THE CALABI-YAU PROBLEM IN RIEMANNIAN 3-MANIFOLDS

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Calabi-Yau problem

It dealt with existence of complete **bounded** minimal surfaces in $\mathbb{R}^3$

Theorem (Jorge-Xavier, Ann. of Math. 1980)

There exists a complete, non-flat, minimal disk between **two parallel planes***.*

Theorem (Nadirashvili, Invent. math. 1996)

There exists a complete minimal disk in a **ball***.*

Open Problems

Nadirashvili's example opened a Pandora's box of new conjectures and questions.

- **Yau:** Are there complete **embedded** minimal surfaces in a ball of $\mathbb{R}^3$?
- **Yau:** Are there complete **proper** minimal immersions $f : \mathbb{D} \rightarrow \mathbb{B}$?
- **Nadirashvili:** Can the limit set be a **Jordan curve** ? or the **whole sphere** ?
- **Xavier:** Is every **convex body** of $\mathbb{R}^3$ the **convex hull** of a complete minimal surface ?
- **Yau:** What is the asymptotic behavior of the **Gaussian curvature** ?
- **Yau:** What is the behaviour of the **spectrum** of the Laplacian operator ?

Complete bounded minimal surfaces

IMMERSED CASE	vs	EMBEDDED CASE
EXISTENCE		NON-EXISTENCE
Jorge-Xavier Nadirashvili		Colding-Minicozzi Meeks-Rosenberg Meeks-Pérez-Ros

Complete bounded embedded minimal surfaces

Theorem (Colding-Minicozzi, Ann. of Math. 2008)

A complete embedded minimal surface with **finite topology** *in* $\mathbb{R}^3$ *must be proper (in* $\mathbb{R}^3$*.)*

Theorem (Meeks-Rosenberg, Duke Math. J. 2005)

If M *is a complete embedded minimal surface with* **injectivity radius** $I_M > 0$, *then* M *is proper.*

Theorem (Meeks-Pérez-Ros, Preprint)

If M *is a complete embedded minimal surface in* $\mathbb{R}^3$ *with* **finite genus** *and a* **countable number of ends**, *then* M *is proper.*

Bounded embedded minimal surface conjecture

Conjecture (Martín-Meeks-Nadirashvili; Meeks-Perez-Ros)

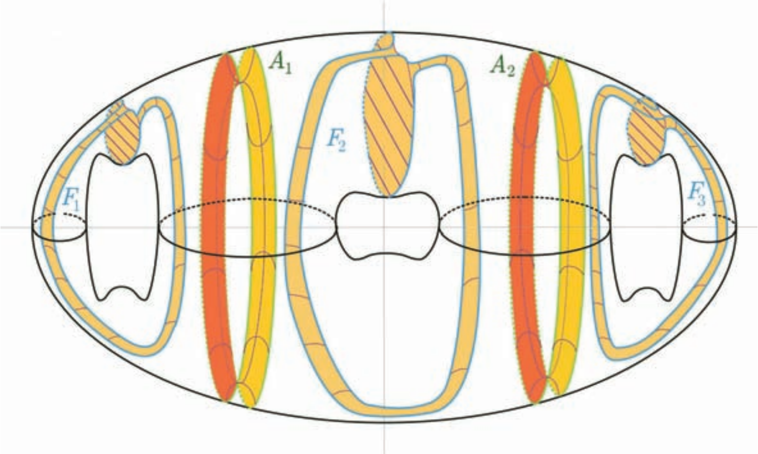
Let **M** *be an open surface.*

1. *There exists a complete proper minimal embedding of* **M** *in* **some** *smooth bounded domain* **D** $\subset \mathbb{R}^3$ *iff the* **number of nonorientable ends is finite** *and* **every end of M has infinite genus.**
2. *There exists a complete proper minimal embedding of* **M** *in* **every** *smooth bounded domain* **D** $\subset \mathbb{R}^3$ *iff* **M is orientable with every end having infinite genus.**

Theorem (Topological Obstruction Theorem (Ferrer, Martín, Meeks))

If **M** *is a nonorientable surface and has an* **infinite number of nonorientable ends**, *then* **M cannot properly embed** *in any smooth bounded domain of* $\mathbb{R}^3$.

