Gradient bounds for elliptic problems singular at the boundary

Tommaso Leonori

Granada, 24 de Enero 2012
Let us consider the following class of second order Hamilton Jacobi equations:

$$-\alpha \Delta u + u + H(x, \nabla u) = 0 \quad \text{in } \Omega,$$

where $\Omega$ is a smooth (say $C^2$) bounded domain in $\mathbb{R}^N$, $N \geq 2$, $\alpha > 0$ and $H(x, p)$ is a Caratheodory function.
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We are interested in considering nonlinear Hamiltonians that are singular at the boundary.

Our aim is to prove gradient bounds for such class of equations.
The typical structure:

\[-\alpha \Delta u + u + F(x) \cdot \nabla u + g(x, \nabla u) = f(x) \quad \text{in } \Omega,\]

\(F(x) \cdot \nabla u\) is a singular transport term, \(|F(x)| \sim \sigma \text{dist}(x, \partial \Omega)\) with \(F(x)\) "directed outward". 

\(g(x, \nabla u)\) is a nonlinear term with "natural growth". 

\(f(x)\) is a locally Lipschitz function (possibly singular at \(\partial \Omega\)). 

Rmk.: No boundary conditions are prescribed!
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Motivation

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\begin{aligned}
    dX_t &= a_t \, dt + \sqrt{2} \, dB_t, \\
    X_0 &= x \in \Omega,
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where \( B_t \) is the Brownian motion, and \( a \in C^0(\Omega, \mathbb{R}^N) \) represents the control.
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“We want to constrain a Brownian motion in a given domain $\Omega$ by controlling its drift”.
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Deterministic case

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In the deterministic case it is enough to require that ($\nu$ is the outward normal at the boundary)

\[a(x) \cdot \nu(x) < 0 \quad x \sim \partial \Omega.\]
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Indeed we have \((\nabla d(x) = -\nu(x))\)

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i.e. the distance to the boundary grows, as \(x(t)\) get close to the boundary.
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and thus if $a$ is bounded there exists a unique solution $\nu$ bounded and $\mathbb{E}(\tau_x) \leq \|\nu\|_{\infty}$. 

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Typical examples of controls are constructed as functions of the distance to the boundary, that are singular at the boundary itself, i.e.

\[ a(x) \sim \psi(d(x)) \quad \text{with} \quad \lim_{d(x) \to 0} |\psi(x)| = +\infty. \]
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This field has a privileged direction which reminds of the control mechanism acting basically in the normal direction.
Gradient bound: Idea of the method

Bernstein method for nonlinear elliptic equations:

Let \( v \) be a solution of \( -\Delta v + v + H(x, \nabla v) = f \) in \( \Omega \) and \( f \) smooth. We want to find an (upper) bound for \( |\nabla v|^2 \) by looking at the equation that it solves. The equation involves the laplacian, so the first step is to write \( \Delta |\nabla v|^2 \).

\[
\partial_{x_i} |\nabla v|^2 = \sum_{k=1}^{N} v_k^2 = \sum_{k=1}^{N} 2v_k v_{ki} = 2 \sum_{k=1}^{N} v_k (v_{ii})_k + v_{ki} v_{ki}.
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Example: $H(x, p) \equiv 0$.

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Thus dropping the second (positive) term on the right hand side, we deduce

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\Delta |\nabla v|^2 \geq 2 |\nabla v|^2 - 2 \nabla v \cdot \nabla f
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Hence $|\nabla v|^2$ is a subsolution for

