COMPLETE BOUNDED NULL CURVES IMMERSED IN $\mathbb{C}^3$ AND $\mathrm{SL}(2,\mathbb{C})$

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ABSTRACT. We construct a simply connected complete bounded Bryant surface in the hyperbolic 3-space $\mathcal{H}^3$. Such a surface in $\mathcal{H}^3$ can be lifted as a complete bounded null curve in $\mathrm{SL}(2,\mathbb{C})$. Using a transformation between null curves in $\mathbb{C}^3$ and null curves in $\mathrm{SL}(2,\mathbb{C})$, we are able to produce the first examples of complete bounded null curves in $\mathbb{C}^3$. As an application, we can show the existence of a complete bounded minimal surface in $\mathbb{R}^3$ whose conjugate minimal surface is also bounded. Moreover, we can show the existence of a complete bounded immersed complex submanifold in $\mathbb{C}^2$.

1. INTRODUCTION

The existence of complete non flat minimal surfaces with bounded coordinate functions, has been the instigator of many interesting articles on the theory of minimal surfaces in $\mathbb{R}^3$ and $\mathbb{C}^3$ over the last few decades. The question of whether there exists a complete bounded complex submanifold in $\mathbb{C}^n$ was proposed by P. Yang in [Y] and answered by P. Jones in [J] where this author present a short and elegant method to construct bounded (embedded) complex curves $X : \mathbb{D}_1 \to \mathbb{C}^3$, where $\mathbb{D}_1$ means the open unit disc of the complex plane. Although these curves are minimal in $\mathbb{C}^3$ (they are holomorphic), their respective projections $\mathrm{Re}\, X$ and $\mathrm{Im}\, X$ are not minimal in $\mathbb{R}^3$. If we pursue this, we need to impose that the complex curve $X : \mathbb{D}_1 \to \mathbb{C}^3$ also satisfies

$$(X_1')^2 + (X_2')^2 + (X_3')^2 = 0 \qquad \bigl(X = (X_1, X_2, X_3)\bigr), \tag{1.1}$$

where $'$ denotes the derivative with respect to the complex coordinate on $\mathbb{D}_1$. From now on, curves of this kind will be called *holomorphic null curves.*

The previous question is closely related to an earlier question by E. Calabi, who asked in 1965 [C] whether or not it is possible for a complete minimal surface in $\mathbb{R}^3$ to be contained in a ball in $\mathbb{R}^3$. Two articles, in particular, have made very important, if not fundamental, contributions to this problem. The first one was by L. P. Jorge and F. Xavier [JX], who constructed examples of complete minimal surfaces in a slab. The second one was by N. Nadirashvili [N], who more recently produced examples contained in a ball. In both cases, the key step was the ingenious use of Runge's classical theorem.

In respect to complete bounded minimal null curves in $\mathbb{C}^n$, the existence of such curves has been an open problem for $n = 3$. For the case $n = 2$, J. Bourgain [Bo] proves that these curves can not exist. Moreover, Jones in [J] proved that for $n \geq 4$ it is possible to construct complete bounded null curves in $\mathbb{C}^n$.

In this paper we give a positive solution to the existence of complete bounded null curves in $\mathbb{C}^3$ and obtain some interesting consequences. To be more precise, we prove the following theorem:

Theorem A. *There is a complete holomorphic null immersion $X : \mathbb{D}_1 \to \mathbb{C}^3$ whose image is bounded. In particular, there is a complete bounded* (*immersed*) *minimal surface in $\mathbb{R}^3$ such that its conjugate minimal surface is also bounded.*

Date: June 05, 2008.

The First author is partially supported by MEC-FEDER Grant no. MTM2007-61775. The second and the third authors are partially supported by Grant-in-Aid for Scientific Research (A) No. 19204005 and Scientific Research (B) No. 14340024, respectively, from the Japan Society for the Promotion of Science.

Here, we denote by $\mathbb{D}_r$ (resp. $\overline{\mathbb{D}}_r$) the open (resp. closed) ball in $\mathbb{C}$ of radius r centered at 0.

Since the projection of X into $\mathbb{C}^2$ gives a holomorphic immersion, we also get the following result, see Section 3.2:

Corollary B. *There is a complete holomorphic immersion $Y : \mathbb{D}_1 \to \mathbb{C}^2$ whose image is bounded.*

We remark that the existence of complete bounded complex submanifolds in $\mathbb{C}^3$ has been shown in [J].

Theorem A is equivalent to the existence of complete bounded null curves in $\mathrm{SL}(2,\mathbb{C})$, and also equivalent to complete bounded Bryant surfaces (mean curvature 1 surfaces) in the hyperbolic 3-space $\mathcal{H}^3$. Here a holomorphic map $\mathcal{B}\colon M \to \mathrm{SL}(2,\mathbb{C})$ from a Riemann surface M to the complex Lie group $\mathrm{SL}(2,\mathbb{C})$ is called *null* if the determinant $\det \mathcal{B}'$ of $\mathcal{B}' = d\mathcal{B}/dz$ vanishes, that is $\det \mathcal{B}' = 0$. where z is a complex coordinate of M. A projection $\beta = \pi \circ \mathcal{B}\colon M \to \mathcal{H}^3$ of a null holomorphic curve is a Bryant surface in $\mathcal{H}^3$, where $\pi\colon \mathrm{SL}(2,\mathbb{C}) \to \mathcal{H}^3 = \mathrm{SL}(2,\mathbb{C})/\mathrm{SU}(2)$ is the projection, see (2.12) in Section 2.2.

Then Theorem A is a corollary to the existence of complete bounded null curve in $\mathrm{SL}(2,\mathbb{C})$ as in Theorem C, see Section 3.1. To state the theorem, we define the matrix norm $|\cdot|$ as

$$|A| := \sqrt{\operatorname{trace} AA^*} \qquad (A^* := {}^t\overline{A}),$$

for 2×2-matrix A (see Appendix A). Note that if $A \in \mathrm{SL}(2,\mathbb{C})$, $|A| \geq \sqrt{2}$, and the equality holds if and only if A is the identity matrix.

Theorem C. *For each real number $\tau > \sqrt{2}$, there is a complete holomorphic null immersion $Y : \mathbb{D}_1 \to \mathrm{SL}(2,\mathbb{C})$ such that $|Y| < \tau$. In particular, there is a complete Bryant surface in $\mathcal{H}^3 = \mathrm{SL}(2,\mathbb{C})/\mathrm{SU}(2)$ of genus zero with one end contained in a given geodesic ball (of radius $\cosh^{-1}\tau$, see Lemma A.2 in Appendix A).*

A projection of immersed null holomorphic curves in $\mathbb{C}^3$ (resp. $\mathrm{SL}(2,\mathbb{C})$) onto Lorentz-Minkowski 3-space L^3 (resp. de Sitter 3-space S^3_1) gives maximal surfaces (resp. CMC-1 surfaces), which may admit singular points. Recently, Alarcon [A] constructed a space-like maximal surface bounded by a hyperboloid in L^3, which is weakly complete in the sense of [UY3] but may not be bounded. It should be remarked that our bounded null curve in $\mathbb{C}^3$ in Theorem A induces a bounded maximal surface in L^3 as a refinement of Alarcon's result:

Corollary D. *There are a weak complete space-like maximal surface in L^3 and a weakly complete space-like* CMC-1 *surface in S^3_1 whose image is bounded.*

The definition of weakly completeness for maximal surfaces and for CMC-1 surfaces (with singularities) are mentioned in the proof in Section 3.3.

Our procedure to prove Theorem C is similar in style to that used by Nadirashvili in [N] (see also [MN] for a more general construction). However, we have to improve the techniques because Nadirashvili's method does not allow us to control the imaginary part of the resulting minimal immersion. In order to do this, we work on a Bryant surface in hyperbolic 3-space $\mathcal{H}^3$ instead of a minimal surface in Euclidean 3-space. On each step of construction, we will apply Runge approximation for very small region of the surface, and so we can treat such a small part of the Bryant surface like as minimal surface in the Euclidean 3-space, which is the first crucial point. We shall give an error estimation between minimal surface and the Bryant surface by using the well-known ODE-technique (see A.2 in Appendix A). Next, we will lift the resulting bounded Bryant surface into a null curve $\mathrm{SL}(2,\mathbb{C})$. Since $\mathcal{H}^3$ is a quotient of $\mathrm{SL}(2,\mathbb{C})$ by $\mathrm{SU}(2)$, the compactness of $\mathrm{SU}(2)$ yields the boundedness of the lifted null curve associated with the bounded Bryant surface. Finally, using a transformation between null curves in $\mathbb{C}^3$ and null curves in $\mathrm{SL}(2,\mathbb{C})$, we

can get a complete bounded null immersed curve in $\mathbb{C}^3$ from the one in $\mathrm{SL}(2,\mathbb{C})$. Section 2 is devoted to explain this equivalence.

