Development and analysis of an integral fluidodynamic model in hollow fibre for different operational modes

Fernando Camacho, Encarnación Jurado, Germán Luzón, Jose M. Vicaria *

Chemical Engineering Department, Faculty of Sciences, University of Granada, Avda. Fuentenueva s/n, 18.071 Granada, Spain

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An integral fluidodynamic model for hollow fibre (HF) has been developed and analysed. The model explains the pressure and flow profiles in three operation modes: open- and full-shell mode with forced circulation in the shell in cocurrent or countercurrent (OSFC); closed- and full-shell mode with forced recirculation in the shell in cocurrent or countercurrent (CSFR) and open- and empty-shell mode (OES). A methodology has been proposed to determine the parameters of the system and to verify the different operational systems proposed. Simple expressions have been developed to evaluate the pressures and the flows in the lumen fibres and in the shell, the transmembrane pressure, and the permeate flow for each operation mode assayed. The behaviour of the HF depends on the geometry of the module, on the operation mode chosen, and on the flows circulating through the lumen fibres and through the shell. The experimental results found with a commercial HF verified the model developed. The use of these expressions led us to choose the HF, the operation mode and the adequate flows that optimise the objective desired.

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1. Introduction

The use of HF devices is growing for sea-water desalination, water clarification, artificial kidneys or hemodialfiltration [1,2], biotechnology applications [3] or catalytic bioreactor [4] due to the advantages of these membrane systems, such as high membrane surface:volume ratio; high density of the catalyst that these reactor systems allow in a small reactor volume [5]; the recovery and reuse of the catalyst from the reaction mixture and the possibility of integrating a catalytic conversion, product separation or concentration and catalyst recovery in a single operation [6].

For an optimal use of the HF for these applications, detailed knowledge of the real functioning of these systems is indispensable, making it necessary to know the flow and pressure profile in the HF. Thus, Berman [7] developed solutions to the Navier–Stokes equations for fluid flow in a rectangular slit with porous walls. The solution was achieved with such assumptions as a steady state, incompressible fluid, laminar flow or the velocity of the fluid leaving the walls of the channel not being position dependent. These assumptions were adopted by later researchers such as Karode [8], who considered the wall velocity proportional to the local transmembrane pressure and compared the model developed, considering the pressure in the fibre to be an independent variable, with the numerical computational fluidodynamics simulation. Also, numerous works have developed theoretic models (many not experimentally verified), which simulate the flow and/or pressure distribution in the different zones of the HF [9–11]. The extreme radius:length ratio of the fibres suggested that a 1-D model could be suitable to simulate the hydrodynamics. Thus, with the use of parameters characteristic of fibres made with Cuprophan, Patkar et al. [12] and Labecki et al. [13] developed a theoretic model to analyse the flow in several different configurations (closed-shell mode, dead-end, cross-fl ow filtration, countercurrent, and cocurrent contacting). The authors systematically considered the lumen and the shell sides as two interpenetrating porous regions and combined Darcy’s law and fluid continuity to give a set of 2-dimensional partial differential equations governing the hydrodynamics. The theoretical simulations made show that both the cocurrent and countercurrent configurations may have implications for high-flux dialysis, because the lumen half-length velocity can be higher in the cocurrent case, thus reducing the concentration polarization at the membrane surface. However, these authors did not analyse the influence of the configuration on the transmembrane flow. Smart et al. [14,15] proposed theoretic models for cocurrent and countercurrent flow configurations to evaluate productivity and selectivity, considering the influence of the pressure, packing density, and fibre diameter. More recently, Kostoglou and...

* Corresponding author. Tel.: +34 958 240445; fax: +34 958 248992.
E-mail address: vicaria@ugr.es (J.M. Vicaria).

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Karabelas [16] developed complex mathematical models that simulate the fluidodynamic of the HF. Also, different works have proposed models to explain the HF bioreactor process dynamics. Most of these are theoretical analysis that offer analytical expressions for the velocities and pressure profiles in the HF bioreactors by solving the coupled momentum and continuity equations in the fibre-lumen matrix and shell; expressions for the flow and pressure profiles in the HF operated in closed-shell or open-shell; mathematical models to predict the coupled hydrodynamics and mass transport in HF [11,12] or models based on the numerical solution of the dimensionless balance equations governing mass transfer within the regions of the reacting system [5].

As commented above, it is necessary to evaluate the pressure profile in the HF to ascertain the transmembrane flow. Several researchers have used empirical equations to explain the pressure drop in the lumen of the fibres [17] but generally the pressure in the lumen compartment is assumed to drop along the fibre length according to the Hagen–Poiseuille equation [18–20]. Greater controversy exits in explaining the pressure profile in the shell. If there is no forced circulation in the shell, researchers usually consider the pressure on the shell to be constant and equal to the semi-sum of the entry and exit pressure, this being experimentally verified when the exit of the lumen is open to the atmosphere. Nevertheless, the experimental results show that the pressure in the shell varies when a recirculation flux is used in the shell. Thus, different researchers have considered that the flow in the shell could be simulated by empirical equations as Ergun’s equation [19,21,22], although other empirical equations have been used [1]. Other authors [18,23,24] have simulated the flux in the shell by considering laminar flow. Also, Ergun’s equation is transformed into the Carman–Kozeny equation, the theoretical pressure drops being a hundred times lower than the experimental data [23]. Most of these models consider that the flow circulating through the shell is constant and disregard the transmembrane flow. When the transmembrane flow is low, the simplification would be acceptable, but it would not be correct with high permeabilities. Another aspect to be considered is the distribution of the fibres in the HF shell. Many researchers consider that the fibres distribution is homogenous or that this aspect do not affect significantly on the HF behaviour. Nevertheless, others indicate that the random packing of the fibres affect to the HF fluidodynamic [21,25–28]. Some researchers have used the Voronoi tessellations to model the non-homogenous flow in the shell side due to the randomly packed bundles [29–32]. Bao and Lipscomb [33] used this methodology to assess the mass transfer for axial flows through randomly packed fibre bundles using a hybrid finite element (FE)–finite difference (FD) method to solve the governing equations. The results obtained by these authors indicate that the mass transfer coefficients can decrease dramatically and that the mass transfer is lower in the low packing regions. Similar conclusions were obtained by Zhen et al. [34] from a theoretical analysis of the shell-side flow distribution made in a randomly packed HF module using the random cell model.

Some authors have used magnetic resonance imaging (MRI) to characterize the velocity profiles within porous media, including hollow fibre modules [35–45]. Frank et al. [46] made experimental measurements of concentration within a hemodialyzer using X-ray computed tomography. They found that although the average values for a cross-section decrease uniformly from lumen inlet to outlet, large variations in concentration occur within a cross-section. They attribute these concentration variations to variations in shell flow finding that the regions of lower shell flow have higher concentrations. The regions of lower shell flow do not always correspond to regions of higher fibre packing.

Also, Eloot and Wachter [47] and Kieffer et al. [48] used Computational Fluid Dynamics (CFD) to visualize the flow in a dialyser. A 3-D finite model of the blood–dialysate interface over the complete length of the dialyser was developed. Also, Ghidossi et al. [17] numerically investigated the flow profile in a HF by CFD as a function of the working conditions and the membrane geometry. Also [8], [49], and [50] developed models with CFD for HF.

