Calabi-Yau domains in three manifolds

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Abstract

We prove that for every smooth compact Riemannian three-manifold $\overline{W}$ with nonempty boundary, there exists a smooth properly embedded one-manifold $\Delta \subset W = \text{Int}(\overline{W})$, each of whose components is a simple closed curve and such that the domain $\mathcal{D} = W - \Delta$ does not admit any properly immersed open surfaces with at least one annular end, bounded mean curvature, compact boundary (possibly empty) and a complete induced Riemannian metric.

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1 Introduction.

A natural question in the global theory of minimal surfaces, first raised by Calabi in 1965 [2] and later revisited by Yau [11, 12], asks whether or not there exists a complete immersed minimal surface in a bounded domain $\mathcal{D}$ in $\mathbb{R}^3$. As is customary, we will refer to this problem as the Calabi-Yau problem for minimal surfaces. In 1996, Nadirashvili [10] provided the first example of a complete, bounded, immersed minimal surface in $\mathbb{R}^3$. However, Nadirashvili's techniques did not provide properness of such a complete minimal immersion in any bounded domain. Under certain restrictions on $\mathcal{D}$ and the topology of an open surface[1] M, Alarcón, Ferrer, Martín, and Morales [1, 7, 8, 9] proved the existence of a complete, proper minimal immersion of M in $\mathcal{D}$. Recently, Ferrer, Martin and Meeks [4] have given a complete solution to the "**proper Calabi-Yau problem for smooth bounded domains**" by demonstrating that for every smooth bounded domain $\mathcal{D} \subset \mathbb{R}^3$ and for every open surface M, there exists a complete proper minimal immersion $f\colon M \to \mathcal{D}$; furthermore, in [4], they proved that such an immersion $f\colon M \to \mathcal{D}$ can be constructed so that for any two distinct ends E_1, E_2 of M, the limit sets $L(E_1)$, $L(E_2)$ in $\partial\mathcal{D}$ are disjoint compact sets[2].

In contrast to the above existence results, in this paper we prove the existence of nonsmooth bounded domains $\mathcal{D}$ in $\mathbb{R}^3$, and more generally, domains $\mathcal{D}$ inside any Riemannian three-manifold, for which some open surface M can not be properly immersed into $\mathcal{D}$ as a complete surface with bounded mean curvature. In this case, we will say that $\mathcal{D}$ is a **Calabi-Yau domain** for M. The result described

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[1]We say that a surface is *open* if it is connected, noncompact and without boundary.
[2]See Definition 2.1 for the definition of the limit set of an end of a surface in a three-manifold.

in the next theorem generalizes the main theorem of Martín, Meeks and Nadirashvili in [6] which demonstrates the existence of nonsmooth bounded domains in $\mathbb{R}^3$ which do not admit any complete, properly immersed minimal surfaces with compact boundary (possibly empty) and at least one annular end.

Theorem 1.1 *Let $\overline{W}$ be a smooth compact Riemannian three-manifold with nonempty boundary and let $W = \mathrm{Int}(\overline{W})$. There exists a properly embedded one-manifold $\Delta \subset W$ whose path components are smooth simple closed curves, such that $\mathcal{D} = W - \Delta$ is a Calabi-Yau domain for any surface with compact boundary (possibly empty) and at least one annular end. In particular, $\mathcal{D}$ does not admit any complete, noncompact, properly immersed surfaces of finite topology, compact boundary and constant mean curvature.*

2 Notation and the description of Δ.

Before proceeding with the proof of the main theorem, we fix some notation.

1. $\mathbb{B}(R) = \{x \in \mathbb{R}^3 \mid |x| < R\}$ and $\mathbb{B} = \mathbb{B}(1)$.
2. $\overline{\mathbb{B}(R)} = \{x \in \mathbb{R}^3 \mid |x| \leq R\}$ and $\overline{\mathbb{B}} = \overline{\mathbb{B}(1)}$.
3. $\mathbb{S}^2(R) = \partial\mathbb{B}(R)$ and $\mathbb{S}^2 = \partial\mathbb{B}$.
4. For $p \in \mathbb{R}^3$ and $\varepsilon > 0$, $\mathbb{B}(p, \varepsilon) = \{x \in \mathbb{R}^3 \mid d(p, x) < \varepsilon\}$ is the open ball of radius ε centered at p.
5. For $n \in \mathbb{N}$, $\mathbb{B}_n = \mathbb{B}(1 - \frac{1}{2^n})$ and $\mathbb{S}^2_n = \partial\mathbb{B}_n$.
6. For any set $F \subset \mathbb{R}^3$, the cone on F is
$$C(F) = \{x \in \mathbb{R}^3 \mid x = ta \ \text{ where } t \in (0, \infty) \text{ and } a \in F\}.$$
7. For any set $F \subset \mathbb{R}^3$ and $\varepsilon > 0$, let $F(\varepsilon) = \{x \in \mathbb{R}^3 \mid d(x, F) \leq \varepsilon\}$ be the closed ε-neighborhood of F, where d is the distance function in $\mathbb{R}^3$.

In the proof of Theorem 1.1, we will need the following definition.

Definition 2.1 *Let $f\colon M \to \mathcal{D}$ be a proper immersion of surface M with possibly nonempty boundary into an open domain $\mathcal{D}$ contained in a three-manifold N with possibly nonempty boundary. The* **limit set** *of M is*
$$L(M) = \bigcap_{\alpha \in I} (\overline{f(M) - f(E_\alpha)}),$$
where $\{E_\alpha\}_{\alpha \in I}$ is the collection of compact subdomains of M and the closure $\overline{f(M) - f(E_\alpha)}$ is taken in N. The **limit set** *$L(e)$* **of an end** *e* **of** *M is defined to be the intersection of the limit sets all properly embedded subdomains of M with compact boundary which represent e. Notice that $L(M)$ and $L(e)$ are closed sets of $\partial\mathcal{D}$, and so each of these limit sets is compact when N is compact.*

First we will prove Theorem 1.1 in the case $\overline{W}$ is the smooth closed Riemannian ball $\overline{\mathbb{B}} \subset \mathbb{R}^3$. In this case, we will construct a properly embedded 1-manifold $\Delta \subset \mathbb{B}$ with path components consisting of smooth simple closed curves such that every proper immersion $f\colon A = \mathbb{S}^1 \times [0, \infty) \to \mathbb{B} - \Delta$ of an annulus with a complete induced metric has unbounded mean curvature; this result will prove Theorem 1.1 in the special case $\overline{W} = \overline{\mathbb{B}}$. The proof of the case of Theorem 1.1 when $\overline{W}$ is a smooth Riemannian ball, or more generally, an arbitrary compact smooth Riemannian manifold with nonempty boundary follows from straightforward modifications of the proof of the $\mathbb{B} - \Delta$ case; these modifications are outlined in the last paragraph of the proof.

