

Exotic Minimal Surfaces

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Abstract

We prove a general fusion theorem for complete orientable minimal surfaces in \mathbb{R}^3 with finite total curvature. As a consequence, complete orientable minimal surfaces of weak finite total curvature with exotic geometry are produced. More specifically, universal surfaces (i.e., surfaces from which all minimal surfaces can be recovered) and space-filling surfaces with arbitrary genus and no symmetries.

1 Introduction

As usual, a surface is said to be *open* if it is non-compact and has empty boundary. An open Riemann surface is said to be *hyperbolic* if and only if it carries a negative non-constant subharmonic function. Otherwise, it is said to be *parabolic*. Compact Riemann surfaces with empty boundary are said to be *elliptic*.

Let M be a Riemann surface possibly with non empty compact boundary, and let $X : M \rightarrow \mathbb{R}^3$ be a conformal minimal immersion. Throughout this paper we will always assume that X extends as a conformal minimal immersion to an open Riemann surface containing M as a proper subset. When X has finite total curvature (FTC for short), Huber and Osserman theorems [5, 11] imply that M has finite conformal type and the Weierstrass data of X extend meromorphically to the topological ends of M . This simply means that $M = M^c - E$, where M^c is a compact Riemann surface and $E \subset M^c - \partial(M^c)$ is a finite set, and the Weierstrass data of X have no essential singularities at points of E . The Riemann surface M^c is called the *Osserman compactification* of M . Likewise, a conformal complete minimal immersion $X : M \rightarrow \mathbb{R}^3$ is said to be of *weak finite total curvature* (WFTC for short) if $X|_\Omega$ has finite total curvature (FTC for short) for any proper region $\Omega \subset M$ with compact boundary and finite topology.

Unlike the FTC case, there exists orientable complete minimal surfaces of WFTC with arbitrary topology and conformal type (see [8]). The aim of this paper is to present some examples of this kind of surfaces with *exotic geometry*.

An interesting question is whether there exists a complete minimal surface from which all minimal surfaces could be recovered. Given an open Riemann surface N , a complete conformal minimal immersion $Y : N \rightarrow \mathbb{R}^3$ is said to be *universal* if it passes by all compact minimal surfaces in \mathbb{R}^3 . In other words, if for *any* compact Riemann surface M with $\partial(M) \neq \emptyset$ and *any* conformal minimal immersion $X : M \rightarrow \mathbb{R}^3$, there is a sequence $\{M_n\}_{n \in \mathbb{N}}$ of regions in N and biholomorphisms $h_n : M \rightarrow M_n$, $n \in \mathbb{N}$, such that $\{Y \circ h_n\}_{n \in \mathbb{N}} \rightarrow X$ uniformly on M . Our first result provides an affirmative answer to this question (see Theorem 4.2):

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Theorem I *There exist parabolic universal minimal surfaces of WFTC.*

Any universal minimal immersion $Y : N \rightarrow \mathbb{R}^3$ is *space-filling* (that is to say, $\overline{Y(N)} = \mathbb{R}^3$). As far as the author knows, all previously known space-filling complete minimal surfaces are hyperbolic, simply connected and highly symmetric. The reason why is that their construction is based either in Schwarzian reflection on a fundamental compact domain or in the classical Björling problem (see [4] for a good setting). However, in Corollary 4.1 we have shown that:

Theorem II *There exists space-filling complete minimal surfaces with WFTC, arbitrary (possibly infinity) genus, parabolic conformal type and no symmetries.*

Both above results are based on a general connected sum construction (or fusion theorem) for complete minimal surfaces with FTC. For a thorough exposition of the details, the following notations are required.

Given two Riemann surfaces M and M^* possibly with non empty boundary, M^* is said to be an *extension* of M if M is a proper subset of M^* , $M \cap \partial(M^*) = \emptyset$ and $M^* - M^\circ$ contains no compact connected components that are disjoint from $\partial(M^*)$, where M° is the topological interior of M in M^* . If M^* is an extension of M and $j : M \rightarrow M^*$ is the inclusion, then $j_* : \mathcal{H}_1(M, \mathbb{R}) \rightarrow \mathcal{H}_1(M^*, \mathbb{R})$ is a group monomorphism, hence up to natural identifications $\mathcal{H}_1(M, \mathbb{R}) \subset \mathcal{H}_1(M^*, \mathbb{R})$.

If $X : M \rightarrow \mathbb{R}^3$ is a conformal minimal immersion and $\gamma \subset M$ is an oriented closed curve, the flux of X on γ is given by $p_X(\gamma) := \int_\gamma \mu(s) ds$, where s is the oriented arclength parameter on γ and $\mu(s)$ is the conormal vector of X at $\gamma(s)$ for all s . Recall that $\mu(s)$ is the unique unit tangent vector of X at $\gamma(s)$ such that $\{dX(\gamma'(s)), \mu(s)\}$ is a positive basis. Since X is a harmonic map, $p_X(\gamma)$ depends only on the homology class of γ and the well defined flux map $p_X : H_1(M, \mathbb{Z}) \rightarrow \mathbb{R}^3$ is a group homomorphism.

Our Fusion Theorem asserts the following (see Theorem 4.1):

Theorem III (Fusion) *Let M_1, M_2, \dots be a finite or infinite sequence of pairwise disjoint Riemann surfaces with finite conformal type and non empty boundary. For each $n \in \mathbb{N}$ let $X_n : M_n \rightarrow \mathbb{R}^3$ be a conformal complete minimal surface of FTC.*

Then, for any $\epsilon > 0$ there exist an open parabolic extension M^ of $\cup_n M_n$ and a conformal complete minimal immersion $Y : M^* \rightarrow \mathbb{R}^3$ of WFTC such that $\|X_n - Y|_{M_n}\|_0 \leq \epsilon/n$ and $p_Y|_{\mathcal{H}_1(M_n, \mathbb{Z})} = p_{X_n}$, where $\|\cdot\|_0$ is the norm of the supremum on M_n , $n \in \mathbb{N}$.*

The main tool for proving this theorem has been the algebraic bridge principle given in [8]. This bridge principle allows good control over the conformal structure, asymptotic behavior and flux map of the resulting surface, and supplies a natural connected sum construction for complete minimal surfaces of finite total curvature in \mathbb{R}^3 . Interesting results of this kind can be found in Kapouleas [7] and Yang [12] works. Theorem III is the core of our existence result for space-filling minimal surfaces. The existence of universal minimal surfaces follows from the separability of the moduli space of complete minimal surfaces with FTC and the Fusion Theorem as well.

2 Preliminaries on Riemann surfaces

As usual, we denote by \mathbb{C} , $\overline{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$ and \mathbb{D} the complex plane, the extended complex plane and the conformal unit disc.

Given a Riemann surface M , $\partial(M)$ will denote the one dimensional topological manifold determined by the boundary points of M . Given $S \subset M$, write S° and \overline{S} for the topological interior and the topological closure of S in M , respectively. A proper connected subset $S \subset M$ is said to be a *region* if it is a topological surface with the induced topology. Open connected subsets of $M - \partial(M)$ are said to be *domains* of M .

Given a point $P \in M - \partial(M)$, we denote by μ_P the *harmonic measure of M with respect to P* . For any Borel measurable set $I \subset \partial(M)$, $\mu_P(I) = u_I(P)$, where u_I is the unique harmonic function on M vanishing on the ideal boundary of M and satisfying $u_I|_I = 1$, $u_I|_{\partial(M)-I} = 0$.

Definition 2.1 A Riemann surface M with $\partial(M) \neq \emptyset$ is said to be *parabolic* if there exists $P \in M - \partial(M)$ such that μ_P is full, i.e. $\mu_P(\partial M) = 1$. If N is an open Riemann surface, N is parabolic in the classical sense if and only if $N - D^\circ$ is parabolic as a Riemann surface with boundary for some (hence for any) closed disc $D \subset N$.

The fact that μ_P is full does not depend on the interior point P ; this follows from the maximum principle. If $P \in \Omega - \partial(\Omega)$ where $\Omega \subset M$ is a proper region, we denote by μ_P^Ω the harmonic measure relative to Ω with respect to P . Notice that parabolic surfaces are exactly those on which the maximum principle holds. See [1] for a good setting.

Let M and M^* be two Riemann surfaces possibly with non empty boundary. The surface M^* is said to be an *annular extension* of M if M^* is an extension of M and the connected components of $M^* - M^\circ$ are either simply or doubly connected, that is to say, homeomorphic to either $[0, 1] \times \mathbb{S}^1$ or $\mathbb{D} - \{E\}$ for some $E \in \mathbb{D}$. In this case $j_* : \mathcal{H}_1(M, \mathbb{R}) \rightarrow \mathcal{H}_1(M^*, \mathbb{R})$ is an isomorphism.

2.1 Approximation results on Riemann surfaces

Let N be a Riemann surface with $\partial(N) = \emptyset$, and let $S \subset N$ denote a subset different from N and satisfying $\overline{S^\circ} \cap S = S$ (for instance, a finite collection of pairwise disjoint regions in N). A connected component V of $N - S$ is said to be *bounded* if \overline{V} is compact.

Definition 2.2 We denote by $\mathfrak{F}^N(S)$ the space of functions $f : S \rightarrow \overline{\mathbb{C}}$ being meromorphic on some open neighborhood of S in N . If S is open then $\mathfrak{F}^N(S)$ coincides with the space of meromorphic functions on S , hence does not depend on N and is simply written $\mathfrak{F}(S)$. We write $\mathfrak{F}_0^N(S)$ (and $\mathfrak{F}_0(S)$ when S is open) for the corresponding space of holomorphic functions.

All these spaces will be endowed with the $\omega(S)$ -topology, namely, the topology of the uniform convergence on S . To be more precise, we shall say that a function $f \in \mathfrak{F}^N(S)$ can be uniformly approximated on S by functions in $\mathfrak{F}(N)$ if there exists $\{f_n\}_{n \in \mathbb{N}} \subset \mathfrak{F}(N)$ such that $\{|f_n|_S - f|\}_{n \in \mathbb{N}} \rightarrow 0$ uniformly on S . In particular all f_n have the same set \mathcal{P}_f of poles on S . Likewise we define the uniform approximation of an $f \in \mathfrak{F}_0^N(S)$ by functions in $\mathfrak{F}_0(N)$.

A complex 1-form θ on S is said to be of type $(1, 0)$ if for any conformal chart (U, z) on N we have that $\theta|_{U \cap S} = f(z)dz$ for some $f : U \cap S \rightarrow \overline{\mathbb{C}}$. For instance, holomorphic and meromorphic 1-forms on N are of type $(1, 0)$.

Definition 2.3 We denote by $\mathfrak{W}^N(S)$ the space of 1-forms of type $(1, 0)$ on S being meromorphic on some open neighborhood of S in N . If S is open then $\mathfrak{W}^N(S)$ coincides with the space of meromorphic 1-forms on S , hence does not depend on N and is simply written $\mathfrak{W}(S)$. We write $\mathfrak{W}_0^N(S)$ (and $\mathfrak{W}_0(S)$ when S is open) for the corresponding space of holomorphic 1-forms.