To prove Theorem C, the following lemma plays a crucial role (see Section 3.4):

Main Lemma. *Let ρ and τ be positive real numbers and $X\colon \overline{\mathbb{D}}_1 \to \mathrm{SL}(2,\mathbb{C})$ a holomorphic null immersion such that*

(1) $X(0) = \mathrm{id}$,
(2) $(\overline{\mathbb{D}}_1, ds^2_X)$ *contains the geodesic disc of radius ρ with center 0, where ds^2_X is the induced metric by X defined in* (2.11)*.*
(3) $|X(z)| \le \tau$ *for $z \in \overline{\mathbb{D}}_1$.*

Then for arbitrary positive numbers ε and s, there exists a holomorphic null immersion $Y = Y_{X,\varepsilon,s}\colon \overline{\mathbb{D}}_1 \to \mathrm{SL}(2,\mathbb{C})$ such that

(a) $Y(0) = \mathrm{id}$,
(b) $(\overline{\mathbb{D}}_1, ds^2_Y)$ *contains the geodesic disc of radius $\rho + s$ with center 0,*
(c) $|Y(z)| \le \tau\sqrt{1+2s^2+\varepsilon}$ *for $z \in \overline{\mathbb{D}}_1$,*
(d) $|Y - X| < \varepsilon$ *and $|\phi_Y - \phi_X| < \varepsilon$ on the open disc $\mathbb{D}_{1-\varepsilon}$, where $\phi_X = X^{-1}X'$, $\phi_Y = Y^{-1}Y'$ and $' = d/dz$ for the complex coordinate z in $\mathbb{D}_1$.*

Remark. A crucial difference between our main lemma and the main lemma in [N] is the estimation of extrinsic radius. In [N], the extrinsic radius is estimated as $\sqrt{r^2+s^2}+\varepsilon$ and the boundedness of the extrinsic radius of the resulting surface reduces to the fact that $\sum_{n=1}^{\infty} n^{-2}$ converges. However, our estimation of the extrinsic radius is multiplicative and reduces to the fact that $\prod_{n=1}^{\infty}(1+n^{-2})$ converges.

We would like to finish this introduction by mentioning a relating result. P. F. X. Müller in [M] introduced a remarkable relationship between complete bounded minimal surfaces in $\mathbb{R}^3$ and martingales.

2. Preliminaries

2.1. **Null curves in $\mathbb{C}^3$.** Let M be a Riemann surface and $\mathcal{F}\colon M \to \mathbb{C}^3$ a null holomorphic *immersion*, and set

$$\Phi = \phi\, dz, \qquad \phi = \phi_{\mathcal{F}} = \mathcal{F}' \qquad \left(' = \frac{d}{dz}\right), \tag{2.1}$$

where z is a local complex coordinate on M. Then $\phi = (\phi_1, \phi_2, \phi_3)$ is a locally defined $\mathbb{C}^3$-valued holomorphic function such that

$$\phi \cdot \phi = 0 \qquad \text{and} \qquad |\phi| > 0, \tag{2.2}$$

where $\cdot$ is the inner product of $\mathbb{C}^3$ defined by

$$\boldsymbol{x} \cdot \boldsymbol{y} = x_1y_1 + x_2y_2 + x_3y_3 \qquad \big(\boldsymbol{x} = (x_1,x_2,x_3), \boldsymbol{y} = (y_1,y_2,y_3)\big), \tag{2.3}$$

and $|\boldsymbol{x}| = \sqrt{\boldsymbol{x}\cdot\overline{\boldsymbol{x}}}$ is the Hermitian norm of $\mathbb{C}^3$. Conversely, if a $\mathbb{C}^3$-valued 1-form $\Phi = \phi\, dz$ on a simply connected Riemann surface M satisfies (2.2),

$$\mathcal{F}_\Phi(z) := \int_{z_0}^{z} \Phi\colon M \longrightarrow \mathbb{C}^3 \tag{2.4}$$

is a holomorphic null immersion, where $z_0 \in M$ is a base point.

We define the *induced metric* $ds^2_{\mathcal{F}}$ of $\mathcal{F}$ as

$$ds^2_{\mathcal{F}} = \frac{1}{2}|\Phi|^2 = \frac{1}{2}|\phi|^2\,|dz|^2 = \frac{1}{2}\lambda^2_{\mathcal{F}}|dz|^2 = \frac{1}{2}\mathcal{F}^*\langle\ ,\ \rangle \qquad (\lambda_{\mathcal{F}} := |\phi|), \tag{2.5}$$

where $\langle\ ,\ \rangle$ is the canonical Hermitian metric of $\mathbb{C}^3$. Note that $ds^2_{\mathcal{F}}$ is a *half* of the pull-back $\mathcal{F}^*\langle\ ,\ \rangle$, and coincides with the induced metric of the minimal immersion

$$\mathrm{Re}\,\mathcal{F}\colon M \longrightarrow \mathbb{R}^3.$$

Since $\mathcal{F}$ is a null curve, we can write

$$\Phi = \frac{1}{2}\bigl((1-g^2), \mathrm{i}(1+g^2), 2g\bigr)\eta\, dz, \qquad \mathrm{i} = \sqrt{-1} \tag{2.6}$$

where g and η are a meromorphic function and a holomorphic function, respectively. We call $(g, \eta\, dz)$ the *Weierstrass data* of $\mathcal{F}$. Using these Weierstrass data, we can write

$$\lambda_{\mathcal{F}} = \frac{1}{\sqrt{2}}(1+|g|^2)|\eta|. \tag{2.7}$$

Here, $g\colon M \to \mathbb{C}\cup\{\infty\}$ can be identified with the Gauss map by the stereographic projection.

2.2. **Null curves in** $\mathrm{SL}(2,\mathbb{C})$**.** A holomorphic map $\mathcal{B}$ from a Riemann surface M into the complex Lie group $\mathrm{SL}(2,\mathbb{C})$ is called *null* if $\det \mathcal{B}' = 0$ holds on M, where $'$ denotes the derivative with respect to a complex coordinate z. Take a null holomorphic *immersion* $\mathcal{B}\colon M \to \mathrm{SL}(2,\mathbb{C})$ and let

$$\Psi = \psi\, dz, \qquad \psi = \psi_{\mathcal{B}} = \mathcal{B}^{-1}\mathcal{B}'. \tag{2.8}$$

where $|\cdot|$ denotes the matrix norm defined in (A.1) in the Appendix. Since $\mathcal{B}$ is a null immersion, Ψ is a holomorphic $\mathfrak{sl}(2,\mathbb{C})$-valued 1-form with

$$\det\psi = 0 \qquad \text{and} \qquad |\psi| > 0, \tag{2.9}$$

Conversely, if an $\mathfrak{sl}(2,\mathbb{C})$-valued holomorphic 1-form $\Psi = \psi\, dz$ on a simply connected Riemann surface M satisfies (2.9), the solution $\mathcal{B}$ of an ordinary differential equation

$$\mathcal{B}^{-1}d\mathcal{B} = \Psi, \qquad \mathcal{B}(z_0) = \mathrm{id} \tag{2.10}$$

is a null holomorphic immersion into $\mathrm{SL}(2,\mathbb{C})$.

We define the *induced metric* $ds^2_{\mathcal{B}}$ of $\mathcal{B}$ as

$$ds^2_{\mathcal{B}} = \frac{1}{2}|\Psi|^2 = \frac{1}{2}|\psi|^2\,|dz|^2 = \frac{1}{2}\lambda^2_{\mathcal{B}}|dz|^2 = \frac{1}{2}\mathcal{B}^*\langle\ ,\ \rangle \qquad (\lambda_{\mathcal{B}} := |\psi|), \tag{2.11}$$

where $\langle\ ,\ \rangle$ is the canonical Hermitian metric of $\mathrm{SL}(2,\mathbb{C})$ induced from the matrix norm (A.1) in the Appendix. Identifying the hyperbolic 3-space $\mathcal{H}^3$ with the set

$$\mathcal{H}^3 = \{aa^*\,;\, a \in \mathrm{SL}(2,\mathbb{C})\} = \mathrm{SL}(2,\mathbb{C})/\,\mathrm{SU}(2)$$

as in (A.3) in Appendix A,

$$\beta := \pi\circ\mathcal{B} = \mathcal{B}\mathcal{B}^*\colon M \longrightarrow \mathcal{H}^3 \tag{2.12}$$

gives a conformal mean curvature one immersion (a CMC-1 surface or a Bryant surface, [Br1, UY1]), where $\pi\colon\ \mathrm{SL}(2,\mathbb{C}) \to \mathcal{H}^3 = \mathrm{SL}(2,\mathbb{C})/\,\mathrm{SU}(2)$ is the projection. The induced metric of β coincides with $ds^2_{\mathcal{B}}$, which is the reason why we add the coefficient $1/2$ in (2.11). Since $\mathcal{B}$ is null, we can write

$$\Psi = \frac{1}{\sqrt{2}}\begin{pmatrix} g & -g^2 \\ 1 & -g \end{pmatrix}\eta\, dz \tag{2.13}$$

where g and η are a meromorphic function and a holomorphic function, respectively. We call $(g, \eta\, dz)$ the *Weierstrass data* of $\mathcal{B}$. Then we can write

$$\lambda_{\mathcal{B}} = \frac{1}{\sqrt{2}}(1+|g|^2)|\eta|. \tag{2.14}$$

The meromorphic function g is called the *secondary Gauss map* of $\mathcal{B}$ ([UY2]).

If a null curve $\mathcal{F}$ in $\mathbb{C}^3$ and a null curve $\mathcal{B}$ in $\mathrm{SL}(2,\mathbb{C})$ are obtained by the same Weierstrass data $(g, \eta\, dz)$, their induced metrics coincide, and then they have the same intrinsic behavior. In this case, we call $\mathcal{B}$ the *cousin* of $\mathcal{F}$. The forms Φ and Ψ in (2.1) and (2.8) are related as

$$\hat{\psi} := \phi = (\phi_1, \phi_2, \phi_3) \longleftrightarrow \psi = \frac{1}{\sqrt{2}}\begin{pmatrix} \phi_3 & \phi_1 + \mathrm{i}\phi_2 \\ \phi_1 - \mathrm{i}\phi_2 & -\phi_3 \end{pmatrix}, \tag{2.15}$$

in which $|\phi| = |\psi|$ holds. Remark that the Weierstrass data (g, ω)in [UY1] coincides with $(g, \sqrt{2}\eta\, dz)$ here.