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\sup_{\overline{\Omega}} |\nabla v|^2 \leq \| \nabla f \|_\infty^2 + \sup_{\partial \Omega} |\nabla v|^2
\]
The model equation we have in mind is the following:

\[
\alpha \Delta u + u + B(x) \cdot \nabla u \, d(x) + c(x) |\nabla u|^2 = f(x) \quad \text{in} \quad \Omega,
\]

where \( \alpha > 0 \), \( B(x) \in W^{1, \infty}(\Omega) \), \( d(x) \in C^2(\Omega) \), \( d \equiv \text{dist}(x, \partial \Omega) \) near \( \partial \Omega \) and \( \nabla d = -\nu \) at \( \partial \Omega \), \( c(x) \in W^{1, \infty}(\Omega) \), without any sign condition! i.e. the hamiltonian \( H(x, p) \) is not coercive with respect to \( |p| \); \( f(x) \in W^{1, \infty}_{\text{loc}}(\Omega) \), possibly singular at \( \partial \Omega \).
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- \( d(x) \in C^2(\Omega) \), \quad d \equiv \text{dist}(x, \partial \Omega) \text{ near } \partial \Omega \text{ and } \nabla d = -\nu \text{ at } \partial \Omega, \)
- \( c(x) \in W^{1,\infty}(\Omega) \), without any sign condition! i.e. the Hamiltonian \( H(x, p) \) is not coercive with respect to \( |p| \);
- \( f(x) \in W^{1,\infty}_{\text{loc}}(\Omega) \), possibly singular at \( \partial \Omega \).
Theorem (T.L., A. Porretta - ARMA 2011)

Let \( c(x) \in W^{1,\infty}(\Omega), B(x) \in W^{1,\infty}(\Omega)^N \) with

\[
B(x) \cdot \nu \geq \sigma > 0, \quad B(x) \cdot \tau = 0 \quad \text{at } \partial \Omega
\]

and \( \sigma > \alpha \) and assume that \( f(x) \in W^{1,\infty}_{loc}(\Omega) \) satisfies near the boundary

\[
|f| \leq \frac{\rho(d)}{d}, \quad |\nabla f| \leq \frac{\rho(d)}{d^2} \quad \text{where } \int_0^1 \frac{\rho(s)}{s} \, ds < \infty.
\]

Then there exists a solution \( u \) of \((E_\alpha)\) in \( u \in C^2(\Omega) \cap W^{1,\infty}(\Omega) \).
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Then there exists a solution \( u \) of \( (E_\alpha) \) in \( u \in C^2(\Omega) \cap W^{1,\infty}(\Omega) \).

Moreover \( u \) is the unique bounded solution and \( \frac{\partial u(x)}{\partial \nu} \to 0 \) as \( x \to \partial \Omega \).
Theorem (T.L., A. Porretta - ARMA 2011)

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Then there exists a solution $u$ of $(E_\alpha)$ in $u \in C^2(\Omega) \cap W^{1,\infty}(\Omega)$.

Moreover $u$ is the unique bounded solution and $\frac{\partial u(x)}{\partial \nu} \to 0$ as $x \to \partial \Omega$.

For $\alpha = \sigma$ the same result holds true under stronger hypothesis on $\rho$, namely

$$\int_0^1 \frac{1}{s} \left( \int_0^s \frac{\rho(\tau)}{\tau} \, d\tau \right) \, ds < \infty.$$
Idea of the Proof.

First, we approximate our equation in order to ”desingularize” it, with solutions that satisfy a Neumann boundary condition at the interior of $\Omega$, 

\[
-\alpha \Delta u_n + u_n + B(x) \cdot \nabla u_n + c(x) |\nabla u_n|^2 = f(x) \quad \text{in} \quad \Omega_n
\]

\[
\frac{\partial u_n}{\partial \nu} = 0 \quad \text{on} \quad \partial \Omega_n
\]

Where $\Omega_n = \{x \in \Omega : d(x) > \frac{1}{n}\}$. 

We focus our attention on the function $w_n = |\nabla u_n|^2 e^{\theta(d)}$ where $\theta$ is a bounded function (but its first derivative, in general, is singular at $d(x) = 0$).
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\begin{align*}
-\alpha \Delta u_n + u_n + \frac{B(x) \cdot \nabla u_n}{d(x)} + c(x)|\nabla u_n|^2 &= f(x) \quad \text{in } \Omega_n, \\
\frac{\partial u_n}{\partial \nu} &= 0 \quad \text{on } \partial \Omega_n,
\end{align*}
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Idea of the Proof.

