HYPERBOLIC COMPLETE MINIMAL SURFACES WITH ARBITRARY TOPOLOGY

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Abstract

We show a method to construct orientable minimal surfaces in $\mathbb{R}^3$ with arbitrary topology. This procedure gives complete examples of two different kinds: surfaces whose Gauss map omits four points of the sphere and surfaces with a bounded coordinate function. We apply also these ideas to construct stable minimal surfaces with high topology which are incomplete or complete with boundary.

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1 Introduction.

The literature on minimal surfaces is vast. Nowadays we know a wealth of orientable complete hyperbolic minimal surfaces, but these examples are somewhat rare.

The most part of them have been obtained by using the Weierstrass representation for minimal surfaces and are parametrized by the unit disc $\overline{D}$.

However, to show examples with non trivial topology is a quite difficult problem: we have to specify first a Riemann surface, then guess the complex theoretic data and finally check period closing and completeness.

The period problem and the completeness are the main difficulty, and this is the reason because we don’t know a lot of examples.

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It is only due to remarkable effort and persistence that this somewhat implicit method has given a satisfactory geometric understanding for some families of such surfaces.

In this paper we present a method which answers the above question. We obtain an analytically clear general construction for orientable complete hyperbolic minimal surfaces with arbitrary topology. As a consequence, this method yields complete non flat examples of two different kinds: surfaces whose Gauss map omits four points of the sphere and surfaces with a bounded coordinate function.

R. Oserman [O], F. Xavier [X] and H. Fujimoto [FU1, FU2] studied the size of the spherical image of an orientable complete nonflat minimal surface. The last author obtained the best possible theorem, and proved that the number of exceptional values of the Gauss map is at most four. Indeed, there are many kinds of complete minimal surfaces whose Gauss map omits four points of the sphere. Among these examples we emphasize the classical Sherk’s doubly periodic surface and those described by K. Voss in [V] (see also [O]). Voss examples are simply-connected and, of course, of hyperbolic type.

In this paper we have proved:

**Theorem A.** For any \( n \in \mathbb{N}, n \geq 1 \), there exist a family of orientable complete non flat minimal surfaces of genus \( n \) and finite topology whose Gauss maps omit four points of the sphere.

Concerning orientable complete non flat examples with a bounded coordinate function, the first result of existence was obtained by L.P. Jorge and F. Xavier [JX1]. Later F.F. Brito [B1] showed a new method to construct the same kind of surfaces. Many other authors, as H. Rosenberg and E. Toubiana [RT], F.J. Lopez [L], C. Costa and P.A.Q. Simoes [CS] and F. Brito [B2], have obtained complete examples with non trivial topology contained in a slab of \( \mathbb{R}^3 \). Their methods are based on those described by Jorge-Xavier and Brito.

We use both Jorge-Xavier and Brito constructions in a different way to obtain:

**Theorem B.** For any \( n \in \mathbb{N}, n \geq 1 \), there exist a family of orientable complete non flat minimal surfaces of genus \( n \), finite topology and contained in a slab of \( \mathbb{R}^3 \).

Finally, we mention that our construction yields also stable examples with arbitrary topology which are incomplete or complete with boundary. We emphasize the following result:

**Theorem C.** For any \( n \in \mathbb{N}, n \geq 1 \), there exist a family of complete non flat stable minimal surfaces of genus \( n \), finite topology and non empty boundary.

In Theorem C we can substitute the hypothesis of completeness for compactness.

We can find families of surfaces satisfying Theorems A, B or C with exactly three ends, and it
is natural to deduce from our method the existence of examples with any number greater than or equal to three of ends.

Furthermore, any surface in above three families can be deformed, by surfaces of the same type and preserving the topology and number of ends along the deformation, into a branched covering of a plane.

For the details, see section 3.

2 Preliminaries on Minimal Surfaces and Riemann Surfaces.

Throughout this section, \( M \) will be a Riemann surface with piecewise analytic boundary \( \partial M \). This means that \( M \) can be considered as a subset of an open Riemann surface \( M_0 \), the conformal structure of \( M - \{ \partial M \} \) is that induced by \( M_0 \) and \( \partial M \) consist of a set of piecewise analytic curves in \( M_0 \), each of them homeomorphic to either \( \mathbb{R} \) or \( S^1 \). The case \( \partial M = \emptyset \) is allowed.

Functions and 1-forms meromorphic (holomorphic) on \( M \) are by definition the restriction of functions and 1-forms meromorphic (holomorphic) on \( M_0 \).

Let \( \Phi_k, k = 1, 2, 3 \) denote three holomorphic 1-forms on \( M \) satisfying:

\[
\left( \sum_{k=1}^{3} \Phi_k^2 \right)(P) = 0 \quad \left( \sum_{k=1}^{3} |\Phi_k|^2 \right)(P) \neq 0
\]  

(1)

for any \( P \in M \), and

\[\text{Re}\left(\int_{\alpha} \Phi_k\right) = 0, \quad k = 1, 2, 3\]

(2)

for any closed curve \( \alpha \subset M \).

Then, given \( P_0 \in M \), the map \( x : M \to \mathbb{R}^3 \) defined by:

\[x(P) = \text{Re}\left(\int_{P_0}^{P} (\Phi_1, \Phi_2, \Phi_3)\right)\]

(3)

provides a conformal minimal immersion of \( M \) in \( \mathbb{R}^3 \). Conversely, up to translations, any conformal minimal immersion \( x \) of \( M \) in \( \mathbb{R}^3 \) is obtained in this way.

Moreover, \( g = (\Phi_2 - i\Phi_1)/(i\Phi_3) \) is the Gauss map of \( x \) and the 1-forms \( \Phi_k, k = 1, 2, 3 \) are determined by \( g \) and the 1-form \( \Phi_3 \).

Furthermore, \( ds^2 = \sum_{k=1}^{3} |\Phi_k|^2 \) is the metric induced by \( x \) on \( M \), that is, if \( z \) is a local conformal parameter on \( M \) and \( \Phi_k(z) = f_k(z)dz \) then \( ds^2 = \sum_{k=1}^{3} |f_k(z)|^2|dz|^2 \).

By definition, \( (M, (\Phi_k)_{k=1,2,3}) \) is the Weierstrass representation of \( x \). For details see [O].

To finish this section, we recall some basic topics on compact Riemann surfaces.
For the remainder, we will suppose that \( M \) is a compact Riemann surface (without boundary) of genus \( n \geq 1 \).