We call a $\mathbb{C}^3$-valued holomorphic 1-form $\Phi = \phi\, dz$ on M a *W-data* if (2.2) holds. If M is simply connected, it provides a null curve in $\mathbb{C}^3$ by (2.4), while a null curve in $\mathrm{SL}(2, \mathbb{C})$ by (2.15) and (2.10).

3. Proofs of the Theorems

3.1. **Correspondence of null curves in $\mathbb{C}^3$ and $\mathrm{SL}(2, \mathbb{C})$.** Firstly, we give a proof of Theorem A in the introduction using Theorem C. It should be remarked that even when a null curve in $\mathbb{C}^3$ is complete and bounded, its cousin in $\mathrm{SL}(2, \mathbb{C})$ may not be bounded in general. So we consider another transformation to prove Theorem A.

Let

$$\mathcal{T}\colon \{(x_1, x_2, x_3) \in \mathbb{C}^3\,;\, x_3 \neq 0\} \to \{(y_{ij}) \in \mathrm{SL}(2, \mathbb{C})\,;\, y_{11} \neq 0\} \tag{3.1}$$
$$\mathcal{T}(x_1, x_2, x_3) = \frac{1}{x_3}\begin{pmatrix} 1 & x_1 + \mathrm{i}x_2 \\ x_1 - \mathrm{i}x_2 & (x_1)^2 + (x_2)^2 + (x_3)^2 \end{pmatrix},$$

which is biholomorphic, and it can be easily checked that $\mathcal{T}$ maps null curves in $\mathbb{C}^3 \setminus \{x_3 = 0\}$ to null curves in $\mathrm{SL}(2, \mathbb{C})$. The map $\mathcal{T}$ is essentially the same as Bryant's transformation of null curves between $\mathbb{C}^3$ and the complex quadric Q^3 [Br2].

Proof of Theorem A via Theorem C. Let $\mathcal{B}\colon M \to \mathrm{SL}(2, \mathbb{C})$ be a complete bounded null immersion defined on a Riemann surface M. By replacing $\mathcal{B}$ by $a\mathcal{B}$ ($a \in \mathrm{SL}(2, \mathbb{C})$), we may assume $\mathcal{B}_{11} \neq 0$ without loss of generality, where $\mathcal{B} = (\mathcal{B}_{ij})$. Then $\mathcal{T}^{-1} \circ \mathcal{B}$ is a bounded holomorphic null immersion of M into $\mathbb{C}^3$. On any bounded set in $\mathbb{C}^3$, the pull-back of canonical Hermitian metric of $\mathrm{SL}(2, \mathbb{C})$ by $\mathcal{T}$ is equivalent to the canonical Hermitian metric of $\mathbb{C}^3$ by the following well-known Lemma 3.1. Hence the induced metric $ds^2_{\mathcal{T}^{-1}\circ\mathcal{B}}$ of $\mathcal{T}^{-1} \circ \mathcal{B}$ is complete because so is the induced metric $ds^2_{\mathcal{B}}$ of $\mathcal{B}$. □

Lemma 3.1. *Let g_1 and g_2 be two Riemannian metrics on a manifold N. For each compact subset K of N, there exist constants $a, b > 0$ such that $ag_1 \leq g_2 \leq bg_1$ on K.*

Remark 3.2. Boundedness of the real part of a null immersion in $\mathbb{C}^3$ (a minimal surface) does not imply the boundedness of its imaginary part (the conjugate surface) in general. In contrast to this fact, *a Bryant surface β is bounded if and only if so is its holomorphic lift $\mathcal{B}$*, because $|\beta| = |\mathcal{B}\mathcal{B}^*| \leq |\mathcal{B}|^2$ and $\operatorname{trace}\beta = \operatorname{trace}\mathcal{B}\mathcal{B}^* = |\mathcal{B}|^2$, where $|\cdot|$ denotes the matrix norm as in Appendix A. Or the *compact* Lie group $\mathrm{SU}(2)$ is considered as the "imaginary part" in $\mathcal{H}^3 = \mathrm{SL}(2, \mathbb{C})/\,\mathrm{SU}(2)$.

3.2. **Proof of Corollary B.** Next, we give a proof of Corollary B in the introduction. Let $\mathcal{F} = (F_1, F_2, F_3) : \mathbb{D}_1 \to \mathbb{C}^3$ be a null holomorphic immersion obtained by Theorem A, and $(g, \eta\, dz)$ the Weierstrass data as in (2.6). Then the projection $\hat{F} := (F_1, F_2)\colon \mathbb{D}_1 \to \mathbb{C}^2$ of $\mathcal{F}$ is a bounded holomorphic map. Moreover, it is a complete immersion. In fact,

$$4\left|\frac{d\hat{F}}{dz}\right|^2 = \left(|1 - g^2|^2 + |1 + g^2|^2\right)|\eta|^2 = 2(1 + |g|^4)|\eta|^2 \geq (1 + |g|^2)^2|\eta|^2 = ds^2_{\mathcal{F}}.$$

Hence $d\hat{F}/dz$ never vanishes and $\hat{F}$ is complete. □

3.3. **Proof of Corollary D.** Let $X = (X_1, X_2, X_3)\colon \mathbb{D}_1 \to \mathbb{C}^3$ be a bounded null immersion as in Theorem A. Then

$$f_X := \mathrm{Re}(\mathrm{i}X_1, X_2, X_3)\colon \mathbb{D}_1 \longrightarrow L^3$$

provides a maximal surface (i.e. with zero mean curvature) with singularities. Since X is an immersion, f_X is considered as a *maxface* in the sense of [UY3]. Moreover, the *lift*

metric as in [UY3, Definition 2.7] coincides with ds^2_X given in (2.5), which is complete. Then in the sense of [UY3, Definition 4.4], f_X is weakly complete.

Similarly, take a null immersion $Y\colon \mathbb{D}_1 \to \mathrm{SL}(2,\mathbb{C})$ as in Theorem C, and let

$$f_Y := Y \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} Y^* \colon \mathbb{D}_1 \longrightarrow S^3_1 = \mathrm{SL}(2,\mathbb{C})/\,\mathrm{SU}(1,1),$$

where S^3_1 is the de Sitter 3-space. Then f_Y gives a CMC-1 (i.e., mean curvature one) surface with singularities. Since Y is an immersion, f_Y is a *CMC-1 face* in the sense of [F] and [FRUYY]. Moreover, since the induced metric of ds^2_Y is complete, f_Y is *weakly complete* in the sense of [FRUYY]. □

3.4. **Proof of Theorem C via Main Lemma.** The proof of Theorem C via Main Lemma follows an standard inductive argument. We construct a suitable sequence $\{X_n\}_{n=0}^{\infty}$ of null immersions of $\overline{\mathbb{D}}_1$ into $\mathrm{SL}(2,\mathbb{C})$ as follows: Take an initial null immersion $X_0\colon \overline{\mathbb{D}}_1 \to \mathrm{SL}(2,\mathbb{C})$ such that $(\overline{\mathbb{D}}_1, ds^2_{X_0})$ is the geodesic disc of radius ρ_0 with center 0, and fix a positive integer $k_0 \geq 1$. For each integer $n > 0$, we apply our Main Lemma inductively supposing that X_{n-1} has been already constructed. Let

$$\varepsilon \leq \frac{1}{(n+k_0)^2}, \qquad s = \frac{1}{n+k_0} \ \left(\leq \frac{1}{8}\right).$$

Since $2s^2 + \varepsilon < 3/(n+k_0)^2$, we can construct X_n such that

(1) $(\mathbb{D}_1, ds^2_{X_n})$ contains a geodesic disc centered at the origin and of radius

$$\rho_n := \rho_0 + \sum_{k=1}^{n} \frac{1}{k+k_0},$$

(2) the inequality

$$|X_n|^2 \leq \tau_0^2 \prod_{k=1}^{n} \left(1 + \frac{3}{(k+k_0)^2}\right)$$

holds on $\overline{\mathbb{D}}_1$, where τ_0 is a positive constant depending only on the initial choice of the null curve X_0,

(3) and $\{X_n\}$ converges to a complete null immersion $X\colon \mathbb{D}_1 \to \mathrm{SL}(2,\mathbb{C})$ uniformly on any compact set of $\mathbb{D}_1$.

As a consequence, X satisfies

$$|X|^2 \leq \tau_0^2 \prod_{n=1}^{\infty} \left(1 + \frac{3}{(k+k_0)^2}\right) < \tau_0^2 \prod_{n=1}^{\infty} \left(1 + \left(\frac{2}{k}\right)^2\right) = \tau_0^2 \frac{\sinh(2\pi)}{2\pi}$$

on $\mathbb{D}_1$. We can choose the initial data of the initial curve X_0 such that τ_0 is arbitrarily close to $\sqrt{2}$. Since k_0 is also arbitrary, we can let $|X| < \tau$ for an arbitrary $\tau > \sqrt{2}$. □

4. Proof of Main lemma

4.1. **Labyrinth.** To prove the main lemma, we will work on Nadirashvili's labyrinth ([N, CR, MN]). We fix the definitions and notations on the Labyrinth: Let N be a (sufficiently large) positive number. For $k = 0, 1, 2, \dots, 2N^2$, we set

$$r_k = 1 - \frac{k}{N^3} \qquad \left(r_0 = 1, r_1 = 1 - \frac{1}{N^3}, \dots, r_{2N^2} = 1 - \frac{2}{N}\right), \tag{4.1}$$

and let

$$\mathbb{D}_{r_k} = \{z \in \mathbb{C}\,;\, |z| < r_k\} \quad \text{and} \quad S_{r_k} = \partial \mathbb{D}_{r_k} = \{z \in \mathbb{C}\,;\, |z| = r_k\}. \tag{4.2}$$