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\begin{cases}
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We focus our attention on the function

\[ w_n = |\nabla u_n|^2 e^{\theta(d)} \]

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Step 1. Boundary behavior. We notice that the condition

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implies that there exists a function $\mu$ such that

$$\mu(x) \nu(x) = \nabla (\nabla u_n \cdot \nu(x)) = D^2 u_n \nu(x) + J(\nu(x)) \nabla u_n \quad \text{on } \partial \Omega_n.$$
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Thus in the direction of \( \nabla u_n \) we have
\[ \mu(x) \nu(x) \cdot \nabla u_n = D^2 u_n \nu(x) \cdot \nabla u_n + J(\nu(x)) \nabla u_n \cdot \nabla u_n \quad \text{on } \partial \Omega_n. \]
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Thus in the direction of \( \nabla u_n \) we have (using the Neumann condition)

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and

$$0 = \frac{1}{2} \nabla |\nabla u_n|^2 \cdot \nu(x) - D^2 d(x) \nabla u_n \cdot \nabla u_n \quad \text{on } \partial \Omega_n.$$
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\]
\[
0 = \frac{1}{2} \nabla|\nabla u_n|^2 \cdot \nu(x) - D^2 d(x) \, \nabla u_n \cdot \nabla u_n \quad \text{on } \partial \Omega_n,
\]
\[
\nabla|\nabla u_n|^2 \cdot \nu \leq 2 \| D^2 d \| \, |\nabla u_n|^2 \quad \text{on } \partial \Omega_n.
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$$\nabla |\nabla u_n|^2 \cdot \nu \leq 2 \| D^2 d \| |\nabla u_n|^2 \quad \text{on } \partial \Omega_n.$$ 

Thus it is easy to see that on $\partial \Omega_n$,

$$\nabla w_n \cdot \nu = \nabla \left( |\nabla u_n|^2 e^{\theta(d)} \right) \cdot \nu = -\theta'(d)w_n + e^{\theta(d)} \nabla |\nabla u_n|^2 \cdot \nu \leq -\theta'(d)w_n + 2 \| D^2 d \| e^{\theta(d)} |\nabla u_n|^2$$
Step 1. Boundary behavior. We notice that the condition
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\[ \mu(x) \nu(x) = \nabla (\nabla u_n \cdot \nu(x)) = D^2 u_n \nu(x) + D\nu(x) \nabla u_n \quad \text{on} \quad \partial \Omega_n. \]
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Thus it is easy to see that on \( \partial \Omega_n \),
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\[ \leq -\theta'(d)w_n + 2 \| D^2 d \| e^{\theta(d)} |\nabla u_n|^2 \]
\[ < 0 \]
Thus (Hopf Lemma) the maximum of \( w_n \) is not achieved at the boundary of \( \Omega_n \).
Step 2. Near $\partial \Omega$. 

Assume that $c(x) \equiv 0$ (to deal with the general case we need to modify the proof). We fix a $\delta > 0$ (small) and we study the equation solved by $w_n = |\nabla u_n|^2 e^{\theta(d)}$ in $\Omega \setminus \Omega_{\delta}$ for $n$ large enough. Notice that $\alpha \Delta |\nabla u_n|^2 = 2\nabla \alpha \Delta u_n \cdot \nabla u_n + 2\alpha |D^2 u_n|^2$. Using that $u_n$ solves $\alpha \Delta u_n = u_n + B(x) \cdot \nabla u_n - f(x)$, it follows that $\alpha \Delta w_n = 2\alpha \theta' (d) \nabla w_n \cdot \nabla d + B(x) \cdot \nabla w_n d - 2DB \nabla u_n \cdot \nabla u_n d e^{\theta(d)} - 2|\nabla u_n| |\nabla f| e^{\theta(d)} + 2\alpha |D^2 u_n|^2$. 

