Some Results for Elliptic Equations with a term $\pm|\nabla u|^q$.

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A classical inequality by Hardy

If $u \in W^{1,2}(\mathbb{R}^N)$ then

$$\Lambda_N \int_{\mathbb{R}^N} \frac{u^2}{|x|^2} \leq \int_{\mathbb{R}^N} |\nabla u|^2,$$

where the optimal constant is

$$\Lambda_N = (\frac{N-2}{2})^2.$$

- Λ_N is not attained in $W^{1,2}(\mathbb{R}^N)$
- The optimal constant for the corresponding inequality in $W_0^{1,2}(\Omega)$ is $\Lambda_N(\Omega) \equiv \Lambda_N$ provides that $0 \in \Omega$. Moreover Λ_N is not attained in $W^{1,2}(\Omega)$

Linear precedents.

It is well known that for problem

$$-\Delta u = f, \quad x \in \Omega, \quad u|_{\partial\Omega} = 0,$$

we have

- if $f \in L^m(\Omega)$, $m > \frac{N}{2}$, then $u \in W_0^{1,2}(\Omega) \cap L^\infty(\Omega)$;
- if $f \in L^m(\Omega)$, $\frac{2N}{N+2} \le m \le \frac{N}{2}$ then $u \in W_0^{1,2}(\Omega) \cap L^{m^{**}}(\Omega)$, $m^{**} = \frac{Nm}{N-2m}$;
- if $f \in L^m(\Omega)$, $1 < m < \frac{2N}{N+2}$ then $u \in W_0^{1,m^*}(\Omega)$, $m^* = \frac{Nm}{N-m}$.

Consider now the following zero-order perturbation of the Laplacian,

$$-\Delta u = \lambda \frac{u}{|x|^2} + f \text{ in } \Omega,\ u = 0 \text{ on } \partial\Omega,$$

where $0 \in \Omega$ bounded domain in $\mathbb{R}^N$ and $0 < \lambda \le \Lambda_N \equiv (\frac{N-2}{2})^2$.

THEOREM.(L. Boccardo, L. Orsina, I.P.) Assume

$$(E) \quad \lambda < \frac{N(m-1)(N-2m)}{m^2},$$

then

- If $f \in L^m(\Omega)$, $\frac{2N}{N+2} \le m < \frac{N}{2}$, $u \in L^{m^{**}}(\Omega) \cap W_0^{1,2}(\Omega)$, $m^{**} = \dfrac{Nm}{N-2m}$.
- If $f \in L^m$, $1 < m < \frac{2N}{N+2}$, $u \in W_0^{1,m^*}(\Omega)$.
- If $m = 1$ in general are no solution.
- If $m > \dfrac{N}{2}$ in general the solution are unbounded.

Semilinear precedents.

Consider the semilinear equation

$$(E) \quad -\Delta u - \lambda \frac{u}{|x|^2} = u^p$$

and $\alpha_{(-)} = \dfrac{N-2}{2} - \sqrt{\Lambda_n - \lambda}$

THEOREM. (H. Brezis, L. Dupaigne, A. Tesei)

- **Let** $0 \le \lambda \le \Lambda_N$. **If** $1 < p < p^+(\lambda) \equiv 1 + \dfrac{2}{\alpha_-}$ **there exists a nontrivial solution to** (E) **such that,**

$$u^p, \frac{u}{|x|^2} \in L^1_{loc}$$

- **Let** $0 < \lambda \le \Lambda_N$ **and** $p \ge p^+(\lambda)$. **If** $u \in L^p_{loc}(B_R(0) \setminus \{0\})$, $u \ge 0$ **satisfies**

$$-\Delta u - \lambda \frac{u}{|x|^2} \ge u^p \text{ in } \mathcal{D}'(B_R(0) \setminus \{0\}),$$

then $u \equiv 0$.

The quasilinear case: Presentation.

We will consider the model problem:

$$-\Delta u \pm |\nabla u|^p = \lambda \frac{u}{|x|^2} + \alpha\, f \text{ in } \Omega, \quad u = 0 \text{ on } \partial\Omega, \quad 1 \leq p \leq 2.$$

The main point under consideration is to clarify the competition of the Hardy potential versus the gradient term.

According with the sign of the term in the gradient we study:

Presentation and plan of the talk.

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  - **Optimal power for existence/nonexistence depending on** λ**.**

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  - Existence in the complementary interval.

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  - Existence in the complementary interval.
- Sign $+$
  - Breaking down the resonance.

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  - Blow-up
  - Existence in the complementary interval.
- Sign $+$
  - Breaking down the resonance.
  - Optimality of the results.

Optimal power for nonexistence

Consider,

$$(PR)\quad \begin{cases} -\Delta u = |\nabla u|^p + \lambda \dfrac{u}{|x|^2} + f & \text{in } \Omega, \\ u > 0 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

where $f \in L^1_{loc}(\Omega)$ $f(x) \geq 0$ in $\Omega \subset \mathbb{R}^N$, smooth bounded domain such that $0 \in \Omega$, $N \geq 3$.

DEFINITION. **We say that $u \in L^1_{loc}(\Omega)$ is a very weak supersolution (subsolution) to equation**

$$-\Delta u = |\nabla u|^p + \lambda \frac{u}{|x|^2} + f \quad \text{in} \quad \Omega,$$

if $\dfrac{u}{|x|^2} \in L^1_{loc}(\Omega)$, $|\nabla u|^p \in L^1_{loc}(\Omega)$ and $\forall \phi \in C_0^\infty(\Omega)$ such that $\phi \geq 0$, we have that

$$\int_\Omega (-\Delta \phi) u \, dx \geq \ (\leq) \int_\Omega (|\nabla u|^p + \lambda \frac{u}{|x|^2} + f) \phi \, dx.$$

If u is a very weak super and sub-solution, then we say that u is a very weak solution.

Optimal power for nonexistence

- If in problem (PR) we replace $|x|^{-2}$ by a weight $g \in L^m(\Omega)$ with $m > \frac{N}{2}$, then there exists λ_0, $0 < \lambda_0 < \lambda_1(g)$ such that for $0 < \lambda < \lambda_0$ problem (PR) has a weak solution for suitable datum f.
- We will see that the weight $|x|^{-2}$ behaves in a very different way.

NOTATION.

We denote

$$\alpha_{(\pm)} = \frac{N-2}{2} \pm \sqrt{\left(\frac{N-2}{2}\right)^2 - \lambda}.$$

$\alpha_{(\pm)}$ are the roots of $\alpha^2 - (N-2)\alpha + \lambda = 0$.

Such roots give the radial solutions $|x|^{-\alpha_{(\pm)}}$ to the equation

$$-\Delta u - \lambda \frac{u}{|x|^2} = 0.$$

Optimal power for nonexistence

LEMMA 1. Assume $u \gneqq 0$ in Ω such that $u \in L^1_{loc}(\Omega)$ and $\dfrac{u}{|x|^2} \in L^1_{loc}(\Omega)$.