Take \( \{Q_0, Q_1, \ldots, Q_k\} \subset M \) a set of \( k+1 \) different points and define \( M_0 = M - \{Q_0, Q_1, \ldots, Q_k\} \).

For the following results, see [F-K], p. 79-90. Let \( H^1_{hol}(M), H^1_{hol}(M_0) \) denote the first holomorphic de Rham cohomology group of \( M, M_0 \) respectively. By definition, this group is the vector space of holomorphic differentials on \( M, M_0 \) respectively, factorized by the subspace of exact holomorphic differentials.

Let \( \{\xi_1, \ldots, \xi_n\} \) be a basis of the \( \mathbb{C} \)-linear vector space of holomorphic 1-forms on \( M \) (remember that \( \dim \mathbb{C}(H^1_{hol}(M)) = n \)). Consider the Weierstrass "gap" sequence at \( Q_0 \):
\[ 1 = q_1 < \ldots < q_n < 2n \]
(see [F-K], p. 81) and take \( \xi_{n+j} \) a meromorphic 1-form on \( M \) with a pole of order \( q_j + 1 \) at \( Q_0 \) and holomorphic on \( M - \{Q_0\} \), \( j = 1, \ldots, n \). Finally label as \( \xi_{2n+j} \) a meromorphic 1-form on \( M \) with simple poles at both \( Q_0, Q_j \) and holomorphic on \( M - \{Q_0, Q_j\} \), \( j = 1, \ldots, k \).

**Theorem 1** ([F-K], p. 90) Each element of \( H^1_{hol}(M_0) \) is uniquely represented by a meromorphic differential in the linear span of the \( 2n + k \) linearly independent differentials \( \{\xi_1, \ldots, \xi_{2n+k}\} \).

Let \( f \) be a non constant meromorphic function on \( M \) with a pole of order \( m_0 \) at \( Q_0 \), \( m_0 > q_n \), and holomorphic on \( M - \{Q_0\} \). Suppose that the canonical divisor of \( df \) is given by:
\[
[df] = \prod_{j=1}^{k} \frac{Q_j^{m_j}}{Q_0^{m_0+1}}
\]

The following lemma is an easy consequence of the Weierstrass "gap" theorem:

**Lemma 1** There exists a meromorphic function \( v \) on \( M \) satisfying:

1. \( v \) is holomorphic on \( M - \{Q_0, Q_1, \ldots, Q_k\} \).
2. The pole order of \( v \) at \( Q_j \) is greater than or equal to \( m_j + 2 \), \( j = 1, \ldots, k \).
3. The pole order of \( v \) at \( Q_0 \) is not a multiple of \( m_0 \). In particular, \( v \) is not a rational function of \( f \).

**Proof:** If \( 1 = p_1^j < \ldots < p_n^j < 2n \) is the Weierstrass "gap" sequence at \( Q_j \), \( j = 1, \ldots, k \), there exists a meromorphic function \( v_j \) holomorphic on \( M - \{Q_j\} \) and with a pole at \( Q_j \) of order \( s_j \), where \( s_j \) is the first non "gap" greater than or equal to \( m_j + 2 \). Analogously, there exists a meromorphic function \( v_0 \) holomorphic on \( M - \{Q_0\} \) and with a pole at \( Q_0 \) of order \( s_0 \), where \( s_0 \) is the first non "gap" at \( Q_0 \) such that \( \gcd(s_0, m_0) = 1 \). To conclude the lemma, define \( v = \sum_{j=0}^{k} v_j \). \( \square \)
3 Main Theorems.

Let $D = \{ z \in \mathbb{C} : |z| < 1 \}$ be the unit disk and let $A$ denote a set such that $D \subset A \subset \overline{D}$. We suppose that the boundary of $A$: $\partial A := A \cup \partial D$ is the union of a countable (possibly void) set of open arcs contained in $\partial D$. It is clear that $\partial A = \emptyset$ if and only if $A = D$ and $\partial A = \partial \overline{D}$ if and only if $A = \overline{D}$. In a natural way $A$ is a Riemann surface with (possibly void) boundary.

Let $x : A \rightarrow \mathbb{R}^3$ be a conformal non flat minimal immersion of $A$ in $\mathbb{R}^3$ and denote by $g$ its Gauss map.

We are going to show a method to produce minimal immersions with arbitrary topology derived from $x$. Firstly, we give a brief outline of the global strategy.

Let $M$ be a compact Riemann surface without boundary of genus $n \geq 1$ and take $G : M \rightarrow \mathbb{C}$, $\mathbb{C} = \mathbb{C} \cup \{ \infty \}$, a non constant meromorphic function. As we will see later, we can choose $G$ in such a way that the set $G^{-1}(\overline{D})$ has a connected component $C$ topologically equivalent to a compact surface of genus $n$ with a finite number of holes. In this case, we define $C_A$ as the topological closure of $C$ in $G^{-1}(A)$. Obviously $C_A$ is a Riemann surface with (possibly void) piecewise analytic boundary. Furthermore $\partial A = \emptyset$ implies $\partial C_A = \emptyset$ and $C = C_A$, and when $A = \overline{D}$ then $C_A$ is a compact Riemann surface with piecewise analytic boundary.

Our main achievement is to construct conformal minimal immersions $y : C_A \rightarrow \mathbb{R}^3$ whose gauss map is equal to $g \circ (G|_{C_A})$. This construction is essentially based on a good choice of both the meromorphic function $G$ on $M$ and the Weierstrass data on $C_A$. The main difficulty is, of course, to solve the period problem.

This procedure will preserve obviously any property, depending on the Gauss map, which is invariant after composing with a branched covering: for instance, the size of the image of the Gauss map and the stability properties of the surface.

Another important preserved property is the completeness.

We are going to explain the details.

The first step consist of fixing a point $Q_0 \in M$ and taking a non constant meromorphic function $f$ on $M$ with a pole of order $m_0$ at $Q_0$ and holomorphic on $M - \{ Q_0 \}$. Labelling as $1 = q_1 < \ldots < q_n < 2n$ the Weierstrass “gap” sequence at $Q_0$, we assume that $m_0 > q_n$. Write the canonical divisor of $df$ by:

$$[df] = \prod_{j=1}^{k} Q_j^{n_j} Q_0^{m_0+1}$$

and let $v$ be the function in Lemma 1.