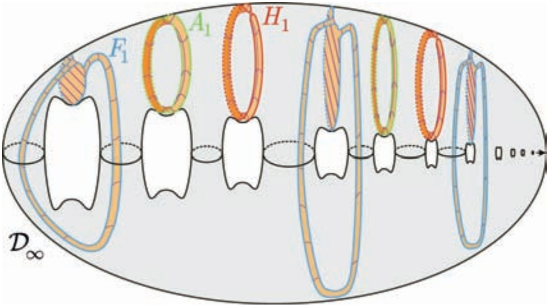
Topological Obstruction Theorem



Bounded embedded minimal surface conjecture

Conjecture (Ferrer, Martín, Meeks)

Let $\mathcal{D}_\infty$ be the bounded domain in $\mathbb{R}^3$ described in the figure, which is **smooth except at one point**. *A necessary and sufficient condition for an open surface* **M** *to admit a complete, proper minimal embedding in $\mathcal{D}_\infty$ is that every end of* **M** *has* **infinite genus**.



Examples with disjoint ends

Theorem (Ferrer, Martín, Meeks)

Let $\mathcal{D}$ be a domain which is convex (possibly $\mathcal{D} = \mathbb{R}^3$) or smooth and bounded. Given any open surface **M**, *there exists a complete proper minimal immersion $f : \mathbf{M} \to \mathcal{D}$.*
If $\mathcal{D}$ is smooth and bounded, then the minimal immersion f can be constructed such that the **limit sets** *of distinct ends of* **M** *are disjoint.*

Remark

The **proof** *of this result is the first key point in my approach with Kapouleas, Meeks and Nadirashvili to solving the* **embedded Calabi-Yau conjecture,** *including the nonorientable case.*

General existence theorem

Theorem (Ferrer, Martín, Meeks)

Let $\mathcal{D}$ be a domain which is convex (possibly $\mathcal{D} = \mathbb{R}^3$) or smooth and bounded. Given any open surface **M**, *there exists a complete proper minimal immersion $f : \mathbf{M} \to \mathcal{D}$.*
If $\mathcal{D}$ is smooth and bounded, then the minimal immersion f can be constructed such that the **limit sets** *of distinct ends of* **M** *are disjoint.*

MAIN TOOLS IN THE PROOF:

- **Density theorem.**
- **Existence of simple exhaustions.**
- **Bridge principle.**

Density theorem

Theorem (Alarcón, Ferrer, Martín)

Let **D** *be a convex domain. Then* **complete** *orientable minimal surfaces of finite topology which are* **properly immersed** *in* **D** *are dense in the space of all properly immersed orientable minimal surfaces in* **D***, endowed with the topology of* **smooth convergence on compact sets***.*

Remark

The above theorem also has been shown to hold in the **nonorientable** *setting by Ferrer, Martín and Meeks.*

Simple exhaustions.

Let **M** be a **noncompact** surface.

A smooth compact exhaustion

$$\mathcal{U} = \{M_1 \subset M_2 \subset \cdots\}$$

of M is called *simple* if:

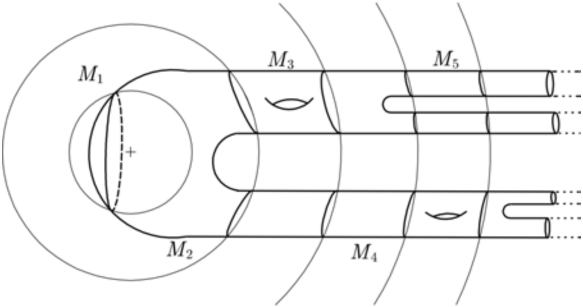
1. M_1 is a disk.

