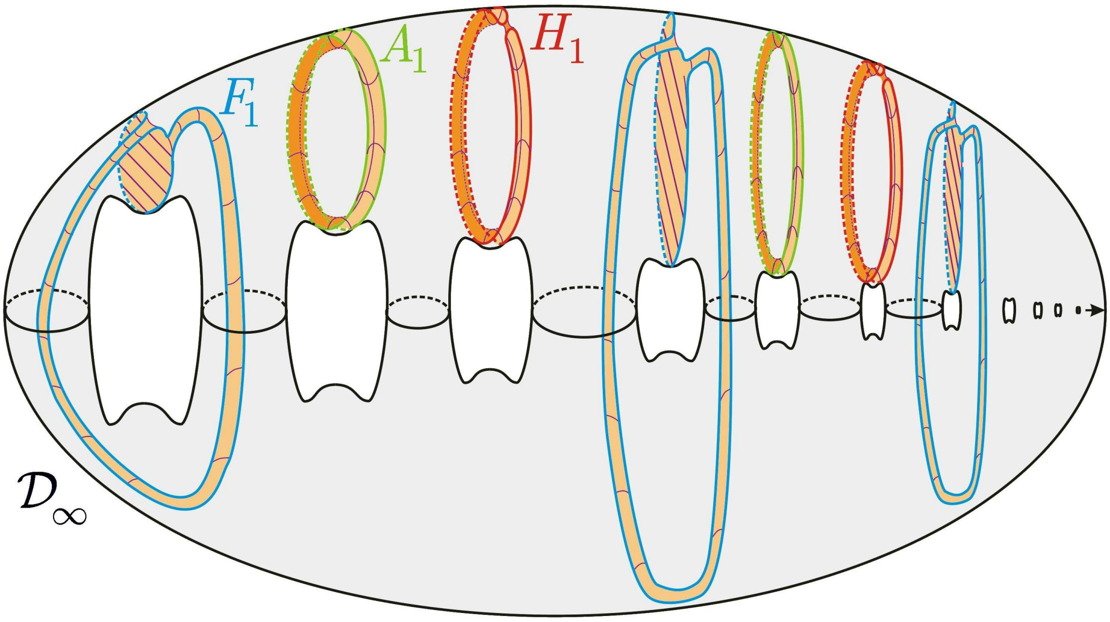

FIGURE 10. The domain $\mathcal{D}_\infty$

strip $N(\gamma(k)) \subset C^k_\infty \subset N - \left[\partial\Sigma_{k-1} \cup \left(\cup_{i=1}^{k-1} N(\gamma(i))\right)\right]$, which is a positive distance from $\partial N \cup \left(\cup_{i=1}^{k-1} N(\gamma(i))\right)$ and so that $N(\gamma(k))$ contains the normal projection to $\partial\mathcal{D}_\infty$ of the bridge along $\gamma(k)$. Then C^m_∞ is the connected component of $N - [\partial\Sigma_m \cup (\cup_{i=1}^m N(\gamma(i)))]$ which contains p_∞ in its closure. Furthermore, each $\gamma(k)$ can be chosen so that it intersects each $V(s_i)$ transversely in at most one point. In particular, we may assume that $N(\gamma(k)) \cap x^{-1}([s_i, s_{i+1}]) \subset \partial\mathcal{D}_\infty$ is either empty, a thin strip which intersects each of the boundary components of $x^{-1}([s_i, s_{i+1}]) \cap \partial\mathcal{D}_\infty$ in a compact arc or a thin strip which intersects only one of the boundary curves of $x^{-1}([s_i, s_{i+1}]) \cap \partial\mathcal{D}_\infty$ and this intersection is a connected arc; the last case occurs when $\gamma(k)$ intersects the boundary of $x^{-1}([s_i, s_{i+1}]) \cap \partial\mathcal{D}_\infty$ in a single point, which happens exactly twice. Let $i(0,k) < i(1,k)$ be the natural numbers so that $\gamma(k)$ intersects $\partial V(s_{i(0,k)})$ and $\partial V(s_{i(1,k)})$ in exactly one point, respectively.

Given a $n \in \mathbb{N}$, assume that Σ_n has been constructed and we will construct Σ_{n+1} satisfying all of the properties mentioned in the previous paragraph. Let $\partial_n \subset \partial\Sigma_n$ be the component of $\partial\Sigma_n$ which corresponds to $\partial\Delta_{n+1} \cap \partial M_n$ and let p_n be a point of ∂_n with largest x-coordinate. Observe that $x(p_n) \in [s_{i(0,n+1)-1}, s_{i(0,n+1)}]$. We next describe in detail how to construct $\gamma(n+1)$.