If u satisfies $-\Delta u - \lambda \dfrac{u}{|x|^2} \geq 0$ in $\mathcal{D}'(\Omega)$ with $\lambda \leq \Lambda_N \Rightarrow \exists C > 0$ and there exists a ball $B_R(0) \subset \Omega$ such that $u(x) \geq C|x|^{-\alpha_-}$ in $B_R(0)$, where $\alpha_- = \frac{N-2}{2} - \sqrt{(\frac{N-2}{2})^2 - \lambda}$.

Outline of the proof. By strong M. P. $u \geq \eta$ in a small ball $B_R(0)$.

- Fix $R > 0$ and consider $w \in W^{1,2}(B_R(0))$ the unique solution to
$$-\Delta w - \lambda \frac{w}{|x|^2} = 0 \quad \text{in} \quad B_R(0), \quad w = \eta \quad \text{on} \quad \partial B_R(0).$$
By an elementary computation, it follows that $w(r) = Cr^{-\alpha_-}$ in $B_R(0)$, with $\alpha_- = \frac{N-2}{2} - \sqrt{(\frac{N-2}{2})^2 - \lambda}$ and $C = \eta R^{\alpha_-}$.

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  $\alpha_- = \frac{N-2}{2} - \sqrt{(\frac{N-2}{2})^2 - \lambda}$ and $C = \eta R^{\alpha_-}$.

- By comparison, we conclude that $u \geq w \quad \text{in} \quad B_R(0)$, then $u \geq C|x|^{-\alpha_-} \quad \text{in} \quad B_R(0)$.

Optimal power for nonexistence

LEMMA 2. *(Necessary condition for existence).* **Consider the equation**

$$(L) \quad -\Delta w - \lambda \frac{w}{|x|^2} = g \text{ in } \Omega,$$

with $g \in L^1_{loc}(\Omega)$, $g(x) \geq 0$ **and** $\lambda \leq \Lambda_N$. **If** (L) **has a very weak supersolution then**

$$|x|^{-\alpha(-)} g \in L^1_{loc}(\Omega).$$

Outline of the proof. **Assume** w **a very weak supersolution to** (L).

Let $B_R(0) \subset \Omega$ **be a ball.**

Consider $g_n \equiv T_n(g)$ **and solve the problem**

$$(L_n) \quad -\Delta w_n - \lambda \frac{w_n}{|x|^2} = g_n \text{ in } B_R(0), \quad w_n = 0 \text{ on } \partial B_R(0).$$

Then, $\mathbf{i})$ $\{w_n\}_{n\in\mathbb{N}}$ **in nondecreasing and** $\mathbf{ii})$ $w_n \leq w$.

Consider ϕ, **the solution to problem**

$$-\Delta\phi - \lambda \frac{\phi}{|x|^2} = 1 \text{ in } B_R(0), \quad \phi = 0 \text{ on } \partial B_R(0),$$

then $\phi(x) \simeq c|x|^{-\alpha(-)}$ **in a neighborhood of** $x = 0$.

Take (formally) ϕ **as a test function in problem** (L_n) **there result**

$$\int_{B_R(0)} w_n dx = \int_{B_R(0)} g_n \phi dx \geq C_2 \int_{B_R(0)} g_n |x|^{-\alpha-} dx,$$

then the result follows by monotone convergence theorem.

Optimal power for nonexistence

THEOREM. *(Main nonexistence result)*. **Assume that $f \geq 0$ and $p_+(\lambda) = \frac{2+\alpha_{(-)}}{1+\alpha_{(-)}}$. Then if $p \geq p_+(\lambda)$, there is not very weak supersolution to equation (PR). In the case where $f \equiv 0$, the unique non negative very weak supersolution is $u \equiv 0$.**

Outline of the proof. **We divide the proof in three steps.**

First step: $p > p_+(\lambda)$. **Assume by contradiction that u is a weak super-solution to (PR).**

Then $-\Delta u - \lambda \dfrac{u}{|x|^2} \gneq 0$ **and hence** $u(x) \geq C|x|^{-\alpha_{(-)}}$ **in** $B_r(0) \subset \mathbb{R}^N$.

Consider $\phi \in \mathcal{C}_0^\infty(B_r(0))$ **and use** $|\phi|^{p'}$ **as a test function in** (PR),

$$\int_{B_r(0)} p'|\phi|^{p'-1}\nabla u \nabla \phi = \int_{B_r(0)} |\nabla u|^p |\phi|^{p'} + \lambda \int_{B_r(0)} \frac{u}{|x|^2}|\phi|^{p'} + \int_{B_r(0)} f|\phi|^{p'},$$

by Hölder and Young inequalities,

$$\int_{B_r(0)} p'|\phi|^{p'-1}\nabla u \nabla \phi \leq \frac{1}{2}\int_{B_r(0)} |\nabla u|^p |\phi|^{p'} + C\int_{B_r(0)} |\nabla \phi|^{p'}, \text{ hence}$$

$$c_1\lambda \int_{B_r(0)} \frac{u|\phi|^{p'}}{|x|^2} dx \leq \int_{B_r(0)} |\nabla \phi|^{p'} dx,$$ **($c_1 > 0$ independent of u and ϕ).**

By the local behavior of u in $B_r(0)$,

$$c_2\lambda \int_{B_r(0)} \frac{|\phi|^{p'}}{|x|^{2+\alpha_{(-)}}} dx \leq \int_{B_r(0)} |\nabla \phi|^{p'} dx.$$

Since $p > p_+(\lambda)$, hence $2 + \alpha_{(-)} > p'$ $\Rightarrow$ a contradiction with the Hardy inequality in $W_0^{1,p'}(B_r(0))$.

Optimal power for nonexistence

Second step: $p = p_+(\lambda)$ **and** $\lambda < \Lambda_N$**.** **As in the first step if u is a very weak super-solution,** $u(x) \geq \dfrac{c_0}{|x|^{\alpha(-)}}$ **in some ball $B_\eta(0) \subset\subset \Omega$, without loss of generality we assume that $\eta = e^{-1}$.**

Notice that in this case $p_+(\lambda)' \equiv 2 + \alpha_{(-)}$, then we need a sharper lower estimate for u

By Lemma 2 we obtain that

$$\int_{B_\eta(0)} |\nabla u|^{p_+(\lambda)} |x|^{-\alpha(-)}\, dx < \infty \text{ and } \int_{B_\eta(0)} \frac{u}{|x|^{2+\alpha(-)}} dx < \infty.$$

Consider $w(x) = |x|^{-\alpha(-)} (\log(\frac{1}{|x|}))^\beta$, $\beta > 0$ to be chosen later.

Since $\lambda < \Lambda_N$, $w \in W^{1,2}(B_\eta(0))$ and in particular $w \in W^{1,p_+(\lambda)}(B_\eta(0))$.

By a direct computation we obtain that for $|x| \leq e^{-1}$, by choosing β small enough,

$$-\Delta w - \lambda \frac{w}{|x|^2} \leq \beta^{\frac{1}{2}} |\nabla w|^{p_+(\lambda)} h(x)$$

where $h(x) = \left(\alpha_{(-)} \log(\frac{1}{|x|}) + \beta((\log(\frac{1}{|x|}))^{-1}\right)^{1-p_+(\lambda)}$**, which is bounded in the ball $B_\eta(0)$.**

By scaling, $u_1 \equiv c_1 u$**,**

$$-\Delta u_1 - \lambda \frac{u_1}{|x|^2} \geq c_1^{1-p} |\nabla u_1|^{p_+(\lambda)}.$$

We have to prove that $u_1 \geq w$.