On the other hand, different operation modes have been studied by different researchers: continuous open-shell mode [9], closed-shell mode with no recirculation in the shell [9–12], cocurrent operation [13,51], countercurrent operation [13,18,20] or cross–flow operation. Some studies are theoretical without experimental corroboration and, though these configurations are governed by the same physical principles, the flow and pressure distributions and mass transfer depend on the operation mode.

The aim of the present work is to develop a general integrated model to simulate the fluidodynamics of the HF. This general model has been particularized for different operation modes. It has been verified that the models developed reproduce the experimental results found. In addition, simple integrated equations have been developed to calculate the transmembrane pressure and flow in the HF. These equations imply better knowledge of the system that allows the choice of the most appropriate configuration for each application. It has also been observed that both the operation mode and the flows have a strong influence on the transmembrane flow.

2. Experimental set-up

Fig. 1 shows a schematic diagram of the HF. The experimental set-up has two peristaltic pumps, a rotameter and a HF called NT1975 supplied by Sorin Biomedica with membranes made in Cuprophan®, a surface area of 1.95 m² and a size pore of 5 kDa. The module has two compartments, one formed by the lumen of the fibres and the other being the space between fibres and the shell cartridge. One peristaltic pump is used to maintain a constant flow of water (20 °C) in the upstream port; the second pump is used to maintain a constant flow of water (20 °C) in the shell in cocurrent or countercurrent flow when these configurations are assayed. The flows assayed are within the range recommended by the manufacturer and the range normally used in these modules. A solution of 1% formaldehyde was used to prevent the bacterial growth in the HF during the experiments.

The experimental system in this work enabled the quick and accurate measurement of pressure in the shell. In the stationary regime, the pressure in different zones of the experimental system was measured by the water column reached in tubes 14 mm in diameter connected to the upstream port and to the different orifices made in the shell of the HF, this allowing the pressure inside the shell cartridge to be measured in the z-direction. The tubes were open and therefore the pressure was measured with respect to the atmospheric pressure. The downstream port was opened to the atmosphere and hence the pressure at the exit of the fibres was considered equal to zero.

When the open–and-empty shell operation mode (OES) (Fig. 2f) was assayed in the HF, the flow was introduced into the lumen of the fibres by the upstream port while the shell was open and without liquid. The shell was open and, therefore, equal to the atmospheric pressure. The lumen-inlet pressure was measured. The flow range assayed was from 0 to 11.5 × 10⁻⁶ m³/s. When the stationary regime was reached, the lumen-inlet pressure was measured together with the water mass collected in 1.5–4.0 min at the downstream port and at the shell port. The assays were made in triplicate.
When the closed- and full-shell operation modes (CSFR) were assayed in the HF, the flow was introduced into the lumen of the fibres by the upstream port. The shell was closed and full. The lumen-inlet pressure and the pressure along the orifices made in the shell were measured when the stationary regime was reached. The flows in the downstream port and in the shell port were calculated from the water mass measured for 1–4 min. The assays were made in triplicate. When closed- and full-shell mode with no recirculation forced in the shell (Fig. 2e) was assayed in the HF, there was no forced flow in the shell and the flow in the upstream port was from 0 to $14.0 \times 10^{-6}$ m$^3$/s.

When the closed- and full-shell mode with forced recirculation in the shell in cocurrent (Fig. 2c) was assayed in the HF, the flow in the

![Fig. 1. Schematic diagram of the hollow-fibre.](image)

Fig. 1. Schematic diagram of the hollow-fibre.

![Fig. 2. Operation modes analysed in the HF. OSFC: (a) open- and full-shell mode with forced circulation in the shell in cocurrent; (b) open- and full-shell mode with forced recirculation in the shell in countercurrent. CSFR: (c) closed- and full-shell mode with forced recirculation in the shell in cocurrent; (d) closed- and full-shell mode with forced recirculation in the shell in countercurrent; (e) closed- and full-shell mode with no recirculation forced in the shell. OES: (f) open- and empty-shell mode.](image)
upstream port was from 0 to $1.40 \times 10^{-6}$ m$^3$/s, and the flow in the shell port was from 0 to $1.56 \times 10^{-6}$ m$^3$/s, both in cocurrent. When closed- and full-shell mode with forced recirculation in the shell in countercurrent (Fig. 2d) was assayed in the HF, the flow in the upstream port was from 0 to $1.40 \times 10^{-6}$ m$^3$/s, and the flow in the shell port was from 0 to $1.56 \times 10^{-6}$ m$^3$/s, both in countercurrent. The flow in the lumen fibres was considered positive. The flow in the shell was considered positive for cocurrent flow and negative for countercurrent flow.

A light microscope (TOPI-CETI) was used to measure the fibres. When the experiments finished, the HF was destroyed, removing transversal sections of fibre to measure their internal diameter and the thickness under the microscope. The measurements showed values of $D_i = 168 \mu$m and $t_w = 18 \mu$m.

### 3. Development of the theoretical models

Integrated models based on module geometry, membrane properties and operating conditions are developed to determine the pressure and flow profiles in the fibre lumen and in the shell for different operation modes (Fig. 2). The suitability of the theoretical models was verified with experimental results found with the HF shown in Fig. 1. These models consider that the HF is horizontally placed.

Several initial assumptions and limiting conditions normally considered by different researchers were assumed: isothermal process; Newtonian incompressible fluid with constant physical properties; laminar flow in stationary flow for the operation modes analysed both in the lumen fibres as well as in the shell (this aspect is verified experimentally); operation at steady state in the configurations analysed; flow rates and pressures are uniformly distributed over the module cross-section and vary only along the $z$-direction [20]; fibre bundles are regarded as a collection of parallel rods, meaning that the packing-density distribution and flow distribution are consistent along the numbers of fibres and the numbers of fibres and $t_w$ thickness of the wall membrane. As can be seen, for the countercurrent flow, $dP_f/dz$ is positive.

On the other hand, the permeation flux through the membranes is proportional to the transmembrane pressure in the laminar regime [9,13] as:

$$V_p = \frac{L_p}{\mu} (P_f - P_S)$$

where $V_p$ is the permeation velocity and $L_p$ is the permeability. It can be seen that the permeate flow is positive when it permeates from the fibres to the shell, and negative in the opposite case. Patkar et al. [12] and Labecki et al. [13], studying the permeability of membranes made in Cuprophan, indicated that $L_p$ is a parameter that can be determined by measuring the permeate flow in the shell at different lumen pressures when the shell is open. This parameter is calculated from the relationship between the total permeate-flow rate obtained with this configuration and the transmembrane pressure.