Then, define $M_0 = M - \{ Q_0, Q_1, \ldots, Q_k \}$ and let $\{ \xi_1, \ldots, \xi_{2n+k} \}$ be the basis of $H^1_{hol}(M_0)$ given in Theorem 1.

We deal with the following family of meromorphic functions:

$$\{ G^A : A = (\lambda, \lambda_1, \ldots, \lambda_{k+2n}) \in \mathbb{C}^{2n+k+1} \}$$
where

\[ G^\Lambda = \lambda v + \sum_{j=1}^{2n+k} \frac{\lambda_j \xi_j}{df} \]

**Remark 1** For any \( \Lambda \in \mathbb{C}^{2n+k+1} \), the function \( G^\Lambda \) is holomorphic on \( M_0 \).

Moreover, if \( \Lambda \neq 0 \) then \( G^\Lambda \) is not constant. Furthermore, when \( \lambda \neq 0 \) then the function \( G^\Lambda \) has a pole at any point \( Q_j, j \in \{0, 1, \ldots, k\} \).

To check the second part of the remark, observe that from Lemma 1 and the definition of the 1-forms \( \xi_1, \ldots, \xi_{2n+k} \) in the preceding section, the function \( v \) has a pole at \( Q_j \) of order greater than the one of \( \sum_{j=1}^{2n+k} \lambda_j \xi_j \) at this point, \( j = 0, 1, \ldots, k \). Hence, \( \lambda \neq 0 \) implies that \( G^\Lambda \) has a pole at \( Q_j, j = 0, 1, \ldots, k \). If \( \Lambda \neq 0 \) and \( \lambda = 0 \), use that \( \xi_1, \ldots, \xi_{2n+k} \) are linearly independent (see theorem 1) to get \( G^\Lambda \neq \text{constant} \).

We denote \(|\Lambda| = |\lambda| + \sum_{j=1}^{2n+k} |\lambda_j|\).

**Lemma 2** There exist a constant \( \varepsilon > 0 \) such that:

If \(|\Lambda| < \varepsilon\) then the set \((G^\Lambda)^{-1}(\mathbb{D})\) has a connected component \( C^\Lambda \) topologically equivalent to a compact surface of genus \( n \) with at most \( k+1 \) holes. Furthermore, if \( \lambda \neq 0 \) then \((G^\Lambda)^{-1}(\mathbb{D})\) has exactly \( k+1 \) holes.

**Proof**: Remember that the function \( G^\Lambda \) is holomorphic on \( M_0 \) and, from Remark 1, \( \lambda \neq 0 \) implies that \( G^\Lambda \) has a non trivial pole at any \( Q_j, j = 0, 1, \ldots, k \).

Let \( \{\overline{D}_j : j = 0, 1, \ldots, k\} \) denote set of closed pairwise disjoint conformal disks such that \( Q_j \in D_j \) (here \( D_j \) is the interior of \( \overline{D}_j \), \( j = 0, 1, \ldots, k \)).

From our choice of \( f \) and \( \xi_j \), the functions \( \xi_j/df \) are holomorphic on \( M_0 \cup \{Q_0\}, j = 1, \ldots, 2n+k \). Hence we can define for each \( j \in \{1, \ldots, 2n+k\} \):

\[ K_j = \text{Maximum} \left\{ |\xi_j/df|(P) : P \in M - \left( \bigcup_{i=1}^{k} D_i \right) \right\} > 0 \]

and analogously

\[ K_0 = \text{Maximum} \left\{ |v|(P) : P \in M - \left( \bigcup_{i=0}^{k} D_i \right) \right\} > 0 \]

Then, take \( K > \text{Maximum} \{K_j : j = 0, 1, \ldots, 2n+k\} > 0 \) and \( 0 < \varepsilon \leq \frac{1}{(2n+k+1)K} \).

Thus, for \(|\Lambda| < \varepsilon\), it is not hard to check that:

\[ M - \left( \bigcup_{i=0}^{k} D_i \right) \subset (G^\Lambda)^{-1}(\mathbb{D}) \]
We are going to study the topology of \((G^\lambda)^{-1}(\mathbb{D})\), \(|\Lambda| < \varepsilon\). The case \(\Lambda = 0\) is trivial, and so we suppose that \(\Lambda \neq 0\).

First we observe that \((G^\lambda)^{-1}(\mathbb{C} - \overline{D})\) contains as many connected components as different poles has \(G^\lambda\).

If \(G^\lambda\) has a pole at \(Q_j, j \in \{0, 1, \ldots, k\}\), then this point is the unique pole lying in \(\overline{D}_j\). Taking into account that \(|G^\lambda|_{\partial D_j} < 1\), there is a connected component of \((G^\lambda)^{-1}(\mathbb{C} - \overline{D})\) including \(Q_j\) and contained in \(D_j\). Hence each pole of \(G^\lambda\) is included in a different connected component of \((G^\lambda)^{-1}(\mathbb{C} - \overline{D})\). If the set \((G^\lambda)^{-1}(\mathbb{C} - \overline{D})\) had another connected component, label as \(\Omega\), containing no pole of \(G^\lambda\), then the function \(\frac{\lambda}{G^\lambda}|_{\partial C}\) would satisfy: it is holomorphic, \(\frac{\lambda}{G^\lambda}|_{\partial C}\) would be constant, a contradiction (recall that \(\Lambda \neq 0\) and remark 1).

Let \(C^\lambda\) be the connected component of \((G^\lambda)^{-1}(\mathbb{D})\) containing \(M - (\bigcup_{i=0}^{k} D_j)\).

It is clear now that \(\partial C^\lambda \subset \bigcup_{i=0}^{k} D_j\) and \(\partial C^\lambda\) has at least as many components as different poles has \(G^\lambda\). On the other hand \(\partial C^\lambda\) is the nodal set of the harmonic function \(\log(|G^\lambda|)\) and so it is the union of a finite number of properly immersed curves which are embedded and analytic except at a discrete (indeed finite) set of points where some of such curves meet in a equiangular way (see [CH] for a more general setting). Note that the finite set of singular points is void if and only if \(G^\lambda\) has no branch points on \(\partial C^\lambda\).