For all $n \in \mathbb{N}$:

2. Each component of $M_{n+1} - \text{Int}(M_n)$ has one boundary component in ∂M_n and at least one boundary component in ∂M_{n+1}.
3. $M_{n+1} - \text{Int}(M_n)$ contains a unique nonannular component which topologically is a pair of pants or an annulus with a handle or a Möbius strip with a disk removed.

If **M** has finite topology with genus g and k ends, then we call the compact exhaustion *simple* if properties 1 and 2 hold, property 3 holds for $n \leq g + k$, and when $n > g + k$, all of the components of $M_{n+1} - \text{Int}(M_n)$ are annular.

Simple exhaustions.



Lemma

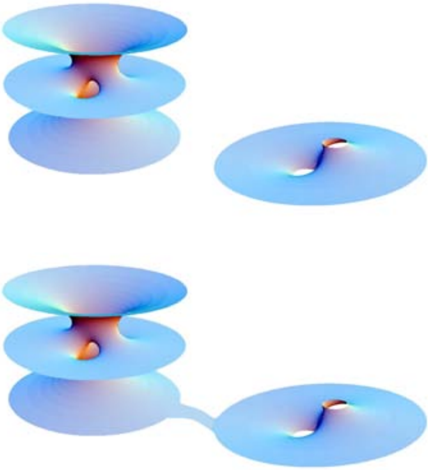
Every open surface admits a **simple exhaustion**.

Bridge principle.

Let $\mathbf{M}$ be a **nondegenerate** minimal surface in $\mathbb{R}^3$, and let $\mathbf{P} \subset \mathbb{R}^3$ be a thin curved rectangle whose two short sides lie along $\partial\mathbf{M}$ and that is otherwise disjoint from $\mathbf{M}$.

Theorem (White)

It is possible to deform $\mathbf{M} \cup \mathbf{P}$ slightly to make a minimal surface with boundary $\partial(\mathbf{M} \cup \mathbf{P})$.



Proof of the theorem in the convex case.

Let $\mathcal{D}$ be *a general convex domain* (not necessarily bounded or smooth). Consider $\{\mathcal{D}_n,\ n \in \mathbb{N}\}$ a smooth exhaustion of $\mathcal{D}$, where $\mathcal{D}_n$ is bounded and strictly convex, for all n. The existence of such a exhaustion is guaranteed by a classical result of **Minkowski.** Let **M** be an open surface and

$$\mathcal{M} = \{M_1 \subset M_2 \subset \cdots \subset M_n \subset \cdots\}$$

a simple exhaustion of **M**. Our purpose is to construct a sequence of minimal surfaces with nonempty boundary

$$\{\Sigma_n\}_{n\in\mathbb{N}}$$

satisfying:

(1_n). $\vec{0} \in \Sigma_n$ and $\partial\Sigma_n \subset \partial\mathcal{D}_n$;

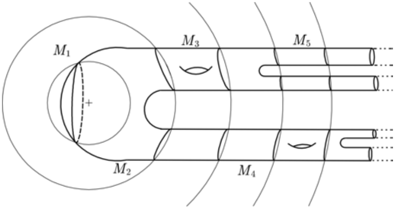
(2_n). $\mathrm{dist}_{\Sigma_n}(\vec{0}, \partial\Sigma_n) \geq \mathrm{dist}_{\Sigma_{n-1}}(\vec{0}, \partial\Sigma_{n-1}) + 1$;

(3_n). $\Sigma_n \cap \overline{\mathcal{D}}_{n-1} \approx \Sigma_{n-1}$ (in order to have a good limit of $\{\Sigma_n\}_{n\in\mathbb{N}}$.)