If we consider Eq. (3), the permeation through the membrane is thus given by:

$$\frac{d\phi_f}{dz} = -k_p(P_f - P_S) \quad k_p = \frac{L_p \pi n D_i}{\mu}$$

the permeate flow in the $z$-position ($\phi_f$) is given by:

$$\phi_f = - \int_0^z \frac{d\phi_f}{dz} dz$$

Deriving Eq. (4), and substituting Eqs. (1) and (2), we get:

$$\frac{d^2 \phi_f}{dz^2} = k_p \frac{\phi_f}{k_p} - \frac{\phi_f}{k_p} \frac{\phi_S}{k_S}$$

If a fluid balance is made between $z = 0$ and $z$, we have:

$$\phi_{f0} - \phi_f = \phi_S - \phi_{S0}$$
On the other hand, the total permeate flow between the upstream and downstream ports of the module ($\phi_{PT}$) is given by:

$$\phi_{PT} = \phi_{P0} - \phi_{SL} = \phi_{P0} - \phi_{SL}$$

(8)

Solving $\phi_{5}$ from Eq. (7), substituting in Eq. (6), and reordering, we get the following non-homogenous second-order linear differential equation:

$$\frac{d^2 \phi_{F}}{dz^2} - \left( \frac{k_p}{k_F} + \frac{k_p}{k_S} \right) \phi_{F} = - \frac{k_p}{k_S} (\phi_{P0} + \phi_{SL})$$

(9)

Considering in this equation the limiting conditions:

$$z = 0 \quad \phi_{F} = \phi_{P0}$$
$$z = L \quad \phi_{F} = \phi_{P0} + \phi_{S0} - \phi_{SL}$$

(10)

The general solution of Eq. (9) would be:

$$\phi_{F} = C_1 \exp \left( \sqrt{\frac{k_p}{k_F} + \frac{k_p}{k_S}} \right) + C_2 \exp \left( - \sqrt{\frac{k_p}{k_F} + \frac{k_p}{k_S}} \right) + \frac{(k_p/k_F)(\phi_{P0} + \phi_{S0})}{(k_p/k_S) + (k_p/k_S)}$$

(11)

Introducing the adimensional numbers, we get:

$$\lambda = \frac{k_F}{k_S} = \frac{1}{n} \left( \frac{D_k}{D_t} \right)^4 = \frac{1}{n} \left( \frac{D_k^2 - n(D_i + 2tw)^2}{nD_t(D_i + 2tw)} \right)^4$$

(12)

$$\delta = \sqrt{\frac{k_p}{k_F} + \frac{k_p}{k_S}L} = \sqrt{\frac{128L}{D_t^4} \left( 1 + \frac{1}{\lambda} \right) L}$$

(13)

applying the limits indicated in Eq. (10) and considering Eq. (8), we find:

$$\phi_{F} = \left( \frac{\lambda \phi_{P0} - \phi_{S0}}{\lambda + 1} \right) \frac{\sinh(\delta(z/L)) + \sinh(\delta(1 - (z/L)))}{\sinh(\delta)} - \phi_{PT} \frac{\sinh(\delta(z/L))}{\sinh(\delta)} + \frac{(\phi_{P0} + \phi_{S0})}{\lambda + 1}$$

(14)

In consideration of Eq. (7), the expression for $\phi_{S}$ is:

$$\phi_{S} = - \left( \frac{\lambda \phi_{P0} - \phi_{S0}}{\lambda + 1} \right) \frac{\sinh(\delta(z/L)) + \sinh(\delta(1 - (z/L)))}{\sinh(\delta)} + \frac{(\phi_{P0} + \phi_{S0})}{\lambda + 1}$$

(15)

Substituting Eq. (14) in Eq. (1) and integrating, considering that in $z = 0, P_{F} = P_{P0}$, we get:

$$P_{F} = P_{P0} - \frac{(\lambda \phi_{P0} - \phi_{S0})}{k_pL} \frac{\lambda \delta}{(\lambda + 1)^2} \frac{\left( \cosh(\delta(z/L)) - 1 - \cosh(\delta(1 - (z/L))) + \cosh(\delta) \right)}{\sinh(\delta)}$$

$$+ \frac{\phi_{PT}}{k_pL} \frac{\lambda \delta}{(\lambda + 1)^2} \frac{\left( \cosh(\delta(z/L)) - 1 - \cosh(\delta(1 - (z/L))) + \cosh(\delta) \right)}{\sinh(\delta)}$$

(16)

Substituting Eq. (14) and Eq. (7) in Eq. (2) and integrating, we have:

$$P_{S} = P_{P0} + \frac{(\lambda \phi_{P0} - \phi_{S0})}{k_pL} \frac{\delta}{(\lambda + 1)^2} \frac{\left( \cosh(\delta(z/L)) - 1 - \cosh(\delta(1 - (z/L))) + \cosh(\delta) \right)}{\sinh(\delta)}$$

$$- \frac{\phi_{PT}}{k_pL} \frac{\delta}{(\lambda + 1)^2} \frac{\left( \cosh(\delta(z/L)) - 1 - \cosh(\delta(1 - (z/L))) + \cosh(\delta) \right)}{\sinh(\delta)}$$

(17)

On the other hand, the transmembrane pressure ($P_{PM}$) can be determined in the $z$-position as:

$$P_{PM} = P_{F} - P_{S} = (P_{P0} - P_{S0}) - \frac{(\lambda \phi_{P0} - \phi_{S0})}{k_pL} \frac{\delta}{(\lambda + 1)^2} \frac{\left( \cosh(\delta(z/L)) - 1 - \cosh(\delta(1 - (z/L))) + \cosh(\delta) \right)}{\sinh(\delta)}$$

$$+ \frac{\phi_{PT}}{k_pL} \frac{\delta}{\sinh(\delta)(\lambda + 1)^2} \frac{\left( \cosh(\delta(z/L)) - 1 - \cosh(\delta(1 - (z/L))) + \cosh(\delta) \right)}{\sinh(\delta)}$$

(18)

Different authors [53] have defined the “inversion point” as the point at which the value of the pressure inside the fibres equals the pressure in the shell. At this point, the flow of the permeate changes direction and is in this position where the maximum permeate flow is reached. Therefore, at this point $P_{PM} = 0$ and the following must be fulfilled:

$$0 = (P_{P0} - P_{S0}) - \frac{(\lambda \phi_{P0} - \phi_{S0})}{k_pL} \frac{\delta}{(\lambda + 1)^2} \frac{\left( \cosh(\delta(z/L)) - 1 - \cosh(\delta(1 - (z/L))) + \cosh(\delta) \right)}{\sinh(\delta)}$$

$$+ \frac{\phi_{PT}}{k_pL} \frac{\delta}{\sinh(\delta)(\lambda + 1)^2} \frac{\left( \cosh(\delta(z/L)) - 1 - \cosh(\delta(1 - (z/L))) + \cosh(\delta) \right)}{\sinh(\delta)}$$

(19)

Eq. (4) and Eq. (5) being equal, and substituting Eq. (18), $\phi_{p}$, we get:

$$\phi_{p} = k_p(P_{P0} - P_{S0}) - \frac{\delta(\lambda \phi_{P0} - \phi_{S0})(\cosh(\delta(z/L)) - 1 - \cosh(\delta(1 - (z/L))) + \cosh(\delta))}{(\lambda + 1)^2 L}$$

$$+ \frac{\phi_{PT} \delta(\cosh(\delta(z/L)) - 1)}{L}$$

(20)

The total permeate flow between the upstream and downstream ports of the module can be determined by integration of Eq. (5):

$$\phi_{PT} = \int_{0}^{L} \phi_{P} \, dz = k_p(P_{P0} - P_{S0})L \frac{\sinh(\delta)}{\delta} - \frac{(\lambda \phi_{P0} - \phi_{S0})}{\lambda + 1} (\cosh(\delta) - 1)$$

(21)
With substitution of Eq. (21) in Eq. (19), the following must be fulfilled at the inversion point:

$$ (P_{T0} - P_{S0}) \cosh(\delta(1/L)) = \frac{(\lambda \phi_0 - \phi_{S0})}{k_L} \left( \frac{\delta}{\lambda + 1} \right) \frac{\cosh(\delta) \cosh(\delta(1/L)) - \cosh(\delta(1 - (1/L)))}{\sinh(\delta)} \tag{22} $$