A 1-form $\theta \in \mathfrak{W}^N(S)$ can be uniformly approximated on S by 1-forms in $\mathfrak{W}(N)$ if there exists $\{\theta_n\}_{n \in \mathbb{N}} \subset \mathfrak{W}(N)$ such that $\{\frac{\theta_n - \theta}{dz}|_{S \cap K}\}_{n \in \mathbb{N}} \rightarrow 0$ in the $\omega(S \cap K)$ -topology for any closed conformal disc (K, z) on N (we are assuming that $z : K \rightarrow z(K) \subset \mathbb{C}$ extends as a conformal parameter beyond K in N). In particular all θ_n have the same set of poles \mathcal{P}_θ on S . As above, we say that $\{\theta_n|_S\}_{n \in \mathbb{N}} \rightarrow \theta$ in the $\omega(S)$ -topology. Likewise we define the uniform approximation of a $\theta \in \mathfrak{W}_0^N(S)$ by 1-forms in $\mathfrak{W}_0(N)$.

Let $\mathfrak{Div}(S)$ denote the free commutative group of divisors of S with multiplicative notation. If $D = \prod_{i=1}^n Q_i^{n_i} \in \mathfrak{Div}(S)$, where $n_i \in \mathbb{Z} - \{0\}$ for all i , the set $\{Q_1, \dots, Q_n\}$ is said to be

the support of D . Let $\mathfrak{Deg} : \mathfrak{Div}(S) \rightarrow \mathbb{Z}$ be the group homomorphism given by the degree map $\mathfrak{Deg}(\prod_{j=1}^t Q_j^{n_j}) = \sum_{j=1}^t n_j$. A divisor $D \in \mathfrak{Div}(S)$ is said to be *integral* if $D = \prod_{i=1}^n Q_i^{n_i}$ and $n_i \geq 0$ for all i . Given $D_1, D_2 \in \mathfrak{Div}(S)$, $D_1 \geq D_2$ if and only if $D_1 D_2^{-1}$ is integral. For any $f \in \mathfrak{F}^N(S)$ we denote by $(f)_0$ and $(f)_\infty$ its associated integral divisors of zeroes and poles in S , respectively, and label $(f) = (f)_0 / (f)_\infty$ as the divisor associated to f on S . Likewise we define $(\theta)_0$ and $(\theta)_\infty$, and write $(\theta) = (\theta)_0 / (\theta)_\infty$ for the divisor of θ in S , $\theta \in \mathfrak{W}^N(S)$.

We need the following extension of Runge's theorem (see [2], [9] and [10] for a good setting):

Theorem 2.1 *Let N be a Riemann surface with $\partial(N) = \emptyset$, and let $S \subset N$ be a finite collection of pairwise disjoint compact regions in N . Let $E \subset N - S$ be a subset meeting each bounded component of $N - S$ in a unique point.*

Then any function $f \in \mathfrak{F}^N(S)$ can be uniformly approximated on S by functions $\{f_n\}_{n \in \mathbb{N}}$ in $\mathfrak{F}(N) \cap \mathfrak{F}_0(N - (E \cup \mathcal{P}_f))$, where $\mathcal{P}_f = f^{-1}(\infty) \subset S$.

2.2 Compact Riemann surfaces

The background of the following results can be found, for instance, in [3].

In the sequel, R will denote an elliptic Riemann surface of genus $\nu \geq 1$.

Label $H_1(R, \mathbb{Z})$ as the 1st homology group with integer coefficients of R . Let $B = \{a_j, b_j\}_{j=1, \dots, \nu}$ be a canonical homology basis of $H_1(R, \mathbb{Z})$, and write $\{\xi_j\}_{j=1, \dots, \nu}$ the associated dual basis of $\mathfrak{W}_0(R)$, that is to say, the one satisfying that $\int_{a_k} \xi_j = \delta_{jk}$, $j, k = 1, \dots, \nu$.

Denote by $\Pi = (\pi_{jk})_{j,k=1, \dots, \nu}$ the Jacobi period matrix with entries $\pi_{jk} = \int_{b_k} \xi_j$, $j, k = 1, \dots, \nu$. This matrix is symmetric and has positive definite imaginary part. We denote by $L(R)$ the lattice over \mathbb{Z} generated by the 2ν -columns of the $\nu \times 2\nu$ matrix (I_ν, Π) , where I_ν is the identity matrix of dimension ν .

Finally, call $J(R) = \mathbb{C}^\nu / L(R)$ the Jacobian variety of R , which is a compact, commutative, complex, ν -dimensional Lie group. Given $E \in R$, denote by $\varphi_E : \mathfrak{Div}(R) \rightarrow J(R)$, $\varphi_E(\prod_{j=1}^s Q_j^{n_j}) = \sum_{j=1}^s n_j {}^t(\int_E^{Q_j} \xi_1, \dots, \int_E^{Q_j} \xi_\nu)$ the Abel-Jacobi map with base point E , where ${}^t(\cdot)$ means matrix transpose. If there is no room for ambiguity we simply write φ .

Abel's theorem asserts that $D \in \mathfrak{Div}(R)$ is the principal divisor associated to a meromorphic function $f \in \mathfrak{W}(R)$ if and only if $\mathfrak{Deg}(D) = 0$ and $\varphi(D) = 0$. Jacobi's theorem says that $\varphi : R_\nu \rightarrow J(R)$ is surjective and has maximal rank (hence a local biholomorphism) almost everywhere, where R_ν denotes the space of integral divisors in $\mathfrak{Div}(R)$ of degree ν .

Riemann-Roch theorem says that $r(D^{-1}) = \mathfrak{Deg}(D) - g + 1 + i(D)$ for any $D \in \mathfrak{Div}(R)$, where $r(D^{-1})$ (respectively, $i(D)$) is the dimension of the complex vectorial space of functions $f \in \mathfrak{F}(R)$ (respectively, 1-forms $\theta \in \mathfrak{W}(R)$) satisfying that $(f) \geq D^{-1}$ (respectively, $(\theta) \geq D$).

A point $Q \in R$ is said to be a *Weierstrass point* if there exists a non constant meromorphic function $h \in \mathfrak{F}(R)$ satisfying that $(h)_\infty \leq Q^\nu$. The number of Weierstrass points in R lies in between $2\nu - 2$ and $\nu(\nu^2 - 1)$.

2.3 Bridge constructions for Riemann surfaces

A Riemann surface M (possibly with non empty compact boundary) is of finite conformal type if and only if it has finite topology and is parabolic. The *Osserman compactification* M^c of M is obtained by filling out the conformal punctures corresponding to the topological ends of M . Moreover, if we attach conformal discs on the holes of M^c we get an elliptic Riemann surface R that we will call a *conformal compactification* of M . Notice that M^c is unique up to biholomorphisms, whereas R depends on the gluing process. With this language,

$$M^c = R - (\cup_{j=1}^b U_j) \text{ and } M = M^c - \{E_1, \dots, E_a\},$$

where U_j , $j = 1, \dots, b$ are open discs in R with pairwise disjoint closures in R and $\{E_1, \dots, E_a\} \subset M^c - \partial(M^c)$.

The following notion of conformal connected sum captures some natural bridge constructions for Riemann surfaces. We include the details just for completeness.

Let M_1, M_2 be two disjoint Riemann surfaces of finite conformal type and non empty boundary, and fix disjoint Jordan arcs $\gamma_i \subset \partial(M_i)$, $i = 1, 2$. Without loss of generality assume that $M_1^c \cap M_2^c = \emptyset$ as well and write $M = M_1 \cup M_2$ and $M^c = M_1^c \cup M_2^c$. Let S be a closed conformal disc disjoint from M^c , and introduce a mark on S consisting of two distinct Jordan arcs $\gamma'_1, \gamma'_2 \subset \partial(S)$. By definition, $\Upsilon = (\{\gamma_1, \gamma_2\}, \{S, \gamma'_1, \gamma'_2\})$ is said to be a *conformal bridge* between M_1 and M_2 .

The surfaces M_1 and M_2 can be connected via Υ as follows. Take a biholomorphism $w : S \rightarrow [0, 1] \times [-\delta, \delta] \subset \mathbb{C}$ such that $\gamma'_i = w^{-1}(s_i)$, where s_i is the segment $\{i - 1\} \times [-\delta, \delta]$, $i = 1, 2$ (the real number δ is uniquely determined by the mark $\{\gamma'_1, \gamma'_2\}$ on $\partial(S)$). Take a closed disc $V_i \subset M_i$ such that $\gamma_i = V_i \cap \partial(M_i)$, $i = 1, 2$, and $V_1 \cap V_2 = \emptyset$, and consider biholomorphisms $w_1 : V_1 \rightarrow [-1, 0] \times [-\delta, \delta]$ and $w_2 : V_2 \rightarrow [1, 2] \times [-\delta, \delta]$ such that $w_i(\gamma_i) = s_i$, $i = 1, 2$. Then simply attach S to M by identifying the points $w^{-1}((i - 1, t))$ and $w_i^{-1}((i - 1, t))$ for any $t \in [-\delta, \delta]$, $i = 1, 2$.

Definition 2.4 We write $M_1 \#_{\Upsilon} M_2$ for the quotient surface, and call it a *connected conformal sum* (or simply, a *conformal sum*) of M_1 and M_2 via Υ (see Figure 1).

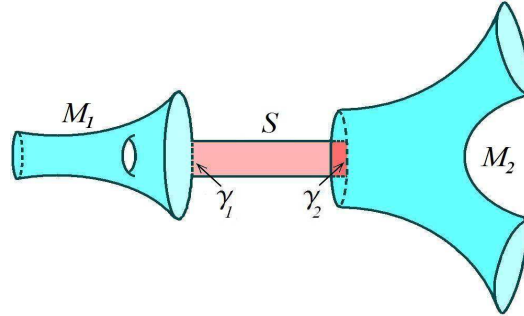


Figure 1: The surface $M_1 \#_{\Upsilon} M_2$

Up to the projection map to the quotient, $\gamma_i = \gamma'_i$, $i = 1, 2$. Furthermore, M_1, M_2 and S are subsets of $M_1 \#_{\Upsilon} M_2$ satisfying $M_i \cap S = \gamma_i$, $i = 1, 2$, and $M_1 \#_{\Upsilon} M_2 = M \cup S$. Adding the natural chart from $V_1 \cup S \cup V_2$ onto $[-1, 2] \times [-\delta, \delta]$ induced by w, w_1 and w_2 , $M_1 \#_{\Upsilon} M_2$ becomes a Riemann surface of finite conformal type and non empty boundary and $H_1(M_1 \#_{\Upsilon} M_2, \mathbb{Z}) = H_1(M_1, \mathbb{Z}) \oplus H_2(M_2, \mathbb{Z})$. A conformal compactification R_{Υ} of $M_1 \#_{\Upsilon} M_2$ is said to be a *conformal compactification of M via Υ* . Obviously these constructions guarantee the uniqueness of neither $M_1 \#_{\Upsilon} M_2$ nor R_{Υ} , because they depend on the gluing processes.