The first step in the construction of Δ is to create a CW-complex structure Λ on the open ball $\mathbb{B}$. Consider the boundary ∂ of the box $[-1, 1] \times [-1, 1] \times [-1, 1] \subset \mathbb{R}^3$. The surface ∂ has a natural structure of a simplicial complex $\mathcal{X}_1$ with faces $\mathcal{F}_1 = \{F_1, F_2, ..., F_6\}$ contained in planes parallel to the coordinate planes, edges $\mathcal{E}_1 = \{E_1, E_2, \ldots, E_{12}\}$ and vertices $\mathcal{V}_1 = \{v_1, v_2, \ldots, v_8\}$. Let $\mathcal{X}_2$ denote the related refined simplicial complex obtained from $\mathcal{X}_1$ by adding vertices to the centers of each of the faces of $\mathcal{F}_1$ and to the centers of each of the edges in $\mathcal{E}_1$, thereby obtaining new collections $\mathcal{F}_2, \mathcal{E}_2, \mathcal{V}_2$ of faces, edges, and vertices. In this subdivision each face of $\mathcal{F}_2$ corresponds to subsquare in one the faces in $\mathcal{F}_1$ with four line segments, each of length one. Note that $\mathcal{F}_2$ has $6 \cdot 4$ faces, $\mathcal{E}_2$ has $2 \cdot 6 \cdot 4$ edges and $\mathcal{V}_2$ has $6 \cdot 4 + 2$ vertices. Continuing inductively the refining of the complex $\mathcal{X}_2$, produces at the n-th stage a simplicial complex $\mathcal{X}_n$ with $6 \cdot 4^{n-1}$ square faces $\mathcal{F}_n$, $2 \cdot 6 \cdot 4^{n-1}$ edges $\mathcal{E}_n$ and $6 \cdot 4^{n-1} + 2$ vertices $\mathcal{V}_n$.

We define the 1-skeleton Γ of Λ as follows:

$$\Gamma = \bigcup_{k=1}^{\infty} \left[C(\mathcal{E}_k) \cap \mathbb{S}^2_k\right] \cup \left[C(\mathcal{V}_k) \cap \left(\overline{\mathbb{B}}_{k+1} - \mathbb{B}_k\right)\right],$$

where $C(\mathcal{E}_k)$ denotes the cone $C(\cup \mathcal{E}_k)$. Extend the proper 1-dimensional CW-complex $\Gamma \subset \mathbb{B}$ to a proper 2-dimensional CW-subcomplex Λ' of Λ as follows. The faces of Λ' are the spherical squares in $\mathbb{S}^2_k - \Gamma$, as k varies in $\mathbb{N}$, together with the set of flat rectangles $C(\alpha) \cap (\mathbb{B}_{k+1} - \overline{\mathbb{B}}_k)$, where α is a 1-simplex in $\Gamma \cap \mathbb{S}^2_k$, as k varies in $\mathbb{N}$ and α varies in $\Gamma \cap \mathbb{S}^2_k$, see Figure 1 below. Let $\mathcal{F}$ denote the set of faces of Λ. Finally, $\mathbb{B} - \Lambda'$ contains an infinite collection $\mathcal{G} = \{G_\alpha\}_{\alpha \in I}$ of components which have the appearance of a cube which is a radial product of a spherical square in some $\mathbb{S}^2_k - \Gamma$ with a small interval of length $2^{-(k+1)}$, together with the special component $\mathbb{B}(\frac{1}{2})$. The set $\mathcal{G}$ is the set of 3-cells in Λ, which completes the construction of the CW-complex structure Λ of $\mathbb{B}$.

Define the related closed, piecewise smooth regular neighborhood $\widehat{N}(\Gamma)$ of Γ:

$$\widehat{N}(\Gamma) = \bigcup_{k=1}^{\infty} \left[\left(C(\mathcal{E}_k) \cap \mathbb{S}^2_k\right)\left(\frac{1}{2^k 10}\right)\right] \cup \left[\left(C(\mathcal{V}_k) \cap \left(\overline{\mathbb{B}}_{k+1} - \mathbb{B}_k\right)\right)\left(\frac{1}{2^k 100}\right)\right].$$

Then let $N(\Gamma) \subset \mathrm{Int}(\widehat{N}(\Gamma))$ be a small smooth closed regular neighborhood of Γ in $\mathbb{B}$ such that its boundary $\partial N(\Gamma)$ intersects each face F in $\mathcal{F}$ transversely in a simple closed curve $\beta(F)$ that bounds a disk $L(F) \subset F$; let $\mathcal{L} = \{L(F) \mid F \in \mathcal{F}\}$. For each open 1-simplex $\alpha \in \Gamma$, let $P(\alpha)$ be the plane perpendicular to α at the midpoint of α. Let $\widetilde{N}(\Gamma) \subset \mathrm{Int}(\widehat{N}(\Gamma))$ be another smooth closed regular neighborhood of Γ with $N(\Gamma) \subset \mathrm{Int}(\widetilde{N}(\Gamma))$ and such that $\partial \widetilde{N}(\Gamma) \cap P(\alpha)$ contains a simple closed curve $\beta(\alpha)$ close to α and which links α. Let $W(\alpha) \subset P(\alpha)$ denote the closed disk with boundary curve $\beta(\alpha)$ and let $\mathcal{W} = \{W(\alpha) \mid \alpha \in \Gamma\}$, see Figure 2.

The set Δ is the collection $\left[\bigcup_{\alpha \in \Gamma} \beta(\alpha)\right] \cup \left[\bigcup_{F \in \mathcal{F}} \beta(F)\right]$. The domain described in Theorem 1.1 is $\mathcal{D} = \mathbb{B} - \Delta$.

We conclude this section with the following immediate consequence of our constructions above.

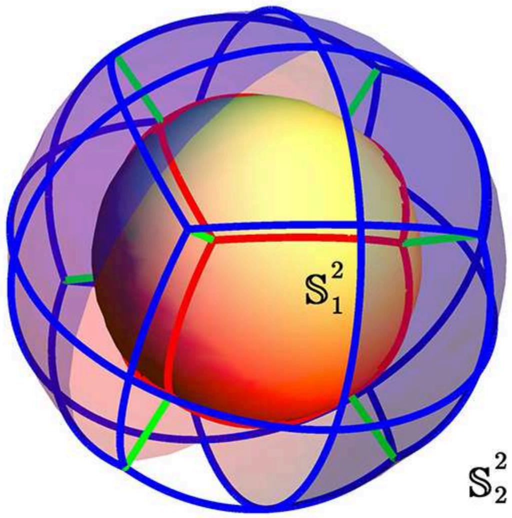

Figure 1: The first two steps in the construction of Λ.

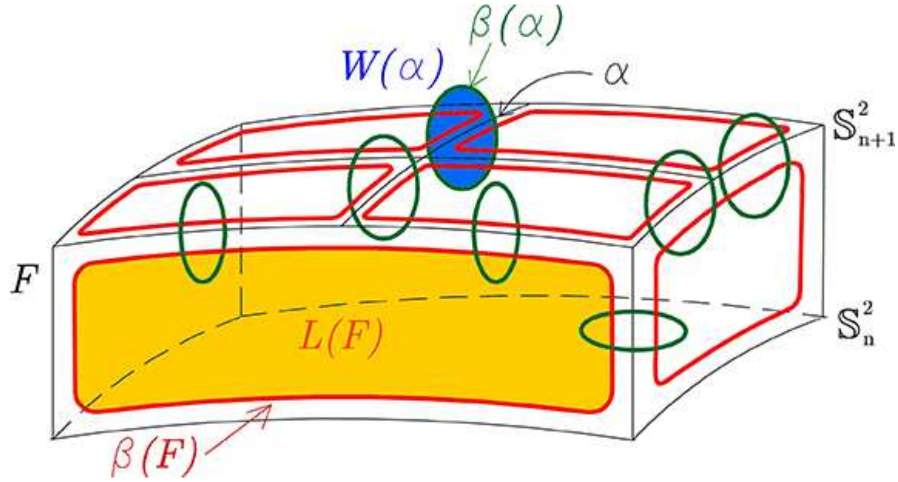

Figure 2: The 1-dimensional simplicial complex Γ, the 1-manifold Δ consisting of closed curves $\beta(F)$ and $\beta(F)$ and the disks $W(\alpha)$ and $L(F)$, where F is a face in Λ and α is a 1-simplex in Λ.