On the other hand, for simulating the behaviour of the system, it is usual to express the equations in adimensional form. The flows are normalized by dividing by a constant reference flow, \( \phi_{T0} \), and the pressures by a constant reference pressure, \( P_{T0} \), leaving all the variables expressed as a function of the imposed variables \( P_{T0}, P_{S0}, \phi_{T0} \) and \( \phi_{S0} \), grouped in the adimensional terms \( k_L P_{T0}/\phi_{T0}, (1 - P_{S0}/P_{T0}), P_{S0}/P_{T0} \) and \( \phi_{S0}/\phi_{T0} \):

$$ \phi_{T} = \frac{k_L P_{T0}/\phi_{T0} (1 - P_{S0}/P_{T0})}{\sinh(\delta)} \left( \frac{\phi_{T0} - \phi_{S0}}{\lambda + 1} \right) \cosh(\delta) \cosh(\delta(1/L)) - \cosh(\delta(1 - (1/L))) \tag{23} $$

$$ \phi_{S} = \frac{\lambda \phi_0 - \phi_{S0}}{\sinh(\delta)} \left( \frac{1 - P_{S0}/P_{T0}}{\lambda + 1} \right) \cosh(\delta) \cosh(\delta(1/L)) + \frac{(1 + \phi_{S0}/\phi_{T0})}{\lambda + 1} \cosh(\delta(1 - (1/L))) \tag{24} $$

$$ \phi_{L} = \frac{k_L P_{T0}/\phi_{T0} (1 - P_{S0}/P_{T0})}{\sinh(\delta)} \left( \frac{\phi_{T0} - \phi_{S0}}{\lambda + 1} \right) \cosh(\delta) \cosh(\delta(1/L)) - \cosh(\delta(1 - (1/L))) \tag{25} $$

$$ \frac{P_S}{P_{T0}} = 1 - \frac{\lambda \phi_0 - \phi_{S0}}{k_L P_{T0}/\phi_{T0} (\lambda + 1)^2} \sinh(\delta) + \frac{\lambda}{\lambda + 1} \left( \frac{1 - P_{S0}/P_{T0}}{(\lambda + 1)^2} \right) \cosh(\delta) \cosh(\delta(1/L)) - \cosh(\delta(1 - (1/L))) \tag{26} $$

$$ \frac{P_S}{P_{T0}} = \frac{\lambda \phi_0 - \phi_{S0}}{k_L P_{T0}/\phi_{T0} (\lambda + 1)^2} \sinh(\delta) - \frac{1}{\lambda + 1} \left( \frac{1 - P_{S0}/P_{T0}}{(\lambda + 1)^2} \right) \cosh(\delta) \cosh(\delta(1/L)) - \cosh(\delta(1 - (1/L))) \tag{27} $$

$$ \frac{P_{TM}}{P_{T0}} = \left( 1 - \frac{P_{S0}}{P_{T0}} \right) \cosh(\delta) \cosh(\delta(1/L)) - \frac{\lambda \phi_0 - \phi_{S0}}{k_L P_{T0}/\phi_{T0} (\lambda + 1)^2} \sinh(\delta) \tag{28} $$

$$ \frac{P_{TM}}{P_{T0}} = \left( 1 - \frac{P_{S0}}{P_{T0}} \right) \cosh(\delta) \cosh(\delta(1/L)) - \frac{\lambda \phi_0 - \phi_{S0}}{k_L P_{T0}/\phi_{T0} (\lambda + 1)^2} \sinh(\delta) \tag{29} $$

### 3.2. Closed- and full-shell operation modes with forced recirculation in the shell (CSFR mode)

In these configuration modes (Fig. 2c–e) \( \phi_{T0} = 0 \), and therefore the entry and exit flows both in the shell as well as in the fibres are equal, \( \phi_{S0} = \phi_{T0} \) and \( \phi_{T0} = \phi_{T0} \). If these limit conditions are considered in the model, the equations become simplified, and the permeate flow in the z-position (Eq. (20)) for this configuration would be:

$$ \phi_{p} = k_L P_{T0}/P_{S0} - \frac{\delta(\lambda \phi_0 - \phi_{S0})}{(\lambda + 1)L} \cosh(\delta) \cosh(\delta(1/L)) - \cosh(\delta(1 - (1/L))) + \cosh(\delta) \tag{30} $$

If \( \phi_{T0} = 0 \) is considered in Eq. (21):

$$ (P_{T0} - P_{S0}) = \frac{(\lambda \phi_0 - \phi_{S0})}{k_L} \frac{\delta}{(\lambda + 1)L} \sinh(\delta) \tag{31} $$

and when Eq. (31) is substituted in Eq. (22), the position of the inversion point corresponds to the solution of:

$$ \cosh(\delta) \cosh(\delta(1/L)) = \cosh(\delta(1 - (1/L))) \tag{32} $$

For the particular case of these configuration modes, the inversion point occurs at \( z/L = 0.5 \), as different authors have suggested and can be seen experimentally.

Also, \( \phi_{FE} \) is the total permeate flow between the fibres and the shell in \( z = L/2 \), where the transmembrane flow changes direction. In this position, the maximum permeate flow is reached and is equal to:

$$ \phi_{FE} = \int_{0}^{L/2} \phi_{p} \, dz = k_L P_{T0}/P_{S0} - \frac{L}{2} \frac{(\lambda \phi_0 - \phi_{S0})/\sinh(\delta)}{(\lambda + 1)} \left( 2 \sinh(0.5\delta) + 0.5\delta \cosh(\delta) - \sinh(\delta) - 0.5\delta \right) \tag{33} $$

In an adimensional form, the equations for this operation mode are:

$$ \phi_{p} = \frac{\lambda \phi_0 - \phi_{S0}}{\sinh(\delta)} + \frac{1}{\lambda + 1} \left( 1 + \frac{\phi_{S0}}{\phi_{T0}} \right) \tag{34} $$

$$ \phi_{S} = \frac{\lambda \phi_0 - \phi_{S0}}{\sinh(\delta)} + \frac{\lambda}{\lambda + 1} \left( 1 + \frac{\phi_{S0}}{\phi_{T0}} \right) \tag{35} $$
On the other hand, the transmembrane pressure is equal to

\[ P_{TF} = \phi_0 L \left( \frac{k_F P_{T0}}{\phi_0} \right) \left( 1 - \frac{P_{SO}}{P_{T0}} \right) \left( \frac{\lambda - \frac{\phi_0}{\phi_0}}{\lambda + 1} \right) \frac{\delta}{\lambda} \left( \cosh(\delta z/L) - 1 - \cosh(\delta(1 - (z/L))) + \cosh(\delta) \right) \frac{\sinh(\delta)}{} (36) \]

Substituting Eq. (45) in Eq. (1) and integrating, we get:

\[ P_F = P_{T0} \left( \frac{\phi_0 L}{\phi_0} \right) \left( \frac{\lambda - \phi_0}{\phi_0} \right) \frac{\delta}{\lambda + 1} \left( 1 - \frac{P_{SO}}{P_{T0}} \right) \left( 1 - \frac{P_{SO}}{P_{T0}} \right) \left( 2 \sinh(0.5\delta) + 0.5\delta \cosh(\delta) - \sinh(\delta) - 0.5\delta \right) \frac{\sinh(\delta)}{} (37) \]

Furthermore, without forced recirculation in the shell, the entry and exit flows of the shell ports are zero, enabling the simplification of the equations of the CSFR model (Fig. 2e). In this configuration the feed stream is introduced by the manifolds into the lumen fibres, a bypass flow from fibres to the shell and back occurs, and finally the flux flows out to the end of the fibre. This configuration is usually used when the hollow fibres are employed in cell cultures. The pressure drop inside the fibres is normally considered to be proportional to the feed flow rate in the case of laminar flow. For this configuration, the literature usually considers that the pressure in the shell is the mean of the pressure existing at the upstream and downstream ports of the module.