This bridge construction can be used for generating parabolic Riemann surfaces of arbitrary topology. Indeed, let $\{M_j\}_{1 \leq j < \sigma}$, $\sigma \in \mathbb{N} \cup \{+\infty\}$, be sequence of pairwise disjoint Riemann surfaces of finite conformal type and non empty boundary, and call $M := \cup_{1 \leq j < \sigma} M_j$. Set $W_1 = M_1$, and working inductively, for each $j < \sigma$ choose a bridge Υ_j between W_j and M_{j+1} and set $W_{j+1} = W_j \#_{\Upsilon_j} M_{j+1}$. By definition, the Riemann surface $\#_{\Upsilon_j \in \Upsilon} M_j := \cup_{0 \leq j < \sigma} W_{j+1}$ is said to be a *conformal sum* of $\{M_j\}_{1 \leq j < \sigma}$ via the multi-bridge $\Upsilon = \{\Upsilon_j\}_{1 \leq j < \sigma}$. Notice that $\#_{\Upsilon_j \in \Upsilon} M_j$ has genus $\sum_{j < \sigma} \nu_j$, where ν_j is the genus of M_j for all j .

An open Riemann surface M^* is said to be a *parabolic completion* of M via Υ if M^* is parabolic and there exists a proper *topological* embedding $\mathcal{I} : \#_{\Upsilon_j \in \Upsilon} M_j \rightarrow M^*$ such that $\mathcal{I}|_M = \text{Id}_M$ and M^* is an annular extension of $\mathcal{I}(\#_{\Upsilon_j \in \Upsilon} M_j)$. In particular M^* has genus $\sum_{j < \sigma} \nu_j$ as well and $\mathcal{I}_* : H_1(\#_{\Upsilon_j \in \Upsilon} M_j, \mathbb{Z}) \rightarrow H_1(M^*, \mathbb{Z})$ is an isomorphism. Up to the group monomorphism induced by the inclusion map, $H_1(M_j, \mathbb{Z})$ is a subset of $H_1(M^*, \mathbb{Z})$, $1 \leq j < \sigma$, and therefore $H_1(M^*, \mathbb{Z})$ is the direct sum $\oplus_{1 \leq j < \sigma} H_1(M_j, \mathbb{Z})$.

Lemma 2.1 *Given $M = \cup_{1 \leq j < \sigma} M_j$ and $\Upsilon = \{\Upsilon_j\}_{1 \leq j < \sigma-1}$ as above, M admits a parabolic completion M^* via Υ .*

Proof: Assume first that $\sigma < +\infty$. In this case $\#_{\Upsilon_j \in \Upsilon} M_j$ is of finite conformal type. Let R be the conformal compactification of $\#_{\Upsilon_j \in \Upsilon} M_j$ and consider a finite subset $E \subset R$ containing all the ends of M and meeting each component of $R - \#_{\Upsilon_j \in \Upsilon} M_j$ in a unique point. It suffices to take $M^* = R - E$ and set \mathcal{I} as the inclusion map.

Suppose now that $\sigma = +\infty$. Fix a closed disc $D \subset M_1 - \partial(M_1)$ and a point $P \in D - \partial(D)$. As above, put $W_1 = M_1$ and $W_{j+1} = W_j \#_{\Upsilon_j} M_{j+1}$ for each $j \geq 1$. Let $c_1^j \subset \partial(W_j)$ and $c_2^j \subset \partial(M_{j+1})$ be the two boundary components (closed Jordan curves) connected by Υ_j , $j \geq 1$.

Let us construct a sequence $N_1 \subset N_2 \subset \dots$ of Riemann surfaces and proper embeddings $\mathcal{I}_j : W_j \rightarrow N_j$, $j \geq 1$, such that:

- (a) $\mathcal{I}_j|_{W_{j-1}} = \mathcal{I}_{j-1}$, $(\cup_{h > j} M_h) \cap N_j = \emptyset$, $\cup_{h \leq j} M_h \subset N_j$ and $\mathcal{I}_j|_{\cup_{h \leq j} M_h}$ is the inclusion map, $j \geq 2$,
- (b) N_j is a Riemann surface of finite conformal type and $\partial(N_j)$ is a Jordan curve homologically equivalent in N_j to $\mathcal{I}_j(c_1^j)$, $j \geq 1$,
- (c) N_j is an annular extension of both $\mathcal{I}_j(W_j)$ and $N_{j-1} \#_{\Upsilon'_{j-1}} M_j$ for a suitable bridge Υ'_{j-1} in N_j connecting $\partial(N_{j-1})$ and c_2^{j-1} , $j \geq 2$.
- (d) $\mu_P^{N_j - D^\circ}(\partial(D)) > \frac{j-1}{j}$, where $\mu_P^{N_j - D^\circ}$ is the harmonic measure of $N_j - D^\circ$ with respect to P , $j \geq 1$.

Let R_1 be an open parabolic annular extension of M_1 , and notice that R_1 is biholomorphic to a finitely punctured compact Riemann surface. Without loss of generality suppose that $R_1 \cap (\cup_{h > 1} M_h) = \emptyset$. Since R_1 is parabolic, we can find a proper region $N_1 \subset R_1$ such that N_1 has just one hole (hence connected boundary), $M_1 \subset N_1 - \partial(N_1)$, $\partial(N_1)$ is homologically equivalent to c_1^1 , and N_1 is an annular extension of M_1 . Set $\mathcal{I}_1 : M_1 \rightarrow N_1$ the inclusion map and observe that $\mu_P^{N_1 - D^\circ}(\partial(D)) > 0$. The above items hold for $j = 1$.

Reasoning inductively, suppose that we have constructed N_j and \mathcal{I}_j , $1 \leq j \leq m-1$, satisfying the above properties. Take a bridge Υ'_{m-1} between N_{m-1} and M_m connecting $\partial(N_{m-1})$ and c_2^{m-1} . Let R_m be an open parabolic annular extension of $N_{m-1} \#_{\Upsilon'_{m-1}} M_m$. As above notice that R_m is a finitely punctured compact Riemann surface, and without loss of generality suppose that $R_m \cap (\cup_{h > m} M_h) = \emptyset$. Let $\mathcal{I}_m : W_m \rightarrow R_m$ be any extension of \mathcal{I}_{m-1} as a proper topological embedding satisfying that $\mathcal{I}_m|_{M_m} = \text{Id}_{M_m}$. Since R_m is parabolic, there exists a proper region $N_m \subset R_m$ with just one hole such that $\mathcal{I}_m(W_m) \subset N_m - \partial(N_m)$, N_m is an annular extension of both $\mathcal{I}_m(W_m)$ and $N_{m-1} \#_{\Upsilon'_{m-1}} M_m$, $\partial(N_m)$ is homologically equivalent to $\mathcal{I}_m(c_1^m)$ and $\mu_P^{N_m - D^\circ}(\partial(D)) > 1 - 1/m$. Considering the natural embedding $\mathcal{I}_m : W_m \rightarrow N_m$, the induction is closed.

Set $M^* = \cup_{j \geq 1} N_j$ and $\mathcal{I} : \#_{\Upsilon_j \in \Upsilon} M_j \rightarrow M^*$, $\mathcal{I}|_{W_j} = \mathcal{I}_j$ for all j . It is not hard to check that the open Riemann surface M^* is an annular extension of $\mathcal{I}(\#_{\Upsilon_j \in \Upsilon} M_j)$. Moreover, if $\mu_P^{M^* - D^\circ}$ is the harmonic measure of $M^* - D^\circ$ with respect to P then $\mu_P^{M^* - D^\circ}(\partial(D)) = \lim_{j \rightarrow \infty} \mu_P^{N_j - D^\circ}(\partial(D)) = 1$, proving that M^* is parabolic and the lemma. \square

3 Preliminaires on minimal surfaces

Throughout this section, N will be an open Riemann surface and $M \subset N$ a finite union of pairwise disjoint regions with compact boundary.

Definition 3.1 Let $\mathcal{E}(N)$ denote the space of conformal complete minimal immersions $X : N \rightarrow \mathbb{R}^3$ of WFTC. Likewise, we write $\mathcal{E}_N(M)$ for the space of conformal complete minimal immersions $X : M \rightarrow \mathbb{R}^3$ of WFTC that extend as a conformal minimal immersion to some neighborhood of M in N .

If M is open then $\mathcal{E}(M) = \mathcal{E}_M(M)$. When M has finite conformal type, $\mathcal{E}_N(M)$ is the space of conformal complete minimal immersions of M in \mathbb{R}^3 with FTC that extend to some neighborhood of M in N . These spaces will be endowed with the following \mathcal{C}^0 topology:

Definition 3.2 A sequence $\{X_n\}_{n \in \mathbb{N}} \subset \mathcal{E}_N(M)$ is said to converge in the $\mathcal{C}^0(M)$ -topology to $X_0 \in \mathcal{E}_N(M)$ if for any proper region $\Omega \subset M$ of finite conformal type, $\{X_n|_\Omega\}_{n \in \mathbb{N}} \rightarrow X_0|_\Omega$ uniformly on Ω . If M has finite conformal type, this topology coincides with the one of uniform convergence on M .

Take $X \in \mathcal{E}_N(M)$ and write (ϕ_1, ϕ_2, ϕ_3) for the complex differential $\partial_z X$. Notice that $\partial_z X \in \mathfrak{W}_0^N(M)^3$. Since X is conformal and minimal, then $\phi_1 = \frac{1}{2}(1/g - g)\phi_3$ and $\phi_2 = \frac{i}{2}(1/g + g)\phi_3$, where $g \in \mathfrak{F}^N(M)$ and up to the stereographic projection coincides with the Gauss map of X . The pair (g, ϕ_3) is known as the *Weierstrass representation* of X (see [11]).

Clearly $X(P) = X(Q) + \operatorname{Re} \int_Q^P (\phi_1, \phi_2, \phi_3)$, $P, Q \in M$. The induced intrinsic metric ds^2 on M and its Gauss curvature \mathcal{K} are given by the expressions:

$$ds^2 = \sum_{j=1}^3 |\phi_j|^2 = \frac{1}{4} |\phi_3|^2 \left(\frac{1}{|g|} + |g| \right)^2, \quad \mathcal{K} = - \left(\frac{4|dg||g|}{|\phi_3|(1+|g|^2)^2} \right)^2. \quad (1)$$

The total curvature of X is given by $c(X) := \int_M \mathcal{K} dA$, where dA is the area element of ds^2 , and the *flux map* of X by the expression $p_X : H_1(M, \mathbb{Z}) \rightarrow \mathbb{R}^3$, $p_X(\gamma) = \operatorname{Im} \int_\gamma \partial_z X$.

By Huber, Osserman and Jorge-Meeks results [5, 11, 6], if X is complete and of FTC then X is proper, M has finite conformal type and the Weierstrass data of X extend meromorphically to M^c .

Remark 3.1 Take $\{X_n, n \in \mathbb{N}\} \cup \{X\} \subset \mathcal{E}_N(M)$ and assume that $\{X_n\}_{n \in \mathbb{N}} \rightarrow X$ in $\mathcal{E}_N(M)$. If $\Omega \subset M$ is a proper region of finite conformal type, it is not hard to see that $\{(X_n - X)|_{\Omega^c}\}_{n \in \mathbb{N}} \rightarrow 0$ uniformly on Ω^c and the Weierstrass data of X_n converge in the $\omega(\Omega^c)$ -topology to the ones of X on Ω^c . Indeed, just observe that by Riemann's removable singularity theorem, $X_n - X$ extends harmonically to the punctures of Ω for all n and $\{X_n - X\}_{n \in \mathbb{N}} \rightarrow 0$ uniformly on Ω^c as well.