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Notice that

\[
\alpha \Delta |\nabla u_n|^2 = 2 \nabla \alpha \Delta u_n \cdot \nabla u_n + 2 \alpha |D^2 u_n|^2.
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Using that $u_n$ solves $\alpha \Delta u_n = u_n + \frac{B(x) \cdot \nabla u_n}{d} - f(x)$,
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$$\alpha \Delta w_n = 2 \alpha \theta'(d) \nabla w_n \cdot \nabla d + \frac{B(x) \cdot \nabla w}{d}$$

$$+ w_n \left[2 + \alpha \left(\theta''(d) - \theta'(d)^2 + \Delta d \theta'(d)\right) - B(x) \cdot \nabla d \frac{\theta'(d)}{d}\right]$$

$$- 2 \frac{DB \nabla u_n \cdot \nabla u_n}{d} e^{\theta(d)} - 2 |\nabla u_n| |\nabla f| e^{\theta(d)} + 2 \alpha |D^2 u_n|^2$$
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\]
\[
- 2 \frac{DB \nabla u_n \cdot \nabla u_n}{d} e^{\theta(d)} - 2 |\nabla u_n| |\nabla f| e^{\theta(d)} + 2\alpha |D^2 u_n|^2
\]
\[
\geq - \frac{\|DB\|}{d} |\nabla u_n|^2 \geq 0.
\]
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$$\alpha \Delta |\nabla u_n|^2 = 2 \nabla \alpha \Delta u_n \cdot \nabla u_n + 2 \alpha |D^2 u_n|^2.$$  

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$$+ w_n \left[ 2 + \alpha \left( \theta''(d) - \theta'(d)^2 + \Delta d \theta'(d) \right) - B(x) \cdot \nabla d \frac{\theta'(d)}{d} \right]$$

$$- 2 \frac{DB \nabla u_n \cdot \nabla u_n}{d} e^{\theta(d)} - 2 |\nabla u_n| |\nabla f| e^{\theta(d)} + 2 \alpha |D^2 u_n|^2 \\ \geq - \frac{\|DB\|}{d} |\nabla u_n|^2$$

Recalling that $B \cdot \nu \geq \sigma > \alpha$
Step 2. Near $\partial \Omega$. Assume that $c(x) \equiv 0$ (to deal with the general case we need to modify the proof).
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$$\alpha \Delta w_n \geq 2 \alpha \theta'(d) \nabla w_n \cdot \nabla d + \frac{B(x) \cdot \nabla w_n}{d} + w_n \left[ 2 + \alpha \left( \theta''(d) - \theta'(d)^2 + \Delta d \theta'(d) \right) + \sigma \frac{\theta'(d)}{d} \right]$$

$$-2 \frac{||DB||}{d} |\nabla u_n|^2 e^{\theta(d)} - 2 |\nabla u_n||\nabla f| e^{\theta(d)} = w_n$$

Recalling that $B \cdot \nu \geq \sigma > \alpha$.
We now choose
\[ \theta(s) = \int_0^s \frac{\rho(\sigma)}{\sigma} d\sigma \]
where, we recall \( \frac{\rho(\sigma)}{\sigma} \) is integrable (i.e. \( \rho(0) = 0, \rho > 0 \)).

\[ \alpha \Delta w_n \geq 2\alpha \theta'(d) \nabla w_n \cdot \nabla d + \frac{B(x) \cdot \nabla w_n}{d} \]
\[ + w_n \left[ 2 + \alpha \left( \theta''(d) - \theta'(d)^2 + \Delta d \theta'(d) \right) + \sigma \frac{\theta'(d)}{d} \right] \]
\[ - 2 \frac{\|DB\|}{d} w_n - 2 \|\nabla u_n\| \|\nabla f\| e^{\theta(d)} \]
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$$\alpha \Delta w_n \geq 2\alpha \theta'(d) \nabla w_n \cdot \nabla d + \frac{B(x) \cdot \nabla w_n}{d}$$