When \(Q_j\) is a pole of \(G^\lambda\), the set \(\alpha_j = \partial C^\lambda \cap D_j\) consist of exactly one piecewise analytic curve homeomorphic to \(S^1\). To see this, observe that \(\alpha_j\) bounds only a compact domain in \(D_j\). Otherwise there would be at least two compacts domains with boundary contained in \(\alpha_j\) and so one of them, label as \(\Omega\), would not contain \(Q_j\). Therefore \(\Omega\) would be a connected component of \((G^\lambda)^{-1}(\mathbb{C} - \overline{D})\) containing no pole of \(G^\lambda\), which is absurd by the maximum principle as above.

Putting together all the results we have obtained, we deduce that \(C^\lambda\) is topologically a compact surface of genus \(n\) minus as many holes (bounded by the curves \(\alpha_j\)) as poles has the function \(G^\lambda\). In particular from Remark 1, \(\Lambda \neq 0\) implies that \(C^\lambda\) has exactly \(k+1\) holes.

In what follows, we assume that \(|\Lambda| < \varepsilon\), where \(\varepsilon\) is the constant given in Lemma (2).

**Remark 2** If we label as \(C^\lambda_A\) the topological closure of \(C^\lambda\) in \((G^\lambda)^{-1}(A)\), then \(C^\lambda_A\) is a Riemann surface with (possibly void) piecewise analytic boundary. Furthermore \(A = \mathbb{D}\) implies \(C^\lambda = C^\lambda_A\) and \(A = \overline{D}\) yields that \(C^\lambda_A\) is a compact Riemann surface with piecewise analytic boundary.

For doing our reasoning, we can take an arbitrary compact Riemann surface \(M\) and any point \(Q_0 \in M\). Furthermore, the Riemann-Roch theorem provides a enormous family of suitable functions \(f\) and \(v\). It is natural to think that this generality yields, by using lemma 2, a large family of genus \(n\) surfaces with an arbitrary high finite number of holes (or ends).

However, the most interesting success should be to obtain surfaces with the lowest possible number of ends. At this point, we have:
Lemma 3  For a suitable choice of the Riemann surface \( M \), the point \( Q_0 \in M \) and the functions \( f \) and \( v \), the surfaces \( C^\Lambda \) have at most three holes. Moreover, if \( \lambda \neq 0 \) then they have exactly three holes.

Proof: Consider the compact genus \( n, n > 0 \), Riemann surface defined by:

\[
M = \{(z, w) \in \mathbb{C}^2 : w^{2n+1} = z^2 - 1\}
\]

with the canonical complex structure.

Label as \( Q_0 = (\infty, \infty), Q_1 = (1, 0), Q_2 = (-1, 0) \) and \( \{P_1, \ldots, P_{2n+1}\} = z^{-1}(0) \). Then take \( f = z \) and \( v = \frac{z^3}{w(z^2 - 1)} \).

It is clear that the canonical divisor of \( df \) and the principal divisor of \( v \) are given by:

\[
[df] = \frac{Q_1^{2n}Q_2^{2n}}{Q_0^{2n+2}}, \quad [v] = \frac{P_1^3 \cdots P_{2n+1}^3}{Q_0^{2n-1}Q_1^{2n+2}Q_2^{2n+2}}
\]

The highest Weierstrass gap of \( M \) at \( Q_0 \) is always less than \( 2n \) and then less than the pole order of \( f \) at \( Q_0 \). Furthermore, \( v \) satisfies the conditions 1, 2 and 3 in lemma 1.

Hence, we can apply above construction and lemma 2 yields surfaces with at most three ends.

\( \Box \)

Let \( \{D_j : j = 0, 1, \ldots, k\} \) be the set of closed pairwise disjoint conformal disks chosen in the proof of Lemma 2. Recall that \( Q_j \in D_j, j = 0, 1, \ldots, k \) and \( \overline{D_j} \cap \overline{D_i} = \emptyset, i \neq j \).

Let \( \{\gamma_1, \ldots, \gamma_{2n}\} \) denote a homology basis of \( M \) contained in \( M - (\bigcup_{j=0}^k \overline{D_j}) \), and label as \( \gamma_{2n+j} = \partial D_j, j = 1, \ldots, k \). It is clear that the set \( \{\gamma_1, \ldots, \gamma_{2n+k}\} \) contains a homology basis of \( C^\Lambda_A \) (in fact is a homology basis when \( C^\Lambda_A \) has \( k+1 \) holes).

Let \( (A, (\Phi_k)_{k=1,2,3}) \) denote the Weierstrass representation of \( x : A \to \mathbb{R}^3 \). Remember that the Gauss map \( g \) of \( x \) and the 1-form \( \Phi_3 \) determine the Weierstrass data, and label as \( h \) the holomorphic function on \( A \) defined by: \( h(z)dz = (\Phi_3/g)(z) \).

For each \( \Delta = (\delta_1, \ldots, \delta_{2n+k}) \in \mathbb{C}^{2n+k} \), define the meromorphic function:

\[
F^\Delta = G^{(0,\Delta)} = \sum_{j=1}^{2n+k} \delta_j \frac{\xi_j}{df}
\]

Take on \( C^\Lambda_A \) the following Weierstrass data:

\[
g^\Lambda = g \circ G^\Lambda, \quad \Phi^\Lambda_{3,\Delta} = Exp(F^\Delta)(h \circ G^\Lambda)df
\]  

(5)
Remark 3 When $C^\Lambda_A$ has $k+1$ holes, (for instance, if $\lambda \neq 0$ (see Remark 1 and Lemma 2)), the zeroes and poles of $df$ (see (4)) are not contained in $C^\Lambda_A$. Hence, taking also into account that $F^\Lambda$ is holomorphic and never vanishes on $M_0$, the three 1-forms associated to $g^\Lambda$ and $\Phi^\Lambda_3$:

$$2\Phi^\Lambda_1 = (g^\Lambda - 1/g^\Lambda)\Phi^\Lambda_3, \quad 2\Phi^\Lambda_2 = i(g^\Lambda + 1/g^\Lambda)\Phi^\Lambda_3, \quad \Phi^\Lambda_3$$

satisfy (1).

Therefore, fixing $P_0 \in C^\Lambda_A$ and taking into account (3), the function $x^\Lambda : C^\Lambda_A \to \mathbb{R}^3$ given by:

$$2x^\Lambda = \text{Real} \int_{P_0} (g^\Lambda - 1/g^\Lambda, i(g^\Lambda + 1/g^\Lambda), 2) \Phi^\Lambda_3$$

defines a multivaluated conformal minimal immersion of $C^\Lambda_A$ in $\mathbb{R}^3$.