We define an annular domain $\mathcal{A}$ as

$$\mathcal{A} := \mathbb{D}_1 \setminus \mathbb{D}_{r_{2N^2}} = \mathbb{D}_1 \setminus \mathbb{D}_{1-\frac{2}{N}}, \tag{4.3}$$

and

$$A := \bigcup_{k=0}^{N^2-1} \mathbb{D}_{r_{2k}} \setminus \mathbb{D}_{r_{2k+1}}, \quad \widetilde{A} := \bigcup_{k=0}^{N^2-1} \mathbb{D}_{r_{2k+1}} \setminus \mathbb{D}_{r_{2k+2}},$$

$$L = \bigcup_{k=0}^{N-1} l_{\frac{2k\pi}{N}}, \qquad \widetilde{L} = \bigcup_{k=0}^{N-1} l_{\frac{(2k+1)\pi}{N}},$$

where l_θ is the ray $l_\theta = \{re^{\mathrm{i}\theta}\,;\, r \geq 0\}$. Let Σ be a compact set defined as

$$\Sigma := L \cup \widetilde{L} \cup S, \qquad S = \bigcup_{j=0}^{2N^2} \partial\mathbb{D}_{r_j} = \bigcup_{j=0}^{2N^2} S_{r_j},$$

and define a compact set Ω by

$$\Omega = \mathcal{A} \setminus U_{1/(4N^3)}(\Sigma),$$

where $U_\varepsilon(A)$ denotes the ε-neighborhood (of the Euclidean plane $\mathbb{R}^2 = \mathbb{C}$) of A. Each connected component of Ω has width $1/(2N^3)$.

For each number $j = 1, \dots, 2N$, we set

$$\omega_j := \big(l_{\frac{j\pi}{N}} \cap \mathcal{A}\big) \cup \big(\text{ connected components of } \Omega \text{ which intersect with } l_{\frac{j\pi}{N}}\big),$$
$$\varpi_j := U_{1/(4N^3)}(\omega_j).$$

Then ω_j's are compact sets.

4.2. **Transformation of Holomorphic data.** The construction of bounded complete minimal surfaces in $\mathbb{R}^3$ provides the following assertion (see [N, MN]):

Lemma 4.1. *Let $N(\geq 4)$ be an integer, and $\Phi = \phi\, dz$ a W-data on $\overline{\mathbb{D}}_1$ (in the sense of the end of Section 2). Then for each $\varepsilon > 0$ and each integer j satisfying $1 \leq j \leq 2N$, there exists a new W-data $\tilde{\Phi} = \tilde{\phi}\, dz$ on $\overline{\mathbb{D}}_1$ satisfying the following properties*:

(a) *On $\mathbb{D}_1 \setminus \varpi_j$, it holds that*

$$|\phi - \tilde{\phi}| < \frac{\varepsilon}{2N^2},$$

(b) *there exists a constant $C > 0$ depending only on Φ such that*

$$\begin{cases} |\tilde{\phi}| \geq C\, N^{3.5} & \text{on } \omega_j, \\ |\tilde{\phi}| \geq C\, N^{-0.5} & \text{on } \varpi_j, \end{cases}$$

(c) *there exists a real unit vector $\boldsymbol{u} = (u_1, u_2, u_3)$ in $\mathbb{R}^3$ such that $|u_3| > 1 - 2/N$ and*

$$\boldsymbol{u} \cdot (\phi - \tilde{\phi}) = 0$$

holds, where $\cdot$ is an inner product as in (2.3).

Proof. Let f_Φ be the real part of null immersion $\mathcal{F}_\Phi : \overline{\mathbb{D}}_1 \to \mathbb{C}^3$ as in (2.4) for $z_0 = 0$. We think in f_Φ as the minimal immersion $\mathcal{F}_{j-1}$ in [N, p. 463]. Then we can construct a new minimal immersion $\mathcal{F}_j$ imitating the corresponding procedure as in [N]. However, our construction of $\mathcal{F}_j$ is much easier. Actually, we do not need to adjust that $\mathcal{F}_j(\varpi_j)$ to be contained in a certain cone with small cone-angle centered at x_3-axis in $\mathbb{R}^3$.

The conditions (a) and (b) follow from [MN, p. 292-3, items (A.1$_i$), (A.2$_i$), and (A.3$_i$)]. The condition (c) follows from [N, (11)]. However, to get the estimate $|\tilde{\phi}| \geq CN^{-0.5}$ on ϖ_j, we need the property that

$$\frac{2}{\sqrt{N}} \leq |g| \leq \frac{\sqrt{N}}{2}$$

as in [MN, (B.3) or p. 293], where g is the meromorphic function in (2.6). For this purpose, the axis for the Lopez-Ros deformation corresponding to the deformation of Weierstrass

data $(g, \eta dz) \mapsto (g/h, h\eta dz)$ with respect to a certain holomorphic function as in [N, (4)] and in [MN, p. 294] might be slightly moved from the x_3-axis with the angle $|\theta| \le 2/\sqrt{N}$, that is

$$\cos\theta \ge 1 - \frac{2}{N}.$$

Finally, we let $\tilde{\Phi}$ to be the Weierstrass data of our $\mathcal{F}_j$. Then it satisfies the desired properties. □

4.3. **A reduction of Main Lemma.** In this subsection, we shall reduce our main lemma to the following

Key Lemma. *Let $\mathcal{B} = \mathcal{B}_0 : \overline{\mathbb{D}}_1 \to \mathrm{SL}(2,\mathbb{C})$ be a null holomorphic immersion satisfying $\mathcal{B}(0) = \mathrm{id}$ such that $(\overline{\mathbb{D}}_1, ds^2_{\mathcal{B}})$ contains a geodesic disc of radius ρ with center 0. For a sufficiently large integer N and each positive numbers ε and s with $s < 1/8$, there exists a sequence of null immersions*

$$\mathcal{B}_j : \overline{\mathbb{D}}_1 \to \mathrm{SL}(2,\mathbb{C}) \qquad (j = 1, \dots, 2N)$$

satisfying the following properties:

(1) $\mathcal{B}_j(0) = \mathrm{id}$,

(2) $|\psi_j - \psi_{j-1}| < \varepsilon/(2N^2)$ *holds on* $\mathbb{D}_1 \setminus \varpi_j$, *where*

$$\Psi_j = \psi_j dz := (\mathcal{B}_j)^{-1} d\mathcal{B}_j \qquad (j = 0, 1, \dots, 2N),$$

(3) *there exists a constant $c > 0$ depending only on $\mathcal{B}_0$ such that*

$$\begin{cases} |\psi_j| \ge c\, N^{3.5} & \text{on } \omega_j, \\ |\psi_j| \ge c\, N^{-0.5} & \text{on } \varpi_j, \end{cases}$$

(4) $\mathbb{D}_1$ *contains a closed geodesic disc $\overline{D}_g$ centered at 0 with radius $\rho + s$ with respect to $ds^2_{\mathcal{B}_{2N}}$. Moreover, it holds that*

$$|\mathcal{B}_{2N}(p)| \le \left(\max_{z \in \overline{\mathbb{D}}_1} |\mathcal{B}_0(z)|\right) \sqrt{1 + 2s^2 + \frac{b}{\sqrt{N}}} \quad \text{for} \quad p \in \partial D_g,$$

where ∂D_g is the boundary of $\overline{D}_g$, and $b > 0$ is a constant depending only on $\mathcal{B}_0$.

Proof. We construct the sequence of null curves $\mathcal{B}_1, \dots, \mathcal{B}_{2N}$ in $\mathrm{SL}(2,\mathbb{C})$ inductively.

Assume that $\mathcal{B}_{j-1}$ ($j \ge 1$) constructed already. Then $\mathcal{B}_j$ is constructed as follows: We set

$$\zeta_j := \left(1 - \frac{2}{N} - \frac{1}{4N^3}\right) e^{\mathrm{i}\pi j/N} \qquad (j = 1, 2, \dots, 2N).$$

Then $\zeta_j \in \partial\varpi_j$, and $|\zeta_j|$ attains the Euclidean distance between the origin $0 \in \mathbb{D}_1$ and ϖ_j, see Figure 1.

Let $\beta_{j-1} = \mathcal{B}_{j-1}\mathcal{B}^*_{j-1} : \overline{\mathbb{D}}_1 \to \mathcal{H}^3$ be a Bryant surface associated with $\mathcal{B}_{j-1}$, and we set

$$H(z) := a\big\{\mathcal{B}_{j-1}(\zeta_j)\big\}^{-1}\mathcal{B}_{j-1}(z)a^*, \qquad \text{and} \qquad h = HH^* : \overline{\mathbb{D}}_1 \to \mathcal{H}^3. \tag{4.4}$$

Here $a \in \mathrm{SU}(2)$ is chosen so that the geodesic line passing through $h(0)$ and $h(\zeta_j)$ lies in x_0x_3-plane in $\mathcal{H}^3 \subset L^4$. (Here, we consider $\mathcal{H}^3$ a hyperboloid in L^4, see (A.2) in Appendix A). We set

$$\frac{1}{\sqrt{2}}\begin{pmatrix} \psi_3 & \psi_1 + \mathrm{i}\psi_2 \\ \psi_1 - \mathrm{i}\psi_2 & -\psi_3 \end{pmatrix} = \psi := H^{-1}\frac{dH}{dz} \qquad \text{and} \qquad \phi = (\psi_1, \psi_2, \psi_3).$$