Let M_1, M_2 be two proper regions in N with finite conformal type and non-empty boundary, call $M = M_1 \cup M_2$ and consider an analytic regular Jordan arc $\beta \subset N$ with endpoints $P_1 \in \partial(M_1)$ and $P_2 \in \partial(M_2)$ and otherwise disjoint from M (see Figure 2). We will always assume that β lies in an open analytic arc β_0 in N such that $\beta_0 - \beta \subset M - \partial(M)$. A proper region $V \subset N$ is said to be an *annular extension* of $M \cup \beta$ in N if $M \cup \beta \subset V^\circ$, $V - (M^\circ \cup \beta)$ contains no connected components with compact closure that are disjoint from $\partial(V)$, and $V - (M^\circ \cup \beta)$ consists of a finite collection of compact annulus and once punctured closed discs. In particular, the induced homomorphism $j_* : H_1(M \cup \beta, \mathbb{Z}) \rightarrow H_1(V, \mathbb{Z})$ is an isomorphism, where $j : M \cup \beta \rightarrow V$ is the inclusion map. See Figure 3. A map $X : M \cup \beta \rightarrow \mathbb{R}^3$ is said to be *smooth* if $X|_{M_j}$, $j = 1, 2$, and $X|_{\beta_0}$ are smooth. For instance, if $Y : N \rightarrow \mathbb{R}^3$ is a smooth map then $X = Y|_{M \cup \beta}$ is smooth.

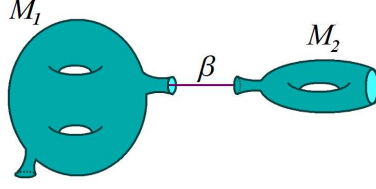


Figure 2: The surfaces M_1 , M_2 and the curve β .

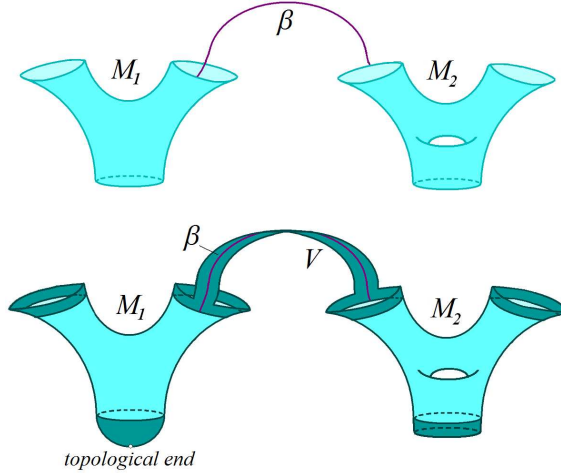


Figure 3: An annular neighborhood V of $M \cup \beta$.

Definition 3.3 We denote by $\mathcal{E}_N(M \cup \beta)$ the space of smooth maps $X : M \cup \beta \rightarrow \mathbb{R}^3$ such that $X_j := X|_{M_j} \in \mathcal{E}_N(M_j)$, $j = 1, 2$, and $X|_\beta$ is a smooth immersion. This space is endowed with the $C^0(M \cup \beta)$ -topology of the uniform convergence on $M \cup \beta$.

It is clear that $Y|_{M \cup \beta} \in \mathcal{E}_N(M \cup \beta)$ for all $Y \in \mathcal{E}(N)$.

Notice that $H_1(M \cup \beta, \mathbb{Z}) = H_1(M, \mathbb{Z}) = H_1(M_1, \mathbb{Z}) \oplus H_1(M_2, \mathbb{Z})$, and for each $X \in \mathcal{E}(M \cup \beta)$ identify the flux map $p_X : H_1(M \cup \beta, \mathbb{Z}) \rightarrow \mathbb{R}^3$ of X with the one of $X|_M$.

The proof of the following results can be found in [8]:

Theorem 3.1 (The Algebraic Bridge Principle) Assume that N is an open Riemann surface of finite conformal type and $N - (M \cup \beta)$ consists of a finite collection of pairwise disjoint once punctured conformal discs. Let X be an arbitrary immersion in $\mathcal{E}_N(M \cup \beta)$.

Then there exists a sequence $\{Y_n\}_{n \in \mathbb{N}} \subset \mathcal{E}(N)$ such that $p_{Y_n}|_{H_1(M \cup \beta)} = p_X$ for all $n \in \mathbb{N}$ and $\{Y_n|_{M \cup \beta}\}_{n \in \mathbb{N}} \rightarrow X$ in the $C^0(M \cup \beta)$ -topology. Furthermore, if C is a positive constant and V an annular extension of $M \cup \beta$ in N , then $\{Y_n\}_{n \in \mathbb{N}}$ can be chosen in such a way that $d_{Y_n}(M \cup \beta, \partial(V)) \geq C$ for all n , where d_{Y_n} is the intrinsic distance in N induced by Y_n .

Theorem 3.2 (General Approximation) Let M be a Riemann surface of finite conformal type and $\partial(M) \neq \emptyset$, and let M^* be an extension of M with $\partial(M^*) = \emptyset$. Consider an immersion $X \in \mathcal{E}_{M^*}(M)$ and a linear extension $q : H_1(M^*, \mathbb{Z}) \rightarrow \mathbb{R}$ of p_X .

Then, there exists a sequence of conformal complete minimal immersions $\{Y_n\}_{n \in \mathbb{N}} \in \mathcal{E}(M^*)$ such that $\{Y_n|_M\}_{n \in \mathbb{N}} \rightarrow X$ in the $C^0(M)$ -topology and $p_{Y_n} = q$.

4 Fusion theorems for minimal surfaces of FTC

Parabolicity is a powerful tool because it ensures the well-posedness (existence, uniqueness, stability...) of interesting geometrical problems. Theorem 3.2 implies the existence of complete minimal surfaces of WFTC with arbitrarily prescribed (non compact) parabolic conformal structure. In particular, there exist complete parabolic minimal surfaces with arbitrary topology. Our interest resides in obtaining fusion theorems for this kind of surfaces.

Given a topological space T and a continuous map $f : T \rightarrow \mathbb{R}^3$, we write $\|f\|_0 = \sup\{\|f(P)\| : P \in T\}$, where $\|\cdot\|$ is the Euclidean norm.

Theorem 4.1 (Fusion) *Let $\{M_j\}_{1 \leq j < \sigma}$ be a sequence of pairwise disjoint Riemann surfaces of finite conformal type and non empty boundary, where $\sigma \in \mathbb{N} \cup \{+\infty\}$, and let M^* be a parabolic completion of $M = \cup_{1 \leq j < \sigma} M_j$. Consider $X_i \in \mathcal{E}_{M^*}(M_i)$, $i \geq 1$, and fix $\epsilon > 0$.*

Then there is $Y \in \mathcal{E}(M^)$ such that $\|Y|_{M_j} - X_j\|_0 \leq \epsilon/j$ and $p_Y|_{H_1(M_j, \mathbb{Z})} = p_{X_j}$ for all $j \geq 1$.*

Proof: Consider the multi-bridge $\Upsilon = \{\Upsilon_j\}_{1 \leq j < \sigma-1}$ and proper embedding $\mathcal{I} : \#_{\Upsilon_j \in \Upsilon} M_j \rightarrow M^*$ such that $\mathcal{I}|_M = \text{Id}_M$ and M^* is an annular extension of $\mathcal{I}(\#_{\Upsilon_j \in \Upsilon} M_j)$. Up to natural identifications, we will assume that \mathcal{I} is the inclusion map and M^* is an annular extension of a conformal sum $\#_{\Upsilon_j \in \Upsilon} M_j$. Fix a point $P \in M_1$.

Like in the proof of Lemma 2.1, we can find an exhaustion $N_1 \subset N_2 \subset \dots$ of M^* by proper regions of finite conformal type and (compact) connected boundary, and a sequence of bridges $\Upsilon'_1, \Upsilon'_2, \dots$ between N_1 and M_2 , N_2 and M_3, \dots , such that: N_1 is an annular extension of M_1 , $\cup_{h \leq j} M_j \subset N_j$, $(\cup_{h > j} M_j) \cap N_j = \emptyset$ and N_{j+1} is an annular extension of $N_j \#_{\Upsilon'_j} M_{j+1}$, $j \geq 1$. When $\sigma < +\infty$ the sequence ends at $N_{\sigma-1} = M^*$.

Let us construct $Y_j \in \mathcal{E}_{M^*}(N_j)$, $1 \leq j < \sigma$, such that: $\|Y_j|_{M_j} - X_j\|_0 \leq \frac{\epsilon}{j^{2j}}$, $p_{Y_j}|_{H_1(M_j, \mathbb{Z})} = p_{X_j}$ and $d(Y_j(P), Y_j(\partial(N_j))) > j$ for all $j \geq 1$, and $\|Y_j|_{N_{j-1}} - Y_{j-1}\|_0 \leq \frac{\epsilon}{j^{2j}}$ and $p_{Y_j}|_{H_1(N_{j-1}, \mathbb{Z})} = p_{Y_{j-1}}$ for all $j \geq 2$.

Indeed, by Theorem 3.2, there is $Y_1 \in \mathcal{E}_{M^*}(N_1)$ such that $\|Y_1|_{M_1} - X_1\|_0 \leq \epsilon/2$, $p_{Y_1}|_{H_1(M_1, \mathbb{Z})} = p_{X_1}$ and $d(Y_1(P), Y_1(\partial(N_1))) > 1$. Reasoning inductively, suppose we have constructed $Y_j \in \mathcal{E}_{M^*}(N_j)$ satisfying the above properties. By Theorem 3.1, there exist $Y_{j+1} \in \mathcal{E}_{M^*}(N_{j+1})$ such that $\|Y_{j+1}|_{M_{j+1}} - X_{j+1}\|_0, \|Y_{j+1}|_{N_j} - Y_j\|_0 \leq \epsilon/(j+1)^{2^{j+1}}$, $p_{Y_{j+1}}|_{H_1(M_{j+1}, \mathbb{Z})} = p_{X_{j+1}}$, $p_{Y_{j+1}}|_{H_1(N_j, \mathbb{Z})} = p_{Y_j}$ and $d(Y_{j+1}(P), Y_{j+1}(\partial(N_{j+1}))) > j+1$, closing the induction.

When $\sigma < +\infty$ the immersion $Y = Y_{\sigma-1} \in \mathcal{E}(M^*)$ solves the theorem.

If $\sigma = +\infty$, there exists a possibly branched conformal minimal immersion $Y : M^* \rightarrow \mathbb{R}^3$ such that $\{Y_m|_{N_j}\}_{m \in \mathbb{N}} \rightarrow Y|_{N_j}$ uniformly on N_j for any $j \geq 1$. Furthermore, $\|Y|_{M_j} - X_j\|_0 \leq \epsilon/j$ and $\|Y|_{N_j} - Y_j\|_0 \leq \epsilon/(j+1)$ for any $j \geq 1$. Let us show that Y has no branch points. Indeed, let (g_j, ϕ_3^j) denote the Weierstrass data of Y_j , $j \geq 1$, and likewise call (g, ϕ_3) as the ones of Y . Obviously, $\{g_j, \phi_3^j\}_{j \in \mathbb{N}} \rightarrow (g, \phi_3)$ uniformly on compact subsets of M^* . Take an arbitrary $P_0 \in M^*$, and consider $j_0 \in \mathbb{N}$ such that $P_0 \in N_{j_0}^\circ$. Up to a rigid motion, $g(P_0) \neq 0, \infty$, hence we can find an closed disc $D \subset N_{j_0}$ such that $P_0 \in D^\circ$ and the functions $g|_D$ and $g_j|_D$, $j \in \mathbb{N}$, are holomorphic and without zeroes. Since Y_j has no branch points, ϕ_3^j has no zeroes on D for all j . By Hurwith theorem, either $\phi_3 = 0$ or ϕ_3 has no zeroes on D as well. In the first case the identity principle gives $\phi_3 = 0$ on M^* , contradicting that $\|Y - X_1\|_0 < \epsilon$ on M_1 when ϵ is small enough, and proving that Y_ϵ is an immersion.