$$+ w_n \left[ 2 + \alpha \left( \frac{\rho'(d)}{d} - \frac{\rho(d)}{d^2} - \frac{\rho'^2(d)}{d^2} - |\Delta d| \frac{\rho(d)}{d} \right) + \sigma \frac{\rho(d)}{d^2} \right]$$

$$- 2 \frac{||DB||}{d} w_n - 2 |\nabla u_n| |\nabla f| e^{\theta(d)}$$
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\[
\alpha \Delta w_n \geq 2\alpha \theta'(d) \nabla w_n \cdot \nabla d + \frac{B(x) \cdot \nabla w_n}{d} \\
+ (\sigma - \alpha) \frac{\rho(d)}{d^2} (1 + o(1)) w_n - 2 |\nabla u_n| |\nabla f| e^{\theta(d)}
\]
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abla u_n\| \|
abla f\| e^{\theta(d)}}{d^2} \]

since \( \|
abla f\| \leq \frac{\rho(d)}{d^2} \)
We now choose
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\[-\alpha \Delta w_n + 2\alpha \theta'(d) \nabla w_n \cdot \nabla d + \frac{B(x) \cdot \nabla w_n}{d} + \frac{(\sigma - \alpha)}{2} \frac{\rho(d)}{d^2} w_n \leq C_0 \frac{\rho(d)}{d^2}.\]
We now choose

$$\theta(s) = \int_0^s \frac{\rho(\sigma)}{\sigma} d\sigma$$

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Thus on the interior maximum points $w_n \leq \frac{2}{(\sigma - \alpha)} C_0$. 

This implies $\sup_{\Omega \setminus \Omega_\delta} |\nabla u_n|^2 \leq \tilde{C}_0 + \sup_{\partial \Omega \delta} |\nabla u_n|^2$. 

Thus on the interior maximum points $w_n \leq \frac{2}{(\sigma - \alpha)} C_0$. 


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where, we recall $\frac{\rho(\sigma)}{\sigma}$ is integrable (i.e. $\rho(0) = 0$, $\rho > 0$).

$$-\alpha \Delta w_n + 2\alpha \theta'(d) \nabla w_n \cdot \nabla d + \frac{B(x) \cdot \nabla w_n}{d} + \frac{(\sigma - \alpha)}{2} \frac{\rho(d)}{d^2} w_n \leq C_0 \frac{\rho(d)}{d^2}.$$ 

Thus on the interior maximum points $w_n \leq \frac{2}{(\sigma - \alpha)} C_0$.

This implies

$$\sup_{\Omega_n \setminus \Omega_\delta} |\nabla u_n|^2 \leq \widetilde{C}_0 + \sup_{\partial \Omega_\delta} |\nabla u_n|^2.$$
For the case $c(x) \neq 0$ we have to deal with

\[ w_n = |\nabla u_n|^2 e^{\theta(d)} (1 + \beta(u_n)) \]

where $\beta$ is a suitable smooth, positive bounded function (computations in this case are much more heavy).
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**Step 3. Interior estimate.** By classical elliptic regularity ([GT]):
\[ \forall K \subset \subset \Omega, \quad \sup_K |\nabla u_n|^2 \leq C(\text{dist} (K, \partial \Omega)). \]
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Thus we deduce that
\[
\exists c > 0 : |\nabla u_n|^2 \leq c \quad \text{in} \ \Omega.
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Uniqueness

When $H(x, \cdot)$ is convex, uniqueness holds in a suitable class of functions. This is a consequence of a classical principle that is well known in the linear case. If there exists $\phi \in C^2(\Omega)$ such that

$$-\alpha \Delta \phi + \phi + H(x, \nabla \phi) \leq 0 \text{ in } \Omega,$$
$$\phi = -\infty \text{ on } \Omega,$$

then uniqueness holds for solutions such that $u = o(|\phi|)$. In the case of equation (E_{\alpha}) it holds with $\phi \sim \log(d)$. Thus bounded solutions are unique!
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z_\varepsilon \geq 0 \quad \Rightarrow \quad u_\varepsilon \geq v \quad \Rightarrow \quad u \geq v
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Regularity and boundary conditions

This statement can be very useful as a regularity result.