Then ϕ is a W-data on $\overline{\mathbb{D}}_1$, and one can easily check that

$$\psi_{j-1} = a\psi a^*. \tag{4.5}$$

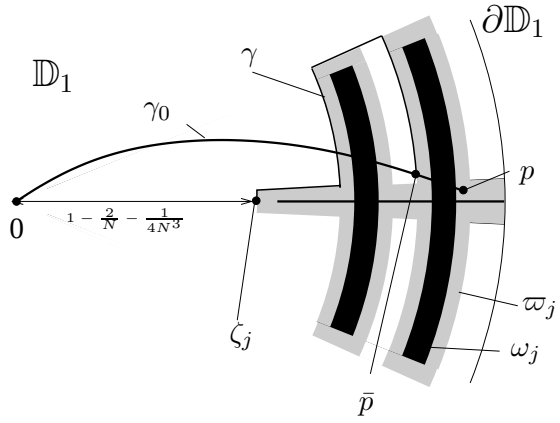

FIGURE 1. The point ζ_j and the curve γ

Applying Lemma 4.1 to ϕ, we get new a W-data $\tilde{\phi} = (\tilde{\phi}_1, \tilde{\phi}_2, \tilde{\phi}_3)$ satisfying (a)–(c). Then we define a new null immersion $\widetilde{H} : \overline{\mathbb{D}}_1 \to \mathrm{SL}(2, \mathbb{C})$ such that

$$\widetilde{H}^{-1}\frac{d\widetilde{H}}{dz} = \tilde{\psi} = \frac{1}{\sqrt{2}}\begin{pmatrix} \tilde{\phi}_3 & \tilde{\phi}_1 + \mathrm{i}\tilde{\phi}_2 \\ \tilde{\phi}_1 - \mathrm{i}\tilde{\phi}_2 & -\tilde{\phi}_3 \end{pmatrix}, \qquad \text{and} \qquad \widetilde{H}(\zeta_j) = \mathrm{id}\,. \tag{4.6}$$

After that, we set

$$\mathcal{B}_j(z) := a^*\big\{\widetilde{H}(0)\big\}^{-1}\widetilde{H}(z)a.$$

Then $\mathcal{B}_j(0) = \mathrm{id}$, which proves (1), and if we set $\psi_j\, dz := \mathcal{B}_j^{-1} d\mathcal{B}_j$, it holds that

$$\psi_j = a\tilde{\psi}a^*. \tag{4.7}$$

By (a), (b) in Lemma 4.1 and (4.5) and (4.7), we get the assertions (2) and (3).

In this way, we can get $\mathcal{B}_{2N}$ inductively. Note that, by (2) and Corollary A.6, it holds that

$$\mathrm{dist}_{\mathcal{H}^3}\big(\beta_j(z), \beta_{j-1}(z)\big) \le \frac{c_1\varepsilon}{2N^2}, \qquad \text{on} \qquad \mathbb{D}_1 \setminus \varpi_j, \tag{4.8}$$

where $\beta_j = \mathcal{B}_j\mathcal{B}_j^*$, $\beta_{j-1} = \mathcal{B}_{j-1}\mathcal{B}_{j-1}^*$ and $\mathrm{dist}_{\mathcal{H}^3}$ stands for the canonical distance function of $\mathcal{H}^3$, see A.1 in Appendix A. Here, the constant c_1 depends only on $\mathcal{B}_0$. *Throughout this proof, we shall denote by*

$$c_1, c_2, \dots$$

constants which depend only on $\mathcal{B}_0$. The properties (2) and (3) yield that $\mathbb{D}_1$ contains a closed geodesic disc $\overline{D}_g$ of radius $\rho + s$ centered at 0 with respect to the induced metric $ds^2_{\mathcal{B}_{2N}}$ by $\mathcal{B}_{2N}$ (see [N, p. 463]). Let $p \in \partial D_g$ and we will show (4) in the statement of the Lemma. If $p \notin \varpi_j$ for all $j = 1, \dots, 2N$, the inequality is easy to show using (4.8) and Lemma A.2 in Appendix A, see [N, MN].

Otherwise, we assume $p \in \varpi_j$ for some j. Let γ_0 be the $ds^2_{\mathcal{B}_{2N}}$-geodesic joining 0 and p, and take $\bar{p} \in \gamma_0 \cap \partial\varpi_j$ such that the geodesic joining $\bar{p}$ and p lies in ϖ_j, see Figure 1. Then the $ds^2_{\mathcal{B}_{2N}}$-distance of p and $\bar{p}$ satisfies

$$\mathrm{dist}_{ds^2_{\mathcal{B}_{2N}}}(\bar{p}, p) \le s + \frac{c_2}{\sqrt{N}},$$

where c_2 is a constant depending only on $\mathcal{B}_0$ (see [MN, p. 296]). Thus, by taking a suitable path γ (as in Figure 1) joining ζ_j and $\bar{p}$ in the complement of ϖ_j, we have

$$\mathrm{dist}_{ds^2_{\mathcal{B}_{2N}}}(\zeta_j, p) \le s + \frac{c_3}{\sqrt{N}}, \tag{4.9}$$

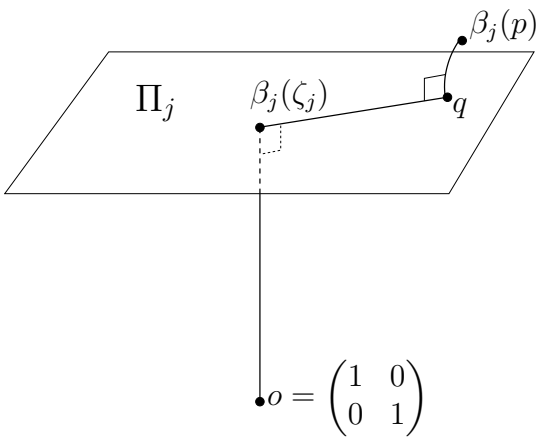

FIGURE 2. the plane Π_j

since $\mathrm{dist}_{ds^2_{\mathcal{B}_{2N}}}(\zeta_j, \bar{p})$ is of order $1/N^2$.

Let Π_j be the totally geodesic plane in $\mathcal{H}^3$ passing through $\beta_j(\zeta_j)$ which is perpendicular to the geodesic joining o and $\beta_j(\zeta_j)$, where $o \in \mathcal{H}^3$ is the point corresponding to the identity matrix id (as in (A.2) in Appendix A). Let q ($\in \Pi_j$) be the foot of perpendicular from $\beta_j(p)$ to the plane Π_j, see Figure 2. Then $\mathrm{dist}_{\mathcal{H}^3}\big(q, \beta_j(p)\big)$ gives the distance of $\beta_j(p)$ and the plane Π_j, and (4) is obtained as a conclusion of the following Lemma:

Lemma 4.2. *Under the situations above, namely, for $p \in \partial D_g \cap \varpi_j$ and for $q \in \Pi_j$ satisfying*

$$\mathrm{dist}_{\mathcal{H}^3}(\beta_j(p), q) = \mathrm{dist}_{\mathcal{H}^3}(\beta_j(p), \Pi_j),$$

it holds that

$$\mathrm{dist}_{\mathcal{H}^3}\big(\beta_j(p), q\big) \le 14s^2 + \frac{c_4}{\sqrt{N}},$$

where c_4 is a constant depending only on $\mathcal{B}_0$.

We shall prove this lemma later, and now finish the proof of Key Lemma:

Proof of Key Lemma, continued. Assume Lemma 4.2 is true. By (4.8) and Lemma A.2, we have

$$\begin{aligned} |\mathcal{B}_j(\zeta_j)|^2 &= 2\cosh\big(\mathrm{dist}_{\mathcal{H}^3}\big(o, \beta_j(\zeta_j)\big)\big) \le 2\cosh\Big(\mathrm{dist}_{\mathcal{H}^3}\big(o, \beta_0(\zeta_j)\big) + \frac{c_1\varepsilon}{N}\Big) \\ &\le \big(\max_{z\in\overline{\mathbb{D}}_1}|\beta_0(z)|\big)^2\left(1 + \frac{c_5}{N}\right), \qquad \left(o := \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \in \mathcal{H}^3\right), \end{aligned}$$

see (A.4) in Appendix A. On the other hand, by (4.9) (4.8) and the fact that the three points $\beta_j(\zeta_j)$, $\beta_j(p)$, q form a right triangle (Figure 2).

$$\mathrm{dist}_{\mathcal{H}^3}\big(\beta_j(\zeta_j), q\big) \le \mathrm{dist}_{\mathcal{H}^3}\big(\beta_j(\zeta_j), \beta_j(p)\big) \le \mathrm{dist}_{ds^2_{\mathcal{B}_{2N}}}(\zeta_j, p) \le s + \frac{c_3}{\sqrt{N}}.$$