Finally, let us see that Y is complete and of WFTC. By Osserman's theorem, the Gauss map of Y_j extends meromorphically to N_j^c , $j \in \mathbb{N}$. Since $\|Y_j - Y|_{N_j}\|_0$ is finite then $Y_j - Y$ extends harmonically to N_j^c and $Y|_{N_j}$ is complete and of finite total curvature for any j . It remains to check that Y is complete. First notice that those curves in M^* diverging to an annular end of some N_j have infinite intrinsic length with respect to Y . Moreover, the fact that $d(Y_j(P), Y_j(\partial(N_j))) > j$ for all j implies that $\lim_{j \in \mathbb{N}} d(Y(P), Y(\partial(N_j))) = +\infty$. This shows that any divergent curve in

M^* that does not diverge to an annular end of some N_j have infinite intrinsic length as well, and so $Y \in \mathcal{E}(M^*)$. Clearly $p_Y|_{H_1(M_j, \mathbb{Z})} = p_{X_j}$ for all $j \geq 1$ and we are done. \square

This fusion theorem can be used for producing minimal surfaces with exotic geometry. We start with the following existence result for space-filling minimal surfaces:

Corollary 4.1 *For each $\nu \in \{0\} \cup \mathbb{N} \cup \{\infty\}$, there exists a space-filling, open, parabolic, complete and minimal surface in \mathbb{R}^3 with genus ν , WFTC and no symmetries.*

Proof: Let $\{v_1, v_2, v_3\} \subset \mathbb{R}^3$ be three linearly independent unit vectors in general position, that is to say, such that $\langle v_{i_1}, v_{j_1} \rangle \neq \pm \langle v_{i_2}, v_{j_2} \rangle$ provided that $\{i_1, j_1\} \neq \{i_2, j_2\}$, where $\langle \cdot, \cdot \rangle$ is the Euclidean metric. Let $\{r_n : n \in \mathbb{N}\}$ be a bijective enumeration of \mathbb{Q} , and write $\Sigma_{i,n} = r_n v_i + \{u \in \mathbb{R}^3 : \langle u, v_i \rangle = 0 \text{ and } \langle u, u \rangle \geq 1/n^2\}$, $i = 1, 2, 3$, $n \in \mathbb{N}$. Consider a conformal parameterization $Y_{i,n} : M_{i,n} \rightarrow \mathbb{R}^3$ of $\Sigma_{i,n}$, where $M_{i,n} \cap M_{j,m} = \emptyset$ provided that $(i, n) \neq (j, m)$.

Let $Z : N \rightarrow \mathbb{R}^3$ denote a conformal parameterization of the Chen-Gackstatter genus one minimal surface, take a closed disc $D \subset N$ and write $Z_0 = Z|_{N_0}$, where $N_0 = N - D^\circ$. Recall that N (hence N_0) has an only topological end, and $Z_0(N_0)$ is asymptotic to the classical Enneper surface. In particular, $Z_0(N_0)$ is not asymptotic to a plane. Call $N_{0,n} = N_0 \times \{n\}$ and set $Z_{0,n} : N_{0,n} \rightarrow \mathbb{R}^3$, $Z_{0,n}((P, n)) = Z_0(P)$ for all $n < \nu + 1$. Let $\{Y_j : N_j \rightarrow \mathbb{R}^3 : j \in \mathbb{N}\}$ denote a bijective enumeration of $\{Y_{i,n} : i = 1, 2, 3, n \in \mathbb{N}\} \cup \{Z_{0,n} : n < \nu + 1\}$.

Let N_j^* be an annular neighborhood of N_j homeomorphic to N_j , and without loss of generality suppose that Y_j can be extended to N_j^* , $j \in \mathbb{N}$. Take a parabolic completion M^* of $\{N_j^*\}_{j \in \mathbb{N}}$ and observe that $Y_j \in \mathcal{E}_{M^*}(N_j)$ for all j . Consider the fusion immersion $Y \in \mathcal{E}(M^*)$ associated to $\{Y_j\}_{j \in \mathbb{N}}$ and any $\epsilon > 0$ via Theorem 4.1.

It is not hard to check that Y is space-filling (we leave the details to the reader). Moreover M^* has genus ν , hence it suffices to check that Y has no symmetries. Reason by contradiction, and suppose there exists a rigid motion $\sigma : \mathbb{R}^3 \rightarrow \mathbb{R}^3$ different from the identity map Id and leaving invariant $Y(M^*)$. Call $\sigma_0 : M^* \rightarrow M^*$ as the intrinsic isometry satisfying that $Y \circ \sigma_0 = \sigma \circ Y$. The embedded planar annular ends of Y have limit normal vector parallel to some v_i , $i \in \{1, 2, 3\}$. As σ_0 maps annular ends onto annular ends with the same geometry, then $\vec{\sigma}$ leaves invariant the system of vectors $\{\pm v_j : j = 1, 2, 3\}$, where $\vec{\sigma}$ is the linear isometry associated to σ . Taking into account that the vectors v_1, v_2 and v_3 are placed in general position, we infer that $\vec{\sigma} = \pm \text{Id}$.

Assume for a moment that $\vec{\sigma} = \text{Id}$, that is to say, σ is a non trivial translation. In this case σ_0 takes annular planar ends on annular planar ends with the same limit normal vector. Fix $i \in \{1, 2, 3\}$, and for each $n \in \mathbb{N}$ let $m(n) \in \mathbb{N}$ denote the unique natural number such that $M_{i,m(n)}$ and $\sigma_0(M_{i,n})$ determine the same annular end. Call $\Omega_{i,n} := \sigma_0(M_{i,n}) \cap M_{i,m(n)}$ and notice that the Euclidean distance $d(Y(\Omega_{i,n}) - r_{m(n)} v_i, \Sigma_i) \rightarrow 0$ as $n \rightarrow \infty$, where $\Sigma_i = \{u : \langle u, v_i \rangle = 0, u \neq 0\}$. This clearly implies that the translation vector of σ must be orthogonal to v_i . However, this can not occur for all $i \in \{1, 2, 3\}$, getting a contradiction.

Suppose now that $\vec{\sigma} = -\text{Id}$, i.e., σ is a symmetry with respect to a point $P_0 \in \mathbb{R}^3$. Reasoning as above, σ preserves the annular ends of Enneper type. However, the Enneper type ends of $Y(M^*)$ lie in a neighborhood of radius ϵ of the genus one Chen-Gakstatter surface $Z(N)$, and this surface has no central symmetries. This contradiction concludes the proof. \square

4.1 Universal minimal surfaces

This section is devoted to the existence problem of universal minimal surfaces.

We start with some notations. Let M be a Riemann surface of finite conformal type with $\partial(M) \neq \emptyset$, and let N be an open Riemann surface. An immersion $Y \in \mathcal{E}(N)$ is said to *pass by* $X \in \mathcal{E}(M)$ if there exist proper regions $\{\Omega_n\}_{n \in \mathbb{N}}$ in N and biholomorphisms $h_n : M \rightarrow \Omega_n$,

$n \in \mathbb{N}$, such that $\{Y \circ h_n\}_{n \in \mathbb{N}} \rightarrow X$ in the $\mathcal{C}^0(M)$ -topology. Note that if Y passes by X then $X(M) \subset \overline{Y(N)}$, but the converse is not necessarily true.

Definition 4.1 *Let N be an open Riemann surface N . An immersion $Y \in \mathcal{E}(N)$ is said to be universal if for any compact Riemann surface M with non empty boundary and any conformal minimal immersion $X : M \rightarrow \mathbb{R}^3$, Y passes by X .*

The next lemma is an elementary consequence of Theorem 4.1:

Lemma 4.1 *Let $\{Y_i : N_i \rightarrow \mathbb{R}^3\}_{1 \leq i < \sigma}$ be a sequence of conformal complete minimal immersions of FTC, where $\sigma \in \mathbb{N} \cup \{+\infty\}$, and assume that $\partial(N_i) \neq \emptyset$ for all i .*

Then there exists an open parabolic Riemann surface M^ and an immersion $Y \in \mathcal{E}(M^*)$ such that Y passes by Y_i for all i .*

Proof: Recall that N_i has finite conformal type for all i , and without loss of generality assume that $N_i \cap N_j = \emptyset$, $i \neq j$. Set $N_{i,j} = N_i \times \{j\}$, $h_{i,j} : N_i \rightarrow N_{i,j}$, $h_{i,j}(P) = (P, j)$, $Y_{i,j} = Y_i \circ h_{i,j}^{-1}$ for all $j \in \mathbb{N}$. Write \mathcal{Y} for the countable family $\{Y_{i,j} : 1 \leq i < \sigma, j \in \mathbb{N}\}$, and take a bijective enumeration $\{X_j : M_j \rightarrow \mathbb{R}^3\}_{j \in \mathbb{N}}$ of \mathcal{Y} . Label M_j^* as an annular neighborhood of M_j homeomorphic to M_j^* where X_j can be extended as a conformal minimal immersion, $j \in \mathbb{N}$, and let M^* denote a parabolic completion of $\{M_j^*\}_{j \in \mathbb{N}}$. Note that $X_j \in \mathcal{E}_{M^*}(M_j)$ for all j , and consider the fusion immersion $Y \in \mathcal{E}(M^*)$ of Theorem 4.1 associated to $\{X_i\}_{i \in \mathbb{N}}$ and $\epsilon > 0$. For any $i < \sigma$ and $j \in \mathbb{N}$, label i_j as the unique natural such that $Y_{i,j} = X_{i_j}$ (hence $N_{i,j} = M_{i_j}$). As $\lim_{j \rightarrow \infty} \|Y|_{M_{i_j}} - X_{i_j}\|_0 = 0$, then $\{Y \circ h_{i,j}\}_{j \in \mathbb{N}} \rightarrow Y_i$ in the $\mathcal{C}^0(N_i)$ -topology, concluding the proof. \square

In order to approach the existence problem of universal minimal surfaces, we need some preliminary results on Riemann surfaces. We start with the following:

Lemma 4.2 *Let R be an elliptic Riemann surface, and let V be an open disc in R . Then there is an $f_V \in \mathfrak{F}(R)$ all of whose branch points lie in V .*

Proof: The proof is trivial when $R = \overline{\mathbb{C}}$. Then we will assume that R has positive genus ν . For the following, it is convenient to go over again the notations and results of Section 2, and specially those of Subsection 2.2.