Moreover, $x^\Lambda$ is well defined on $C^\Lambda_A$ if and only if (2) holds, that is,

$$\text{Real} \left( \int_{\gamma_j} \Phi^\Lambda_i \right) = 0, \quad i = 1, 2, 3, j = 1, \ldots, 2n + k \quad (6)$$

The following theorem answers this problem:

**Theorem 2** There exist an open subset $U \subset \mathbb{C}$, $0 \in U$, and a differentiable (in fact real analytic and in general non-constant) map $\Psi = (\Psi_1, \Psi_2) : U \to \mathbb{C}^{2n+k} \times \mathbb{C}^{2n+k}$ such that, $|\lambda, \Psi_1(\lambda)| < \varepsilon$, $\forall \lambda \in U$, and the minimal immersion $x^\lambda : C^\Lambda_A(\lambda, \Psi_1(\lambda)) \to \mathbb{R}^3$ given by $x^\lambda = x^\lambda(\lambda, \Psi_1(\lambda), \Psi_2(\lambda)$ is well defined (i.e., it has no real periods).

**Proof:** Let $\Sigma$ denote the open set:

$$\Sigma = \{ (\Lambda, \Delta) \in \mathbb{C}^{2n+k+1} \times \mathbb{C}^{2n+k} : |\Lambda| < \varepsilon \}$$

We will use the notation:

$$\lambda_j = x_{2j-1} + ix_{2j}, \delta_j = y_{2j-1} + iy_{2j}, \quad j = 1, \ldots, 2n + k$$

where $x_i, y_i \in \mathbb{R}$, $i = 1, \ldots, 4n + 2k$. For each $j = 1, \ldots, 2n + k$, $i \in \{1, 2, 3\}$, define $p^{i,j} : \Sigma \to \mathbb{R}$ as follows:

$$p^{i,j}(\Lambda, \Delta) = \text{Real} \left( \int_{\gamma_j} \Phi^\Lambda_i \right)$$
Define also for each $j \in \{1, \ldots, 2n + k\}$:

$$p^{4,j}(\Lambda, \Delta) = \text{Im} \left( \int_{\gamma_j} \Phi_4^{\Lambda, \Delta} \right)$$

Let $P : \Sigma \longrightarrow \mathbb{R}^{4(2n+k)}$ be the function given by:

$$P = (p^{i,1}, \ldots, p^{i,2n+k})_{i=1,2,3,4}$$

and write for $i \in \{1,2,3,4\}$:

$$A_i = \begin{pmatrix}
\frac{\partial p^{i,1}}{\partial x_1} & \frac{\partial p^{i,1}}{\partial x_2} & \cdots & \frac{\partial p^{i,1}}{\partial x_{4n+k}} \\
\frac{\partial p^{i,2}}{\partial x_1} & \frac{\partial p^{i,2}}{\partial x_2} & \cdots & \frac{\partial p^{i,2}}{\partial x_{4n+k}} \\
\vdots & \vdots & \ddots & \vdots \\
\frac{\partial p^{i,2n+k}}{\partial x_1} & \frac{\partial p^{i,2n+k}}{\partial x_2} & \cdots & \frac{\partial p^{i,2n+k}}{\partial x_{4n+k}}
\end{pmatrix}, \quad B_i = \begin{pmatrix}
\frac{\partial p^{i,1}}{\partial y_1} & \frac{\partial p^{i,1}}{\partial y_2} & \cdots & \frac{\partial p^{i,1}}{\partial y_{4n+k}} \\
\frac{\partial p^{i,2}}{\partial y_1} & \frac{\partial p^{i,2}}{\partial y_2} & \cdots & \frac{\partial p^{i,2}}{\partial y_{4n+k}} \\
\vdots & \vdots & \ddots & \vdots \\
\frac{\partial p^{i,2n+k}}{\partial y_1} & \frac{\partial p^{i,2n+k}}{\partial y_2} & \cdots & \frac{\partial p^{i,2n+k}}{\partial y_{4n+k}}
\end{pmatrix}$$

Since $\Phi_i^{(0,0)} = N_i df$, where $N_i$ is a constant, $i = 1, 2, 3$, then $P((0,0)) = 0$. So, if we prove that:

$$\text{det}(D)(0,0) \neq 0 \quad (7)$$

where $D$ is the matrix:

$$\begin{pmatrix}
A_1 & B_1 \\
A_2 & B_2 \\
A_3 & B_3 \\
A_4 & B_4
\end{pmatrix}$$

then the lemma holds by applying the Implicit Function Theorem to the function $P$ at $(0,0)$.

Assume that the determinant in (7) vanishes. Thus, the columns of the matrix $D$, label as $\{v_1, \ldots, v_{4(2n+k)}\}$, yield a sistem of $4(2n+k)$ linearly dependent vectors in $\mathbb{R}^{4(2n+k)}$. Hence, there exist $r_l, s_l \in \mathbb{R}$, $l = 1, \ldots, 2(2n+k)$ (with at least one of them non zero), such that:

$$\sum_{l=1}^{2(2n+k)} (r_l v_{1} + s_l v_{2(2n+k)+l}) = 0,$$

that is:
\[
\sum_{l=1}^{2(n+k)} \left( r_l \frac{\partial p_{l,j}}{\partial x_l} + s_l \frac{\partial p_{l,j}}{\partial y_l} \right)(0,0) = 0
\] (8)

for any \( i \in \{1, 2, 3, 4\} \) and \( j \in \{1, \ldots, 2n+k\} \).

On the other hand:

\[
\frac{\partial p_{i,j}}{\partial x_{2l-1}} = \text{Real} \left( \int_{\gamma_j} \frac{\partial \Phi_i^{l,\Delta}}{\partial \lambda_l} \right), \quad \frac{\partial p_{i,j}}{\partial y_{2l-1}} = \text{Real} \left( \int_{\gamma_j} \frac{\partial \Phi_i^{l,\Delta}}{\partial \delta_l} \right)
\]

\[
\frac{\partial p_{i,j}}{\partial x_{2l}} = \text{Im} \left( \int_{\gamma_j} \frac{\partial \Phi_i^{l,\Delta}}{\partial \lambda_l} \right), \quad \frac{\partial p_{i,j}}{\partial y_{2l}} = \text{Im} \left( \int_{\gamma_j} \frac{\partial \Phi_i^{l,\Delta}}{\partial \delta_l} \right)
\]

for \( i \in \{1, 2, 3\} \) and \( l, j \in \{1, \ldots, 2n+k\} \), and analogously:

\[
\frac{\partial p_{i,j}^{l,\Delta}}{\partial x_{2l-1}} = \text{Im} \left( \int_{\gamma_j} \frac{\partial \Phi_3^{l,\Delta}}{\partial \lambda_l} \right), \quad \frac{\partial p_{i,j}^{l,\Delta}}{\partial y_{2l-1}} = \text{Im} \left( \int_{\gamma_j} \frac{\partial \Phi_3^{l,\Delta}}{\partial \delta_l} \right)
\]

\[
\frac{\partial p_{i,j}^{l,\Delta}}{\partial x_{2l}} = -\text{Real} \left( \int_{\gamma_j} \frac{\partial \Phi_3^{l,\Delta}}{\partial \lambda_l} \right), \quad \frac{\partial p_{i,j}^{l,\Delta}}{\partial y_{2l}} = -\text{Real} \left( \int_{\gamma_j} \frac{\partial \Phi_3^{l,\Delta}}{\partial \delta_l} \right)
\]

for \( l, j \in \{1, \ldots, 2n+k\} \).

Thus, labeling as \( z_l = r_{2l-1} - ir_{2l}, \) \( w_l = s_{2l-1} - is_{2l}, \) \( l = 1, \ldots, 2n+k, \) the formulae in (8) imply:

\[
\text{Real} \left( \sum_{l=1}^{2(n+k)} z_l \int_{\gamma_j} \frac{\partial \Phi_i^{l,\Delta}}{\partial \lambda_l}(0,0) + w_l \int_{\gamma_j} \frac{\partial \Phi_i^{l,\Delta}}{\partial \delta_l}(0,0) \right) = 0
\] (9)

for \( i \in \{1, 2, 3\} \) and \( l, j \in \{1, \ldots, 2n+k\} \). and

\[
\text{Im} \left( \sum_{l=1}^{2(n+k)} z_l \int_{\gamma_j} \frac{\partial \Phi_i^{l,\Delta}}{\partial \lambda_l}(0,0) + w_l \int_{\gamma_j} \frac{\partial \Phi_i^{l,\Delta}}{\partial \delta_l}(0,0) \right) = 0
\] (10)

for \( l, j = 1, \ldots, 2n+k. \) But from (5) and for each \( l, j \in \{1, \ldots, 2n+k\} \) we have:

\[
\frac{\partial \int_{\gamma_j} \Phi_1^{l,\Delta}}{\partial \lambda_l}(0,0) = \left( \frac{h(g - 1/g)}{2} \right)^l \int_{\gamma_j} \xi_l, \quad \frac{\partial \int_{\gamma_j} \Phi_1^{l,\Delta}}{\partial \delta_l}(0,0) = \left( \frac{h(g - 1/g)}{2} \right)^l \int_{\gamma_j} \xi_l
\]
\[
\frac{\partial}{\partial \lambda_l} \int_{\gamma_j} \Phi_2^\Lambda,\Delta(0,0) = i \left( \frac{h(g + 1/g)}{2} \right)'(0) \int_{\gamma_j} \xi_l, \quad \frac{\partial}{\partial \lambda_l} \int_{\gamma_j} \Phi_3^\Lambda,\Delta(0,0) = i \left( \frac{h(g + 1/g)}{2} \right)'(0) \int_{\gamma_j} \xi_l
\]

\[
\frac{\partial}{\partial \lambda_l} (0,0) = \Phi_2^\Lambda,\Delta(0,0) = i \left( \frac{h(g + 1/g)}{2} \right)'(0) \int_{\gamma_j} \xi_l
\]

and so (9) and (10) become:

\[
\text{Real} \left( \sum_{l=1}^{2n+k} (z_l(h(g - 1/g))' + w_l(h(g - 1/g))(0)) \int_{\gamma_j} \xi_l \right) = 0
\]

(11)

\[
\text{Im} \left( \sum_{l=1}^{2n+k} (z_l(h(g + 1/g))' + w_l(h(g + 1/g))(0)) \int_{\gamma_j} \xi_l \right) = 0
\]

(12)

for any \( j \in \{1, \ldots, 2n+k\} \). From (12), the meromorphic 1-form on \( M \) defined by \( \sum_{l=1}^{2n+k} (z_lh'(0) + w_lh(0)) \xi_l \) is exact, and hence by theorem 1 we obtain:

\[ z_lh'(0) + w_lh(0) = 0, \quad l = 1, \ldots, 2n+k \]

From (1), \( h(0) \neq 0 \) and so \( w_l = -z_lh'(0)/h(0), \quad l = 1, \ldots, 2n+k \). Substituting \( w_l \) for this expression, the equations (11) become:

\[
\text{Real} \left( \sum_{l=1}^{2n+k} z_lh(0)g'(0)(1 + 1/g^2(0)) \int_{\gamma_j} \xi_l \right) = 0
\]

(13)

\[
\text{Im} \left( \sum_{l=1}^{2n+k} z_lh(0)g'(0)(1 - 1/g^2(0)) \int_{\gamma_j} \xi_l \right) = 0
\]
for $j = 1, \ldots, 2n + k$.

**Assertion.** Without loss of generality, we can assume that $g'(0) \neq 0$ and $g(0) \in \mathbb{R} - \{1, -1\}$.

To see this, take a point $z_0 \in \mathbb{D}$ such that $g'(z_0) \neq 0$, and let $L$ be a Möbius transformation leaving $\mathbb{D}$ invariant such that $L(0) = z_0$ (for instance, $L(z) = (z + z_0)/(\overline{z_0}z + 1)$). Then substitute $x$ for the minimal immersion $x \circ L$. The new Gauss map is $g \circ L$ and $(g \circ L)'(0) \neq 0$.

Moreover, after composing with a suitable rigid motion, we can also suppose that $g(0) \in \mathbb{R} - \{1, -1\}$.

Taking into account the former assertion and (13), we get:

$$
\sum_{i=1}^{2n+k} z_l \int_{\gamma_l} \xi_l = 0
$$

for $j = 1, \ldots, 2n + k$. Therefore the 1-form $\sum_{i=1}^{2n+k} z_l \xi_l$ is exact on $M$ and by using once again theorem 1 we get $z_l = 0, l = 1, \ldots, 2n + k$. This implies $w_l = 0, l = 1, \ldots, 2n + k$ and so $r_l, s_l = 0, l = 1, \ldots, 2(2n + k)$, which is absurd.