Lemma 2.2 *Let E be one of the following:*

1. *a 1-simplex, face or 3-cell in Λ;*
2. *a disk in either $\mathcal{W}$ or $\mathcal{L}$;*
3. *a component of $\widetilde{N}(\Gamma) - \cup\mathcal{W}$.*

If for some $\delta \in (0, \frac{1}{4})$, $E \cap [\mathbb{B} - \mathbb{B}(1-\delta)] \neq \emptyset$, then E is contained in a ambient ball B_E of radius 4δ.

3 $L(A)$ is a path connected subset of $\mathbb{S}^2$ with more than one point.

In this and the following sections, $f\colon A \to \mathcal{D}$ will denote a counterexample to Theorem 1.1 which, after a small smooth perturbation, we will assume to be a fixed properly immersed annulus diffeomorphic to $\mathbb{S}^1 \times [0,1)$ satisfying:

1. The supremum of the absolute mean curvature of A is less than a fixed constant $H_0 > 10$;
2. f is transverse to the disks in $\mathcal{W}$ and to the surface $\partial\widetilde{N}(\Gamma)$;
3. f is in general position with respect to Λ, i.e., f is disjoint from the set of vertices $\mathcal{V}$ of Λ, transverse to the closed faces of Λ and so, it is also transverse to $\mathbb{S}^2_k$ for each $k \in \mathbb{N}$.

Lemma 3.1 *If $f\colon \Sigma \to \mathcal{D}$ is a properly immersed surface with compact boundary and $\mathbf{e}$ is an end of Σ, then the limit set $L(\mathbf{e})$ of the end $\mathbf{e}$ is path connected.*

Proof. This is a standard result, but for the sake of completeness, we present its proof. Let $p, q \in L(\mathbf{e})$ be distinct points. Let $\mathcal{D}_1 \subset \mathcal{D}_2 \subset \ldots \subset \mathcal{D}_n \subset \ldots$ be a smooth compact exhaustion of $\mathcal{D}$. After replacing by subsequences, we may assume that there is a sequence of pairs of points p_n, q_n which lie in the component of $\Sigma - \mathrm{Int}(f^{-1}(\mathcal{D}_n))$ which represents $\mathbf{e}$ and such that $\lim_{n\to\infty} f(p_n) = p$ and $\lim_{n\to\infty} f(q_n) = q$.

Let $\sigma_n\colon [0,1] \to \Sigma - \mathrm{Int}(f^{-1}(\mathcal{D}_n))$ be paths with $\sigma_n(0) = p_n$ and $\sigma_n(1) = q_n$. Since the space $\mathcal{C}([0,1], \overline{\mathbb{B}})$ of continuous maps of $[0,1]$ into $\overline{\mathbb{B}}$ is a compact metric space in the sup norm, a subsequence of the paths $f \circ \sigma_n$ converges to a continuous map $f \circ \sigma$ of $[0,1]$ to $\partial\mathcal{D} = \left[\mathbb{S}^2 \cup \Delta\right] \subset \overline{\mathbb{B}}$ with $f \circ \sigma(0) = p$ and $f \circ \sigma(1) = q$. Since $f \circ \sigma([0,1]) \subset L(\mathbf{e})$ also holds, $L(\mathbf{e})$ is path connected. □

Lemma 3.2 *If $L(A) \cap \Delta \neq \emptyset$ or if $L(A)$ consists of a single point in $\mathbb{S}^2$, then A has finite area.*

Proof. By Theorems 3.1 and 3.1' in [3], the bounded mean curvature hypothesis and the properness hypothesis on f imply that if f composed with the inclusion map of $\mathcal{D}$ into $\mathbb{R}^3$ is proper outside of a point in $\mathbb{S}^2$ or outside of a component of Δ, then the surface A has finite area. Since $L(A)$ is path connected and the path components of $\partial\mathcal{D}$ are $\mathbb{S}^2$ or a simple closed curve in Δ, then the lemma follows. □

Lemma 3.3 *If $F\colon A \to \mathbb{R}^3$ is a complete immersion of $\mathbb{S}^1 \times [0,\infty)$ with bounded mean curvature, then A has infinite area.*

Proof. Suppose that A has finite area and we will obtain a contradiction. Since A is a complete annulus of finite area, there exists a sequence γ_n of pairwise disjoint, piecewise smooth, closed embedded geodesics with a single corner, which are topologically parallel to ∂A and whose lengths tend to 0 as n tends to infinity. Assume that the index ordering of the geodesics γ_n agrees with the relative distances of these curves to ∂A. Replace A by the subend $A(\gamma_1)$ with $\partial A(\gamma_1) = \gamma_1$. By the Gauss-Bonnet formula applied to the subannulus $A(\gamma_1, \gamma_n)$ with boundary $\gamma_1 \cup \gamma_n$, the total Gaussian curvature of $A(\gamma_1, \gamma_n)$ is greater than -4π. Since the Gaussian curvature function K_A of A is pointwise bounded from above by H_0^2, then the integral $\int_{A(\gamma_1)} K_A^+ \, dA$, where $K_A^+(x) = \max\{K_A(x), 0\}$, is finite because A has finite area. Hence, after replacing A by a subend of A, we may assume that $\int_{A(\gamma_1)} K_A^+ \, dA < \pi$. So, we conclude that $\int_{A(\gamma_1,\gamma_n)} K_A^- \, dA > -5\pi$, for all n, where $K_A^-(x) = \min\{K_A(x), 0\}$.

On the other hand, since the area of A does not grow at least linearly with the distance from ∂A, the norm of the second fundamental form of A is unbounded on A. By standard rescaling arguments (see for example [5]), there exists a divergent sequence $p_n \in A(\gamma_1)$ of blow-up points on the scale of the second fundamental form with norm of the second fundamental form at p_n being $\lambda_n > n$, and intrinsic neighborhoods $B_A(p_n, \frac{\lambda_n}{10})$ such that a subsequence of the rescaled surfaces $\lambda_n \left[f(B_A(p_n, \frac{\lambda_n}{10})) - p_n\right]$ converges in the C^2-norm to a minimal disk D in $\mathbb{R}^3$ satisfying:

1. The norm of the second fundamental form of D is at most 1 and equal to 1 at the origin.
2. D is a graph over the projection to its tangent plane at the origin.
3. The total curvature of D is $-\varepsilon$ for some $\varepsilon > 0$. Hence for n large, the integral of the function K_A^- on $B_A(p_n, \frac{\lambda_n}{10})$ is less than $-\frac{\varepsilon}{2}$.

By property 3 above, we conclude that $\lim_{n\to\infty} \int_{A(\gamma_1,\gamma_n)} K_A^- \, dA = -\infty$, which contradicts our earlier observation that $\int_{A(\gamma_1,\gamma_n)} K_A^- \, dA$ is bounded from below by -5π. $\square$

The next lemma is an immediate consequence of Lemmas 3.2 and 3.3.

Lemma 3.4 *$L(A)$ is a path connected compact subset of $\mathbb{S}^2$ containing two distinct points x and y. In particular, the immersion f can be seen as a proper immersion in $\mathbb{B}$.*

In the next sections, we will analyze how certain subdomains of the immersed annulus $f(A)$ intersects certain specific two-dimensional subsets of $\mathcal{D}$, for which we need the following definitions.