Any bounded solution of \((E_\alpha)\) is \(W^{1,\infty}(\Omega)\).

Notice that this result of uniqueness/regularity holds true without knowing any information on the solution at the boundary!

Since the solution belongs to \(W^{1,\infty}(\Omega)\), there exists the trace at \(\partial\Omega\) and thus, for any \(x_0 \in \partial\Omega\) we can rescale the equation near the boundary, we make a blow-up and it follows that the solution satisfies

\[\lim_{x \to x_0 \in \partial\Omega} \frac{\partial}{\partial \nu}(u(x)) = 0.\]

This in particular means that the homogeneous Neumann boundary condition is intrinsic in the equation.
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Optimality of $\sigma \geq \alpha$: the Fichera condition

In the linear framework we can observe that the condition $\sigma \geq \alpha$ is optimal.
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In the linear framework we can observe that the condition $\sigma \geq \alpha$ is optimal. Indeed for linear equations as

$$a_{ij}\partial_{ij}^2 v + b_j v_j + cv = f \quad \text{in } \Omega$$

you can prescribe Dirichlet boundary data in the set

$$\Gamma_d = \left\{ x \in \partial \Omega : a_{ij}(x)\nu(x)\nu(x) > 0 \text{ or } \sum_j \left( b_j - \sum_i \partial_{x_i} a_{ij} \right) \nu_j > 0 \right\}$$

Assume that $c(x) \equiv 0$ in $(E_\alpha)$ and multiply the equation by $d(x)$, hence we have:

$$-\alpha d(x)\Delta u + d(x)u + B(x) \cdot \nabla u - d(x)f(x) = 0 \quad \text{in } \Omega.$$

Thus if $\sigma < \alpha$ our estimate should depend on the boundary value of $u$!
Generalizations

\[ -\alpha \Delta u + u + H(x, \nabla u) = 0 \quad \text{in } \Omega , \]
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\[-\alpha \Delta u + u + H(x, \nabla u) = 0 \quad \text{in } \Omega\,, \text{ where } H(x, p)\]
satisfies a local natural growth condition, and general assumptions, as

\[|H(x, p) - p \cdot H_p(x, p)| \leq C_0 |p|^2 + \frac{\rho(d)}{d}\,\]

\[H_x(x, p) \cdot \frac{p}{|p|} \geq -\frac{\rho(d)}{d^2} |p| - \frac{\rho(d)}{d} |p|^2 - \frac{\rho(d)}{d^2}\,\]

\[H_p(x, p) \cdot \nu(x) \geq \frac{\sigma}{d} - C_1 |p|,\]

and either

\[\sigma > \alpha\,, \quad \text{and} \quad \int_0^1 \frac{\rho(t)}{t} dt < \infty\,,\]

or

\[\sigma = \alpha\,, \quad \text{and} \quad \int_0^1 \frac{1}{t} \left(\int_0^t \frac{\rho(\tau)}{\tau} d\tau\right) dt < \infty\,.\]
Generalizations

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- Weighted Lipschitz estimates (Hölder-type estimates, blow-up solutions...
Stability (first order equation)

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are we able to prove the existence of a Lipschitz solution for the equation

$$(E_0) \quad u + \frac{B(x) \cdot \nabla u}{d(x)} + c(x)|\nabla u|^2 = f(x) \quad \text{in } \Omega$$

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In order to give a positive answer to such a question, we have to straight some hypotheses on the nonlinear term.
Stability (first order equation)

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Two ingredients are needed:

- In order to get interior gradient bound, \(c(x)\) has to be positive in \(\Omega\) (possibly vanishing at \(\partial \Omega\));
- An approximation that involves a vanishing transport term, i.e., the solutions of \((E_0)\) are limit of \(u - \alpha \Delta u + \alpha \nu \cdot \nabla u d(x) + B(x) \cdot \nabla u d(x) + c(x) |\nabla u|^2 = f(x) \) in \(\Omega\).
Stability (first order equation)