Claim 4.1 *There exists a $\tau_0 \in \mathfrak{W}(R) \cap \mathfrak{W}_0^R(R - V)$ without zeroes in $R - V$.*

Proof: Fix $E \in V$ and take a non zero $\theta \in \mathfrak{W}_0(R)$. Put $(\theta) = D_1 \cdot D$, where $D_1 \in \mathfrak{Div}(R - V)$ and $D \in \mathfrak{Div}(V)$. By Jacobi's theorem, we can find an open disc $U \subset V$ such that $\varphi_E : U_\nu \rightarrow \varphi_E(U_\nu)$ is a diffeomorphism, where U_ν is the set of divisors in R_ν with support in U . Since $J(R)$ is a compact additive Lie Group and $\varphi_E(U_\nu) \subset J(R)$ is an open subset, one has $n_0 \varphi_E(U_\nu) = J(R)$ for large enough $n_0 \in \mathbb{N}$. Therefore, there is $D_2 \in U_\nu$ such that $\varphi_E(D_2^{n_0}) = \varphi_E(D_1) = \varphi_E(D_1 E^m)$, where $m = n_0 \nu - \mathfrak{Deg}(D_1)$. By Abel's theorem there exists $f_0 \in \mathfrak{F}(R)$ such that $(f_0) = \frac{D_2^{n_0}}{D_1 E^\nu}$. It suffices to set $\tau_0 = f_0 \theta$. \square

Fix a non Weierstrass point $Q \in V$, and label $\mathcal{U}_Q \subset \mathfrak{W}(R)$ as the complex vectorial subspace of meromorphic 1-forms with $(\theta) \geq Q^{-\nu-1}$. By Riemann-Roch theorem, $\dim_{\mathbb{C}} \mathcal{U}_Q = 2\nu$ and the map $\mathcal{G} : \mathcal{U}_Q \rightarrow \mathbb{C}^{2\nu}$, $\mathcal{G}(\tau) = (\int_c \tau)_{c \in B}$, is a linear isomorphism.

As usual write $B = \{a_j, b_j\}_{1 \leq j \leq \nu}$ for a canonical basis of $H_1(R, \mathbb{Z})$, and choose the representative curves a_j, b_j , $j = 1, \dots, \nu$, in $R - \overline{V}$.

Claim 4.2 *Let $W \subset R$ be an open disc containing \overline{V} and disjoint from a_j, b_j for all j . Then, for any function $h \in \mathfrak{F}_0^R(R - W)$ never vanishing on $R - W$, there exists $f \in \mathfrak{F}_0(R - \{Q\})$ never vanishing on $R - \{Q\}$ such that $\log(h/f)$ has a well defined branch on $R - W$.*

Proof: Take $\tau \in \mathcal{U}_Q$ such that $dh/h - \tau$ has vanishing periods along a_j, b_j for all j , and observe that $\frac{1}{2\pi i} \int_{a_j} \tau, \frac{1}{2\pi i} \int_{b_j} \tau \in \mathbb{Z}$ for all j . Set $h_0 = \int (dh/h - \tau) \in \mathfrak{F}_0^R(R - W)$ and $f = e^{\int \tau} \in \mathfrak{F}_0(R - \{Q\})$. Finally, note that f never vanishes on $R - \{Q\}$ and $\log(h/f) = h_0 \in \mathfrak{F}_0^R(R - W)$. \square

Let σ be a non null exact 1-form in $\mathfrak{W}_0(R - \{Q\})$, and let $W \subset R$ be an open disc containing \overline{V} and all the zeroes of σ_0 in $R - \{Q\}$.

Claim 4.3 *There exists $\kappa \in \mathfrak{W}(R - \{Q\}) \cap \mathfrak{W}_0^R(R - V)$ without zeroes on $R - V$ and $g_0 \in \mathcal{F}_0^R(R - W)$ such that $\sigma|_{R-W} = e^{g_0}(\kappa|_{R-W})$.*

Proof: Set $h = (\sigma/\tau_0)|_{R-W}$, where τ_0 is the 1-form given in Claim 4.1. If necessary, choose the representative curves $a_j, b_j, j = 1, \dots, \nu$, for B in $R - W$. By the previous claim, there is $f \in \mathfrak{F}_0(R - \{Q\})$ never vanishing on $R - \{Q\}$ such that $g_0 := \log(h/f)$ is a well defined holomorphic map on $R - W$. Label $\kappa = f\tau_0 \in \mathfrak{W}(R - \{Q\}) \cap \mathfrak{W}_0^R(R - V)$, and note that κ has no zeroes on $R - V$. Finally, observe that $\sigma|_{R-W} = (e^{g_0}\kappa)|_{R-W} \in \mathfrak{W}_0^R(R - W)$. \square

Claim 4.4 *The linear map $\mathcal{L}_0 : \mathfrak{F}_0(R - \{Q\}) \rightarrow \mathbb{C}^{2\nu}$, $\mathcal{L}_0(h) = (\int_c h e^{g_0} \kappa)_{c \in B}$, is surjective.*

Proof: Endow $\mathfrak{F}_0(R - \{Q\})$ with the topology of the uniform convergence on compact subsets of $R - \{Q\}$, and observe that \mathcal{L}_0 is continuous. Take a basis $\{\theta_j\}_{j=1, \dots, 2\nu}$ of \mathcal{U}_Q , set $h_j = \theta_j / (e^{g_0} \kappa) \in \mathfrak{F}_0^R(R - V)$ for each j , and observe that $\{(\int_c h_j e^{g_0} \kappa)_{c \in B}\}_{j=1, \dots, 2\nu}$ is a basis of \mathbb{C}^n .

On the other hand, Theorem 2.1 implies that $\mathfrak{F}_0(R - \{Q\})$ is dense in $\mathfrak{F}_0^R(R - W)$ with respect to the $\omega(R - W)$ -topology, and so h_j lies in the closure of $\mathfrak{F}_0(R - \{Q\})$ in $\mathfrak{F}_0^R(R - W)$ for all j . By a continuity argument \mathcal{L}_0 is surjective and we are done. \square

Consider $\{g_n\}_{n \in \mathbb{N}} \subset \mathfrak{F}_0(R - \{Q\})$ such that $\{g_n|_{R-W}\}_{n \in \mathbb{N}} \rightarrow g_0|_{R-W}$ in the $\omega(R - W)$ -topology (use Theorem 2.1). Set $\mathcal{L}_n : \mathfrak{F}_0(R - \{Q\}) \rightarrow \mathbb{C}^{2\nu}$, $\mathcal{L}_n(h) = (\int_c h e^{g_n} \kappa)_{c \in B}$, and observe that \mathcal{L}_n is a continuous linear operator for all $n \in \mathbb{N} \cup \{0\}$. Furthermore, $\{\mathcal{L}_n\}_{n \in \mathbb{N}} \rightarrow \mathcal{L}_0$ in the weak topology, that is to say, $\{\mathcal{L}_n(h)\}_{n \in \mathbb{N}} \rightarrow \mathcal{L}_0(h)$ for all $h \in \mathfrak{F}_0(R - \{Q\})$. By Claim 4.4 there exists $\{f_j\}_{j=1, \dots, 2\nu} \subset \mathfrak{F}_0(R - \{Q\})$ such that $\{\mathcal{L}_n(f_j)\}_{j=1, \dots, 2\nu}$ generates $\mathbb{C}^{2\nu}$, n large enough (up to removing finitely many terms, for all $n \in \mathbb{N} \cup \{0\}$). Define $\mathcal{Q}_n : \mathbb{C}^{2\nu} \rightarrow \mathbb{C}^{2\nu}$, $\mathcal{Q}_n(\{x_j\}_{j=1, \dots, 2\nu}) = (\int_c e^{g_n + \sum_{j=1}^{2\nu} x_j f_j} \kappa)_{c \in B}$, $n \in \mathbb{N} \cup \{0\}$, and notice that $\{\mathcal{Q}_n\}_{n \in \mathbb{N}} \rightarrow \mathcal{Q}_0$ as analytic maps on compact subsets of $\mathbb{C}^{2\nu}$. Since the Jacobian of \mathcal{Q}_n at $\mathbf{0} = (0)_{j=1, \dots, 2\nu}$ is different from zero for all $n \in \mathbb{N} \cup \{0\}$, there exists an Euclidean ball $B_0 \subset \mathbb{C}^{2\nu}$ centered at $\mathbf{0}$ such that $\mathcal{Q}_n : B_0 \rightarrow \mathcal{Q}_n(B_0)$ is a diffeomorphism for all $n \in \mathbb{N} \cup \{0\}$. Furthermore, as $\mathcal{Q}_0(\mathbf{0}) = \mathbf{0}$ then $\mathbf{0} \in \mathcal{Q}_n(B_0)$ for large enough n (without loss of generality, for all n). Take $(y_j^n)_{j=1, \dots, 2\nu} \in B_0$ such that $\mathcal{Q}_n((y_j^n)_{j=1, \dots, 2\nu}) = \mathbf{0}$ and set $\sigma_n = e^{g_n + \sum_{j=1}^{2\nu} y_j^n f_j} \kappa \in \mathfrak{W}_0(R - \{Q\})$, $n \in \mathbb{N}$. The 1-form σ_n have no periods and never vanish on $R - V$, hence the function $F_n = \int \sigma_n \in \mathfrak{F}_0^R(R - V)$ has no branch points on $R - V$, $n \in \mathbb{N}$.

To finish, fix $n_0 \in \mathbb{N}$ and use Theorem 2.1 to find $\{H_k\}_{k \in \mathbb{N}} \subset \mathfrak{F}(R) \cap \mathfrak{F}_0(R - \{Q\})$ such that $\{H_k\}_{k \in \mathbb{N}} \rightarrow F_{n_0}$ in the $\omega(R - V)$ -topology. By Hurwitz theorem, we can suppose that dH_n never vanishes on $R - V$ for all n . It suffices to choose $f_V = H_n$ for some $n \in \mathbb{N}$. \square

Given a polynomial \mathbf{p} with complex coefficients in the variables z and w , we denote by $\text{Deg}_z(\mathbf{p})$ and $\text{Deg}_w(\mathbf{p})$ the degree of \mathbf{p} in z and w , respectively.

Let R be an elliptic Riemann surface of genus $\nu \geq 1$. For any $f \in \mathcal{F}(R)$, write $\text{Deg}(f)$ for the degree of f as meromorphic function on R . Let $Q \in R$ be a non Weierstrass point, and for each $n \geq \nu + 1$, let $f_n \in \mathfrak{F}(R) \cap \mathfrak{F}_0(R - \{Q\})$ denote a non zero function with $\text{Deg}(f_n) = n$ and polar divisor $(f_n)_\infty = Q^n$. Label $z = f_{\nu+1}$ and $w = f_{\nu+2}$. We know there is an irreducible complex polynomial \mathbf{p} in the variables z and w with $\text{Deg}_z(\mathbf{p}) - 1 = \text{Deg}_w(\mathbf{p}) = \nu + 1$ satisfying $\mathbf{p}(z(P), w(P)) = 0$ for all $P \in R$. Furthermore, R is biholomorphic to the algebraic curve $C_{\mathbf{p}} := \{(z, w) \in \overline{\mathbb{C}}^2 : \mathbf{p}(z, w) = 0\}$ (up to this biholomorphism we will consider $R = C_{\mathbf{p}}$), and the pair

$\{z, w\}$ generates the field of meromorphic functions $\mathcal{F}(R)$. The last means that any $f \in \mathfrak{F}(R)$ is of the form $f = \mathbf{p}_1(z, w)/\mathbf{p}_2(z, w)$ for suitable polynomials \mathbf{p}_1 and \mathbf{p}_2 without common factors and having $\text{Deg}_w(\mathbf{p}_i) \leq \nu$.