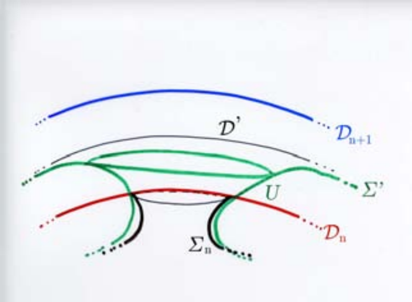
(4_n). $\Sigma_n \cap \overline{\mathcal{D}}_i$ is homeomorphic to M_i, for $i = 1, \ldots, n$.

The sequence $\{\Sigma_n\}_{n\in\mathbb{N}}$ reproduces the topological model of the simple exhaustion

$\mathcal{M} = \{M_1 \subset M_2 \subset \cdots \subset M_n \subset \cdots\}$

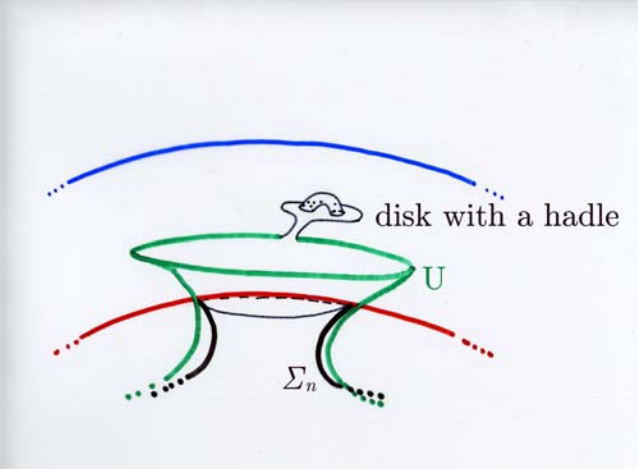


Adding handles. Consider the surface Σ_n. We want to add an annulus with a handle in a component of its boundary.

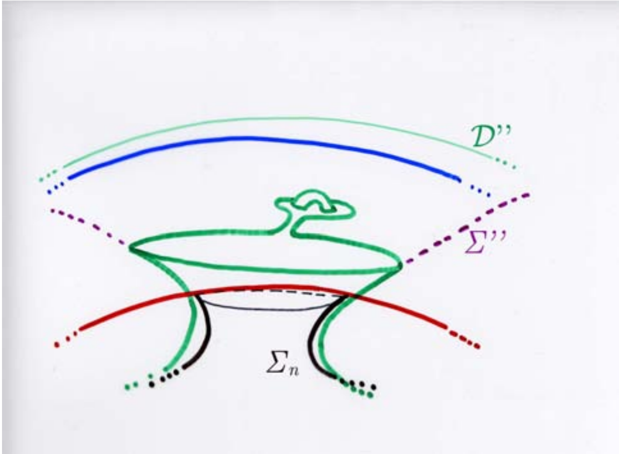


- $\mathcal{D}'$ is a convex domain satisfying $\overline{\mathcal{D}}_n \subset \mathcal{D}' \subset \overline{\mathcal{D}'} \subset \mathcal{D}_{n+1}$.
- Σ' is complete and proper in $\mathcal{D}'$. Take $U \subset \Sigma'$ such that $\mathrm{dist}_U(\vec{0}, \partial U) \geq \mathrm{dist}_{\Sigma_n}(\vec{0}, \partial \Sigma_n) + 1$.

We use the bridge principle to add a **minimal disk with a handle** near the boundary of **U**.

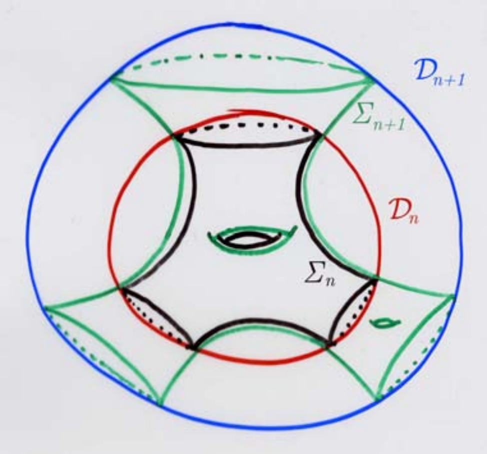


Finally, we consider a **convex domain** $\mathcal{D}''$ with $\overline{\mathcal{D}_{n+1}} \subset \mathcal{D}''$ and we apply the density theorem to obtain Σ''.