3.3. Open- and empty-shell operation mode (OES mode)

In this configuration (Fig. 2f) the feed stream is introduced by the manifolds into the lumen fibres, the retentate flows out to the end of the fibre and the permeate flows through the membrane wall to the shell side, leaving the HF. This configuration is usually used to determine physical characteristics of the membrane, such as permeability. Very often the pressure drop inside fibres is considered to be proportional to the feed flow rate in the case of laminar flow. If the permeation is large, however, this assumption cannot be true because the flow rate changes along the fibres [9].

To determine the equations that enable us to model this configuration mode, we considered the shell to be open in this operation mode, so that \( P_S = 0 \). In this case, if we consider Eq. (2), Eq. (6) becomes:

\[ \frac{d^2 \phi_T}{dz^2} - k_F \frac{\phi_T}{k_F} = 0 \] (41)

with limit conditions:

\[
\begin{align*}
    z = 0 & \quad \phi_T = \phi_{T0} \\
    z = L & \quad \phi_T = \phi_{T0} - \phi_T L
\end{align*}
\] (42)

for which the general solution is:

\[ \phi_T = C_1 \exp \left( \frac{k_F}{k_F} z \right) + C_2 \exp \left( - \frac{k_F}{k_F} z \right) \] (43)

With the introduction of the adimensional number \( \delta_1 \), an adaptation of the adimensional number \( \delta \) to this new configuration:

\[ \delta_1 = \sqrt{\frac{k_F L}{128D_1}} \] (44)

and, when we apply the limits:

\[ \phi_T = \frac{\phi_{T0} - \phi_T L}{\sinh(\delta_1)} \] (45)

Substituting Eq. (45) in Eq. (1) and integrating, we get:

\[ P_F = P_{T0} \left( \frac{\phi_0 L}{\phi_0} \right) \left( \frac{\lambda - \phi_0}{\phi_0} \right) \frac{\delta}{\lambda + 1} \left( \cosh(\delta_1) - \cosh(\delta_1(1 - (z/L))) \right) \frac{k_F L}{\sinh(\delta_1)} \] (46)

And in \( L \):

\[ P_L = P_{T0} \left( \frac{2\phi_0 L}{\phi_0} - \phi_T L \right) \delta_1 \left( \cosh(\delta_1) - \cosh(\delta_1(1 - (z/L))) \right) \frac{k_F L}{\sinh(\delta_1)} \] (47)

On the other hand, the transmembrane pressure is equal to \( P_F \) since the shell is open. Also, in consideration of Eq. (5), the permeate flow in the \( z \)-position would be:

\[ \phi_T = \frac{k_F P_{T0}}{L} \left( \frac{\phi_0 L}{\phi_0} \right) \left( \frac{\lambda - \phi_0}{\phi_0} \right) \frac{\delta}{\lambda + 1} \left( \cosh(\delta_1) - \cosh(\delta_1(1 - (z/L))) \right) \frac{\sinh(\delta_1)}{} \] (48)
The total permeate flow between the upstream and downstream ports of the module \( (\phi_{PT}) \) can be determined by integration of Eq. (48):

\[
\phi_{PT} = \int_0^L \phi_P \, dz = k_P P_{F0} \frac{L}{\delta_1} \sinh(\delta_1) - \phi_{F0}(\cosh(\delta_1) - 1)
\]  

In an adimensional form, the equations for this configuration are:

\[
\frac{\phi_F}{\phi_{F0}} = \frac{\sinh(\delta_1(z/L)) \cosh(\delta_1) + \sinh(\delta_1(1 - z/L))}{\sinh(\delta_1)} - \frac{k_P L P_{F0} \sinh(\delta_1(z/L))}{\phi_{F0} \delta_1} \tag{50}
\]

\[
\frac{\phi_L}{\phi_{F0}} = \frac{k_P L P_{F0} \cosh(\delta_1(z/L)) - \delta_1(\cosh(\delta_1) - \cosh(\delta_1(z/L)) - \cosh(\delta_1(1 - z/L)))}{\sinh(\delta_1)} \tag{51}
\]

\[
\frac{\phi_T}{\phi_{F0}} = \frac{k_P L P_{F0} \sinh(\delta_1)}{\phi_{F0} \delta_1} - (\cosh(\delta_1) - 1) \tag{52}
\]

\[
\frac{P_F}{P_{F0}} = P_{F0}^{1M} = 1 - \frac{\delta_1}{k_P L P_{F0}/\phi_{F0}} \left( \frac{\cosh(\delta_1) \cosh(\delta_1(z/L)) - \cosh(\delta_1(1 - z/L)))}{\sinh(\delta_1)} \right) + \cosh \left( \delta_1 \frac{z}{L} \right) \tag{53}
\]

The equations of this model developed for \( P_S = 0 \) can also be derived from the general model while taking into account a series of conditions such as that \( P_S = 0 \), the absence of flow entry in the shell \( (\phi_S = 0) \), and the flow exit in the shell is due exclusively to the permeate flow \( (\phi_{PT} = \phi_{PT}) \). It also needs to be considered within of the general model that \( \delta \approx \delta_1 \), and also because \( \lambda \) (defined as \( k_S/k_F \)) has a high value.

4. Results and discussion

The suitability of the models proposed can be verified from the experimental results found with different configurations.

4.1. Determination of the model parameters and experimental verification of the model proposed

\( k_F \) and \( k_P \) can be determined by the experiments made on the HF with the OES mode where \( P_S = 0 \) and \( P_L = 0 \). In Eq. (47), considering that \( P_L = 0 \), solving the value of the total flow of the permeate \( \phi_{PT} \), and making it equal to Eq. (49), we get:

\[
\phi_{F0} = \delta_1 \frac{\sinh(\delta_1) \cosh(\delta_1)}{\cosh^2(\delta_1) - 1} - \frac{k_P L P_{F0}}{\phi_{F0}} \tag{54}
\]

This expression enables the determination of the flow in the upstream port as a function of \( P_{F0} \). Substituting this expression in Eq. (47) and reordering, the expression of the total permeate flow between the upstream and downstream ports is:

\[
\frac{\phi_{PT}}{\phi_{F0}} = \frac{\cosh(\delta_1) - 1}{\cosh(\delta_1)} \tag{55}
\]

Eqs. (54) and (55) show that the entry and permeate flows in this operation mode are a function of a single external variable \( (P_{F0}) \) and of the geometric parameters of HF. The relation between the two flows is independent of \( P_{F0} \) and is given by:

\[
\frac{\phi_{PT}}{\phi_{F0}} = \frac{\cosh(\delta_1) - 1}{\cosh(\delta_1)} \tag{56}
\]