Optimal power for nonexistence

Fixed c_0 satisfying $u(x) \geq \dfrac{c_0}{|x|^{\alpha_-}}$ in $|x| \leq \eta = e^{-1}$, chose $c_1 > 0$ such that $c_1 c_0 \geq 1$.

Then for a suitable small β we have:

- $c_1^{1-p_+(\lambda)} \geq ||h||_\infty \beta^{\frac{1}{2}}$.
- $u_1(x) \geq w(x)$ **for** $|x| = e^{-1}$ **and** $-\Delta u_1 - \lambda \dfrac{u_1}{|x|^2} \geq \beta^{\frac{1}{2}} h(x) |\nabla u_1|^{p_+(\lambda)}$.

CLAIM: $u_1 \geq w$. **If $v = w - u_1$ one can check that**

- $v \in W^{1,p_+(\lambda)}(B_\eta(0))$, $v \leq 0$ **on** $\partial B_\eta(0)$ **and**
$$\int_{B_\eta(0)} \frac{|v|}{|x|^{2+\alpha(-)}} dx < \infty, \quad \int_{B_\eta(0)} |\nabla v|^{p_+(\lambda)} |x|^{-\alpha(-)} dx < \infty.$$
- $-\Delta v - \lambda \dfrac{v}{|x|^2} \leq p_+(\lambda) h(x) \beta^{\frac{1}{2}} |\nabla w|^{p_+(\lambda)-2} \nabla w \nabla v \equiv a(x) \nabla v$ **where the vector field**
$$a(x) = -\beta^{\frac{1}{2}} p_+(\lambda) \frac{x}{|x|^2} \in L^q(B_\eta(0))$$ **for all $q < N$.**

  Notice that a is not in the hypothesis by Alaa-Pierre.
- **To overcame this lack of summability we start by applying the Kato's type inequality by Brezis-Ponce, then**

$$(1) \quad -\Delta v_+ - \lambda \frac{v_+}{|x|^2} + p_+(\lambda)\beta^{\frac{1}{2}} \langle \frac{x}{|x|^2}, \nabla v_+ \rangle \leq 0 \text{ and } \int_{B_\eta(0)} \frac{|\nabla v_+|^{p_+}}{|x|^{\alpha(-)}} dx < \infty.$$

Optimal power for nonexistence

Since $\frac{\alpha_{(-)}}{p_+(\lambda)} < \frac{N-2}{2}$, by Hardy-Sobolev inequality v_+ satisfies

$$\int_{B_\eta(0)} \frac{v_+^{p_+(\lambda)}}{|x|^{p_+(\lambda)+\alpha_{(-)}}}dx < \infty. \Rightarrow \exists \sigma_1 > 2+\alpha_{(-)}, \text{ such that } \int_{B_\eta(0)} \frac{v_+}{|x|^{\sigma_1}}dx < \infty.$$

For β small enough, $\gamma = \frac{\beta^{\frac{1}{2}} p_+(\lambda)}{2} < \dfrac{N-2}{2}$ and then the weight $|x|^{-2\gamma}$ is an admissible weight to have Caffarelli-Kohn-Nirenberg inequalities.

We consider the equivalent inequality,

$$-\textbf{div}(|x|^{-2\gamma}\nabla v_+) - \lambda\frac{v_+}{|x|^{2(\gamma+1)}} = |x|^{-2\gamma}\Big(- \Delta v_+ + p_+(\lambda)\langle \frac{x}{|x|^2}, \nabla v_+\rangle - \lambda \frac{v_+}{|x|^2}\Big) \leq 0.$$

The idea should be to use as a test function in (1), $\varphi = \dfrac{1}{|x|^a} - \dfrac{1}{\eta^a}$,

$a = \frac{N-2(\gamma+1)}{2} - \sqrt{\big(\frac{N-2(\gamma+1)}{2}\big)^2 - \lambda}$, the solution to problem

$$\begin{cases} -\textbf{div}(|x|^{-2\gamma}\nabla \varphi) - \lambda\dfrac{\varphi}{|x|^{2(\gamma+1)}} = \dfrac{1}{|x|^{2(\gamma+1)}} & \text{in } \quad B_\eta(0), \\ \varphi = 0 & \text{on } \quad \partial B_\eta(0), \end{cases}$$

Formally we reach the inequality $\displaystyle\int_{B_\eta(0))} \frac{v_+}{|x|^{2(1+\gamma)}}dx \leq 0$, hence $v_+ \equiv 0 \Leftrightarrow u_1 \geq w$.

As φ has not the required regularity we use an approximation argument.

Optimal power for nonexistence

To finish the proof in this case we use the same argument as in the first step. More precisely for all $\phi \in \mathcal{C}_0^\infty(B_r(0))$, $0 < r << \eta$ we have

$$c_1 \int_{B_r(0)} \frac{u_1 |\phi|^{p'_+}}{|x|^2} dx \leq \int_{B_r(0)} |\nabla \phi|^{p'_+} dx$$

where $c_1 > 0$ is independent of ϕ. Using the result of the claim and by the fact that $p'_+ = \alpha_{(-)} + 2$ we obtain that,

$$c_2 \int_{B_r(0)} \frac{|\phi|^{p'_+}}{|x|^{p'_+}} (\log(\frac{1}{|x|})^\beta dx \leq \int_{B_r(0)} |\nabla \phi|^{p'_+} dx$$

a contradiction with Hardy inequality in $W_0^{1,p'_+}(B_r(0))$. Hence the result follows.

Optimal power for nonexistence

Third step: $p = p_+(\lambda)$ **and** $\lambda = \Lambda_N$

In this case $\alpha_{(-)} = \frac{N-2}{2}$ **and** $p_+(\lambda) = \frac{N+2}{N}$**, hence** $u(x) \geq c|x|^{-\alpha_{(-)}}$ **and**

$$\int_{B_\eta(0)} |\nabla u|^{p_+(\lambda)} |x|^{-\alpha_{(-)}}\,dx < \infty.$$

We consider $\phi \in \mathcal{C}_0^\infty(B_\eta(0))$ **such that** $\phi \geq 0$ **and** $\phi = 1$ **in** $B_{\eta_1}(0)$**, then by the regularity of** u **we obtain** $\int_{B_\eta(0)} |\nabla(\phi u)|^{p_+(\lambda)} |x|^{-\alpha_{(-)}}\,dx < \infty$**. Since** $\frac{\alpha_{(-)}}{p_+(\lambda)} = \frac{N(N-2)}{2(N+2)} < \frac{N-2}{2}$**, we can apply Caffarelli-Kohn-Nirenberg inequalities to obtain that**

$$C_1 \int_{B_\eta(0)} (\phi u)^{p_+(\lambda)} |x|^{-\alpha_{(-)}}\,dx \leq \int_{B_\eta(0)} |\nabla(\phi u)|^{p_+(\lambda)} |x|^{-\alpha_{(-)}}\,dx < \infty.$$

$\int_{B_{\eta_1}(0)} u^{p_+(\lambda)} |x|^{-\alpha_{(-)}}\,dx < \infty$ **for some** $\eta_1 < \eta$

In particular,

$$\int_{B_{\eta_1}(0)} \frac{u^{p_+(\lambda)}}{|x|^{\alpha_{(-)}+p_+(\lambda)}}\,dx < \infty.$$

Using the fact that $u(x) \geq c|x|^{-\alpha_{(-)}}$ **there result that**

$$\int_{B_{\eta_1}(0)} \frac{|x|^{-\alpha_{(-)}p_+(\lambda)}}{|x|^{\alpha_{(-)}+p_+(\lambda)}}\,dx < \infty.$$

Since $\alpha_{(-)} + p_+(\lambda) + \alpha_{(-)}p_+(\lambda) = N$**, we reach a contradiction.**

End of the proof.

