Bifurcation and concentration for a degenerate elliptic boundary value problem

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EPFL

Summary

1. Differentiability and bifurcation
2. A degenerate elliptic boundary value problem
3. Bifurcation and concentration
4. Weaker degeneracy

Fréchet differentiability

H is a real Hilbert space

F is *Fréchet* differentiable at $u \in H$

if $\exists T \in B(H,H)$ such that
$\lim_{\|w\| \to 0} \frac{F(u+w)-F(u)-Tw}{\|w\|} = 0$

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$\iff$
$\exists T \in B(H, H)$ such that
$\lim_{t \to 0} \frac{F(u+tv) - F(u)}{t} = Tv$, uniformly for v in bounded subsets of H

Hadamard differentiability

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$\lim_{n\to\infty} \frac{F(u+t_n v_n)-F(u)}{t_n} = Tv$ for all $v \in H$
for all $\{t_n\} \subset \mathbb{R}\backslash\{0\}$ with $t_n \to 0$
and for all $\{v_n\} \subset H$ with $v_n \to v$

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and for all $\{t_n\} \subset \mathbb{R}\backslash\{0\}$ with $t_n \to 0$
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H a real Banach space, $F : H \to H$ with $F(0) = 0$.

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$\lambda \in \mathbb{R}$ is called a *bifurcation point* if
there exists $\{(\lambda_n, u_n)\} \subset \mathbb{R} \times H$ such that

$$F(u_n) = \lambda_n u_n \text{ and } u_n \neq 0 \text{ for all } n \in \mathbb{N},$$

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Let $B_F \subset \mathbb{R}$ denote the set of all bifurcation points.

Necessary conditions for bifurcation

Theorem Let $F : H \to H$ be a function such that $F(0) = 0$ and F is w-Hadamard differentiable at $u = 0$ with $F'(0) = F'(0)^*$. If $\mu \in (\Lambda^e, \infty) \backslash \sigma(F'(0))$ where $\Lambda^e = \sup \sigma_e(F'(0))$ and

$$\lim \sup_{\|u\| \to 0} \frac{\langle F(u) - F'(0)u, u \rangle}{\|u\|^2} < d(\mu, \sigma(F'(0))),$$

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We have an example where $F : L^2(\Omega) \to L^2(\Omega)$ is both Hadamard and w-Hadamard differentiable with $F'(0) = I$ but $B = [a, b]$ where $a < 1 < b$.

Sufficient conditions for bifurcation

(H1) $\psi \in C^1(H, \mathbb{R})$ with $\psi(u) = \psi(-u)$ and $\psi(0) = 0$ such that

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Theorem Suppose (H1), (H2) and (H3).
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(B) $(\Lambda^e_+, \infty) \cap \sigma(F'(0)) \subset B_F$
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If F is w-Hadamard differentiable at $u = 0$,
then $(\Lambda^e_+, \infty) \cap \sigma(F'(0)) = (\Lambda^e_+, \infty) \cap B_F$.

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(D2) $f \in C^1(\mathbb{R})$ with $f(0) = 0, f'(0) = 1,$
$\sup\{|f'(s)| : s \in \mathbb{R}\} = M < \infty$

Finite energy solutions

$$E(u) = \frac{1}{2}\int_\Omega A\,|\nabla u|^2\,dx - \lambda\int_\Omega F(u)dx < \infty$$

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so H is a Hilbert space with
$\langle u, v\rangle_A = \int_\Omega A(x)\nabla u \cdot \nabla v dx$.

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Consider $A(x) = |x|^2$ and $f(s) = s - g(s)$ where $g'(0) = 0$

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$-\nabla \cdot \{|x|^2 \nabla u\} = \lambda f(u)$
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$-\Delta v - \frac{\mu}{r^2} v + \frac{\lambda}{r} g(\frac{v}{r}) = 0$
where $\mu = \lambda + 1 - N$

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If $g(s) = |s|^\sigma s$, the problem is
$-\Delta v - \frac{\mu}{r^2} v + \frac{\lambda}{r^{\sigma+2}} |v|^\sigma v = 0$
$v \in H_0^1(\Omega)$

Solutions of bvp

A solution of bvp is a pair $(\lambda, u) \in \mathbb{R} \times H$ such that
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Define $K(u)$ and $G(u) \in H$ by

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$\mu \in \sigma(K) \cap (\frac{4}{N^2}, \infty) \iff$
the linear boundary value problem

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$\Sigma = \{\frac{1}{\mu} : \mu \in \sigma(K) \cap (\frac{4}{N^2}, \infty)\}$ is the set of all eigenvalues of this linearisation of bvp.

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(ii) If f is odd with $\sup_{s\in\mathbb{R}} |f(s)| < \infty$ and
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There is bifurcation to the right at every $\lambda \in \Sigma$,
vertical bifurcation at every $\lambda \in (\frac{N^2}{4}, \infty)$
and $B \cap (0, \infty) = \Sigma \cup [\frac{N^2}{4}, \infty)$.