The experimental results found with this operation mode are used to verify the suitability of the model proposed. The experimental measurements were made as indicated in Section 2. In fact, in the experimental range assayed, the results found with this operation mode show that \( \phi_{F0} \) and \( \phi_{PT} \) have a linear relationship with \( P_{F0} \) (Fig. 3).

\[ \text{Fig. 3. } \phi_{F0} \text{ and } \phi_{PT} \text{ vs. } P_{F0} \text{ in OES mode (} \bigcirc \text{ = } \phi_{F0}, \bullet \text{ = } \phi_{PT}). \]
The Reynolds number in the shell can be defined as \( \text{Re} = \frac{\rho u \cdot D_m}{\mu} \)

Making it equal to Eq. (56), \( \delta_1 = 0.101 \) (obtained by iterative methods). When the value of \( \delta_1 \), the length of the fibres (\( L = 0.26 \) m) and the numerical value of the slopes as well as their definition, Eqs. (55) and (56), are taken into account:

\[
k_p = 4.93 \times 10^{-11} \text{ m}^2/\text{Pa s} \quad k_f = 3.24 \times 10^{-10} \text{ m}^4/\text{Pa s}
\]

If the viscosity of the fluid used (water 20\(^\circ\)C, \( \mu = 10^{-3} \) Pa s) and the number of fibres of the HF (\( n = 10500 \)) are known, and from the definition of \( k_f \) (Eq. (1)), the internal diameter of the fibres is determined, this being equal to \( D_i = 188 \) \( \mu \)m. The verification of this parameter is complex. The manufacturer indicates nominal values for the internal diameter of 200 \( \mu \)m and for the wall thickness 7.5 \( \mu \)m (Table 1). The measurements made by microscope may have substantial error because the fibres are plastic and easily deformed both longitudinally as well as transversally when they are sectioned. This causes that the lumen of the fibre and the thickness of the wall membrane to vary. Therefore, it can be concluded only that the value of the internal diameter found from the model was within the expected order of magnitude.

Similarly, from the definition of \( k_f \) (Eq. (4)), the membrane permeability is found to be equal to \( L_P = 7.87 \times 10^{-15} \) m. Labecki et al. [13] distinguished between membrane permeability and the apparent membrane permeability. To determine the apparent permeability, they considered the relation between the permeate flow and the transmembrane pressure defined as the pressure drop between the lumen-inlet pressure and the shell pressure in OES mode (method used to determine the value of \( L_P \) in this work). Simulations by this author demonstrated that for membranes with values of apparent membrane permeability of lower than \( 10^{-13} \) m, the permeability and the apparent membrane permeability were equal. Liao et al. [54] used a similar methodology to determine the hydraulic permeability of hemodialyzers in a somewhat more complex experimental system.

On the other hand, if the density and viscosity of the fluid used (water, \( \rho = 1000 \text{ kg/m}^3, \mu = 10^{-3} \text{ Pa s} \)) is known, the Reynolds number in the lumen of the fibres (\( \text{Re} = 4 \delta_0/\mu(D_i + 2tw) \)) can be determined for the operation mode used. For the maximum flow tested (\( \phi_{\delta_0} = 15.6 \times 10^{-5} \) m\(^3\)/s), \( \text{Re} = 7.2 \), this being a laminar regime.

For the determination of the \( k_s \) value, there must be circulation through the shell. The experimental results found for the CSFR mode indicate that the pressure in the shell (\( P_s \)) varied linearly with the \( z \)-direction, as can be seen, for example, in Fig. 4a, a representation that enables us to determine the value of \( dP_s/dz \) for each experiment made. As Eq. (2) indicates, the representation of the experimental value of \( dP_s/dz \) vs. \( \phi_{\delta_0} \) enabled the determination of the value of \( k_s \) (Fig. 4b). The \( k_s \) value was \( (2.04 \pm 0.04) \times 10^{-9} \) m\(^3\)/Pa s. Once the parameters of the model are known, the adimensional numbers of the system are evaluated, \( \delta = 0.109 \) and \( \lambda = 6.33 \).

The \( k_s \) value enables us to calculate the hydraulic diameter of the shell from Eq. (2), giving a value of \( D_h = 3.00 \times 10^{-4} \). From the \( D_h \) definition the thickness of the fibre is evaluated (\( t_w = 2.20 \times 10^{-6} \) m). As Table 1 shows, the values of \( t_w \) and \( D_h \) found from applying the theoretical model to the experimental data have the same order of magnitude as the values determined by microscopic examination.

On the other hand, if the mean flow velocity in the shell \( (u_s) \) is defined as \( u_s = \phi_{\delta_0}/S \), where \( S \) is the section of free passage of fluid through the shell, then:

\[
S = \frac{\pi}{4}(D_i^2 - n(D_i + 2tw)^2)
\]

The Reynolds number in the shell can be defined as \( \text{Re} = u_s \rho D_h/\mu \), and, as the density and viscosity of the fluid used (water, \( \rho = 1000 \text{ kg/m}^3, \mu = 10^{-3} \text{ Pa s} \)) were known, the Reynolds number in the shell for the maximum flow tested (\( \phi_{\delta_0} = 15.6 \times 10^{-5} \) m\(^3\)/s) was \( \text{Re} = 9.65 \), this being a laminar regime.

The goodness of the model proposed can be experimentally verified for CSFR mode. A reordering of Eq. (31) gives:

\[
\phi_{\delta_0}/(\phi_{\delta_0} + \phi_{\delta_0}) = \frac{k_p L \sinh(\delta)}{\delta \sinh(\delta)} \left( \frac{P_{\delta_0} - P_{\delta_0}}{\phi_{\delta_0} + \phi_{\delta_0}} \right) + \frac{1}{\delta + 1}
\]

Similarly, when Eqs. (38) and (39) are applied at \( z = L \), and after reordering:

\[
\frac{\phi_{\delta_0}}{\phi_{\delta_0} + \phi_{\delta_0}} = \frac{k_p L \sinh(\delta)}{\delta \sinh(\delta)} \left( \frac{P_{\delta_0} - P_{\delta_0}}{\phi_{\delta_0} + \phi_{\delta_0}} \right) + \frac{1}{\delta + 1} \left( 1 - \frac{\delta \sinh(\delta)}{2 \sinh(\delta)} \right)
\]
Fig. 4. Experimental results obtained with CSFR mode. (a) $P_s$ vs. $z$ (CSFR mode, cocurrent ($F_0 = 3.12 \times 10^{-6}$ m$^3$/s, $S_0 = 7.44 \times 10^{-6}$ m$^3$/s, $S_0/F_0 = 2.38$), CSFR mode, countercurrent ($F_0 = 8.85 \times 10^{-6}$ m$^3$/s, $S_0 = -7.63 \times 10^{-6}$ m$^3$/s, $S_0/F_0 = -0.86$)); (b) $dP_s/dz$ vs. $\bar{S}_0$.

Fig. 5 verifies that the equations of the model adequately reproduce the results found in experiments in CSFR mode, both cocurrent and countercurrent operation.