Some remarks

- Notice that $p_+(\lambda) < 2$ and
  - $p_+(\lambda) \to 2$ if $\lambda \to 0$
  - $p_+(\lambda) \to \frac{N+2}{N}$ if $\lambda \to \Lambda_N$.

  Therefore we find a discontinuity with the known result for $\lambda = 0$. (See for instance, Hansson-Maz'ya-Verbitsky paper).

- If $1 < p \leq \frac{N}{N-1}$, then there is not very weak positive solution in $\mathbb{R}^N$.

  By contradiction. Assume $1 < p \leq \frac{N}{N-1}$ and u a positive solution.

  By using the strong maximum principle, for any compact set $K \subset \Omega$ there exists a positive constant $c(K)$ such that $u \geq c(K)$. Let $\phi \in \mathcal{C}_0^\infty(\Omega)$, then using $|\phi|^{p'}$ as a test function and by using Young inequalities we obtain that

  $$\int_{\mathbb{R}^N} |\nabla \phi|^{p'} dx \geq c_1 \lambda \int_{\mathbb{R}^N} \frac{u}{|x|^2} |\phi|^{p'} dx.$$

  Since $p' > N$, then $Cap_{1,p'}(K) = 0$ for any compact set of $\mathbb{R}^N$.

  Thus, $\exists \{\phi_n\} \subset \mathcal{C}_0^\infty(\mathbb{R}^N)$ such that $\phi \geq \chi_K$ and $||\nabla \phi_n||_{L^{p'}(\mathbb{R}^N)} \to 0$ as $n \to \infty$.

  Hence by substituting in the last inequality we reach a contradiction. (See Alaa-Pierre).

- In bounded domains there are no restriction on p from below.

Complete blow-up

As a consequence of the non existence result, the following blow-up behavior for approximated problems could be obtained.

THEOREM. Assume that $p \geq p_+(\lambda)$. If $u_n \in W_0^{1,p}(\Omega)$ is a solution to problem

$$\begin{cases} -\Delta u_n = |\nabla u_n|^p + \lambda a_n(x) u_n + \alpha f & \text{in } \Omega, \\ u_n > 0 & \text{in } \Omega, \\ u_n = 0 & \text{on } \partial\Omega, \end{cases}$$

with $f \geq 0$, $f \not\equiv 0$ and $a_n(x) = \dfrac{1}{|x|^2 + \frac{1}{n}}$, then $u_n(x_0) \to \infty, \forall x_0 \in \Omega$.

Idea of the proof. If in some point the limit is finite, Harnack inequality provide an estimate that allow us to construct a *local solution* in contradiction to the nonexistence theorem.

The existence of such solution requires the following result.

LEMMA. Assume that $\{u_n\}$ is a sequence of positive functions such that $\{u_n\}$ is uniformly bounded in $W^{1,p}_{loc}(\Omega)$ for some $2 \geq p > 1$ with $u_n \rightharpoonup u$ weakly in $W^{1,p}_{loc}(\Omega)$ and such that $u_n \leq u$ for all $n \in \mathbb{N}$. Assume that $-\Delta u_n \geq 0$ in $\mathcal{D}'(\Omega)$ and that, if $p < 2$, sequence $\{T_k(u_n)\}$ is uniformly bounded in $W^{1,2}_{loc}(\Omega)$ for k fixed. Then $\nabla T_k(u_n) \to \nabla T_k(u)$ strongly in $(L^2_{loc}(\Omega))^N$.

Existence in $\mathbb{R}^N$: $p_-(\lambda) < p < p_+(\lambda)$ and $\lambda < \Lambda_N$

For $\alpha_{(+)}$ and $\alpha_{(-)}$ as above, consider

$$p_-(\lambda) \equiv \frac{2+\alpha_{(+)}}{1+\alpha_{(+)}} \qquad \text{and} \qquad p_+(\lambda) \equiv \frac{2+\alpha_{(-)}}{1+\alpha_{(-)}}.$$

THEOREM A. Assume that $p_-(\lambda) < p < p_+(\lambda)$ then

$$-\Delta u = |\nabla u|^p + \lambda \frac{u}{|x|^2}$$

has a very weak solution $u > 0$ in $\mathbb{R}^N$.

Proof. We search a solution in the form $u(x) = A|x|^{-\beta}$.

By a direct computation we obtain that $\beta = \dfrac{2-p}{p-1}$ and

$$\beta^p A^{p-1} = \beta(N-\beta-2) - \lambda.$$

To have $A > 0$ we need $\beta \in (\alpha_{(-)}, \alpha_{(+)})$ which is equivalent to $p_-(\lambda) < p < p_+(\lambda)$.

Notice that

$$u, \frac{u}{|x|^2} \in L^1_{loc}(\mathbb{R}^N) \text{ and since } p > p_-(\lambda) > \frac{N}{N-1}, \quad |\nabla u|^p \in L^1_{loc}(\mathbb{R}^N) \quad \square$$

Remark. The solution u in Theorem A is in the space $W^{1,2}_{loc}(\mathbb{R}^N)$ if and only if $p > \frac{N+2}{N}$.

For all $\lambda \in [0, \Lambda_N)$, $\quad \frac{N+2}{N} \in (p_-(\lambda), p_+(\lambda))$

If $\lambda = \Lambda_N$ then $\frac{N+2}{N} = p_-(\lambda) = p_+(\lambda)$.

Existence Dirichlet Problem: $1 < p < p_+(\lambda)$ and $\lambda < \Lambda_N$

To find solution to Dirichlet problem:

1. **Is needed a supersolution and then comparison arguments as in**
   N.E. Alaa, M. Pierre, SIAM J. Math. Anal. Vol 24 no. 1 (1993), 23-35.
2. **The datum must be** ***small in some class of functions,*** **as in the case $\lambda = 0$**

The precise statement is the next.

THEOREM B. Assume that $1 < p < p_+(\lambda)$. There exist c_0 such that if $c < c_0$ and $f(x) \le \dfrac{1}{|x|^2}$, then problem

$$\begin{cases} -\Delta w = |\nabla w|^p + \lambda \dfrac{w}{|x|^2} + c\, f \text{ in } \Omega, \\ w = 0 \text{ on } \partial\Omega, \end{cases}$$

has a positive solution $w \in W_0^{1,2}(\Omega)$.

Outline of the proof.