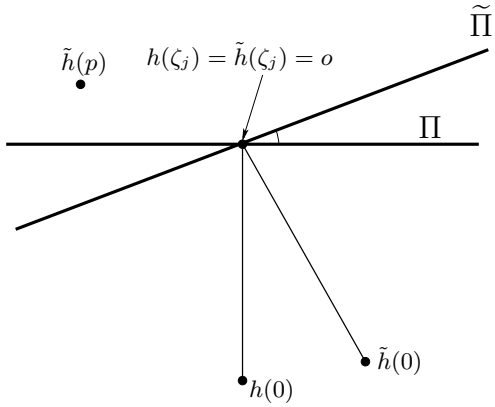

FIGURE 3. Planes Π and $\widetilde{\Pi}$

Thus, applying the hyperbolic Pythagorean theorem for the hyperbolic right triangle $o\beta_j(\zeta_j)q$ (see Figure 2), we have

$$\begin{aligned}\cosh \mathrm{dist}_{\mathcal{H}^3}\big(o,q\big) &= \cosh \mathrm{dist}_{\mathcal{H}^3}\big(0,\beta_j(\zeta_j)\big)\cosh \mathrm{dist}_{\mathcal{H}^3}\big(\beta_j(\zeta_j),q\big)\\ &\le \frac{1}{2}\big(\max_{z\in\overline{\mathbb{D}}_1}|\mathcal{B}_0(z)|\big)^2\left(1+\frac{c_5}{N}\right)\cosh\left(s+\frac{c_3}{\sqrt{N}}\right)\\ &\le \frac{1}{2}\big(\max_{z\in\overline{\mathbb{D}}_1}|\mathcal{B}_0(z)|\big)^2\left(1+s^2+\frac{c_6}{\sqrt{N}}\right)\end{aligned}$$

for a sufficiently large N. The last inequality is obtained by the inequality $\cosh x \le 1+x^2$ for $x\in[0,2]$, and $s+c_3/\sqrt{N}<2$ for a sufficiently large number N. Hence by Lemma 4.2,

$$\begin{aligned}\frac{1}{2}\big|\mathcal{B}_j(p)\big|^2 &= \cosh \mathrm{dist}_{\mathcal{H}^3}\big(o,\beta_j(p)\big) \le \cosh\big[\mathrm{dist}_{\mathcal{H}^3}\big(o,q\big)+\mathrm{dist}_{\mathcal{H}^3}\big(q,\beta_j(p)\big)\big]\\ &= \cosh\big(\mathrm{dist}_{\mathcal{H}^3}(o,q)\big)\cosh\big(\mathrm{dist}_{\mathcal{H}^3}(q,\beta_j(p))\big)\\ &\qquad\qquad + \sinh\big(\mathrm{dist}_{\mathcal{H}^3}(o,q)\big)\sinh\big(\mathrm{dist}_{\mathcal{H}^3}(q,\beta_j(p))\big)\\ &\le \exp\left(\mathrm{dist}_{\mathcal{H}^3}(q,\beta_j(p))\right)\cosh \mathrm{dist}_{\mathcal{H}^3}\big(o,q\big)\\ &\le \exp\left(14s^2+\frac{c_4}{\sqrt{N}}\right)\frac{1}{2}\big(\max_{z\in\overline{\mathbb{D}}_1}|\mathcal{B}_0(z)|\big)^2\left(1+s^2+\frac{c_6}{\sqrt{N}}\right)\\ &\le \frac{1}{2}\big(\max_{z\in\overline{\mathbb{D}}_1}|\mathcal{B}_0(z)|\big)^2\left(1+2\left(14s^2+\frac{c_4}{\sqrt{N}}\right)\right)\left(1+s^2+\frac{c_6}{\sqrt{N}}\right)\\ &\le \frac{1}{2}\big(\max_{z\in\overline{\mathbb{D}}_1}|\mathcal{B}_0(z)|\big)^2\left\{1+s^2+2\left(14s^2+\frac{c_4}{\sqrt{N}}\right)s^2+\frac{c_7}{\sqrt{N}}\right\}\\ &\le \frac{1}{2}\big(\max_{z\in\overline{\mathbb{D}}_1}|\mathcal{B}_0(z)|\big)^2\left(1+2s^2+\frac{c_7}{\sqrt{N}}\right),\end{aligned}$$

here we used the inequality $\exp x \le 1+2x$ for $x\in[0,1]$, and $14s^2+c_4/\sqrt{N}<1/2$ for $s<1/8$ and a sufficiently large N. Thus, using (4.8), we have (4). □

Proof of Lemma 4.2. Let $\tilde{h}=\widetilde{H}\widetilde{H}^*$. Note that h and $\tilde{h}$ are congruent to β_{j-1} and β_j respectively, and $h(\zeta_j)=\tilde{h}(\zeta_j)=o$ because of (4.4) and (4.6). Let $\widetilde{\Pi}$ be the totally geodesic plane through o which is perpendicular to the geodesic joining o and $\tilde{h}(0)$, see Figure 3. Let ι be the isometry of $\mathcal{H}^3$ such that $\iota\circ\beta_j=\tilde{h}$. Then it holds that

$$\widetilde{\Pi}=\iota(\Pi_j),$$

and it is sufficient to estimate the distance between $\widetilde{\Pi}$ and $\tilde{h}(p)$. However, our estimation is not direct.

Firstly, we consider the plane Π passing through o which is perpendicular to the geodesic ray $oh(0)$, which is expressed as

$$\Pi = \{(x_0, x_1, x_2, x_3) \in L^4\,;\, x_3 = 0\} \cap \mathcal{H}^3 \tag{4.10}$$

because of the definition of H in (4.4). Then it can be expected that $h(0)$ must be sufficiently close to $\tilde{h}(0)$ because $\beta_{j-1}(\zeta_j)$ is very close to $\beta_j(\zeta_j)$, and that the distance of $\tilde{h}(p)$ and Π might be close to that of $\tilde{h}(p)$ and $\widetilde{\Pi}$. According to this observation, we shall firstly estimate the distance between $\tilde{h}(p)$ and Π, next give the estimation of the angle of Π and $\widetilde{\Pi}$, and will get the conclusion:

Step 1: *The estimation of the distance of* $\tilde{h}(p)$ *and* Π. We set

$$F(z) = \int_{\zeta_j}^{z} \psi(z)\,dz, \qquad \widetilde{F}(z) = \int_{\zeta_j}^{z} \tilde{\psi}(z)\,dz,$$

where the integration is done in the path γ through the boundary of ϖ_j as in Figure 1. Then by Theorem A.4 in Appendix A, we have

$$H(p) = \mathrm{id} + F(p) + \Delta, \qquad \widetilde{H}(p) = \mathrm{id} + \widetilde{F}(p) + \widetilde{\Delta},$$

where

$$|\Delta| \le \left[\left(\max_{\gamma} |H|\right) \int_{\zeta_j}^{p} |\psi|\,|dz|\right]^2, \quad |\widetilde{\Delta}| \le \left[\left(\max_{\gamma} |\widetilde{H}|\right) \int_{\zeta_j}^{p} |\tilde{\psi}|\,|dz|\right]^2.$$

Here, since the Euclidean length of γ is estimated by c_8/N, we have

$$|F(p)| \le \int_{\zeta_j}^{p} |\psi|\,|dz| \le \sqrt{2}\int_{\zeta_j}^{p} ds_H \le \frac{c_9}{N}. \tag{4.11}$$

On the other hand (4.9) yields

$$|\widetilde{F}(p)| \le \int_{\zeta_j}^{p} |\tilde{\psi}|\,|dz| = \sqrt{2}\int_{\zeta_j}^{p} ds_{\widetilde{H}} \le \sqrt{2}\left(s + \frac{c_3}{\sqrt{N}}\right), \tag{4.12}$$

Moreover, by (4.8), it holds that $\mathrm{dist}_{ds_H^2}(\zeta_j, p) \le c_{10}/N$, and we have

$$\max_{\gamma} |\widetilde{H}|^2 \le 2\left(1 + \frac{c_{11}}{N}\right), \quad \text{and then} \quad |\Delta| \le \frac{c_{12}}{N^2}.$$

On the other hand, by (4.9), we have

$$\max_{\gamma} |H|^2 \le 2\cosh\left(s + \frac{c_3}{\sqrt{N}}\right) \le 2\left(1 + s^2 + \frac{c_{13}}{\sqrt{N}}\right),$$

and then,

$$\begin{aligned} |\widetilde{\Delta}| &\le \left(\max_{\gamma} |\widetilde{H}|^2\right) |\widetilde{F}(p)|^2 \le 4\left(1 + s^2 + \frac{c_{13}}{\sqrt{N}}\right)\left(s + \frac{c_3}{\sqrt{N}}\right)^2 \\ &\le 4(1+s^2)s^2 + \frac{c_{14}}{\sqrt{N}} \le 4.5s^2 + \frac{c_{14}}{\sqrt{N}} \end{aligned} \tag{4.13}$$

because $s \le 1/8$. We let

$$\begin{aligned} \begin{pmatrix} x_0 + x_3 & x_1 + \mathrm{i}x_2 \\ x_1 - \mathrm{i}x_2 & x_0 - x_3 \end{pmatrix} &:= \tilde{h}(p) = (\mathrm{id} + \widetilde{F} + \widetilde{\Delta})(\mathrm{id} + \widetilde{F}^* + \widetilde{\Delta}^*) \\ &= \mathrm{id} + \widetilde{F} + \widetilde{F}^* - F - F^* + \delta, \\ \text{where} \quad \delta &= F + F^* + \widetilde{F}\widetilde{F}^* + \widetilde{\Delta} + \widetilde{\Delta}^* + \widetilde{\Delta}\widetilde{F}^* + \widetilde{F}\widetilde{\Delta}^* + \widetilde{\Delta}\widetilde{\Delta}^*. \end{aligned} \tag{4.14}$$

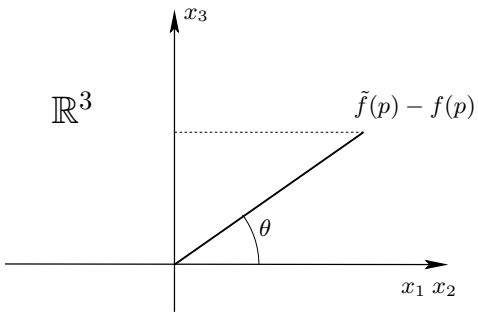

FIGURE 4. The angle θ

We can choose N sufficiently large so that $s + c_3/\sqrt{N} < 1/7$ because of $s < 1/8$. So we have

$$|\widetilde{\Delta}\widetilde{F}^* + \widetilde{F}\widetilde{\Delta}^*| \leq 2|\widetilde{\Delta}||\widetilde{F}| = \frac{9\sqrt{2}}{7}s^2 + \frac{c_{15}}{\sqrt{N}} < 2s^2 + \frac{c_{15}}{\sqrt{N}}.$$