4.2. Analysis of the pressure and flow profiles. Simulation

4.2.1. OES mode

Considering in Eq. (53) that $P_R = 0$ for the experimental device, and thus:

$$\frac{\phi_0}{\phi_0 + \phi_0} = \frac{k_L P_{F0}}{\phi_0} = \frac{\delta_1 (\sinh(\delta_1)^2 - 1)}{\sinh(\delta_1) \cosh(\delta_1)}$$

(60)

Substituting this equation in Eqs. (50), (51), and (53), these are reduced to:

$$\frac{\phi}{\phi_0} = \cosh(\delta_1) \sinh(\delta_1 (1 - (z/L))) + \sinh(\delta_1 (z/L))$$

(61)

$$\frac{\phi L}{\phi_0} = \delta_1 (\cosh(\delta_1) \cosh(\delta_1 (1 - (z/L))) - \cosh(\delta_1 (z/L)))$$

(62)

$$\frac{P_F}{P_{F0}} = \frac{P_{TM}}{P_{TM}} = 1 - \frac{(\sinh(\delta_1)^2 + \cosh(\delta_1 (z/L)) \cosh(\delta_1 (1 - (z/L)))) - 1}{(\cosh(\delta_1)^2 - 1)}$$

(63)

In this operation mode, the pressure drop is practically linear, since the permeate flow is small with respect to the flow that circulates through the fibres. Fig. 6 shows the $\phi_1/\phi_0$ and $\phi_L/\phi_0$ profiles with the $z$-direction for OES mode. As can be seen, $\phi_1/\phi_0$ progressively diminishes in the $z$-direction. For the OES mode, with the substitution of $\delta_1$ in Eq. (56), $\phi_{\nu}/\phi_0 = 0.00512$. Experimentally, it is found that the permeate flow augmented with $\phi_0$, and that in all the experiments made the value of $\phi_{TM}/\phi_0$ remained practically constant and equal to 0.52%, as the model indicated. In Fig. 6, it can be seen that the value of $\phi_{TM}$ corresponds to the experimental value indicated (99.48%).
On the other hand, the flow permeating through the membrane is also reduced in the $z$-direction, as seen with $\phi_L/\phi_{0L}$ in Fig. 6.

4.2.2. CSFR mode

With the CSFR mode, the HF operates under the condition of zero net volume flow of filtrate when it works in the stationary state. These operation modes are usually used, for example, to physically immobilize an enzyme in the shell [55–57], provided that the membrane is permeable to substrates and products. The driving force is the transmembrane pressure.

For the CSFR mode, when $P_{SL} = 0$ is considered in Eq. (38):

$$\frac{k_L P_{F0}}{\Phi_{F0}} = \frac{\lambda \delta}{(\lambda + 1)^2} \left( \frac{\Phi_{F0}}{\Phi_{F0} + \Phi_{SO}} \right) \frac{2(\cosh(\delta) - 1)}{\sinh(\delta)} + \delta \left( 1 + \frac{\Phi_{SO}}{\Phi_{F0}} \right)$$

(64)
With Eq. (64) substituted in Eq. (38), the pressure inside the fibres depends only on the $\phi_{S0}/\phi_{T0}$ and on the parameters of the system:

$$\frac{P_S}{P_{F0}} = 1 - \frac{(\lambda - (\phi_{S0}/\phi_{T0}))(\text{cosh}(\delta(z/L)) - \text{cosh}(\delta(1 - (z/L)))) + \text{cosh}(\delta - 1) + \delta \sinh(\delta)(1 + (\phi_{S0}/\phi_{T0}))(z/L)}{2(\text{cosh}(\delta) - 1)(\lambda - (\phi_{S0}/\phi_{T0})) + \delta \sinh(\delta)(1 + (\phi_{S0}/\phi_{T0}))}$$  \hfill (65)

The pressure drop in the z-direction is practically linear, since the permeate flow is small with respect to the flow that circulates through the fibres.

On the other hand, when $P_{S0}/P_{T0}$ is solved from Eq. (31), and Eq. (64) is considered:

$$\frac{P_{S0}}{P_{T0}} = 1 - \frac{(\lambda - (\phi_{S0}/\phi_{T0}))(\text{cosh}(\delta) - 1)}{\lambda/\delta(\lambda + 1)2(\text{cosh}(\delta) - 1)(\lambda - (\phi_{S0}/\phi_{T0})) + \delta \sinh(\delta)(1 + (\phi_{S0}/\phi_{T0}))}$$  \hfill (66)

This expression indicates that $P_{S0}/P_{T0}$ is a function of $\phi_{S0}/\phi_{T0}$.

With the substitution of Eq. (64) and Eq. (66) in Eq. (39), the pressure in the shell depends only on the working conditions imposed ($\phi_{S0}/\phi_{T0}$) and on the parameters of the system:

$$\frac{P_S}{P_{T0}} = 1 + \frac{(\lambda - (\phi_{S0}/\phi_{T0}))(1/\lambda)\text{cosh}(\delta(z/L)) - (1/\lambda)\text{cosh}(\delta(1 - (z/L)))) - \text{cosh}(\delta + 1) - \delta \sinh(\delta)(1 + (\phi_{S0}/\phi_{T0}))(z/L)}{2(\text{cosh}(\delta) - 1)(\lambda - (\phi_{S0}/\phi_{T0})) + \delta \sinh(\delta)(1 + (\phi_{S0}/\phi_{T0}))}$$  \hfill (67)

In all the experiments made in the CSFR mode the value of $P_{S}$ in $z=L/2$ is the semi-sum of $P_{S0}$ and $P_{S0}$. Given the minor transmembrane flow that occurs, and given that $P_{T0} = 0$, the profile of $P_{S}$ vs. $z$ is practically linear and it is in $z=L/2$ where the value of $P_{S(L/2)} = P_{T0}/2$ is made equal to $P_{S(L/2)}$, for any value of $\phi_{S0}/\phi_{T0}$ tested.

Fig. 7 enables the analysis of $\phi_{S}$ and $\phi_{T}$ (Eqs. (34) and (35)) as a function of $\phi_{S0}/\phi_{T0}$. It bears noting in these figures that for the countercurrent operation, for example at $\phi_{S0}/\phi_{T0} = -10$, the values of the flow in the shell and in the fibres would be $\phi_{S0}/\phi_{T0} = -10$ and $\phi_{S0}/\phi_{T0} = 1$ for $z=0$; $\phi_{S0}/\phi_{T0} = -9.9967$ and $\phi_{S0}/\phi_{T0} = 0.0037$ for $z=L/2$; and $\phi_{S0}/\phi_{T0} = -10$ and $\phi_{S0}/\phi_{T0} = 1$ for $z=L$. Also, it bears mentioning that for $\phi_{S0}/\phi_{T0} = \lambda = 6.33$, $\phi_{S}$ and $\phi_{T}$ remains constant in the z-direction, without any permeation.

As indicated in Section 3.2, in CSFR mode, the maximum permeate flow ($\phi_{T}$) is reached in $z=L/2$, which is where the inversion point is situated. Substituting Eq. (31) in Eq. (37) and considering Eq. (64), we get:

$$\frac{\phi_{PE}}{\phi_{T0}} = \frac{(\lambda - (\phi_{S0}/\phi_{T0}))}{\lambda + 1} \left(1 - 2 \frac{\sin(\delta)}{\sinh(\delta)} \right)$$  \hfill (68)

This equation indicates that the permeate flow depends on $\phi_{S0}/\phi_{T0}$ and on the geometric parameters of the system, as reflected in Fig. 8. In this figure, it is also observed that $\phi_{S0}/\phi_{T0}$ is higher when $\phi_{S0}/\phi_{T0}$ is lower (countercurrent operation). When the value of $\phi_{S0}/\phi_{T0}$ rises, $\phi_{PE}/\phi_{T0}$ becomes lower and with $\phi_{S0}/\phi_{T0} = \lambda = 6.33$ the value of $\phi_{PE}/\phi_{T0}$ becomes zero. When $\phi_{S0}/\phi_{T0} > \lambda$, $\phi_{PE}/\phi_{T0}$ becomes negative, indicating that the fluid of the shell passes to the fibres in the first half of the module and in the opposite sense in the second.