Assume $\overline{w} \in W_0^{1,p}(\Omega)$ is a positive super-solution for the data $f(x) \equiv \frac{1}{|x|^2}$ and c small.

Consider $a_n(x) = \frac{1}{|x|^2 + \frac{1}{n}} \uparrow |x|^{-2}$, $f_n = \min\{f, n\} \uparrow f$. and problem,

$$(TP) \quad \begin{cases} -\Delta v_n = \lambda a_n(x) v_n + \dfrac{|\nabla v_n|^p}{1 + \frac{1}{n}|\nabla v_n|^p} + c f_n \text{ in } \Omega, \\ v_k = 0 \text{ on } \partial\Omega, \end{cases}$$

Existence Dirichlet Problem: $1 < p < p_+(\lambda)$ and $\lambda < \Lambda_N$

By classical theory (TP) has a unique positive solution $v_n \in W_0^{1,2}(\Omega) \cap L^\infty(\Omega)$.

Moreover by the comparison principle in Alaa-Pierre paper,

$$v_n \le v_{n+1} \text{ and } v_n \le \overline{w}, \ \forall n$$

Hence $\overline{v} = \lim_{n\to\infty} v_n \le w$.

Take as test function $\phi_n = (1+v_n)^s - 1$, $0 < s < \frac{p(N-1)-N}{2-p} < 1$,

$$\int_\Omega \frac{|\nabla v_n|^2}{(1+v_n)^{1-s}} dx \le C_1, \quad \int_\Omega |\nabla v_n|^p (1+v_n)^s dx \le C_2,$$

Therefore, in particular

$$\frac{1}{k}\int_\Omega |\nabla T_k v_n|^2 \le C_3, \quad \int_\Omega |\nabla v_n|^p \le C_4.$$

Then by using $\phi(T_k v_n - T_k v)$ as a test function, where $\phi(s) = s\exp^{\frac{1}{4}s^2}$, and the convergence arguments by Boccardo-Gallouët-Orsina we obtain that

$$\nabla T_k v_n \to \nabla T_k v \text{ as } n \to \infty \text{ strongly in } W_0^{1,2}(\Omega).$$

With the test function $\psi_n = (1 + G_k(v_n))^s - 1$, $G_k(t) = t - T_k(t)$, we prove

$$\limsup_{k\to\infty} \int_{v_n \ge k} |\nabla v_n|^p dx \le \limsup_{k\to\infty} \int_\Omega |\nabla G_k(v_n)|^p (1 + G_k(v_n))^s dx = 0,$$

uniformly in n.

By Vitali Lemma $\nabla v_n \to \nabla u, \quad n \to \infty,$ strongly in $L^p(\Omega)$.

Hence u is a very weak solution to problem

If the super-solution has finite energy the arguments are easier.

Existence Dirichlet Problem: $1 < p < p_+(\lambda)$ and $\lambda < \Lambda_N$

The construction of the super-solution is performed in two steps

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- $i)\ p_- < p < p_+(\lambda)$.

  Consider ς, the solution to

$$\begin{cases} -\Delta\varsigma = 0 \text{ in } \Omega, \\ \varsigma = u \text{ on } \partial\Omega, \end{cases}$$

  where u is the radial solution obtained in Theorem A.

  Then $\varsigma \in \mathcal{C}^\infty(\Omega)$ and $0 < c_1 \leq \varsigma \leq c_2$.

  One can check that for t small enough $\overline{w} = t(u - \varsigma)$, is a super-solution. Notice that $\overline{w} \in W_0^{1,p}(\Omega)$, $\overline{w} \geq 0$ in Ω.

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- $ii)\ 1 < p \le p_-$. We start by getting a super-solution in $\Omega = B_R(0)$. For general Ω we perform the same arguments as in the first case using the super-solution in a big ball.
  Since $p \le p_-$, $\exists \beta \in (\alpha_{(-)}, \alpha_{(+)})$, close to $\alpha_{(-)}$ and such that $p(\beta + 1) < \beta + 2$.
  Define $\overline{w}(x) \equiv A(|x|^{-\beta} - R^{-\beta})$ with β close to $\alpha_{(-)}$, then $\overline{w} \in W_0^{1,2}(B_R(0))$ and
  $$-\Delta\overline{w} - \lambda\frac{\overline{w}}{|x|^2} = A(\beta(N - \beta - 2) - \lambda)|x|^{-\beta-2} + \frac{A}{|x|^2}.$$
  Since $\beta \in (\alpha_{(-)}, \alpha_{(+)})$, then $\beta(N-\beta-2)-\lambda > 0$, hence if $A^{p-1} = \dfrac{\beta(N-\beta-2)-\lambda}{\beta^p}$,
  $$-\Delta\overline{w} - \lambda\frac{\overline{w}}{|x|^2} \ge |\nabla\overline{w}|^p + \frac{A}{|x|^2}.$$
  So, if $c_0 = A$, $\overline{w} \in W_0^{1,2}(B_R(0))$. is a super-solution in $B_R(0)$ for all $c < c_0$.

Existence Dirichlet Problem: $\lambda \equiv \Lambda_n$ and $p < \frac{N+2}{N}$

This critical case is more involved. As above we find a super-solution in a ball

- **Consider** $w(x) = \left|\frac{x}{r}\right|^{-\frac{N-2}{2}} \left(\log(\frac{r}{|x|})\right)^{1/2}$.
  $w \in W_0^{1,q}(B_r(0))$ **for all** $q < 2$.
- **For suitable positive constant** c_1**,** $c_1 w$ **is a super-solution in the ball** $B_R(0)$ **to**

$$\begin{cases} -\Delta w &= |\nabla w|^p + \Lambda_N \dfrac{w}{|x|^2} + c_0 f \text{ in } B_1(r), \\ w &= 0 \text{ on } \partial B_r(0). \end{cases}$$

where $|x|^2 f$ **is bounded and** c_0 **is small**

- **The natural framework to find the solution is the Hilbert space** H**, complection of** $\mathcal{C}_0^\infty(B_r(0))$ **respect to the norm**

$$||\phi||^2_{H(B_r(0))} = \int_{B_r(0)} |\nabla \phi|^2 dx - \Lambda_N \int_{B_r(0)} \frac{\phi^2}{|x|^2} dx.$$

In fact we find a solution is such space.

(We avoid technical details).

Breaking down the resonance: existence for all $\lambda > 0$

Consider

$$(PA) \quad \begin{cases} -\Delta u + |\nabla u|^q = \lambda\, g(x) u + f & \text{in } \Omega, \\ u \geq 0 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

where $1 \leq q \leq 2$, $\lambda \in \mathbb{R}$ and $f \in L^m(\Omega)$ with $m \geq 1$. We will assume that g is an *admissible weight* in the sense that the

$$(H1) \quad g \geq 0 \text{ and } g \in L^1(\Omega) \cap W^{-1,q'}(\Omega) \quad q' = \frac{q}{q-1}.$$

Call

$$\lambda_1(g,q) = \inf_{\phi \in W_0^{1,q}(\Omega)\setminus\{0\}} \frac{\left(\int_\Omega |\nabla \phi|^q \, dx\right)^{\frac{1}{q}}}{\int_\Omega g|\phi| \, dx} > 0.$$

Examples.