Definition 3.5 Suppose $F\colon \Sigma \to \mathcal{D}$ is a smooth proper immersion of a surface with compact boundary which is transverse to the disks in $\mathcal{W}$, to $\partial\widetilde{N}(\Gamma)$ and is in general position with respect to Λ. Suppose γ is a simple closed curve in Σ. Then:

1. γ is an $\underline{X_1\text{-type curve}}$, if γ is a component of $F^{-1}(\cup\mathcal{W})$.
2. γ is an $\underline{X_2\text{-type curve}}$, if γ is a component of $F^{-1}(\partial\widetilde{N}(\Gamma))$. Note that in this case $\gamma \subset \left[\partial\widetilde{N}(\Gamma) - \cup\mathcal{W}\right]$ and so curves of X_1-type and X_2-type are disjoint.
3. $\gamma \subset \Sigma$ is an $\underline{X_3\text{-type curve}}$, if γ is a component of $F^{-1}(\cup\mathcal{L})$. Notice that in this case γ is contained in a face of Λ.

Definition 3.6 Given the fixed immersion $f\colon A \to \mathcal{D}$, then:

1. X_1 is the set of X_1-type curves parallel to ∂A and X_2 is the set of X_2-type curves parallel to ∂A.

2. X_3 is the set of X_3-type curves in A which are disjoint from $(\cup X_1) \cup (\cup X_2)$.

3. By Lemma 4.1 below, the countable set X can be expressed as $X = X_1 \cup X_2 \cup X_3 = \{\gamma_i \mid i \in \mathbb{N}\}$, where the natural ordering of the simple closed, pairwise-disjoint curves γ_i in A by their relative distances from ∂A agrees with the ordering of the index set $\mathbb{N}$.

4. A_n denotes the compact subannulus in A with $\partial A_n = \partial A \cup \gamma_n$; note $A_1 \subset A_2 \subset \ldots \subset A_n \subset \ldots$ is a smooth compact exhaustion of A.

5. For $n, j \in \mathbb{N}$, $A(n, j)$ denotes the compact subannulus of A with boundary curves γ_n and γ_{n+j}.

6. $A(k) = \cup_{j=1}^{\infty} A(k, j)$ is the end representative of A with boundary γ_k.

4 Placement properties of $\partial A(k, 1)$ for k large.

Lemma 4.1 *For k large, there exists at least one curve in X in the region $\mathbb{B}_{k+2} - \overline{\mathbb{B}}_{k-1}$. In particular, the set X is infinite.*

Proof. Assume that $f(\partial A)$ is contained in $\mathbb{B}_n$ and we will prove that $\mathbb{B}_{k+2} - \overline{\mathbb{B}}_{k-1}$ contains an element in X, whenever $k > n$. Since $f\colon A \to \mathbb{B}$ is proper and transverse to the spheres $\mathbb{S}^2_i$ for every i, then for $i \geq n$, $f^{-1}(\mathbb{S}^2_i)$ contains a simple closed curve α_i which is a parallel to ∂A. If $f(\alpha_k) \cap (\cup \mathcal{L}) \neq \emptyset$, then either $\alpha_k \in X_3$ or α_k intersects an element γ of $X_1 \cup X_2$, where $f(\gamma)$ is contained in $\left[\mathbb{B}_{k+1} - \overline{\mathbb{B}}_{k-1}\right] \subset \left[\mathbb{B}_{k+2} - \overline{\mathbb{B}}_{k-1}\right]$. Similarly, if $f(\alpha_{k+1}) \cap (\cup \mathcal{L}) \neq \emptyset$, then either $\alpha_{k+1} \in X_3$ or α_{k+1} intersects an element γ of $X_1 \cup X_2$, whose image $f(\gamma)$ must be contained in $\left[\mathbb{B}_{k+2} - \overline{\mathbb{B}}_{k}\right] \subset \left[\mathbb{B}_{k+2} - \overline{\mathbb{B}}_{k-1}\right]$. Hence, we may assume that $f(\alpha_k)$ and $f(\alpha_{k+1})$ are both disjoint from $\cup \mathcal{L}$ and so, $[f(\alpha_k \cup \alpha_{k+1})] \subset \mathrm{Int}(\widetilde{N}(\Gamma))$.

Let $D^k_{\mathcal{W}}$ be the collection of disks in $\mathcal{W}$ which are contained in $\mathbb{B}_{k+1} - \overline{\mathbb{B}}_k$ and let Σ^k be the compact domain which is closure of the component of $\widetilde{N}(\Gamma) - (\cup D^k_{\mathcal{W}})$ which contains $f(\alpha_k)$ in its interior. Let $A(\alpha_k, \alpha_{k+1})$ be the subannulus of A with boundary $\alpha_k \cup \alpha_{k+1}$. Then $(f|_{A(\alpha_k, \alpha_{k+1})})^{-1}(\partial \Sigma^k)$ contains a simple closed curve γ which is parallel to ∂A and which is an element of $X_1 \cup X_2 \subset X$. The existence of γ completes the proof of the assertion. □

Lemma 4.2 *There exists a small $\eta_1 > 0$ such that for any $\eta \in (0, \eta_1]$, if $D \subset A$ is a compact disk with $f(\partial D) \subset \mathbb{B}(z, \eta)$ for some $z \in \mathbb{S}^2$ and D contains a point p such that the distance $d(f(p), z) \geq 1$, then:*

1. *The disk D contains a X_i-type curve β, for $i = 1, 2$ or 3, and $f(\beta)$ lies in $\mathbb{B}(z, 1/2) - \overline{\mathbb{B}}(z, 2\eta)$.*

2. *The curve β can be chosen so that the disk $D(\beta) \subset D$ bounded by β contains p. In particular, $f(D(\beta))$ contains a point of distance at least $\frac{1}{2}$ from its boundary and every point in $D(\beta)$ has intrinsic distance at least η from ∂D.*

Proof. Recall that for any face F in $\mathcal{F}$, $C(F)$ denotes the cone over F. Clearly, for $\eta_1 > 0$ sufficiently small and $\eta \in (0, \eta_1]$, there exist faces F_1, F_2, F_3 and F_4 in $\mathcal{F}$, such that: $\mathbb{B}(z, 2\eta) \subset \mathrm{Int}(C(F_1))$, $C(F_i) \subset \mathrm{Int}(C(F_{i+1}))$, for $i = 1, 2, 3$ and $C(F_4) \subset \mathbb{B}(z, 1/2)$.

At this point we can follow the proof of Lemma 4.1 where the annulus $D - \{p\}$ plays the role of A and the piecewise smooth disk $\partial C(F_i)$ plays the role of $\mathbb{S}^2_{k-2+i}$. Then we obtain an X_i-type curve β parallel to ∂D in $D - \{p\}$ and whose image $f(\beta)$ is in the open region between $\partial(C(F_1))$ and $\partial(C(F_4))$, which is contained $\mathbb{B}(z, 1/2) - \overline{\mathbb{B}}(z, 2\eta)$. This is the desired curve. $\square$

Before stating the next assertion, we need some notation.

Definition 4.3 *Given a curve γ_k in X, we define $\chi_1(f(\gamma_k))$ to be the union of all closed 3-cells in Λ which intersect $f(\gamma_k)$. Similarly, given $i \in \mathbb{N}$ we define $\chi_{i+1}(f(\gamma_k))$ as the union of all closed 3-cells in Λ which intersect $\chi_i\left(f(\gamma_k)\right)$.*

In what follows, we shall use the observation that for $i = 1$ and 2, the set $\chi_i(f(\gamma_k))$ is a piecewise smooth compact ball, whose boundary sphere is a union of faces in $\mathcal{F}$ and it is in general position with respect to the immersion f.