Remark 4.1 If $R = \overline{\mathbb{C}}$, we also have that $R \cong C_{\mathbf{p}_0} := \{(z, w) \in \overline{\mathbb{C}}^2 : \mathbf{p}_0(z, w) = 0\}$ for $\mathbf{p}_0(z, w) = w^2 - (z - a_1)(z - a_2)$, where $a_1, a_2 \in \mathbb{C}$, $a_1 \neq a_2$.

Definition 4.2 For each $v = (\nu, k, s) \in (\mathbb{N} \cup \{0\}) \times \mathbb{N}^2$, write \mathcal{W}_v for the space of couples (\mathbf{p}, F) such that:

- $\mathbf{p}(z, w)$ is an irreducible complex polynomial in (z, w) with $\text{Deg}_z(\mathbf{p}) - 1 = \text{Deg}_w(\mathbf{p}) = \nu + 1$.
- The algebraic curve $C_{\mathbf{p}}$ has genus ν , (∞, ∞) is the only pole of z and w as meromorphic functions on $C_{\mathbf{p}}$, and $(0, 0) \in C_{\mathbf{p}}$.
- $F = ((\mathbf{p}_{1,j}, \mathbf{p}_{2,j}))_{j=1,2,3}$, where $\mathbf{p}_{1,j}$ and $\mathbf{p}_{2,j}$ are complex polynomials in (z, w) with no common factors so that $\text{Deg}_w(\mathbf{p}_{i,j}) \leq \nu$, $i = 1, 2$, $j = 1, 2, 3$.
- Setting $f_j : C_{\mathbf{p}} \rightarrow \overline{\mathbb{C}}$, $f_j(P) := \mathbf{p}_{1,j}(z(P), w(P))/\mathbf{p}_{2,j}(z(P), w(P))$, we have that $\sum_{j=1}^3 f_j^2 = 0$ on $C_{\mathbf{p}}$ and $\text{Deg}(g) = k$, where $g := f_3/(f_1 - if_2)$.
- The polar set $E_{\mathbf{p},F}$ of the vectorial 1-form Fdz on $C_{\mathbf{p}}$ has s points, $(0, 0) \notin E_{\mathbf{p},F}$, and $\sum_{j=1}^3 |f_j|^2 |dz|^2$ has no zeroes on $C_{\mathbf{p}} - E_{\mathbf{p},F}$.
- The meromorphic 1-form $f_j dz \in \mathfrak{M}(R)$ has no real periods on $C_{\mathbf{p}} - E_{\mathbf{p},F}$, $j = 1, 2, 3$.

We also set $\mathcal{A}_v = \mathbb{R}^3 \times \mathcal{W}_v$.

For any two complex polynomials $\mathbf{p}_1(z, w) = \sum_{i,j} a_{i,j} z^i w^j$ and $\mathbf{p}_2(z, w) = \sum_{i,j} b_{i,j} z^i w^j$, we set $d(\mathbf{p}_1, \mathbf{p}_2) = \sum_{i,j} |a_{i,j} - b_{i,j}|$. We endow \mathcal{W}_v with the topology induced by the metric $d^7 \equiv d \times (d \times d)^3$, and likewise equip $\mathcal{A}_v = \mathbb{R}^3 \times \mathcal{W}_v$ with the topology induced by the metric $d_0 \times d^7$, where d_0 is the Euclidean metric in \mathbb{R}^3 .

Given $v = (\nu, k, s)$ and $(\mathbf{p}, F) \in \mathcal{W}_v$ as above, elementary algebraic arguments show that $\text{Deg}_z(\mathbf{p}_{h,j})$, $h = 1, 2$, $j = 1, 2, 3$, admit an universal upper bound depending only on k and ν . Notice also that for any $y = (x, (\mathbf{p}, F)) \in \mathcal{A}_v$, the well defined immersion

$$X_y : C_{\mathbf{p}} - E_{\mathbf{p},F} \rightarrow \mathbb{R}^3, \quad X_y(q) = x + \text{Re}\left(\int_{(0,0)}^q F dz\right)$$

lies in $\mathcal{E}(C_{\mathbf{p}} - E_{\mathbf{p},F})$ for all $(\mathbf{p}, F) \in \mathcal{W}_v$.

For each $v = (\nu, k, s) \in (\mathbb{N} \cup \{0\}) \times \mathbb{N}^2$, let \mathcal{E}_v denote the moduli space of conformal complete minimal immersions $X : M \rightarrow \mathbb{R}^3$ such that M is a s -punctured genus ν elliptic Riemann surface and X has total curvature $-4\pi k$. It is clear that $X_y \in \mathcal{E}_v$ for any $y \in \mathcal{A}_v$.

Equip \mathcal{E}_v with the following topology: a sequence $\{X_n : M_n \rightarrow \mathbb{R}^3\}_{n \in \mathbb{N}} \subset \mathcal{E}_v$ is said to be convergent in the \mathcal{C}_*^0 topology to $X_0 : M_0 \rightarrow \mathbb{R}^3$, where $X_0 \in \mathcal{E}_v$, if for any compact region $\Omega_0 \subset M_0$ there exist compact regions $\Omega_n \subset M_n$ and biholomorphisms $h_n : \Omega_0 \rightarrow \Omega_n$, $n \in \mathbb{N}$, such that $\{X_n \circ h_n\}_{n \in \mathbb{N}} \rightarrow X_0|_{\Omega_0}$ in the $\mathcal{C}^0(\Omega_0)$ -topology.

Lemma 4.3 The map $\Delta_v : \mathcal{A}_v \rightarrow \mathcal{E}_v$, $\Delta_v(y) := X_y$ is surjective and continuous.

Proof: For the surjectivity, take an arbitrary immersion $X : M \rightarrow \mathbb{R}^3$ in \mathcal{E}_v . By Osserman's theorem, $M = R - \{Q_1, \dots, Q_s\}$, where R is an elliptic genus ν Riemann surface, and the Weierstrass data of X extend meromorphically to R . Fix a non Weierstrass point $Q \in M$ and as above take $z, w \in \mathfrak{F}(R)$ with $\text{Deg}(z) = \text{Deg}(w) - 1 = \nu + 1$ and $(z)_\infty = Q^{\nu+1}$, $(w)_\infty = Q^{\nu+2}$. Fix also $Q_0 \in M - \{Q\}$, and without loss of generality suppose $z(Q_0) = w(Q_0) = 0$. Label $\mathbf{p}(z, w)$ as the irreducible polynomial in (z, w) such that $R = C_{\mathbf{p}}$, and write $\partial_z X/dz = (f_j(z, w))_{j=1,2,3}$, where $f_j \in \mathfrak{F}(R)$ is a rational function of the form $\mathbf{p}_{1,j}(z, w)/\mathbf{p}_{2,j}(z, w)$ and $\text{Deg}_w(\mathbf{p}_{i,j}) \leq \nu$, $i = 1, 2$, $j = 1, 2, 3$. As the meromorphic Gauss map $g = f_3/(f_1 - if_2)$ has degree k on R , then $(\mathbf{p}, F = (\mathbf{p}_{1,j}, \mathbf{p}_{2,j})_{j=1,2,3}) \in \mathcal{W}_v$. It is clear that $X = \Delta_v((X(Q_0), (\mathbf{p}, F)))$.

To check that Δ_v is continuous, take $\{y_n = (x_n, (\mathbf{p}_n, F_n))\}_{n \in \mathbb{N} \cup \{0\}} \in \mathcal{A}_v$ such that $\{y_n\}_{n \in \mathbb{N}} \rightarrow y_0$, and fix an arbitrary compact region $\Omega_0 \subset C_{\mathbf{p}_0} - E_{\mathbf{p}_0, F_0}$. We have to find compact regions $\Omega_n \subset C_{\mathbf{p}_n} - E_{\mathbf{p}_n, F_n}$ and biholomorphisms $h_n : \Omega_0 \rightarrow \Omega_n$, such that $\{X_{y_n} \circ h_n\}_{n \in \mathbb{N}} \rightarrow X_{y_0}|_{\Omega_0}$ in the $C^0(\Omega_0)$ -topology.

Remark 4.2 Recall that z and w are meromorphic functions on $C_{\mathbf{p}_n} = \{(z, w) \in \overline{\mathbb{C}}^2 : \mathbf{p}_n(z, w) = 0\}$ for all $n \in \mathbb{N} \cup \{0\}$, hence they depend on n . This lack of notation does not affect our exposition.

Let U_0 be an open subset of $C_{\mathbf{p}_0} - (\{(0, 0)\} \cup \Omega_0)$ containing $E_{\mathbf{p}_0, F_0}$. By Lemma 4.2, there is a meromorphic function $z_0 : C_{\mathbf{p}_0} \rightarrow \overline{\mathbb{C}}$ with all its branch points in U_0 . Write $z_0 = \mathbf{q}_1(z, w)/\mathbf{q}_2(z, w)$, where $\mathbf{q}_1, \mathbf{q}_2$ are polynomials with no common factors and $\text{Deg}_w(\mathbf{q}_i) \leq \nu$, $i = 1, 2$, and choose w_0 any function in $\{z, w\}$ so that $\{z_0, w_0\}$ generates $\mathfrak{F}(C_{\mathbf{p}_0})$. Let $z^l w^j$ be the effective monomial (i.e., with non zero coefficient) in \mathbf{q}_1 and \mathbf{q}_2 with maximum degree as meromorphic function on $C_{\mathbf{p}_0}$. Since $\text{Deg}(w) - 1 = \text{Deg}(z) = \nu + 1$ and $\text{Deg}_w(\mathbf{q}_i) \leq \nu$, $i = 1, 2$, this monomial always exists and is unique. Furthermore, as z and w have an unique pole at the same point (namely, (∞, ∞)) of $C_{\mathbf{p}_0}$, then $\text{Deg}(z_0) = l(\nu + 1) + j(\nu + 2)$. The same argument shows that $z_n : C_{\mathbf{p}_n} \rightarrow \overline{\mathbb{C}}$, $z_n = \mathbf{q}_1(z, w)/\mathbf{q}_2(z, w)$, has $\text{Deg}(z_n) = \text{Deg}(z_0)$ as meromorphic function on $C_{\mathbf{p}_n}$ for n large enough (without loss of generality, for all $n \in \mathbb{N}$).

In the sequel we write $a = \text{Deg}(z_n)$ (which does not depend on n) and $E_n = E_{\mathbf{p}_n, F_n}$, $n \in \{0\} \cup \mathbb{N}$. We also label B_n as the branch point set of z_n on $C_{\mathbf{p}_n}$ for all $n \in \{0\} \cup \mathbb{N}$. For any $P \in C_{\mathbf{p}_0}$, denote b_P as the branching number of $z_0 : C_{\mathbf{p}_0} \rightarrow \overline{\mathbb{C}}$ at P , and for each $\zeta \in \overline{\mathbb{C}}$ write $a_\zeta = \sum_{P \in z_0^{-1}(\zeta)} b_P$.