- $g \in L^m(\Omega)$ **with** $m > \frac{N}{q}$.
- $g(x) \equiv \dfrac{1}{|x|^2}$**, the Hardy potential and** $q > \dfrac{N}{N-1}$.

Existence of solutions for all $\lambda > 0$

The main result is the following.

THEOREM. **Assume $1 < q \leq 2$, $f \in L^1(\Omega)$ and the hypothesis (H1) holds for g, then there exists $u \in W_0^{1,q}(\Omega)$ a weak solution to problem (PA) for all parameter $\lambda > 0$.**

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The proof is done in three steps.

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- f and g in L^r with $r > \frac{N}{q}$.
- $f \in L^1(\Omega)$ and g in L^r with $r > \frac{N}{q}$.

Existence of solutions for all $\lambda > 0$

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The proof is done in three steps.

- f and g in L^r with $r > \frac{N}{q}$.
- $f \in L^1(\Omega)$ and g in L^r with $r > \frac{N}{q}$.
- $f \in L^1(\Omega)$ and g verifying $(H1)$.

First step: f and g in L^r with $r > \frac{N}{q}$

THEOREM a. **Assume that $f, g \in L^r(\Omega)$, with $r > \frac{N}{q}$, are positive functions, then for all $\lambda > 0$ there exists $u \in W_0^{1,2}(\Omega) \cap L^\infty(\Omega)$ a positive weak solution to problem (PA).**

Outline of the proof.

(I) For every fixed $k > 0$ consider $v \in W_0^{1,2}(\Omega) \cap L^\infty(\Omega)$ such that $-\Delta v = \lambda k g(x) + f$ in Ω and denote $M = \|v\|_{L^\infty}$. Then zero is a subsolution and v is a supersolution to problems

$$(PT_n) \quad \begin{cases} w_0 = 0, \\ -\Delta w_n + \dfrac{|\nabla w_n|^q}{1 + \frac{1}{n}|\nabla w_n|^q} = \lambda g(x) T_k w_{n-1} + f, \\ w_n \in W_0^{1,2}(\Omega), \end{cases}$$

for all $n \in \mathbb{N}$. As a consequence of the arguments in Boccardo-Murat-Puel, we find a sequence of nonnegative solutions $\{w_n\}$ to problems (PT_n).

It follows that $-\Delta w_n \le \lambda k g(x) + f = -\Delta v$, so by weak comparison principle, we conclude that $0 \le w_n \le v \le M$, uniformly in n, then in particular, $w_n \in W_0^{1,2}(\Omega) \cap L^\infty(\Omega)$.

First step: f and g in L^r with $r > \frac{N}{2}$

Call $H_n(\nabla w_n) = \dfrac{|\nabla w_n|^q}{1 + \frac{1}{n}|\nabla w_n|^q}$.

Take w_n as a test function in (PT_n),

$$\int_\Omega |\nabla w_n|^2\,dx + \int_\Omega H_n(\nabla w_n)\,w_n\,dx = \lambda \int_\Omega gT_k w_{n-1} w_n\,dx + \int_\Omega f\,w_n\,dx$$

Applying Poincaré and Young's inequality we obtain a positive constant $C(k, g, f, \Omega)$ such that

$$\alpha \int_\Omega |\nabla w_n|^2\,dx \le C(k, g, f, \Omega),$$

therefore $w_n \rightharpoonup u_k$ weakly in $W_0^{1,2}(\Omega)$ with $u_k \in W_0^{1,2} \cap L^\infty(\Omega)$ and $u_k \le M$.

First step: f and g in L^r with $r > \frac{N}{q}$

Convergence claim.- $w_n \to u_k$ **strongly in** $W_0^{1,2}(\Omega)$.

Outline of the proof of the convergence claim.-

Since $q \le 2\ \forall \epsilon \le 1$ **there exists** $C_\epsilon > 0$ **such that**

$$s^q \le \epsilon s^2 + C_\epsilon, \quad s \ge 0.$$

Let $\phi(s) = s \exp^{\frac{1}{4}s^2}$**, which verifies** $\phi'(s) - |\phi(s)| \ge \dfrac{1}{2}$.

Take $\phi(w_n - u_k)$ **as test function in** (PT_n) **and using the same kind of arguments that in Boccardo-Gallouët-Orsina. we obtain that**

$$\frac{1}{2}\int_\Omega |\nabla w_n - \nabla u_k|^2\, dx \le \int_\Omega (\phi'(w_n - u_k) - \epsilon|\phi(w_n - u_k)|)|\nabla w_n - \nabla u_k|^2\, dx \le o(1),$$

whence $w_n \to u_k$ **in** $W_0^{1,2}(\Omega)$.

In particular

$$H_n(\nabla w_n) \to |\nabla u_k|^q \quad \text{in} \quad L^1(\Omega).$$

Therefore

$$(AP1) \quad -\Delta u_k + |\nabla u_k|^q = \lambda g(x)\, T_k u_k + f \quad \text{in} \quad \Omega, \quad u_k \in W_0^{1,2}(\Omega).$$

First step: f and g in L^m with $r > \frac{N}{q}$

(II) Taking $T_m u_k$ as test function in $(AP1)$,

$$\int_\Omega |\nabla T_m u_k|^2\,dx + \int_\Omega |\nabla \Psi_m u_k|^q\,dx \le \lambda \int_\Omega g(x) T_m u_k u_k\,dx + \int_\Omega f\,T_m u_k\,dx$$

$$\le m\epsilon\lambda \left(\int_\Omega g(x) u_k\,dx\right)^q + \lambda m C(\epsilon) + C(\overline{\epsilon}) \|f\|^2_{L^{\frac{N}{2}}} + \overline{\epsilon}\,|\Omega|\,m^{\frac{2N}{N-2}}$$

$$\le \frac{\epsilon m \lambda}{C(q,g)} \int_\Omega |\nabla u_k|^q\,dx + C(\epsilon, \overline{\epsilon}, \lambda, \Omega, m, f).$$

where

$$\Psi_m(s) = \int_0^s T_m(t)^{\frac{1}{q}}\,dt$$

Since

$$\int_\Omega |\nabla \Psi_m u_k|^q\,dx \ge \int_{\{u_k \ge m\}} |\nabla \Psi_m u_k|^q\,dx \ge m \int_{\{u_n \ge m\}} |\nabla u_k|^q\,dx,$$

then

$$\int_\Omega |\nabla u_k|^q\,dx \le \int_\Omega |\nabla T_m u_k|^2\,dx + m \int_{\{u_k \ge m\}} |\nabla u_k|^q\,dx \le$$
$$\frac{\epsilon m \lambda}{C(q,g)} \int_\Omega |\nabla u_k|^q\,dx + C(\epsilon, \overline{\epsilon}, \lambda, \Omega, m, f).$$

First step: f and g in L^r with $r > \frac{N}{q}$

Fixed $m \geq 1$, and choosing ϵ small enough we conclude that

$$u_k \rightharpoonup u \text{ weakly} \quad \text{in} \quad W_0^{1,q}(\Omega).$$

Since $f,\ g \in L^r(\Omega)$ with $r > \frac{N}{q}$, the sequence $\{u_k\}$ is uniformly bounded in $L^\infty(\Omega)$, so

$$u_k \rightharpoonup u \quad \text{in} \quad W_0^{1,2}(\Omega) \text{ with } u \in W_0^{1,2}(\Omega) \cap L^\infty(\Omega).$$

To finish the proof we use the same arguments as in the *convergence claim* to obtain

$$u_k \to u \quad \text{in} \quad W_0^{1,2}(\Omega).$$

Then u is a positive solution to (PA).