Lemma 4.4 *For k large, we have $f(\gamma_{k+1}) \subset \chi_3(f(\gamma_k))$ or $f(\gamma_k) \subset \chi_3(f(\gamma_{k+1}))$. Furthermore, given $\eta > 0$, there exists an integer $k(\eta)$ such that for any $k \geq k(\eta)$ one has:*

1. *$f(A(k)) \subset \left[\mathbb{B} - \overline{\mathbb{B}}(1-\eta)\right]$ and each X_i-type curve γ, $i = 1, 2$ or 3, in $A(k,1)$ is contained in a ball $\mathbb{B}(y(\gamma), \eta)$ for a suitable point $y(\gamma) \in \mathbb{S}^2$.*
2. *There is a point $z(k) \in \mathbb{S}^2$ such that $f(\gamma_k \cup \gamma_{k+1}) \subset \mathbb{B}(z(k), \eta)$.*
3. *Every simple closed curve $\gamma \subset \left[A(k,1) - f^{-1}(\mathbb{B}(z(k), \eta))\right]$ bounds a disk in $A(k,1)$.*

Proof. In order to prove the first statement of the lemma, we distinguish four cases, depending on the position of $f(\gamma_k)$. We will use the fact that by Lemma 2.2, for $k \to \infty$, the curve $f(\gamma_k)$ becomes arbitrarily close to a point $z(k) \in \mathbb{S}^2$.

Case A: $f(\gamma_k) \subset D \in \mathcal{W}$.

In this case $f(A(k,1))$ enters a component C of $\widetilde{N} - \cup\mathcal{W}$ near $f(\gamma_k)$. Consider the compact component Z of $\left(f|_{A(k)}\right)^{-1}(\overline{C}) \subset A(k)$ with boundary component γ_k and let α_k be the boundary curve of $Z - \gamma_k$ which is parallel to $\partial A(k) = \gamma_k$. By the definition of X_1 and X_2, $\alpha_k = \gamma_{k+j} \in [X_1 \cup X_2] \subset X$, for some $j \geq 1$. By definition of X, $\gamma_{k+1} \subset A(k,j)$ and intersects the domain Z. If $\gamma_{k+1} \subset Z$, then clearly $f(\gamma_{k+1}) \subset \overline{C} \subset \chi_2(f(\gamma_k))$ and we are done. Otherwise, $f(\gamma_{k+1})$ must not be contained in $\widetilde{N}(\Gamma)$. This means that γ_{k+1} belongs to X_3 and so it is contained in a face F of Λ, which intersects C. Hence, $F \subset \chi_2(f(\gamma_k))$ which implies $f(\gamma_{k+1}) \subset \chi_2(f(\gamma_k))$.

Case B: $f(\gamma_k) \subset \left[\partial\widetilde{N}(\Gamma) - \cup\mathcal{W}\right]$ and the annulus $f(A(k,1))$ enters $\widetilde{N}(\Gamma)$ near $f(\gamma_k)$.

In this case, the arguments in Case A apply to show that $f(\gamma_{k+1}) \subset \chi_2(f(\gamma_k))$.

Case C: $f(\gamma_k) \subset \left[\partial\widetilde{N}(\Gamma) - \cup\mathcal{W}\right]$ and the annulus $f(A(k,1))$ enters $\mathbb{B} - \widetilde{N}(\Gamma)$ near $f(\gamma_k)$.

First, note that if $f(\gamma_{k+1})$ intersects $\chi_2(f(\gamma_k))$, then $f(\gamma_{k+1}) \subset \chi_3(f(\gamma_k))$. Thus, we may assume that $f(\gamma_{k+1})$ lies outside the compact piecewise smooth ball $\chi_2(f(\gamma_k))$. Consider the compact component Z of $(f|_{A(k,1)})^{-1}(\chi_2(f(\gamma_k)))$ containing γ_k in its boundary. Let $\alpha_k \neq \gamma_k$ be the boundary curve of Z which is parallel in $A(k)$ to γ_k; recall that $A(k)$ is the end of A with boundary γ_k. If $f(\alpha_k)$ intersects $\cup\mathcal{L}$, then $f(\alpha_k)$ is contained in a disk $D \in \mathcal{L}$; in this case, since α_k lies between γ_k and γ_{k+1} and it is parallel to $\partial A(k)$, then $\alpha_k \in X_3$, which is contrary to the definition of γ_{k+1}. Thus, $f(\alpha_k) \subset \partial(\chi_2(f(\gamma_k)))$ and is disjoint from $\cup\mathcal{L}$, and so $f(\alpha_k) \subset \mathrm{Int}(\widetilde{N}(\Gamma))$. Let $A(\gamma_k, \alpha_k) \subset A(k,1)$ be the subannulus with boundary curves $\gamma_k \cup \alpha_k$. As $f(A(\gamma_k, \alpha_k))$ enters

$\mathbb{B} - \widetilde{N}(\Gamma)$ nears $f(\gamma_k)$ and $f(\alpha_k) \subset \mathrm{Int}(\widetilde{N}(\Gamma))$, then ours previous separation arguments imply that there exists a curve $\beta \subset \left(f\left|_{A(\gamma_k,\alpha_k)}\right.\right)^{-1}(\partial\widetilde{N}(\Gamma) - \cup\mathcal{W})$ which is parallel to γ_k. Since $\beta \in X_2$ and $\beta \neq \gamma_{k+1}$, we arrive at a contradiction. This contradiction proves Case C.

Case D: $f(\gamma_k) \subset D \in \mathcal{L}$.

If $f(\gamma_{k+1}) \subset \partial\widetilde{N}(\Gamma)$ or $f(\gamma_{k+1}) \subset \widehat{D} \in \mathcal{W}$, then the arguments in our previously considered cases imply that $f(\gamma_k) \subset \chi_3(f(\gamma_{k+1}))$. Hence, we may assume that $f(\gamma_{k+1}) \subset D' \in \mathcal{L}$ as well.

If $\chi_1(f(\gamma_k)) \cap \chi_1(f(\gamma_{k+1})) \neq \emptyset$, then $f(\gamma_{k+1}) \subset \chi_2(f(\gamma_k))$. Hence, we can assume that $\chi_1(f(\gamma_k)) \cap \chi_1(f(\gamma_{k+1})) = \emptyset$. Recall that $f|_{A(k,1)}$ is in general position with respect to $\partial\left(\chi_1(f(\gamma_k))\right)$ and $\partial\left(\chi_1(f(\gamma_{k+1}))\right)$. Let Z_i be the component of $(f|_{A(k,1)})^{-1}(\chi_1(f(\gamma_i)))$ with boundary component γ_i and let $\alpha_i \neq \gamma_i$ be the boundary component of Z_i which is parallel to γ_i, for $i = k, k+1$, respectively. Since α_k and α_{k+1} lie in $\mathrm{Int}(A(k,1))$, then by definition of X, both $f(\alpha_k)$ and $f(\alpha_{k+1})$ are disjoint from $\cup\mathcal{L}$. Moreover, as $f(\alpha_i) \subset \chi_1(f(\gamma_i))$, for $i = k,\ k+1$, then $f(\alpha_k \cup \alpha_{k+1}) \subset \mathrm{Int}(\widetilde{N}(\Gamma))$. Let $A(\alpha_k, \alpha_{k+1})$ be the subannulus of $A(k,1)$ with boundary $\alpha_k \cup \alpha_{k+1}$.