Case A: $V(s_{i(0,n+1)}) \cap \partial\Sigma_n = \emptyset$. In this case $\gamma(n+1)$ can be constructed from a small perturbation of the union of an arc β_0 joining p_n to $V(s_{i(0,n+1)})$, where β_0 is contained in $C^\infty_n \cap x^{-1}([s_{i(0,n+1)-1}, s_{i(0,n+1)}])$, and an arc $\beta_1 \subset \left(V(s_{i(0,n+1)}) \cup \mathbf{a}\right)$ with one end point in ∂W_{n+1}.

Case B: $V(s_{i(0,n+1)}) \cap \partial\Sigma_n \neq \emptyset$. First consider an arc $\beta_0 \subset C_n^\infty \cap x^{-1}([s_{i(0,n+1)-1}, s_{i(0,n+1)}])$ joining p_n to a point q_1 of $V(s_{i(0,n+1)}) \cap \partial N(\gamma(j_1))$, for some $j_1 < n$. Then, we consider α_1 the connected component of $\partial N(\gamma(j_1))$ containing q_1 and which is contained in $x^{-1}\left([s_{i(0,n+1)}, s_{i(1,j_1)}]\right)$. If $V(s_{i(1,j_1)+1}) \cap \partial\Sigma_n = \emptyset$, then there is an arc $\sigma_1 \subset C_\infty^n \cap x^{-1}([s_{i(1,j_1)}, s_{i(1,j_1)+1}])$ connecting the end point of α_1 to a point in $\partial V(s_{i(1,j_1)+1}) \subset C_\infty^n$. As in Case A we can choose an arc $\beta_1 \subset \left(V(s_{i(1,j_1)+1}) \cup \mathbf{a}\right)$ with one end point in ∂W_{n+1} so that $\gamma(n+1)$ is a small perturbation of $\beta_0 \cup \alpha_1 \cup \sigma_1 \cup \beta_0$.

If $V(s_{i(1,j_1)+1}) \cap \partial\Sigma_n \neq \emptyset$, then we consider the arc σ_1 in $\partial V(s_{i(1,j_1)}) - N(\gamma(j_1))$ connecting the end point of α_1 to a point q_2 in $\partial N(\gamma(j_2)) \cap V(s_{i(1,j_1)}$ for some $j_2 < n$. In this situation, let α_2 be the connected arc of $\partial N(\gamma(j_2)) \cap x^{-1}([s_{i(1,j_1)}, s_{(1,j_2)}])$ starting at q_2. Repeating this process a finite number of times we arrive to a curve $\gamma(j_k)$ so that $V(s_{i(1,j_k)+1}) \cap \partial\Sigma_n = \emptyset$. Then we proceed like in the previous paragraph. We consider an arc $\sigma_k \subset C_\infty^n \cap x^{-1}([s_{i(1,j_k)}, s_{i(1,j_k)+1}])$ connecting the end point of the corresponding arc α_k to a point in $\partial V(s_{i(1,j_k)+1}) \subset C_\infty^n$. Finally, we can choose an arc $\beta_1 \subset \left(V(s_{i(1,j_k)+1}) \cup \mathbf{a}\right)$ with one end point in ∂W_{n+1} so that $\gamma(n+1)$ is a small perturbation of $\beta_0 \cup \alpha_1 \cup \sigma_1 \cup \alpha_2 \cup \sigma_2 \cup \cdots \cup \alpha_k \cup \sigma_k \cup \beta_1$.

It is important to notice that the compact embedded minimal surfaces Σ_n, $n \in \mathbb{N}$, satisfy that for any $r \in (0,1)$ the boundary of Σ_n intersects $\{x \leq r\}$ in the same set of arcs and closed curves, for n sufficiently large. So, there is a bound on the area of $\Sigma_n \cap \{x \leq r\}$, independent of n. Since the surfaces Σ_n are embedded and stable, then a subsequence of them converges on compact subsets of $\overline{\mathcal{D}_\infty} - \{p_\infty\}$ to a limit minimal surface Σ with boundary and which is properly embedded in $\overline{\mathcal{D}_\infty} - \{p_\infty\}$ and so that $\Sigma \cap \mathcal{D}_\infty$ has the topology of M. By boundary regularity, the limit surface Σ is smooth. Moreover, if we choose our connecting bridges sufficiently thin, then we can guarantee that the limit surface is unique.

□

Remark 1. *If we combine the arguments in the previous proof with the density theorem (including the nonorientable version) one can show that every open surface M admits a complete proper minimal immersion in $\mathcal{D}_\infty$ which is properly isotopic to the minimal embedding given in Proposition 3. Similarly, Proposition 2 can be adapted to produce complete proper minimal immersions of a given nonorientable open surface M with $n \in \mathbb{N}$ nonorientable ends into $\mathcal{D}_n$ in such a way that the immersion is properly isotopic to the minimal embedding provided by the proposition and such that the limit sets of distinct ends are disjoint (if M has orientable ends, then the immersion lies in $\mathcal{D}_1$). Taking Theorem 6 into account, this last result is sharp.*

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