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Theorem (T.L., A. Porretta - ARMA 2011)

Assume that $B(x) \in W^{1,\infty}(\Omega)^N$ is such that $B(x) \cdot \nu > 0$, and $f(x) \in W^{1,\infty}_{loc}(\Omega)$ satisfies near the boundary

$$|f| \leq \frac{\rho(d)}{d}, \quad |\nabla f| \leq \frac{\rho(d)}{d^2} \quad \text{where} \quad \int_0^1 \frac{\rho(s)}{s} \, ds < \infty.$$ 

Moreover suppose that $c(x) \in W^{1,\infty}_{loc}(\Omega)$ is a positive function that satisfies the following condition near $\partial \Omega$:

$$|\nabla c(x)|^2 \leq \frac{\rho(d)}{d^2} c(x).$$
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Assume that $B(x) \in W^{1,\infty}(\Omega)^N$ is such that $B(x) \cdot \nu > 0$, and $f(x) \in W^{1,\infty}_{loc}(\Omega)$ satisfies near the boundary

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Moreover suppose that $c(x) \in W^{1,\infty}_{loc}(\Omega)$ is a positive function that satisfies the following condition near $\partial \Omega$:

$$|\nabla c(x)|^2 \leq \frac{\rho(d)}{d^2} c(x).$$

Then there exists $u \in W^{1,\infty}(\Omega)$ which is a viscosity solution of $(E_0)$ and $\frac{\partial u}{\partial \nu} = 0$ (in the viscosity sense) at $\partial \Omega$. 
Application/motivation:

A stochastic control problem with state constraint.
A stochastic control problem with state constraint.

Let’s go back to the model introduced by J.M. Lasry and P.L. Lions, and let us consider the SDE:

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dX_t &= a_t dt + \sqrt{2} dB_t \\
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We have already noticed that the class of controls that confine the process inside \( \Omega \) a.s. for any \( t \) is not empty.
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Among these controls, we want to select one that satisfies a criterion of optimality.
Thus, let $\mathcal{A}$ be the class of all (feedback) controls that keep the process $X_t$ inside the domain $\Omega$ for any time $t > 0$ a.s..
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$\mathbb{E}$ is the expected value, $C_q > 0$ and $\frac{1}{q'} + \frac{1}{q} = 1$, $q \in (1, 2)$,

$$J(x, a) = \mathbb{E} \int_0^\infty \left\{ f(X_t) + C_q |a(X_t)|^{q'} + C_q |a(X_t)|^q e^{-t} \right\} dt$$
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Application to a stochastic control problem

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\begin{align*}
-\Delta u + u + |\nabla u|^q &= f(x) \\
nu(x) &\to +\infty \text{ as } d(x) \to 0
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Some results about large solutions

It has been proved: ([LL])

- existence and uniqueness of the solution $u \in W^{2,p}_{loc}(\Omega)$, $\forall p > 1$;
- asymptotic estimates on $u(x)$, as $d(x) \to 0$: $u(x) \sim C^* d(x)^{-2-q}$ if $1 < q < 2$, $C^* = \frac{(q-1)}{2-q}$;
- the unique optimal control is $a(x) = -q |\nabla u(x)|^{q-2} \nabla u(x)$;
- ([PV])

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\lim_{x \to x_0 \in \partial \Omega} d(x)^{\frac{1}{q-1}} \nabla u(x) = (q-1)^{-1} \nu(x_0),
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The results on the first order, in particular, say that the solution and the gradient (and consequently the control) depend only on the distance to the boundary.

\[ u(x) \sim \psi(d(x)) \]

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where \( \psi(s) \) is the solution of the ODE

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\begin{cases}
-\psi''(s) + |\psi'(s)|^q = 0 \\ s \in (0, 1), 1 < q < 2 \\
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Goal

Our main objectives are:

▶ to give a more precise picture of the behavior of the gradient (and consequently of the control) near $\partial \Omega$;