Consider the collection of disks $D^k_{\mathcal{W}}$ in $\mathcal{W}$ which are contained in the interior of $\chi_2(f(\gamma_k)) - \chi_1(f(\gamma_k))$. Then $\widetilde{N}(\Gamma) - \cup D^k_{\mathcal{W}}$ contains a connected domain whose closure Σ^k in $\mathbb{B}$ satisfies $f(\alpha_k) \subset \mathrm{Int}(\Sigma^k)$ and $f(\alpha_{k+1}) \subset \mathbb{B} - \Sigma^k$. Our previous separation arguments imply that there is a simple closed curve β in $(f|_{A(\alpha_k,\alpha_{k+1})})^{-1}(\partial\Sigma^k)$ which is parallel to γ_k. But $\beta \subset \mathrm{Int}(A(k,1)$ and $\beta \in X_1 \cup X_2$, which is a contradiction. This contradiction completes the proof of the first statement of the lemma.

Item 1 in the lemma is a straightforward consequence of the fact that, as $\to \infty$, then $f(A(k))$ uniformly converges to $\mathbb{S}^2$. Moreover, given a X_i-type curve $\gamma \subset A(k)$, $i = 1,2,3$, the Euclidean diameter of $f(\gamma)$ goes to zero (as $k \to \infty$) and is arbitrarily close to a point $y(\gamma)$ in $\mathbb{S}^2$. Item 2 in the lemma follows from the observation that as $k \to \infty$, the sets $\chi_3\left(f(\gamma_k)\right)$ are arbitrarily close to $f(\gamma_k)$, which in turn, lie arbitrarily close to points $z(k) \in \mathbb{S}^2$. These observations imply that there exists an integer $j(\eta)$ such that for $k \geq j(\eta)$, items 1 and 2 in Lemma 4.4 hold.

In order to obtain item 3, we define $k(\eta) = j(\frac{\eta}{900})$. By definition of $j(\frac{\eta}{900})$, for $k \geq k(\eta)$, $f(\gamma_k \cup \gamma_{k+1}) \subset \mathbb{B}(z(k), \frac{\eta}{900})$ and $f(A(k)) \subset \left[\mathbb{B} - \overline{\mathbb{B}}(1 - \frac{\eta}{900})\right]$. It remains to check that each simple closed curve β in a component K of $(f_{A(k,1)})^{-1}(\mathbb{B} - \mathbb{B}(z(k), \eta))$ bounds a disk in $A(k,1)$; note that $\overline{K} \subset \mathrm{Int}(A(k,1))$. Observe that $\frac{\eta}{900}$ is sufficiently small so that there exist faces F_1, F_2, F_3 and F_4 in $\mathcal{F}$, such that: $\mathbb{B}(z(k), \frac{\eta}{900}) \subset \mathrm{Int}(C(F_1)), C(F_i) \subset \mathrm{Int}(F_{i+1})$, for $i = 1,2,3$ and $C(F_4) \subset \mathbb{B}(z(k), \eta)$.

If $\beta \subset K$ does not bound a disk in $A(k,1)$, then it is parallel to γ_k in $A(k,1)$. Let $A(\gamma_k, \beta)$ denote the subannulus of $A(k,1)$ with boundary $\gamma_k \cup \beta$. Then the arguments in the proof of Lemma 4.2 imply that there exists a simple closed curve $\gamma' \subset \mathrm{Int}(A(\gamma_k, \beta))$ which is parallel to γ_k, $f(\gamma') \subset \mathbb{B}(z(k), \eta)$ and γ' is an X_i-type curve, for $i = 1, 2$ or 3. In particular, $\gamma' \in X$ which is impossible. Thus, every simple closed curve in $A(k,1)$ whose image under f lies outside of $\mathbb{B}(z(k), \eta)$ bounds a disk in $A(k,1)$. This completes the proof of the lemma. $\square$

The next lemma directly follows from the mean curvature comparison principle.

Lemma 4.5 *Suppose $\Sigma \subset A$ is a compact domain such that: $f(\partial\Sigma)$ is contained in $\mathbb{B}(z, \eta)$, where $z \in \mathbb{S}^2$ and $\eta < \frac{1}{H_0}$. Then either $f(\Sigma) \subset \mathbb{B}(z, \eta)$ or $f(\Sigma)$ contains a point outside of $\mathbb{B}(z, \frac{1}{H_0})$.*

5 Proof of the Theorem 1.1.

By Lemma 3.4, $L(A) \subset \mathbb{S}^2$ contains at least two distinct points x and y. We next prove that the limit set of f is the entire sphere $\mathbb{S}^2$.

Lemma 5.1 $L(A) = \mathbb{S}^2$.

Proof. By Lemma 3.4, there are distinct points $x, y \in L(A) \subset \mathbb{S}^2$. Arguing by contradiction, suppose that there exists a point $p \in \mathbb{S}^2 - L(A)$. The definition of limit point and the fact that $f\colon A \to \mathcal{D}$ is proper with $L(A) \subset \mathbb{S}^2$ imply there exists an $\varepsilon > 0$ such that $\mathbb{B}(p, 10\varepsilon) \cap f(A) = \text{Ø}$. By properness of f in $\mathbb{B}$, then for n large, we have $f(\overline{A - A_n}) \subset \left[\mathbb{B} - \overline{\mathbb{B}(1-\varepsilon)}\right]$.

Note that for some $\delta \in (0, \frac{1}{8}\varepsilon)$ sufficiently small, there exists a compact embedded annulus of revolution $E(\delta) \subset \left[(\overline{\mathbb{B}} - \mathbb{B}(1-\delta)) \cap \mathbb{B}(p,\varepsilon)\right]$ with boundary circles in $\mathbb{S}^2 \cup \left[\mathbb{S}^2(1-\delta)\right]$ and such that the radial projection $r(E(\delta)) \subset \mathbb{S}^2$ is contained in the disk $\mathbb{B}(p,\varepsilon) \cap \mathbb{S}^2$. Furthermore, $E(\delta)$ is also chosen to have mean curvature greater than H_0 and with mean curvature vector outward pointing from the domain in $\left[\overline{\mathbb{B}} - \mathbb{B}(1-\delta)\right] - E(\delta)$ which is contained in $\mathbb{B}(p,\varepsilon)$; for instance, $E(\delta)$ can be chosen to be a piece of a suitably scaled compact embedded annulus in some nodoid of constant mean curvature one, see Figure 3 Left. Assume now that ε is also chosen less than $\frac{1}{10}d(x,y)$.

Assume that n and j are chosen sufficiently large so that:

1. $f(A(n,j)) \subset \left[\mathbb{B} - \overline{\mathbb{B}(1-\delta)}\right]$.

2. Any circle in $\mathbb{S}^2 - \{x, y\}$ which represents the generator of the first homology group $\mathbb{H}_1(\mathbb{S}^2 - \{x,y\})$ and whose distance from x and y is at least δ, intersects the radial projection $r(f(A(n,j))) \subset \mathbb{S}^2$. This property holds since x and y are limit points of $f(A)$.