Choose for each $\zeta \in z_0(B_0 \cup E_0)$ an open disc $D_\zeta \subset \overline{\mathbb{C}}$ centered at ζ so that:

- $\{\overline{D}_\zeta : \zeta \in z_0(B_0 \cup E_0)\}$ is a family of pairwise disjoint closed discs,
- $z_0^{-1}(D_\zeta)$ consists of $a - a_\zeta$ conformal discs,
- if $P \in z_0^{-1}(\zeta)$ and U_P is the connected component of $z_0^{-1}(D_\zeta)$ containing P , then $z_0|_{U_P} : U_P \rightarrow D_\zeta$ is a branched covering of b_P sheets, and
- $\overline{U}_P \subset U_0$ when $P \in E_0 \cup B_0$.

Since $\{d(\mathbf{p}_n, \mathbf{p}_0)\}_{n \in \mathbb{N}} \rightarrow 0$, B_0 is the limit set of B_n as $n \rightarrow \infty$ in $\overline{\mathbb{C}}^2$. In other words, if $\{P_n\}_{n \in \mathbb{N}} \subset \overline{\mathbb{C}}^2$ is a convergent sequence such that $P_n \in B_n$ for all n , then its limit lies in B_0 , and any point of B_0 is the limit of a sequence of this kind. Likewise, if we write $F_n = ((\mathbf{p}_{1,j}^n, \mathbf{p}_{2,j}^n))_{j=1,2,3}$ one has $\{d(\mathbf{p}_{i,j}^n, \mathbf{p}_{i,j}^0)\}_{n \in \mathbb{N}} \rightarrow 0$ for all i, j , and so E_0 is the limit set of $\{E_n\}_{n \in \mathbb{N}}$ in $\overline{\mathbb{C}}^2$ as well.

For each $P \in z_0^{-1}(z_0(B_0 \cup E_0))$, choose $Q_P^n \in z_n^{-1}(z_0(P))$, $n \in \mathbb{N}$, so that $\{Q_P^n\}_{n \in \mathbb{N}} \rightarrow P$ as points of $\overline{\mathbb{C}}^2$. By elementary topology, and up to removing finitely many terms of the sequence $\{y_n\}_{n \in \mathbb{N}}$ if necessary, we can suppose that:

- (i) $z_n^{-1}(D_\zeta)$ is a collection of $a - a_\zeta$ pairwise disjoint open discs on $C_{\mathbf{p}_n}$ for all $\zeta \in z_0(B_0 \cup E_0)$ and $n \in \mathbb{N} \cup \{0\}$.

- (ii) For any $P \in z_0^{-1}(z_0(B_0 \cup E_0))$ and $n \in \mathbb{N} \cup \{0\}$, $z_n|_{U_P^n} : U_P^n \rightarrow D_{z_0(P)}$ is a branched covering of b_P sheets, where U_P^n is the component of $z_n^{-1}(D_{z_0(P)})$ containing Q_P^n .¹

Set $W = \overline{\mathbb{C}} - \cup_{\zeta \in z_0(B_0 \cup E_0)} D_\zeta$, $W_n = z_n^{-1}(W) \subset C_{\mathbf{p}_n}$ and $\pi_n := z_n|_{W_n} : W_n \rightarrow W$, $n \in \mathbb{N} \cup \{0\}$. Fix $\zeta_0 \in W$ and choose $P_0^n \in z_n^{-1}(\zeta_0)$, $n \in \mathbb{N} \cup \{0\}$, so that $\{P_0^n\}_{n \in \mathbb{N}} \rightarrow P_0^0$ as points of $\overline{\mathbb{C}}^2$. Basic monodromy arguments give that $(\pi_n)_*(\Pi_1(W_n)) = (\pi_0)_*(\Pi_1(W_0)) \subset \Pi_1(W)$, where $\Pi_1(W_n)$ is the fundamental group of W_n with base point P_0^n , $\Pi_1(W)$ is the one of W with base point ζ_0 , and $(\pi_n)_* : \Pi_1(W_n) \rightarrow \Pi_1(W)$ is the group homomorphism induced by π_n , $n \in \mathbb{N}$.

For each $n \in \mathbb{N}$, let $\lambda_n : W_0 \rightarrow W_n$ denote the unique biholomorphism such that $\lambda_n(P_0^n) = P_0^0$ and $z_n \circ \lambda_n = z_0|_{W_0}$.

Label $J = \{P \in z_0^{-1}(B_0 \cup E_0) : b_P = 0\}$, and notice that $z_n|_{U_P^n} : U_P^n \rightarrow D_{z_0(P)}$ is a biholomorphism for all $P \in J$ and $n \in \mathbb{N}$. Call $V_n = \cup_{P \in J} U_P^n$ and $\hat{W}_n = W_n \cup V_n$, $n \in \mathbb{N} \cup \{0\}$. Let $\hat{\lambda}_n : \hat{W}_0 \rightarrow \hat{W}_n$ denote the natural extension of λ_n satisfying $z_n \circ \hat{\lambda}_n = z_0|_{\hat{W}_0}$, $n \in \mathbb{N}$. Since $\cup_{P \in E_0 \cup B_0} \overline{U_P} \subset U_0$, then $\Omega_0 \subset \hat{W}_0 - \partial(\hat{W}_0)$. Set $\Omega_n = \hat{\lambda}_n(\Omega_0)$ and write $h_n = \hat{\lambda}_n|_{\Omega_0} : \Omega_0 \rightarrow \Omega_n$, $n \in \mathbb{N}$. The facts that $\{d(\mathbf{p}_n, \mathbf{p}_0)\}_{n \in \mathbb{N}} \rightarrow 0$ and $\{d(\mathbf{p}_{i,j}^n, \mathbf{p}_{i,j}^0)\}_{n \in \mathbb{N}} \rightarrow 0$ for all i, j , imply that $\{w \circ \hat{\lambda}_n\}_{n \in \mathbb{N}} \rightarrow w|_{\hat{W}_0}$ and $\{F_n(z \circ \hat{\lambda}_n, w \circ \hat{\lambda}_n)\}_{n \in \mathbb{N}} \rightarrow F_0(z|_{\hat{W}_0}, w|_{\hat{W}_0})$ in the $\omega(\hat{W}_0)$ -topology. Taking into account that $\{x_n\}_{n \in \mathbb{N}} \rightarrow x_0$, we deduce that $\{X_{y_n} \circ h_n\}_{n \in \mathbb{N}} \rightarrow X_{y_0}|_{\Omega_0}$ in the $\mathcal{C}^0(\Omega_0)$ -topology, concluding the proof. \square

Now we can state the main result of this subsection.

Theorem 4.2 (Existence of universal surfaces) *There exist parabolic complete universal minimal surfaces of WFTC in \mathbb{R}^3 .*

Proof: Set $\mathcal{A} = \cup_{v \in (\mathbb{N} \cup \{0\}) \times \mathbb{N}^2} \mathcal{A}_v$ and $\mathcal{E} = \cup_{v \in (\mathbb{N} \cup \{0\}) \times \mathbb{N}^2} \mathcal{E}_v$ endowed with the corresponding direct sum topologies, and define $\Delta : \mathcal{A} \rightarrow \mathcal{E}$, $\Delta|_{\mathcal{A}_v} = \Delta_v$.

Notice that \mathcal{A}_v is separable, take a dense countable subset $\mathcal{D}_v \subset \mathcal{A}_v$ and denote by $\mathcal{S}_v = \Delta_v(\mathcal{D}_v)$. Lemma 4.3 says that $\mathcal{S} := \cup_{v \in (\mathbb{N} \cup \{0\}) \times \mathbb{N} \times \mathbb{N}} \mathcal{S}_v$ is a dense countable subset of \mathcal{E} as well.

For each $X : M \rightarrow \mathbb{R}^3$ in \mathcal{S} , fix a countable basis B_X of the topology on M formed by open discs bounded by Jordan curves in M , and call $\mathcal{S}_X = \{X|_{M-D} : D \in B_X\}$. Finally set $\mathcal{S}_0 = \cup_{X \in \mathcal{S}} \mathcal{S}_X$.

By Lemma 4.1, there exists an open parabolic Riemann surface M^* and an immersion $Y \in \mathcal{E}(M^*)$ passing by X for all $X \in \mathcal{S}_0$. Let us show that Y is universal.

Let M_0 be a compact genus ν Riemann surface with non empty boundary, label $s > 0$ as the number of components in $\partial(M_0)$. Let $X_0 : M_0 \rightarrow \mathbb{R}^3$ be a conformal minimal immersion that extends as a conformal minimal immersion to some open Riemann surface N containing M_0 . Let M_0^* be a compact annular extension of M_0 in N , and construct a conformal compactification R of M_0^* . Consider a finite subset $E \subset R - M_0$ so that $R - E$ is an annular extension of M_0 and notice that $X_0 \in \mathcal{E}_{R-E}(M_0)$. Then take a sequence $\{X_n\}_{n \in \mathbb{N}} \subset \mathcal{E}(R - E)$ converging to X_0 in the $\mathcal{C}^0(M_0)$ -topology (use Theorem 3.2). Note that $X_n \in \mathcal{E}_{v_n}$, where $v_n = (\nu, k_n, s)$ for some $k_n \in \mathbb{N}$. Fix $Q_0 \in M_0$, and use Lemma 4.3 to find $y_n = (X_n(Q_0), (\mathbf{p}_n, F_n)) \in \mathcal{A}_{v_n}$ such that $X_n = \Delta_{v_n}(y_n)$. By the density of \mathcal{S}_{v_n} in \mathcal{E}_{v_n} (see Lemma 4.3), there exists $\{\hat{X}_{j,n} : N_{j,n} \rightarrow \mathbb{R}^3\}_{j \in \mathbb{N}} \subset \mathcal{S}_{v_n}$, regions $W_{j,n} \subset N_{j,n}$ and biholomorphisms $h_{j,n} : M_0 \rightarrow W_{j,n}$, $j \in \mathbb{N}$, such that $\{\hat{X}_{j,n} \circ h_{j,n}\}_{j \in \mathbb{N}} \rightarrow X_n|_{M_0}$ in the $\mathcal{C}^0(M_0)$ -topology. Choose a disc $D_{j,n} \in B_{X_{j,n}}$ disjoint from $W_{j,n}$, call $X_{j,n} = \hat{X}_{j,n}|_{N_{j,n}-D_{j,n}} \in \mathcal{S}_{X_{j,n}} \subset \mathcal{S}_0$ and observe that $\{X_{j,n} \circ h_{j,n}\}_{j \in \mathbb{N}} \rightarrow X_n|_{M_0}$ in the $\mathcal{C}^0(M_0)$ -topology too. Finally, take $j_n \in \mathbb{N}$ such that $\|X_{j_n,n} \circ h_{j_n,n} - X_n|_{M_0}\|_0 < 1/n$ and label $h_n = h_{j_n,n}$, $n \in \mathbb{N}$. Since $\{X_{j_n,n} \circ h_n\}_{n \in \mathbb{N}} \rightarrow X_0$ in the $\mathcal{C}^0(M_0)$ -topology and Y passes by $X_{j_n,n}$ for all n , then Y passes by X_0 and we are done. \square

¹Notice that $U_P = U_P^0$ for all $P \in z_0^{-1}(z_0(B_0 \cup E_0))$.

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