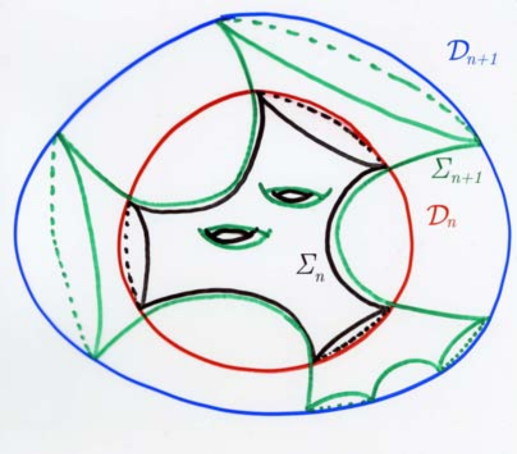


$$\Sigma_{n+1} \subseteq \Sigma'' \cap \overline{\mathcal{D}}_{n+1}.$$

Adding handles.



Adding pair of pants.



Nonexistence theorems

Theorem (Martín, Meeks, Nadirashvili, American J. Math. 2007)

Given D a bounded domain of space, there exists a countable family of horizontal simple closed curves $\{\sigma_n\}_{n\in\mathbb{N}}$, $\sigma_n \subset D$ $\forall n$, so that:

(I) $\widetilde{D} = D \setminus \left(\bigcup_{n\in\mathbb{N}} \sigma_n \right)$ *is a domain;*

(II) *There are* **no complete proper minimal surfaces with at least one annular end** *in* $\widetilde{D}$.

Let D a bounded domain and $\overline{D}$ its closure. We can assume

$$\overline{D} \subset \{(x_1, x_2, x_3) \in \mathbb{R}^3 \ : \ 0 \leq x_3 \leq 1\}$$

and $\overline{D}$ contains points at heights 0 and 1.

For $t \in (0,1)$ **denote**:

- $P_t \equiv$ horizontal plane $x_3 = t$.
- $C_t \stackrel{\text{def}}{=} D \cap P_t$.
- $\{C_{t,i}\}_{i \in I_t}$ the connected components of C_t (I_t countable).

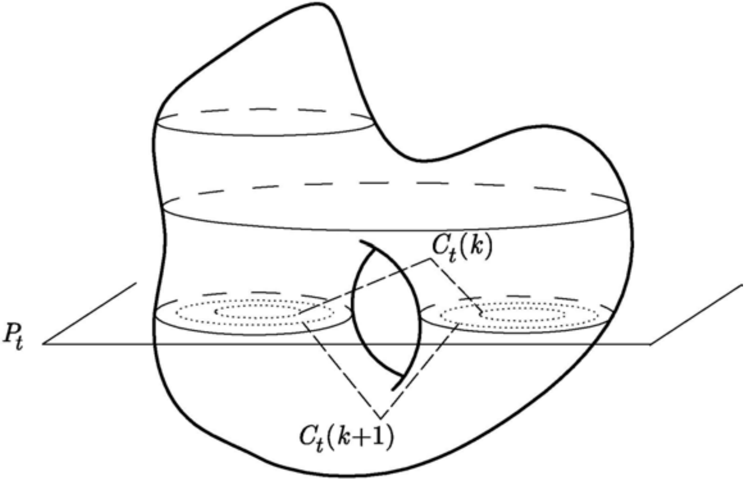
For each t and each $i \in I_t$, choose an exhaustion of $C_{t,i}$ by **smooth compact domains** $C_{t,i,k}$, $k \in \mathbb{N}$, and where:

- $C_{t,i,k} \subset \text{Int}\left(C_{t,i,k+1}\right), \quad \forall k \in \mathbb{N}$,
- $\displaystyle\sup_{x \in \partial C_{t,i,k}} \text{dist}(x, \partial C_{t,i}) < \frac{1}{k}, \quad \forall k \in \mathbb{N}.$

Finally, let

$$C_t(k) \stackrel{\text{def}}{=} \bigcup_{i \in I_t} C_{t,i,k}.$$

Nonexistence theorems



Now consider the following sequence of ordered rational numbers:

$$Q = \left\{ \frac{1}{2}, \frac{1}{3}, \frac{2}{3}, \frac{1}{4}, \frac{2}{4}, \frac{3}{4}, \ldots, \frac{1}{n}, \frac{2}{n}, \ldots, \frac{n-1}{n}, \ldots \right\}.$$

Let t_k the k-th rational number in Q.
Define $\mathcal{F}$ to be the collection of boundary curves to all of the domains $C_{t_k}(k)$, $k \in \mathbb{N}$, and define

$$\widetilde{D} \stackrel{\text{def}}{=} D - \mathcal{F}.$$

- $\widetilde{D}$ is open and connected.

Nonexistence theorems

Definition

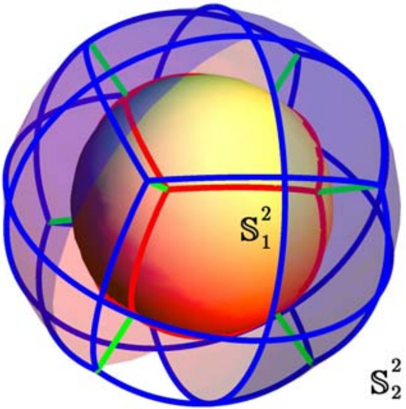
Given $\mathcal{D}$ a domain inside a Riemannian 3-manifold and $\mathbf{M}$ a noncompact surface with boundary (possibly empty), we will say that $\mathcal{D}$ is a **Calabi-Yau domain** for $\mathbf{M}$ iff $\mathbf{M}$ can not be properly immersed into $\mathcal{D}$ as a complete surface with bounded mean curvature.

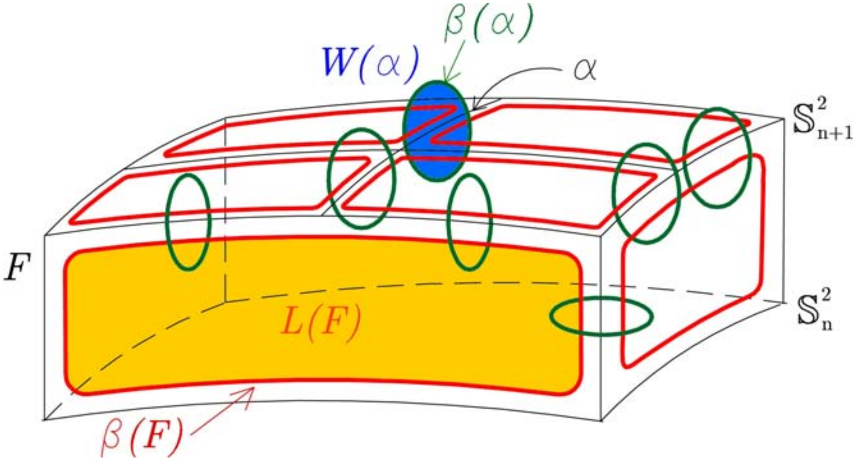
Theorem (Martín, Meeks, Preprint 2009)

Let $\overline{W}$ be a smooth compact Riemannian 3-manifold with nonempty boundary and let $W = \mathrm{Int}(\overline{W})$. There exists a properly embedded 1-manifold $\Delta \subset W$, such that $\mathcal{D} = W - \Delta$ is a Calabi-Yau domain for any surface with compact boundary (possibly empty) and at least one annular end. In particular, $\mathcal{D}$ does not admit any complete, noncompact, properly immersed surfaces of finite topology, compact boundary and constant mean curvature.