3. The radial projection of each of the two boundary curves of $f(A(n,j))$ has diameter less than ε. This condition is possible to achieve since each of the components of $\partial f(A(n,j))$ has image on either a disk component of $\mathcal{W}$, a face of $\mathcal{F}$ or a component of $\partial \widetilde{N}(\Gamma) - \cup\mathcal{W}$, and each of these components and faces is contained ambient balls of radius 4δ by Lemma 2.2, which in turn have radial projections of diameter less than ε.

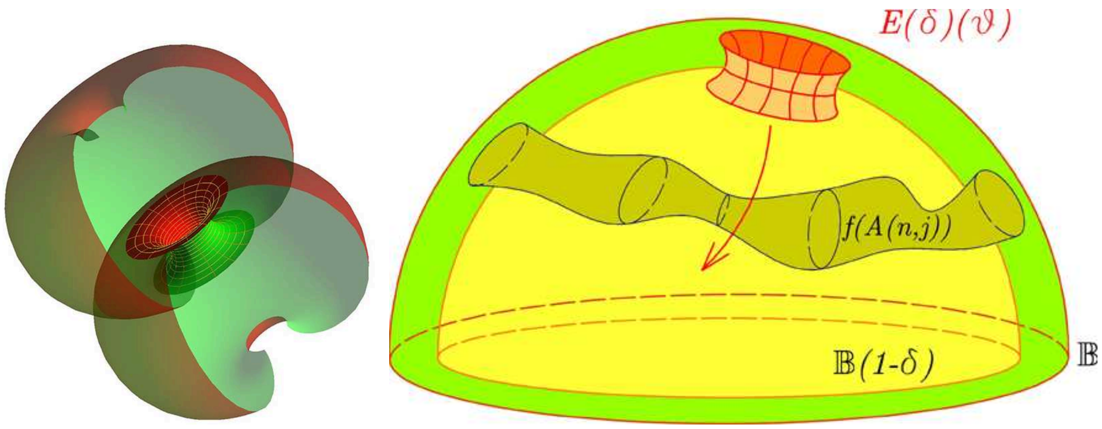

Figure 3: Left: This figure shows a domain on the nodoid corresponding to a scaling of $E(\delta)$. Right: Since all of the $E(\delta)(\vartheta)$ are disjoint from $\partial f(A(n,j))$, a first point of contact in $E(\delta)(\vartheta_0) \cap f(A(n,j))$ occurs an interior point of $f(A(n,j))$.

By the above three properties and our choices of ε and δ, we can choose a circle $S^1 \subset \mathbb{S}^2 - \{x,y\}$ which intersects $r(f(A(n,j)))$, and such that the ε-neighborhood $S^1(\varepsilon)$ of S^1 is disjoint from the radial projection $r(\partial f(A(n,j))) \subset \mathbb{S}^2$ and each component of $\mathbb{S}^2 - r(S^1(\varepsilon))$ contains points of $r(f(A(n,j)))$. Let L be an oriented radial ray which is an axis for the circle S^1. For $\vartheta \in [0, 2\pi)$, consider the family of annuli $E(\delta)(\vartheta)$ obtained by rotating $E(\delta)$ counterclockwise around L by the angle ϑ. By elementary separation properties, there is a smallest $\vartheta_0 \in (0, 2\pi)$ such that $E(\delta)(\vartheta_0) \cap f(A(n,j)) \neq \emptyset$. Since all of the $E(\delta)(\vartheta)$ are disjoint from $\partial f(A(n,j))$, a first point of contact in $E(\delta)(\vartheta_0) \cap f(A(n,j))$ occurs an interior point of $f(A(n,j))$, which must have absolute mean curvature on A at least equal to the minimum of the mean curvature of $E(\delta)(\vartheta_0)$ (see Figure 3 Right). But the mean curvature of $E(\delta)(\vartheta_0)$ is greater than the absolute mean curvature function of A. This contradiction completes the proof of Lemma 5.1. □

The next lemma follows immediately from the arguments presented in the proof of Lemma 5.1; also see Figure 3 Right. We note that the constant H_0 in the statement of the next lemma is the same constant which is the strict upper bound on the supremum of the absolute mean curvature of $f\colon A \to \mathbb{B}$.

Lemma 5.2 *Given any $\varepsilon \in (0, \frac{1}{4})$, there exists an $\eta_0 \in (0, \frac{\varepsilon}{10})$ that also depends on H_0 such that the following statements hold. For any $\eta \in (0, \eta_0]$ and for any immersion $g\colon \Sigma \to \mathbb{B} - \mathbb{B}(1-\eta)$ of a compact surface with boundary and absolute mean curvature less than H_0 such that $g(\partial\Sigma) \subset \left[\overline{\mathbb{B}}(x,\eta) \cup \overline{\mathbb{B}}(y,\eta)\right]$ for two points $x, y \in \mathbb{S}^2$ with $d(x,y) \geq \varepsilon$, then either $g(\Sigma) \subset \left[\overline{\mathbb{B}}(x,\eta) \cup \overline{\mathbb{B}}(y,\eta)\right]$ or $g(\Sigma)$ is ε-close to every point in $\mathbb{S}^2$. (Note that it may be the case that $g(\partial\Sigma)$ is contained entirely in one of the balls $\overline{\mathbb{B}}(x,\eta), \overline{\mathbb{B}}(y,\eta)$.)*

Lemma 5.3 *Given an $\varepsilon \in (0, \frac{1}{2H_0})$, there exists an $n(\varepsilon) \in \mathbb{N}$ such that for each $k \geq n(\varepsilon)$, there exists a point $y(k) \in \mathbb{S}^2$ with $f(A(k,1)) \subset \mathbb{B}(y(k), \varepsilon)$ and $f(A(n(k)) \subset \left[\mathbb{B} - \overline{\mathbb{B}}(1-\varepsilon)\right]$.*

Proof. Fix $\varepsilon \in (0, \frac{1}{2H_0})$ and let $\eta = \min\{\eta_0, \eta_1\}$ where η_0 is given in Lemma 5.2 and depends on ε and H_0 and η_1 given in Lemma 4.2. Let $k(\eta)$ be the related integer given in Lemma 4.4. We claim that for $k \geq k(\eta)$, $f(A(k,1)) \subset \mathbb{B}(z(k), \eta)$ and that $f(A(k(\eta)) \subset \left[\mathbb{B} - \overline{\mathbb{B}}(1-\varepsilon)\right]$, and so, by setting $n(\varepsilon) = k(\eta)$, this claim will complete the proof of the lemma. By Lemma 4.4, $f(A(k(\eta))) \subset \left[\mathbb{B} - \overline{\mathbb{B}}(1-\eta)\right] \subset \mathbb{B} - \overline{B}(1-\varepsilon)$ and so it remains to verify that $f(A(k,1)) \subset \mathbb{B}(z(k), \eta)$.

Suppose that $f(A(k,1))$ contains a point outside of $\mathbb{B}(z(k), \eta)$. Let K be a nonempty component in $(f_{A(1,k)})^{-1}(\mathbb{B} - \mathbb{B}(z(k), \eta))$. By Lemma 4.5, there is a point on K which lies outside of $\mathbb{B}(z(k), \frac{1}{H_0})$. Since $\varepsilon \in (0, \frac{1}{2H_0})$ and $\eta \leq \eta_0$, Lemma 5.2 implies that the distance between every point of $\mathbb{S}^2$ and K is at most ε. In particular, there exists a point $p \in K$ such that $f(p)$ has distance greater than 1 from $z(k)$.