We can now apply the Implicit Function Theorem and obtain the existence of a open subset $U \subset \mathbb{C}$, $0 \in U$, and a differentiable (in fact real analytic) map

$$
\Psi = (\Psi_1, \Psi_2) : U \to \mathbb{C}^{2n+k} \times \mathbb{C}^{2n+k}
$$

such that $P((\lambda, \Psi_1(\lambda)), \Psi_2(\lambda)) = 0$. This means that the 1-form $\Phi^{(\lambda, \Psi_1(\lambda)), \Psi_2(\lambda)}_i$ is exact and $\Phi^{(\lambda, \Psi_1(\lambda)), \Psi_2(\lambda)}_i, i = 1, 2$, have no real periods on $C^{(\lambda, \Psi_1(\lambda))}_A$. Since $(\lambda, \Psi(\lambda)) \in \Sigma$, we deduce that $|\lambda, \Psi_1(\lambda))| < \varepsilon$, $\lambda \in U$.

Note also that when $\lambda \neq 0$, lemma 2 yields that $C^{(\lambda, \Psi_1(\lambda))}_A$ has $k + 1$ holes. In particular from remark 3 the three 1-forms $\Phi^{(\lambda, \Psi_1(\lambda)), \Psi_2(\lambda)}_i, i = 1, 2, 3$ satisfy (1). In general $\Psi \neq 0$, but this matter is not significant because these facts remain true even in the case $\Psi = 0$ (see remark 4).

Hence, $x^{\lambda} = x^{(\lambda, \Psi_1(\lambda)), \Psi_2(\lambda)} : C^{(\lambda, \Psi_1(\lambda))}_A \to \mathbb{R}^3$, $\lambda \in U - \{0\}$, defines a conformal minimal immersion.

This concludes the theorem. \(\square\)

In what follows and for the sake of simplicity, we write for $\lambda \in U - \{0\}$: $\Phi^{\lambda}_i, i = 1, 2, 3, C^{\lambda}_A$, $G^{\lambda}$, $h^{\lambda}$ and $h^{\lambda}$ instead of $\Phi^{(\lambda, \Psi_1(\lambda)), \Psi_2(\lambda)}_i, i = 1, 2, 3,$ $C^{(\lambda, \Psi_1(\lambda))}_A$, $G^{(\lambda, \Psi_1(\lambda))}_A$, $G^{(\lambda, \Psi_1(\lambda))}$, $G^{(\lambda, \Psi_1(\lambda))}_A$, $G^{(\lambda, \Psi_1(\lambda))}_A$, $g^{(\lambda, \Psi_1(\lambda))}$ and $h \circ G^{(\lambda, \Psi_1(\lambda))}$, respectively.

**Remark 4** When $\lambda \in U - \{0\}$ then $\Lambda \neq 0$. Thus lemma 1 and the comments which follow remark 1 imply that the pole order of the meromorphic function $G^{\lambda}$ at $Q_0$ is not a multiple of the one of $f$ at $Q_0$. Hence the function $G^{\lambda}$ is not a rational function of $f$ and so $x^{\lambda}$ is not a (unbranched) covering of $x$. 

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Note that from lemma 2, $C_\lambda$ is topologically equivalent to a genus $n$ compact surface with $k + 1$ holes. In addition, $C_\lambda^A$ is a surface with (possibly void) boundary.

The minimal immersions $\{x^\lambda\}_{\lambda \in U \setminus \{0\}}$ preserve some properties of $x$. The most important one is the completeness. In the case of non empty boundary, completeness means that any divergent curve in $C_\lambda^A$ has infinity length.

**Theorem 3** If $x : A \rightarrow \mathbb{R}^3$ is complete then, for any $\lambda \in U \setminus \{0\}$, the immersion $x^\lambda : C_\lambda^A \rightarrow \mathbb{R}^3$ is complete.

**Proof:** The metric $ds^2$ on $C_\lambda^A$ induced by $x^\lambda$ is given by:

$$ds^2 = \sum_{i=1}^{3} |\Phi_\lambda^i|^2$$

We know that $C^\lambda$ has $k + 1$ holes, and each hole contains in its interior one (and only one) point $Q_i$, $i = 0, 1, \ldots, k$. This means that:

$$C^\lambda = M - (\cup_{j=0}^{k} \overline{E_j^\lambda})$$

where $\overline{E_j^\lambda}$ is topologically a closed disk contained in $M$, $Q_j \in E_j^\lambda$ (where $E_j^\lambda$ is the interior of $\overline{E_j^\lambda}$) and $\overline{E_i^\lambda} \cap \overline{E_j^\lambda} = \emptyset$, $i \neq j$ (in fact $\overline{E_j^\lambda} \subset D_j$, $j = 0, 1, \ldots, k$). Note also that, by definition, $C_\lambda^A - C^\lambda$ is the union of a set of piecewise analytic curves lying in $\cup_{j=0}^{k} \partial E_j^\lambda$.

From the definition of $F^\Delta$, $\Delta \in \mathbb{C}^{2n+k}$, the poles of this function lie in $\{Q_0, Q_1, \ldots, Q_k\}$, and so the same holds for $F^{\Psi_2(\lambda)}$, $\lambda \in U \setminus \{0\}$. Hence we can find two positive constants $k_1(\lambda)$ and $k_2(\lambda)$ in such a way that:

$$k_1(\lambda) < |\exp \left( F^{\Psi_2(\lambda)}(Q) \right) | < k_2(\lambda), \; \forall Q \in C_\lambda^A$$

Therefore, $ds^2$ is complete if and only if the metric $ds_1^2 = |\exp(-2F^{\Psi_2(\lambda)}(Q))|ds^2$ is complete. From the definition of $\Phi_1^i$, $i = 1, 2, 3$, we have:

$$ds_1^2 = \frac{|h^\lambda|^2(1 + |g^\lambda|^2)^2}{2|g^\lambda|^2} |df|^2$$

On the other hand, observe that $G^\lambda$ is holomorphic on $M_0$ and that $df$ has no zeroes on $M_0$. Hence we deduce that the meromorphic function $df/dG^\lambda$ has no zeroes on $M_0$, and so from (14) there exists a positive constant $k(\lambda)$ satisfying:

$$k(\lambda) < \frac{|df|^2}{|dG^\lambda|^2}(Q), \; \forall Q \in C_\lambda^A$$
As the immersion \( x \) is complete, the metric \( ds_0^2 \) on \( A \) defined by:
\[
ds_0^2 = \frac{|h|^2 (1 + |g|^2)^2}{2|g|^2} |dz|^2
\]
is complete. Therefore, the (finitely branched) metric \((G^\lambda)^*(ds_0^2)\) on \( C^\lambda \) is complete too. Taking into account (16) and (17), we have:
\[
ds_1^2 = (G^\lambda)^*(ds_0^2) \frac{|df|^2}{|dG^\lambda|^2} > k(\lambda)(G^\lambda)^*(ds_0^2)
\]
and so \( ds_1^2 \) is complete. The theorem holds. \( \square \)