In many cases (as may be the use of this system in enzymatic reactions where the substrates and products are permeable and circulate through the lumen fibres and the enzyme remains physically retained in the shell) it would be useful to work with countercurrent flows, reducing the relation $\phi_{S0}/\phi_{T0}$, so that for a value of $\phi_{S0}/\phi_{T0} = -\lambda$ the permeate flow is doubled with respect to when there is no forced recirculation in the shell, as can be easily deduced from Eq. (68).

Also, with substitution of Eq. (64) and Eq. (66) in Eq. (40), the transmembrane pressure varies with the z-direction, depending only on $\phi_{S0}/\phi_{T0}$:

$$\frac{P_{TM}}{P_{T0}} = \frac{(\lambda + 1)(1 - (1/\lambda)\phi_{S0}/\phi_{T0}))\text{cosh}(\delta(1 - (z/L)))) - \text{cosh}(\delta(z/L))}{2(\text{cosh}(\delta) - 1)(\lambda - (\phi_{S0}/\phi_{T0})) + \delta \sinh(\delta)(1 + (\phi_{S0}/\phi_{T0}))}$$  \hfill (69)

Fig. 9 shows that $P_{TM}$ varies with $\phi_{S0}/\phi_{T0}$ in the z-direction. As indicated above, regardless of the value of $\phi_{S0}/\phi_{T0}$ assayed, the inversion point appears at $z=L/2$. As might be expected from the analysis of flow made, for $\phi_{S0}/\phi_{T0} = \lambda = 6.33$, $P_{TM} = 0$.
Particularizing Eq. (64) when there is no recirculation forced in the shell, considering that \( \phi_{00} = 0 \), given that for this operation mode the inversion point appears at \( z = L/2 \) and thus \( P_{S(L/2)}/P_{F0} = 0.5 \), and reordering, we get:

\[
\phi_{F0} = \frac{(\lambda + 1)^2 kFL \sinh(\delta)}{(2\lambda^2 \delta (\cosh(\delta) - 1) + \lambda \delta^2 \sinh(\delta))} P_{F0} \tag{70}
\]

This indicates that \( \phi_{F0} \) is directly proportional to \( P_{F0} \) when there is no recirculation forced in the shell. Eq. (70) acceptably fits the experimental results found (Fig. 10).

\[
\phi_{S0}/\phi_{F0} (-)
\]

\[
\phi_{S0}/\phi_{F0} - \phi_{S0}/\phi_{F0} (-)
\]
Fig. 9. \( \frac{P_{TM}}{P_{FO}} \) profile in CSFR mode: \( \frac{\phi_{SO}}{\phi_{FO}} = 10 \), \( \frac{\phi_{SO}}{\phi_{FO}} = 6.33 \), \( \frac{\phi_{SO}}{\phi_{FO}} = 5 \), \( \frac{\phi_{SO}}{\phi_{FO}} = 0 \), \( \frac{\phi_{SO}}{\phi_{FO}} = 5 \), \( \frac{\phi_{SO}}{\phi_{FO}} = 10 \).

Fig. 10. Experimental verification of Eq. (70) (symbols = experimental points, line = simulation with the parameters values).

On the other hand, for \( \frac{\phi_{SO}}{\phi_{FO}} = 0 \), Eq. (68) becomes:

\[
\frac{\phi_{PE}}{\phi_{FO}} = \frac{\lambda}{\lambda + 1} \left( 1 - \frac{2 \sinh(0.5\delta)}{\sinh(\delta)} \right) = 0.00128
\]

5. Conclusions

Integral fluidodynamic models for hollow fibre have been developed and analysed for different operation modes as a function of the geometric parameters and of the entry flows to the fibres and to the shell. The experimental results found with a commercial HF verify the models developed. It has been demonstrated that the operation mode strongly influences the transmembrane flow, and that greater transmembrane flow is always achieved with countercurrent operation than with cocurrent operation for an equal relation between flows that circulate through the lumen fibres and the shell. On the other hand, the direction of the transmembrane flow varies according to the value of the \( \frac{\phi_{SO}}{\phi_{FO}} \) assayed, a situation that could influence the residence time of the fluid in the fibres or in the shell. This aspect is of great importance, for example, when the shell of the HF is used as an enzymatic reactor in recirculation systems.

Therefore, the operation mode used in the HF not only favours better homogenization in the shell and cleanliness of the membrane, as different authors have pointed out, recommending that the direction of the recycle flow be periodically alternated, but it also decisively influences the behaviour of the system.

Thus, a high permeate flow is reached with OES mode, this operation system being recommended when the HF is used as a filtering medium. Nevertheless, the theoretical study indicates that higher transmembrane flows could be achieved with CSFR in countercurrent operation under certain conditions.
Nomenclature

\( C_1, C_2, C_3, C_4 \) integration constants

\( D_c \) internal diameter of the shell (m)

\( D_h \) hydraulic diameter (m)

\( D_i \) internal diameter of the fibres (m)

\( k_F \) constant defined in Eq. (1) (m\(^4\)/Pa s)

\( k_P \) constant defined in Eq. (4) (m\(^2\)/Pa s)

\( k_S \) constant defined in Eq. (2) (m\(^2\)/Pa s)

\( L \) fibre length (m)

\( L_P \) membrane permeability (m)

\( n \) numbers of fibres

\( P_F \) pressure in the lumen fibres (measured with respect to the atmospheric pressure) (N/m\(^2\))

\( P_S \) pressure in the shell (measured with respect to the atmospheric pressure) (N/m\(^2\))

\( P_{TM} \) transmembrane pressure (N/m\(^2\))

Re Reynolds number (dimensionless)

S section of free passage of the fluid through the shell (m\(^2\))

\( t_w \) thickness of the wall membrane (m)

\( u_s \) mean velocity in the shell-side (m/s)

\( V_p \) permeation velocity (m/s)

\( z \) position in the direction of the flow (m)

Symbols

\( \delta \) parameter defined in Eq. (13) (dimensionless)

\( \delta_1 \) parameter defined in Eq. (44) (dimensionless)

\( \lambda \) parameter defined in Eq. (12) (dimensionless)

\( \mu \) viscosity (kg/m s)

\( \rho \) density (kg/m\(^3\))

\( \phi_F \) flow in the lumen fibres (m\(^3\)/s)

\( \phi_P \) permeate flow in the z-position by unit of length (m\(^3\)/s m)

\( \phi_{PE} \) total permeate flow exchanged between the fibres and the shell (m\(^3\)/s)

\( \phi_{PT} \) total permeate flow between the upstream and downstream ports of the module (m\(^3\)/s)

\( \phi_S \) flow in the shell-side (m\(^3\)/s)

Subscripts

0 value of variable in \( z = 0 \)

\( L/2 \) value of variable in \( z = L/2 \)

L value of variable in \( z = L \)

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References