By the third statement in Lemma 4.4, each boundary curve of K bounds a disk in $A(k,1)$. From the simple topology of an annulus we find that exactly one boundary curve of K bounds a disk $D \subset A(k,1)$ and such that $K \subset D$. Next we apply Lemma 4.2 to find an X_i-type curve $\beta_1 \subset D$ which bounds a subdisk $D(\beta_1)$ which contains the point p and which satisfies the other properties in that lemma. In particular, we may assume the intrinsic distance from $D(\beta_1)$ to ∂D is at least η. By the second statement in Lemma 4.4, $f(\beta_1) \subset \mathbb{B}(y(\beta_1), \eta)$ for some point $y(\beta_1) \in \mathbb{S}^2$. By our previous arguments there exists a point $p_1 \in D(\beta_1)$ such that the distance from $f(p_1)$ to $y(\beta_1)$ is greater than one. So, we can apply Lemma 4.2 again to obtain a subdisk $D(\beta_2)$, $D \supset D(\beta_1) \supset D(\beta_2)$, where the intrinsic distance from $\partial D(\beta_1)$ to $D(\beta_2)$ is at least η. Repeating these arguments, induction gives the existence of a sequence of disks $D \supset D(\beta_1) \supset \ldots \supset D(\beta_n) \supset \ldots$ such that the intrinsic distance from $D(\beta_n)$ to ∂D is at least $n\,\eta$. Since D is compact, we obtain a contradiction which proves our earlier claim that $f(A(k,1)) \subset \mathbb{B}(z(k),\eta)$ for $k \geq k(\eta)$. As we have already observed, this claim then proves the lemma. $\square$

We now complete the proof of Theorem 1.1. Fix some $\varepsilon' \in (0, \frac{1}{2H_0})$ and let $\eta_0 \in (0, \frac{\varepsilon'}{10})$ be the related number given in Lemma 5.2. Set $\varepsilon = \eta_0$ and let $n(\varepsilon)$ be the integer given in Lemma 5.3. By Lemma 5.3, for each $i \in \mathbb{N}$, $f(A(n(\varepsilon),i)) \subset [\mathbb{B} - \mathbb{B}(1-\eta_0)]$, $f(\gamma_{n(\varepsilon)}) \subset \mathbb{B}(y(n(\varepsilon)),\eta_0)$ and $f(\gamma_{n(\varepsilon)+i}) \subset \mathbb{B}(y(n(\varepsilon)+i),\eta_0)$. Since the limit set of $A(n(\varepsilon))$ is all of $\mathbb{S}^2$, there exists a smallest $j \in \mathbb{N}$ such that the distance between $y(n(\varepsilon))$ and $y(n(\varepsilon)+j)$ is greater that ε'. By Lemma 5.2 and taking into account that $\mathbb{B}(y(n(\varepsilon)),\eta_0)$ and $\mathbb{B}(y(n(\varepsilon)+j),\eta_0)$ are disjoint, then we conclude that $f(A(n(\varepsilon),j))$ must be ε' close to every point of $\mathbb{S}^2$.

On the other hand, given $k \in \mathbb{N}$, $n(\varepsilon) \leq k < n(\varepsilon)+j$, we know (by Lemma 5.3) that $f(A(k,1)) \subset \mathbb{B}(y(k),\varepsilon)$, for a suitable $y(k) \in \mathbb{S}^2$. Moreover, the choice of j implies that $f(\gamma_k) \subset \mathbb{B}(y(n(\varepsilon)),\varepsilon'+\varepsilon)$, for k satisfying $n(\varepsilon) \leq k < n(\varepsilon)+j$. So, by the triangle inequality we deduce $f(A(k,1)) \subset \mathbb{B}(y(n(\varepsilon)),\varepsilon'+2\varepsilon) \subset \mathbb{B}(y(n(\varepsilon)),2\varepsilon')$ for any k satisfying $n(\varepsilon) \leq k < n(\varepsilon)+j$. This implies $f(A(n(\varepsilon),j)) \subset \mathbb{B}(y(n(\varepsilon)),2\varepsilon')$ which is impossible since $2\varepsilon' < \frac{1}{10}$ and we have already seen that $f(A(n(\varepsilon),j))$ must be ε' close to every point of $\mathbb{S}^2$. This contradiction completes the proof of Theorem 1.1 in the case $\overline{W} = \overline{\mathbb{B}}$.

For the general case where $\overline{W}$ is a smooth compact Riemannian manifold with nonempty boundary, small modifications of the proof of Theorem 1.1 in the special case $\overline{W} = \overline{\mathbb{B}} \subset \mathbb{R}^3$ also demonstrate that there exists a properly embedded 1-manifold $\Delta_W \subset W$, whose path components are smooth simple closed curves, such that $\mathcal{D} = W - \Delta_W$ is a Calabi-Yau domain for any open surface with at least one annular end. In carrying out these modifications in the smooth compact 3-manifold $\overline{W}$, it is convenient, to place the 1-manifold $\Delta_{\overline{W}}$ in the union of small pairwise disjoint closed ε-neighborhoods of the boundary components of $\overline{W}$ which have a natural product structure derived from the distance function to the boundary component. The product structure simplifies the construction of the related 1-complex $\Gamma_{\overline{W}}$ which has one component in each of the ε-neighborhoods of each boundary component of $\overline{W}$. Also note that the properness of any proper immersion of $A = \mathbb{S}^1 \times [0,\infty)$ into $\mathcal{D}$ guarantees that A has a end representative which maps into the ε-neighborhood of exactly one of the boundary components of $\overline{W}$. This discussion completes the proof of Theorem 1.1.

Remark 5.4 The reader familiar with the paper [6] might consider the question: Are the domains $\mathcal{D}_{\mathcal{F}} \subset \mathbb{R}^3$ [6], obtained by removing a infinite proper family $\mathcal{F}$ of horizontal circles from $\mathbb{B}$, Calabi-Yau domains for surfaces with at least one annular end? The answer to this question is no because for at least one such $\mathcal{D}_{\mathcal{F}}$ constructed in [6], there exists a proper, conformal, complete embedding $f\colon \mathbb{R}^2 \to \mathcal{D}$ with absolute mean curvature function less than 1, $f(\mathbb{R}^2)$ is a surface of revolution with axis the x_3-axis, $f(\mathbb{R}^2)$ has intrinsic linear area growth and has limit set $L(\mathbb{R}^2) = \mathbb{S}^2$. The mean curvature function of $f(\mathbb{R}^2)$ in this case contains points of mean curvature arbitrarily close to 1 and also arbitrarily close to -1. In this case for $\mathcal{D}$, the circles in $\mathcal{F} \subset \mathbb{B}$ are chosen to have axis the x_3-axis; the surface has the appearance of taking an infinite connected sum of the spheres $\mathbb{S}^2_k$, $k \in \mathbb{N}$, defined at the beginning of Section 2, joined by small catenoidal type necks centered along points along the x_3-axis which limit to the north and south poles of $\mathbb{S}^2$.

We conclude the paper with the following conjecture.

Conjecture 5.5 *Let $\Delta \subset \mathbb{B}$ be the properly embedded one-manifold given in the proof of Theorem 1.1. If $\overline{B}$ is a smooth compact Riemannian three-ball and $F\colon \overline{B} \to \mathbb{B}$ is a smooth diffeomorphism, then $\mathcal{D} = B - F^{-1}(\Delta)$ is a Calabi-Yau domain for* **any** *noncompact surface with compact boundary* *(possibly empty). In particular, $\mathcal{D} = \mathbb{B} - \Delta$ does not admit any complete, properly immersed open surfaces with bounded mean curvature.*

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