Then by (4.11), (4.12) and (4.13) and the assumption $s \leq 1/8$, we have

$$|\delta| \leq \Big(|\widetilde{F}\widetilde{F}^*| + (|\widetilde{\Delta}| + |\widetilde{\Delta}^*|) + |\widetilde{\Delta}\widetilde{F}^* + \widetilde{F}\widetilde{\Delta}^*|\Big) + |\widetilde{\Delta}\widetilde{\Delta}^*| + |F| + |F^*|$$

$$\leq (2+9+2)s^2 + (4.5)^2 s^4 + \frac{c_{16}}{\sqrt{N}} \leq 14s^2 + \frac{c_{16}}{\sqrt{N}}.$$

We set

$$f(z) := F(z) + F^*(z) = \int_{\zeta_j}^{z} (\psi(z) + \psi^*(z))dz,$$
$$\tilde{f}(z) := \tilde{F}(z) + \tilde{F}^*(z) = \int_{\zeta_j}^{z} (\tilde{\psi}(z) + \tilde{\psi}^*(z))dz.$$

The Minkowski 4-space L^4 can be identified with the space of 2×2-hermitian matrices $\mathrm{Herm}(2)$, as in (A.2) in Appendix A. The Euclidean 3-space $\mathbb{R}^3$ is isometrically embedded in $\mathrm{Herm}(2)$ by

$$\mathbb{R}^3 \ni (x_1, x_2, x_3) \mapsto \begin{pmatrix} x_3 & x_1 + \mathrm{i}x_2 \\ x_1 - \mathrm{i}x_2 & -x_3 \end{pmatrix} \in \mathrm{Herm}(2).$$

Then $f(z)$ and $\tilde{f}(z)$ can be considered as minimal immersions in $\mathbb{R}^3 (\subset \mathrm{Herm}(2))$ induced by ψ and $\tilde{\psi}$ respectively (see (2.13) and (2.15)). Then, (c) of Lemma 4.1 implies that the angle θ between x_1x_2-plane in $\mathbb{R}^3$ and the vector $\tilde{f}(p) - f(p) \in \mathbb{R}^3$ is less than $2/\sqrt{N}$, see Figure 4. Applying this fact, we shall now estimate the distance between Π in (4.10) and $h(p)$: Since the distance between $h(p) = (x_0, x_1, x_2, x_3)$ and its foot of perpendicular to the plane Π is equal to $\sinh^{-1}|x_3|$, (4.14) yields that

$$\begin{aligned}
\mathrm{dist}_{\mathcal{H}^3}\big(\tilde{h}(p), \Pi\big) &= \sinh^{-1}|x_3| \leq |x_3| = \Big|\text{the } x_3\text{-component of } \tilde{h}(p)\Big| \\
&= \Big|\text{the } x_3\text{-component of } f(p) - \tilde{f}(p) + \delta\Big| \\
&\leq |\text{the } x_3\text{-component of } \delta| + \Big|\text{the } x_3\text{-component of } f(p) - \tilde{f}(p)\Big| \\
&\leq |\delta| + |f(p) - \tilde{f}(p)| \sin\theta \leq 14s^2 + \frac{c_{17}}{\sqrt{N}},
\end{aligned}$$

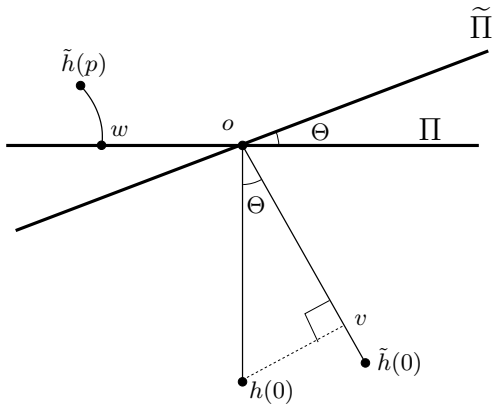

FIGURE 5. Planes Π and $\widetilde{\Pi}$

where we used the inequality

$$\begin{aligned}|x_3| &= \left|\text{the } x_3\text{-component of } \begin{pmatrix} x_0+x_3 & x_1+\mathrm{i}x_2 \\ x_1-\mathrm{i}x_2 & x_0-x_3 \end{pmatrix}\right| \\ &\le \sqrt{2(x_0^2+x_1^2+x_2^2+x_3^2)} = \left|\begin{pmatrix} x_0+x_3 & x_1+\mathrm{i}x_2 \\ x_1-\mathrm{i}x_2 & x_0-x_3 \end{pmatrix}\right|.\end{aligned}$$

Step 2: *The estimation of the angle between of* Π *and* $\widetilde{\Pi}$. By (4.8), we have

$$\mathrm{dist}_{\mathcal{H}^3}\bigl(h(0),\tilde{h}(0)\bigr) \le \frac{c_1\varepsilon}{2N^2}. \tag{4.15}$$

Next, applying (4.8), we have that

$$\begin{aligned}\mathrm{dist}_{\mathcal{H}^3}(o,\beta_0(\zeta_j)) &\le \mathrm{dist}_{\mathcal{H}^3}(o,\beta_{j-1}(\zeta_j)) + \sum_{k=1}^{j-1}\mathrm{dist}_{\mathcal{H}^3}(\beta_k(\zeta_j),\beta_{k-1}(\zeta_j)) \\ &\le \mathrm{dist}_{\mathcal{H}^3}(o,\beta_{j-1}(\zeta_j)) + \frac{c_1\varepsilon}{N},\end{aligned}$$

that is, there exists a positive constant c_{18} such that

$$\begin{aligned}\mathrm{dist}_{\mathcal{H}^3}(h(0),o) &= \mathrm{dist}_{\mathcal{H}^3}(h(0),h(\zeta_j)) = \mathrm{dist}_{\mathcal{H}^3}(o,\beta_{j-1}(\zeta_j)) \\ &\ge \mathrm{dist}_{\mathcal{H}^3}(o,\beta_0(\zeta_j)) - \frac{c_1\varepsilon}{N} \ge c_{18} > 0,\end{aligned} \tag{4.16}$$

if N is sufficiently large.

Now, let v be the foot of perpendicular from $h(0)$ to the geodesic $o\,\tilde{h}(0)$. Then we have

$$\mathrm{dist}_{\mathcal{H}^3}(h(0),v) \le \mathrm{dist}_{\mathcal{H}^3}(h(0),\tilde{h}(0)), \tag{4.17}$$

since the triangle $ovh(0)$ is a right triangle (see Figure5). Let Θ be the angle between the geodesics $oh(0)$ and ov at o, which coincides with the angle between Π and $\widetilde{\Pi}$, see Figure 5. Applying the sine law in hyperbolic geometry, we have

$$\frac{\sinh \mathrm{dist}_{\mathcal{H}^3}(o,h(0))}{\sin(\pi/2)} = \frac{\sinh \mathrm{dist}_{\mathcal{H}^3}(h(0),v)}{\sin\Theta}.$$

By (4.15), (4.16) and (4.17), we get

$$\frac{2\Theta}{\pi} \le \sin\Theta = \frac{\sinh \mathrm{dist}_{\mathcal{H}^3}(h(0),v)}{\sinh \mathrm{dist}_{\mathcal{H}^3}(o,h(0))} \le \frac{\sinh \mathrm{dist}_{\mathcal{H}^3}(h(0),\tilde{h}(0))}{c_{18}} \le \frac{c_1\varepsilon}{2c_{18}N^2},$$

where we use the fact $0 \leq \Theta \leq \pi/2$. This yields that $\Theta \leq c_{19}/N$. Let $w \in \Pi$ be the closest point from $\tilde{h}(p)$ to Π. Then

$$\mathrm{dist}_{\mathcal{H}^3}(o, w) \leq \mathrm{dist}_{\mathcal{H}^3}\big(o, \tilde{h}(p)\big) \leq s + \frac{c_3}{\sqrt{N}}$$

because of (4.9). Thus, by Step 1, we have

$$\begin{aligned}\mathrm{dist}_{\mathcal{H}^3}\big(\tilde{h}(p), \widetilde{\Pi}\big) &\leq \mathrm{dist}_{\mathcal{H}^3}\big(w, \tilde{h}(p)\big) + \mathrm{dist}_{\mathcal{H}^3}\big(w, \widetilde{\Pi}\big) \\ &\leq 14s^2 + \frac{c_{14}}{\sqrt{N}} + \left(s + \frac{c_3}{\sqrt{N}}\right)\frac{c_{20}}{N} \leq 14s^2 + \frac{c_{21}}{\sqrt{N}}.\end{aligned}$$

Since c_{21} depends only on $\mathcal{B}_0$, we have the conclusion. □

We would like to finish this section with four open questions.

Problem 1. Are there complete null curves properly immersed in the unit ball of $\mathbb{C}^3$? This problem is equivalent to ask for the existence complete Bryant surfaces properly immersed in a suitable domain of $\mathcal{H}^3$.

Next problem seems to be intrinsically related with the previous one.

Problem 2. Is it possible to construct complete embedded null curves in a ball of $\mathbb{C}^3$? As we mentioned in the introduction, Jones [J] constructed examples of complete bounded holomorphic curves $X : \mathbb{D}_1 \to \mathbb{C}^3$, but these examples do not satisfy the nullity condition (1.1). Recently, T. Colding and W. P. Minicozzi II in [CM] have proved that the answer to this question in the case of $\mathbb{R}^3$ is "no". Hence, it would be interesting to know whether the situation is similar in the complex case or not.

The next two problems belong to hyperbolic geometry.

Problem 3. Are there bounded complete minimal surface in $\mathcal{H}^3$?