Second step $g \in L^r$, $r > \frac{N}{q}$, $f \in L^1$

We will use the following elementary lemma.

Lemma. $\forall \epsilon > 0,\ \forall k > 0,\ \exists C_\epsilon$ such that

$$s\,T_k(s) \leq \epsilon \Psi_k^q(s) + C_\epsilon, \quad s \geq 0$$

being $\Psi_k(s) = \int_0^s T_k(t)^{\frac{1}{q}}\,dt$

Notice that

$$\Psi_k(s) = \begin{cases} \frac{q}{q+1} s^{\frac{q+1}{q}} & \text{if} \quad s < k, \\ \frac{q}{q+1} k^{\frac{q+1}{q}} + (s-k)k^{\frac{1}{q}} & \text{if} \quad s > k. \end{cases}$$

We will prove the next result.

Theorem b. Assume that $f \in L^1(\Omega)$ and $g \in L^r(\Omega)$ with $r > \frac{N}{q}$, then for all $\lambda \in \mathbb{R}$, problem (PA) has a positive solution $u \in W_0^{1,q}(\Omega)$.

Second step $g \in L^r$, $r > \frac{N}{q}$, $f \in L^1$

Outline of the proof. Consider a sequence $f_n \in L^\infty(\Omega)$ such that $f_n \uparrow f$ in $L^1(\Omega)$.
By Theorem a of step 1, $\exists \{u_n\}_{n\in\mathbb{N}}$, solutions to problems

$$(PT) \begin{cases} -\Delta u_n + |\nabla u_n|^q = \lambda\, g(x)\, u_n + f_n & \text{in } \Omega, \\ u_n > 0 & \text{in } \Omega, \\ u_n = 0 & \text{on } \partial\Omega. \end{cases}$$

Take $T_k u_n$ as test function in (PT), then

$$\int_\Omega |\nabla T_k u_n|^2\, dx + \int_\Omega |\nabla u_n|^q T_k u_n\, dx = \lambda \int_\Omega g(x)\, u_n T_k u_n dx + \int_\Omega f_n T_k u_n\, dx.$$

By Poincaré and Young inequalities, if $0 < \epsilon << \dfrac{\lambda_1(g,q)}{\lambda}$, $\exists C_\epsilon > 0$

$$\int_\Omega |\nabla T_k u_n|^2\, dx + \beta \int_\Omega |\nabla \Psi_k u_n|^q\, dx \le \lambda\, C'(g,\Omega,\epsilon) + k\|f_n\|_{L^1}.$$

Second step $g \in L^r$, $r > \frac{N}{q}$, $f \in L^1$

Then for every $k > 0$,

$$\int_\Omega |\nabla T_k u_n|^2 \quad \leq \quad C(\lambda, \epsilon, \Omega, f, k) \text{ uniformly in } n \in \mathbb{N},$$

$$\int_\Omega |\nabla \Psi_k u_n|^q \quad \leq \quad C(\lambda, \epsilon, \Omega, f, k) \text{ uniformly in } n \in \mathbb{N}.$$

Using the definition of Ψ_k, we conclude that $\exists u \in W_0^{1,q}(\Omega)$ such that $u_n \rightharpoonup u$ weakly in $W_0^{1,q}(\Omega)$.

Since $\{u_n\}$ is uniformly bounded in $L^p(\Omega), \forall p < q^*$, uniformly in n we have,

$$(**) \begin{cases} |\{x \in \Omega, \text{ such that } k-1 < u_n(x) < k\}| \to 0, \text{ as } k \to \infty \\ |\{x \in \Omega, \text{ such that } u_n(x) > k\}| \to 0 \text{ as } k \to \infty. \end{cases}$$

Consider $G_k(s) = s - T_k(s)$ and $\psi_{k-1}(s) = T_1(G_{k-1}(s))$.

Notice that $\psi_{k-1}(u_n)|\nabla u_n|^q \geq |\nabla u_n|^q \chi_{\{u_n \geq k\}}$

Second step $g \in L^r$, $r > \frac{N}{q}$, $f \in L^1$

Claim. $u_n \to u$ strongly in $W_0^{1,q}(\Omega)$.

- Use $\psi_{k-1}(u_n)$ as test function in (PT), then

$$\int_\Omega |\nabla \psi_{k-1}(u_n)|^2\, dx + \int_\Omega \psi_{k-1}(u_n)|\nabla u_n|^q\, dx = \int_\Omega (\lambda g(x) u_n + f_n)\psi_{k-1}(u_n) dx.$$

  And then

$$(***) \quad \limsup_{k\to\infty} \int_{\{u_n \geq k\}} |\nabla u_n|^q\, dx \leq \limsup_{k\to\infty} \int_{\{u_n > (k-1)\}} (\lambda g(x) u_n + f_n) dx = 0$$

  by using also $(**)$ in the right hand side,

- Next we prove that $T_k u_n \to T_k u$ in $W_0^{1,2}(\Omega)$.
  Take $\phi(T_k u_n - T_k u)$ as a test function in (PT) with $\phi(s) = s \exp^{\frac{1}{4}s^2}$.
  Notice that $\phi(T_k u_n - T_k u) \to 0$ strongly in $L^p(\Omega)$, $p \geq 1$. Then

$$\int_\Omega (\lambda g(x) u_n + f_n)\phi(T_k u_n - T_k u)\, dx \to 0 \text{ as } n \to \infty.$$

Second step $g \in L^r$, $r > \frac{N}{q}$, $f \in L^1$

Using the same computation as in the *convergence claim* in the proof of Theorem of first step, we conclude $T_k u_n \to T_k u$ strongly in $W_0^{1,2}(\Omega)$.

To finish the proof, it is sufficient to show that

$$|\nabla u_n|^q \to |\nabla u|^q \quad \text{strongly in } L^1(\Omega).$$

Since the sequence converges a.e. in Ω, by Vitali's theorem it is sufficient to check the equi-integrability. Consider $E \subset \Omega$ a measurable set, then,

$$\int_E |\nabla u_n|^q \, dx \le \int_E |\nabla T_k u_n|^q \, dx + \int_{\{u_n \ge k\} \cap E} |\nabla u_n|^q \, dx.$$

For every $k > 0$, one has that $T_k(u_n) \to T_k(u)$ strongly in $W_0^{1,2}(\Omega)(\Omega)$, therefore the integral $\int_E |\nabla T_k(u_n)|^q \, dx$ is uniformly small if $|E|$ is small enough. By $(***)$

$$\int_{\{u_n \ge k\} \cap E} |\nabla u_n|^q \, dx \le \int_{\{u_n \ge k\}} |\nabla u_n|^q \, dx \to 0 \text{ as } k \to \infty \text{ uniformly in } n.$$

The equintegrability of $|\nabla u_n|^q$ follows immediately.