▶ to study second order effects;

▶ look at the role played by the geometry of the domain.
Main result


Let $\Omega$ be regular and let $H(\varsigma)$ be the mean curvature of $\partial \Omega$ computed at $\varsigma$ and $\bar{x} = \text{Proj}(x, \partial \Omega)$. Then $\forall 1 < q < 2$, as $d(x) \to 0$, 

\[ \partial u(x) \partial \nu = \left( q - 1 \right) - \frac{q}{q-1} d(x) \left[ 1 + \left( N - 1 \right) H(x) d(x) \right] + o(d(x)) \]

and

\[ \begin{cases} \partial u(x) \partial \tau \in L^\infty(\Omega) & \text{if } \frac{3}{2} < q \leq 2, \\ \partial u(x) \partial \tau = O(\left| \log d(x) \right|) & \text{if } q = \frac{3}{2}, \\ \partial u(x) \partial \tau = O(d(x)^{2q-3}) & \text{if } 1 < q < \frac{3}{2}. \end{cases} \]
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1. is singular at the boundary;
2. is mainly directed in the normal direction, pointing inside;
3. in the tangential directions, vanishes as \( d(x) \to 0 \);
4. it has maximum intensity near the points where the boundary is more "curved" (i.e., on the hypersurfaces parallel to \( \partial \Omega \), it achieves its maximum where the mean curvature is maximal).
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Idea of the proof.

We introduce a corrector term, \( u - S \), where

\[
S = d - 2 - q^{-1} \sum_{k=0}^{m} \sigma_k(x) d_k(x),
\]

with \( m > 0 \), \( \sigma_0 = C^* \).

Then we define \( z = u - S \) and we look at the equation solved by \( z \), i.e.

\[
-\Delta z + z + |\nabla z + \nabla S|^{q-2} - |\nabla S|^{q-2} = f(x) + g(x).
\]

The coefficients \( \sigma_k \) are chosen such that \( f(x) + g(x) \) is smooth.

Our aim is to prove a global Lipschitz estimate for \( z \).
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\[
\frac{\partial u(x)}{\partial \nu} - \frac{\alpha C^*}{d^{\alpha+1}(x)} + \sum_{k=1}^{[\alpha]+1} \left[ \frac{(k-\alpha)\sigma_k(x)}{d^{\alpha-k+1}(x)} - \frac{\nabla \sigma_{k-1}(x) \cdot \nu}{d^{\alpha-k+1}(x)} \right] \in L^\infty(\Omega)
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In particular it is easy to see that

$$\sigma_1 = \frac{(q - 1)^{-\frac{2-q}{q-1}}}{3 - 2q} \frac{\Delta d(x)}{2}$$

and recalling that $\Delta d(x) \big|_{\partial \Omega} = (N - 1)H(x)$ we deduce the result of the Theorem.
Thus our aim is to prove a **global Lipschitz estimate** for the (unique) solution of

$$-\Delta z + z + |\nabla z + \nabla S|^q - |\nabla S|^q = f(x) + g(x) \quad \text{in} \quad \Omega$$

where the right hand side is smooth.
Key point: gradient bounds

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\[-\Delta z + z + |\nabla z + \nabla S|^q - |\nabla S|^q = f(x) + g(x) \text{ in } \Omega\]

where the right hand side is smooth. Actually, the first order expansion of the gradient ([PV]) implies that

\[|\nabla z + \nabla S|^q - |\nabla S|^q \sim -\frac{q}{q-1} \frac{\nabla z \cdot \nabla d}{d} + H_0(x, \nabla z),\]

where \(H_0(x, \rho) = O(d^{\frac{2-q}{q-1}}|\nabla z|^2).\)
Thus our aim is to prove a global Lipschitz estimate for the (unique) solution of

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where $H_0(x, p) = O(d^{\frac{2-q}{q-1}} |\nabla z|^2)$. Thus we are in the hypotheses of the previous Theorem. \blacksquare
Gracias!