Problem 4. Are there bounded complete surface with Gaussian curvature less than -1 in $\mathcal{H}^3$?

Like as in [N], our construction of Bryant surface in $\mathcal{H}^3$ can be taken to be of negative Gaussian curvature if we begin to prove our main theorem via an umbilic point free initial Bryant surface. However, we can expect -1 since it is equal to the curvature of $\mathcal{H}^3$.

Appendix A. Several estimations of solutions of ODE

A.1. **Hermitian matrix norm.** We define the Hermitian norm $|\cdot|$ of $M_2(\mathbb{C})$, the set of 2×2 matrices of complex coefficients by

$$|A| := \sqrt{\sum_{i,j=1}^{2} |a_{ij}|^2} = \sqrt{\mathrm{trace}(AA^*)} \qquad \big(A = (a_{ij}) \in M_2(\mathbb{C})\big), \tag{A.1}$$

where $A^* = {}^t\overline{A}$. By definition, $|A| = |uAu^*|$ holds for any unitary matrix $u \in \mathrm{SU}(2)$. Moreover, one can easily verify the following properties:

(1) $|AB| \leq |A|\,|B|$ for all $A, B \in M_2(\mathbb{C})$.
(2) $|Ax| \leq |A|\,|x|$ holds for $A \in M_2(\mathbb{C})$ and $x = {}^t(x_1, x_2) \in \mathbb{C}^2$,

where $|x|^2 := |x_1|^2 + |x_2|^2$.

On the other hand, we identify the set $\mathrm{Herm}(2)$ of 2×2-Hermitian matrices with the Minkowski 4-space L^4 as

$$L^4 \ni (x_0, x_1, x_2, x_3) \longleftrightarrow \begin{pmatrix} x_0 + x_3 & x_1 + \mathrm{i}x_2 \\ x_1 - \mathrm{i}x_2 & x_0 - x_3 \end{pmatrix} \in \mathrm{Herm}(2). \tag{A.2}$$

The hyperbolic 3-space $\mathcal{H}^3$ is considered as the hyperboloid in L^4, which is identified as

$$\mathcal{H}^3 = \{x \in \mathrm{Herm}(2)\,;\, \det x = 1, \mathrm{trace}\, x > 0\} = \{aa^*\,;\, a \in \mathrm{SL}(2, \mathbb{C})\}. \tag{A.3}$$

We denote by $\mathrm{dist}_{\mathcal{H}^3}$ the distance function of $\mathcal{H}^3$. Then

(A.4) $\mathrm{dist}_{\mathcal{H}^3}(o,p) = \cosh^{-1} p_0,$ where

$$o = (1,0,0,0) \leftrightarrow \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad p = (p_0,p_1,p_2,p_3) \leftrightarrow \begin{pmatrix} p_0+p_3 & p_1+\mathrm{i}p_2 \\ p_1-\mathrm{i}p_2 & p_0-p_3 \end{pmatrix}$$

holds, where id is the identity matrix.

Proposition A.1. *For A, $B \in \mathrm{SL}(2,\mathbb{C})$, it holds that*

$$|A^{-1}B| = 2\cosh \mathrm{dist}_{\mathcal{H}^3}(AA^*, BB^*),$$

where AA^* and BB^* are considered as points in $\mathcal{H}^3$ by (A.3)*.*

Since an $\mathrm{SL}(2,\mathbb{C})$ action

$$\mathcal{H}^3 \ni p \longmapsto apa^* \in \mathcal{H}^3 \qquad a \in \mathrm{SL}(2,\mathbb{C})$$

is an isometry of $\mathcal{H}^3$, this proposition is essentially reduced to the following lemma:

Lemma A.2. *For each $A \in \mathrm{SL}(2,\mathbb{C})$,*

$$|A|^2 = 2\cosh \mathrm{dist}_{\mathcal{H}^3}(o, AA^*),$$

where $o \in \mathcal{H}^3$ is the point as in (A.4)*.*

Proof. It is sufficient to show for the case that AA^* is a diagonal matrix. Let

$$p = AA^* = \begin{pmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{pmatrix} \qquad (\lambda > 0).$$

Then by (A.4) and (A.1) and (A.2), we have $2\cosh \mathrm{dist}_{\mathcal{H}^3}(\mathrm{id}, p) = \lambda + \lambda^{-1} = |A|^2$, which proves the assertion. □

A.2. **Several estimations of the solution of ODE.** Let $\alpha\colon [0,a] \to \mathfrak{sl}(2,\mathbb{C})$ be a smooth function into the Lie algebra of $\mathrm{SL}(2,\mathbb{C})$, and consider an initial value problem

(A.5) $$\mathcal{B}^{-1}\mathcal{B}' = \alpha, \qquad \mathcal{B}(0) = \mathrm{id}, \qquad (\mathcal{B}' = d\mathcal{B}/dt).$$

Theorem A.3. *If $\mathcal{B}\colon [0,a] \to \mathrm{SL}(2,\mathbb{C})$ satisfies* (A.5)*, it holds that*

$$|\mathcal{B}(t)| \le \sqrt{2}\,\exp\left(\int_0^t |\alpha|\,dt\right) \qquad (t \in [0,a]).$$

Proof. Since $\big||\mathcal{B}(t+h)| - |\mathcal{B}(t)|\big| \le |\mathcal{B}(t+h) - \mathcal{B}(t)|$, we have that

$$\frac{d}{dt}|\mathcal{B}(t)| \le |\mathcal{B}'(t)| = |\mathcal{B}\alpha| \le |\mathcal{B}|\,|\alpha|,$$

which yields the assertion because of $|\mathcal{B}(t)| \neq 0$. □

Theorem A.4. *If $\mathcal{B}\colon [0,a] \to M_2(\mathbb{C})$ satisfies* (A.5)*, it holds that*

$$|Z(t)| \le \left(\big(\max_{t\in[0,a]} |\mathcal{B}(t)|\big)\int_0^t |\alpha(t)|\,dt\right)^2 \qquad \left(Z(t) := \mathcal{B}(t) - \mathrm{id} - \int_0^t \alpha(t)dt\right).$$

Proof. We set $A = Z\mathcal{B}^{-1}$. Differentiating $Z = A\mathcal{B}$, we have $A'F = \left(\int_0^t \alpha(u)\,du\right)\alpha$. Then

$$Z(t) = A(t)\mathcal{B}(t) = \left\{\int_0^t \left(\int_0^s \alpha(u)\,du\right)\alpha(s)\mathcal{B}^{-1}(s)\,ds\right\}\mathcal{B}(t).$$

Since $\mathcal{B} \in \mathrm{SL}(2,\mathbb{C})$, we have $|\mathcal{B}^{-1}| = |\mathcal{B}|$, and can easily get the assertion. □

We now let α_0 and α_1 be smooth functions into $\mathfrak{sl}(2,\mathbb{C})$ defined on $[0,a]$, and consider initial value problems

$$\begin{aligned} X^{-1}X' &= \alpha_0, \qquad X(0) = \mathrm{id} \\ Y^{-1}Y' &= \alpha_1, \qquad Y(0) = \mathrm{id}\,. \end{aligned} \tag{A.6}$$

For any continuous function $\gamma : [0,a] \to M_2(\mathbb{C})$, we define the supremum norm of γ by

$$\|\gamma\| := \max_{t\in[0,a]} |\gamma(t)|. \tag{A.7}$$

Proposition A.5. *If $X, Y : [0,a] \to \mathrm{SL}(2,\mathbb{C})$ satisfy* (A.6)*, it holds that*

$$Y(t) - X(t) = \left\{ \int_0^t X(u)\big(\alpha_1(u) - \alpha_0(u)\big)Y^{-1}(u)\,du \right\} Y(t).$$

In particular,

$$|Y(a) - X(a)| \le \big(\sqrt{2}m\big)^3 \|\alpha_0 - \alpha_1\|$$

holds, where

$$m = \max\left\{ \exp\left(\int_0^a |\alpha_0(t)|dt\right), \exp\left(\int_0^a |\alpha_1(t)|dt\right) \right\}.$$

Proof. Take A as $Y - X = AY$. Differentiating it, we have $A' = X(\alpha_1 - \alpha_0)Y^{-1}$, which implies the first conclusion. Since $|Y| = |Y^{-1}|$, we have

$$|X - Y| \le \|\alpha_0 - \alpha_1\| \left(\max_{t\in[0,a]} |X|\right) \left(\max_{t\in[0,a]} |Y|\right)^2.$$

By Theorem A.3 we have $|X|, |Y| \le \sqrt{2}m$, which yields the conclusion. □

Corollary A.6. *If $X, Y : [0,a] \to \mathrm{SL}(2,\mathbb{C})$ satisfy* (A.6)*,*

$$d_{\mathcal{H}^3}(XX^*, YY^*) \le \mu\|\alpha_0 - \alpha_1\| \qquad \left(\mu = 4m^4\left(\sqrt{2} + 2m^4\|\alpha_0 - \alpha_1\|\right)\right)$$

holds, where $\|\cdot\|$ is the supremum norm as in (A.7)*.*

Proof. Since $|\,\mathrm{id}\,| = \sqrt{2}$ and $|X^{-1}| = |X| \le \sqrt{2}m$, Proposition A.5 implies that

$$|X^{-1}Y| = |X^{-1}(Y - X) + \mathrm{id}\,| \le (\sqrt{2}m)(\sqrt{2}m)^3\|\alpha_0 - \alpha_1\| + \sqrt{2}.$$

Since

$$2\big(1 + \mathrm{dist}_{\mathcal{H}^3}(XX^*, YY^*)\big) \le 2\cosh \mathrm{dist}_{\mathcal{H}^3}(XX^*, YY^*) = |X^{-1}Y|^2,$$

we can easily get the assertion. □

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