As the Gauss map \( g^\lambda \) of \( x^\lambda \) is given by \( g^\lambda = g \circ G^\lambda \), where \( G^\lambda \) is a meromorphic function, \( x^\lambda \) has any property of \( x \), depending on the gauss map, which is preserved by a branched covering. We emphasize the following ones:

**Corollary 1** If \( x : \mathbb{D} \rightarrow \mathbb{R}^3 \) complete and non flat and \( g \) omits four points of the sphere, then, for any \( \lambda \in U - \{0\} \), \( x^\lambda : C^\lambda \rightarrow \mathbb{R}^3 \) is complete and non flat and \( g^\lambda \) omits four points too.

**Proof:** For any \( \lambda \in U - \{0\} \), the Gauss maps \( g^\lambda \) and \( g \) of \( x^\lambda \) and \( x \), respectively, have the same image. Moreover, by theorem 3 the completeness is preserved. \( \square \)

Complete non flat minimal surfaces whose Gauss map omits four points of the sphere are critical from the point of view of Fujimoto’s theorem [FU1, FU2]. The classical example is Sherk’s doubly periodic minimal surface. We can apply the former corollary to the universal covering of this surface and so construct complete examples with arbitrary topology whose Gauss map omits also four points.

On the other hand, a large family of surfaces of this type was shown by K. Voss [V] (see also [O]). Voss examples are simply connected, and the corollary (1) works for them too.

**Corollary 2** Suppose that \( x : \mathbb{D} \rightarrow \mathbb{R}^3 \) is complete and non flat. Assume also that the third 1-form \( \Phi_3 \) of the Weierstrass representation of \( x \) is bounded, that is, \( \Phi_3(z) = h(z)dz \) and \( |h(z)| < K \), \( \forall z \in \mathbb{D} \).

Then, for any \( \lambda \in U - \{0\} \), the minimal immersion \( x^\lambda : \mathbb{D} \rightarrow \mathbb{R}^3 \) is complete and its third coordinate function is bounded.

**Proof:** Without loss of generality, we can suppose that the homology basis \( \{\gamma_1, \ldots, \gamma_{2n}\} \) of \( M \) satisfies:
\[
\bigcup_{j=1}^{2n} \gamma_j \subset M - \bigcup_{i=0}^{k} \overline{D}_i
\]
and

\[ \Omega = M - \left( \bigcup_{j=1}^{2n} \gamma_j \right) \]

is simply-connected.

Take \( w \) a conformal parameter on \( \Omega \), and observe that (15) and our hypothesis implies that the function \( \Phi^3_\lambda(w)/dw \) is bounded on \( C^\lambda - \left( \bigcup_{j=1}^{2n} \gamma_j \right) \), and so the same holds for the function \( x^\lambda_3 \). As the set \( \bigcup_{j=1}^{2n} \gamma_j \) is compact, \( x^\lambda_3 \) is in fact bounded on \( C^\lambda \).

For the completeness, use theorem 3.

By using different methods, L.P. Jorge and F. Xavier [JX1] and later F. F. Brito [B1] discovered examples of complete non flat minimal immersions \( x : \mathbb{D} \rightarrow \mathbb{R}^3 \) with \( \Phi_3(z) = h(z)dz \), \( h \) bounded. Hence, corollary 2 yields complete non flat examples of high topology with the third coordinate function bounded.

**Corollary 3** If \( x : A \rightarrow \mathbb{R}^3 \) is non flat and stable, then for any \( \lambda \in U - \{0\} \), the immersion \( x^\lambda : C^\lambda_A \rightarrow \mathbb{R}^3 \) is also non flat and stable.

Furthermore, if \( \partial A \neq \emptyset \) and \( x \) is complete, the same holds for \( x^\lambda \).

**Proof:** The index of a minimal surface can be computed in terms of its Gauss map (see [FC]). In fact, the index of \( x \) is the index of the quadratic form associated to the Jacobi operator \( \Delta + |\nabla g| \) on \( A \). Recall that stability means index zero.

If this operator has index zero, then the same holds for the corresponding operator associated to the meromorphic function \( g \circ G \), where \( G \) is any (branched) covering of \( A \) (see, for instance, [FCS]).

From theorem 3, the completeness is preserved. This conclude the corollary.

A way to construct simply-connected stable minimal surfaces consist of taking a piece of a minimal surface whose spherical image has area less than \(-2\pi\) (see [BC]).

Particularly interesting is the case of the helicoid. Let \( A \) be one of the two pieces of the helicoid, bounded both by the same straight line, whose gauss map covers a closed hemisphere \( H_a = \{ u \in S^2 : <u,a> \geq 0 \}, a \in S^2 \). It is clear that conformally \( A = \overline{\mathbb{D}} - \{ P_0 \} \), where \( P_0 \in \partial \mathbb{D} \), and \( x : A \rightarrow \mathbb{R}^3 \) is complete with boundary. Furthermore, this surface is stable because the function \( f = <g,a> \) is positive and satisfies \( \Delta f + |\nabla g|f = 0 \) (see [FCS]). Then, the surfaces arising from \( x \) in corollary 3 are orientable, stable, complete with boundary and of arbitrary topology.

It is interesting to note that it is able to deforme (when \( \lambda \) tend to zero) any surface \( x^\lambda \) in above three large families, by surfaces of the same kind, into a branched covering of a plane. The topology and the number of ends are preserved along this deformation.

Finally, we mention the following interesting consequence of lemma 3:
Corollary 4. There exist orientable non-flat minimal surfaces with three ends and arbitrary genus $n > 0$ in $\mathbb{R}^3$ of the following kinds: complete surfaces whose Gauss map omits three points of the sphere, complete surfaces with a bounded coordinate function and stable surfaces which are incomplete or complete with boundary.

References


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