Remarks

A and f were normalised so that

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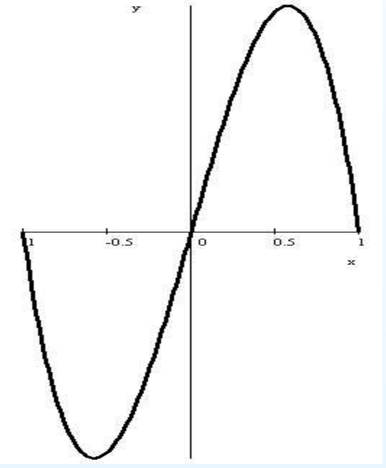
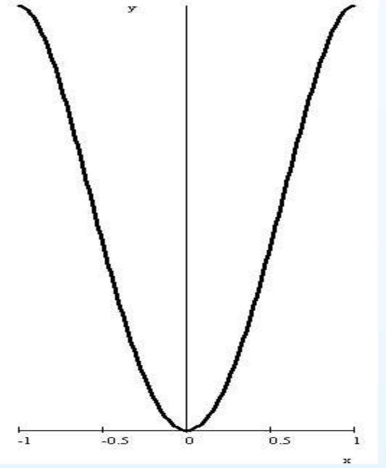
Then $f = F'$ satisfies the conditions of the previous theorems.

Condition (F)

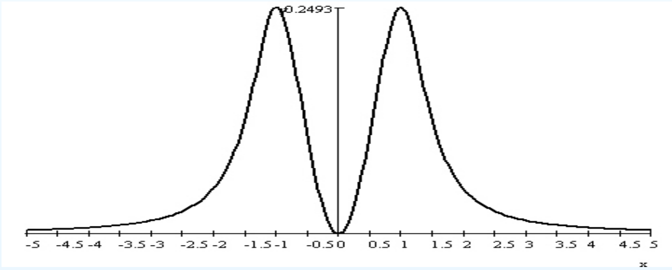


(a) function f

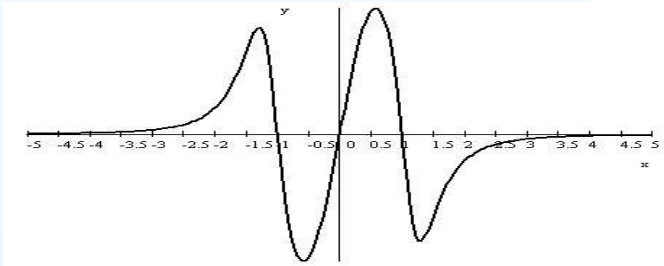


(b) function F

Extension of f



(c) extension of F



(d) extension of $f = F'$

Bounded solutions

Theorem Let (D1) and (F) hold. Then $\Sigma \cup [\frac{N^2}{4}, \infty) \subset B$.
For any $\lambda \in \Sigma \cup [\frac{N^2}{4}, \infty)$, there exists a sequence of solutions $\{(\lambda_n, u_n)\} \subset (0, \infty) \times [H_0 \backslash \{0\}]$ such that

$$\text{for all } n \in \mathbb{N}, |u_n(x)| \leq T \text{ a.e. on } \Omega,$$

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Hence $|u_n|_p \to 0$ as $n \to \infty$ for all $p \in [1, \infty)$.

Radial case

$\Omega = B = \{x \in \mathbb{R}^N : |x| < 1\}$ and $A(x) = C(|x|)$ where
(R) $C \in C^1([0,1])$ with $C(r) > 0$ for all $r \in (0,1], C(0) = 0$ and
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and we have concentration at the origin:
(ii) for any $\varepsilon \in (0,1), u_n \to 0$ uniformly on $\{x \in \mathbb{R}^N : \varepsilon \le |x| \le 1\}$.

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BVP becomes

$$-\{D(s)w'(s)\}' = \frac{\lambda}{N^2} f(w(s)) \text{ for } 0 < s < 1$$

where $D(s) = s^{2(1-\frac{1}{N})} C(s^{\frac{1}{N}})$ and

$$w \in X = \{w \in L^2_{loc}(0,1) : \int_0^1 s^2 w'(s)^2 ds < \infty \text{ and } w(1) = 0\}$$

Nodal properties

If $\lambda > 0$ and $w \in X \backslash \{0\}$ satisfies

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Weaker degeneracy

(D1)$_t$ $A \in C(\overline{\Omega})$ with $A(x) > 0$ for all $x \in \overline{\Omega} \backslash \{0\}$ and
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We can still define a Hilbert space $(H_A, \langle \cdot, \cdot \rangle_A)$ by
$H_A = \{u \in L^2(\Omega) :$
$\int_\Omega A(x) \left| \nabla u(x) \right|^2 dx < \infty$ and $u = 0$ on $\partial \Omega\}$

with $\langle u, v \rangle_A = \int_\Omega A(x) \nabla u(x) \cdot \nabla v(x) dx.$

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(iv) If $t = 2, H_A$ is NOT compactly embedded in $L^2(\Omega)$.

The boundary-value problem

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$\Lambda \in \mathbb{R}$ is a bifurcation point if there is a sequence $\{(\lambda_n, u_n)\} \subset \mathbb{R} \times [H_A \backslash \{0\}]$ of solutions such that $\lambda_n \to \Lambda$ and $|u_n|_2 \to 0$.

Equation in H_A

As before, define $K(u)$ and $G(u) \in H_A$ by

$$\langle K(u), v\rangle_A = \int_\Omega uv\,dx \text{ and}$$

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$\Sigma = \{\frac{1}{\mu_i}\}$ is the set of all eigenvalues.

Regular bifurcation

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Recall that for $t = 2$ and under some extra assumptions on f
$B \cap (0, \infty) = \Sigma \cup [\frac{N^2}{4}, \infty)$.