Nonexistence theorems

$$\mathbb{S}^2_n := \partial\left(\mathbb{B}(0, 1 - \frac{1}{2^n})\right)$$




β(α)
W(α)
α
$\mathbb{S}^2_{n+1}$
F
L(F)
$\mathbb{S}^2_n$
β(F)

Assume there exists $f : \mathbf{A} = \mathbb{S}^1 \times [0, +\infty[\longrightarrow \mathcal{D}$ complete proper minimal immersion with **bounded mean curvature.**
Define $L(\mathbf{A}) \stackrel{\text{def}}{=} \overline{f(\mathbf{A})} - f(\mathbf{A})$.

- $L(\mathbf{A})$ is closed and path connected.
- $L(\mathbf{A}) \subset \partial\mathcal{D} = \partial\mathbb{S}^2 \cup \Delta$.

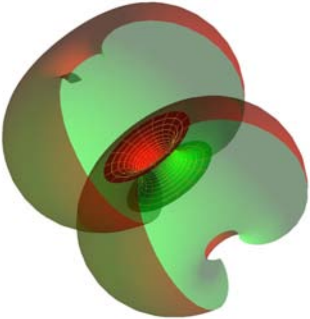
Lemma (Harvey-Lawson, Invent. math. 1975)

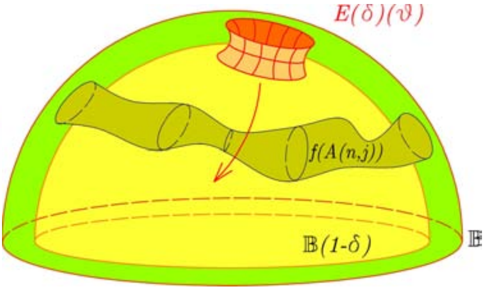
If $L(\mathbf{A}) \cap \Delta \neq \emptyset$ or $L(\mathbf{A})$ consists of a single point, then $\mathbf{A}$ has **finite area.**

Lemma

If $F : \mathbf{A} \to \mathbb{R}^3$ is a **complete** *immersion with* **bounded mean curvature**, *then $\mathbf{A}$ has* **infinite area.**

$L(\mathbf{A})$ is a path connected set in $\mathbb{S}^2$ containing **two different points.** The immersion can be seen as a proper immersion $f : \mathbf{A} \to \mathbb{B}$ with bounded mean curvature.





Left: This figure shows a domain on the nodoid corresponding to a scaling of $E(\delta)$.
Right: Since all of the $E(\delta)(\vartheta)$ are disjoint from $\partial f(A(n,j))$, a first point of contact in $E(\delta)(\vartheta_0) \cap f(A(n,j))$ occurs an interior point of $f(A(n,j))$.

Complete bounded null curves in $\mathbb{C}^n$

A complex curve $X : \mathbb{D}_1 \to \mathbb{C}^n$ is called **null** if $\sum_{j=1}^{n} (X_j')^2 = 0$, where

$X = (X_1, \ldots, X_n)$,

Theorem (Bourgain, Duke Math. J. 1993)

There are **no complete bounded** *null curves (with finite topology) in* $\mathbb{C}^2$*.*

Theorem (Jones, Proc. A. M. S. 1979)

There are complete bounded null curves $X : \mathbb{D}_1 \to \mathbb{C}^n$*, for* $n \geq 4$*. Furthermore,* X *is an* **embedding***.*

Theorem (Martín, Umehara, Yamada, Cal. Var. and PDE's 2009)

There is a **complete bounded** *null curve in* $\mathbb{C}^3$*.*