Final step general weight g

We assume that $f \in L^1(\Omega)$, g verifies (D). Consider $g_n(x) = \min\{g(x), n\} \in L^\infty(\Omega)$.
By Theorem b above, $\exists \{u_n\}_{n\in\mathbb{N}}$, $u_n \geq 0$, solutions to problems

$$(PA_n) \quad \begin{cases} -\Delta u_n + |\nabla u_n|^q &= \lambda\, g_n(x)\, u_n + f \quad \text{in} \quad \Omega, \\ u_n &> 0 \quad \text{in} \quad \Omega, \\ u_n &= 0 \quad \text{on} \quad \partial\Omega. \end{cases}$$

Consider $T_k u_n \in W_0^{1,q}(\Omega) \cap L^\infty(\Omega)$ as test function,

$$\int_\Omega |\nabla T_k u_n|^2\, dx + \int_\Omega |\nabla \Psi_k u_n|^q\, dx \leq k\lambda \int_\Omega g_n(x) u_n\, dx + k \int_\Omega f\, dx.$$

Since

$$\int_\Omega |\nabla \Psi_k u_n|^q\, dx \geq \int_{\{u_n \geq k\}} |\nabla \Psi_k u_n|^q\, dx \geq k \int_{\{u_n \geq k\}} |\nabla u|^q\, dx,$$

then as above

$$\int_\Omega |\nabla T_k(u_n)|^2\, dx + k \int_{\{u_n \geq k\}} |\nabla u_n|^q\, dx \leq k\epsilon\lambda \left(\int_\Omega g_n(x) u_n\, dx \right)^q + k \int_\Omega f\, dx + \lambda k C(\epsilon, \Omega).$$

And

$$\int_\Omega |\nabla u_n|^q\, dx \leq \frac{k\epsilon\lambda}{C(g,q)} \int_\Omega |\nabla u_n|^q\, dx + k \int_\Omega f\, dx + \lambda k C(\epsilon, \Omega),$$

Final step general weight g

Hence $u_n \rightharpoonup u$ weakly in $W_0^{1,q}(\Omega)$.

Using the hypothesis on g it follows that $g_n(x)u_n \to g(x)u$ strongly in $L^1(\Omega)$.

Moreover, to prove that

$$u_n \to u \text{ strongly in } W_0^{1,q}(\Omega).$$

we take again $\phi(T_k u_n - T_k u)$, with $\phi(s) = s\exp^{\frac{1}{4}s^2}$ as test function in (PA_n).

The same arguments as in the *convergence claim* give the strong convergence and allow us to conclude the proof of the main Theorem.

COROLLARY

1. Assume that $g \in L^m(\Omega)$ with $m \geq \frac{qN}{(q-1)N+1}$, then for all $f \in L^1(\Omega)$ and $\lambda \geq 0$, problem (PA) has a positive solution $u \in W_0^{1,q}(\Omega)$ in the distributional sense.
2. Define

$$\lambda_1(g,q) = \inf_{\phi \in W_0^{1,q}(\Omega)\setminus\{0\}} \frac{\displaystyle\int_\Omega |\nabla\phi|^q\,dx}{\displaystyle\int_\Omega g|\phi|^q\,dx},$$

then if $\lambda_1(g,q) > 0$, it follows that $C(g,q) > 0$ and then problem (PA) has a positive solution $u \in W_0^{1,q}(\Omega)$ for all $f \in L^1(\Omega)$ and $\lambda \geq 0$.

Some remarks

1. The existence result obtained means that resonance phenomenon can not occurs if we add $|\nabla u|^q$ as an absorption term. Without the presence of this term, positive solution exists just by assuming that λ is less than the infimum of the spectrum of the operator $-\Delta$ with the corresponding weight and under a suitable condition of f.
2. The same existence result holds if f is a bounded positive Radon measure such that $f \in L^1(\Omega) + W^{-1,2}(\Omega)$, ($f$ is absolutely continuous respect to capacity). In this case, the solution means a renormalized solution.
   The result follows using the same approximation arguments.
3. By the classical regularity theory of renormalized solution we get easily that if u is a positive solution to problem (PA), then $u \in W_0^{1,q}(\Omega) \cap W_0^{1,p}(\Omega)$ for all $p < \frac{N}{N-1}$.

Optimality of the results: Hardy Potential

Consider the problem

$$(PH)\quad \begin{cases} -\Delta u + |\nabla u|^q &= \lambda \dfrac{u}{|x|^2} + f \quad \text{in} \quad \Omega, \\ u &> 0 \quad \text{in} \quad \Omega, \\ u &= 0 \quad \text{on} \quad \partial\Omega. \end{cases}$$

Hardy potential is an admissible weight if $2 \geq q > \frac{N}{N-1}$.

Hence in this interval of values of q we have the main existence theorem.

Hardy potential, $g(x) \equiv \dfrac{1}{|x|^2}$, verifies,

$$(H2) \quad g \geq 0 \text{ and } g \in L^1(\Omega) \text{ with } \lambda_1(g,2) = \inf_{\phi \in W_0^{1,2}(\Omega)\setminus\{0\}} \frac{\displaystyle\int_\Omega |\nabla \phi|^2\, dx}{\displaystyle\int_\Omega g|\phi|^2\, dx} > 0.$$

In fact, $\lambda_1(g,2) = (\dfrac{N-2}{2})^2$.

It is easy to check that by $(H2)$, for all $\overline{\lambda} < \lambda_1(g,2)$, there exists a unique $\varphi \in W_0^{1,2}(\Omega)$, $\varphi > 0$ weak solution to problem

$$(AuX) \qquad -\Delta\varphi = \overline{\lambda} g(x)\varphi + g(x) \quad \text{in} \quad \Omega, \qquad \varphi = 0 \quad \text{on} \quad \partial\Omega.$$

Optimality of the results: Hardy Potential.

The first result is the following one.

THEOREM. Assume that $0 < \lambda < (\frac{N-2}{2})^2$ and $1 < q \le 2$, let φ be the solution to problem (AuX). Suppose f is a positive function such that $\int_\Omega f\varphi\, dx < \infty$, then there exists u solution to (PH) such that $\int_\Omega |\nabla u|^q\, dx < \infty$ and $\int_\Omega |\nabla u|^p\, dx < \infty, \forall p < \frac{N}{N-1}$.

- If $q > \frac{N}{N-1}$ then the result holds for all $f \in L^1(\Omega)$
- The new feature is that for $1 < q \le \frac{N}{N-1}$ the existence requires some extra summability on f.

We will see that for $\lambda > (\frac{N-2}{2})^2$ and $1 < q \le \frac{N}{N-1}$ there in not solution.

Optimality of the results: Hardy Potential.

THEOREM. Assume that $q < q_2 \equiv \frac{N}{N-1}$, if $\lambda > \Lambda_N = \dfrac{(N-2)^2}{4}$, then problem (PH) has no positive very weak positive supersolution in the sense that $v, \dfrac{v}{|x|^2}, |\nabla v|^q \in L^1_{loc}(\Omega)$ and

$$\int \Big(v(-\Delta \phi) + |\nabla v|^q \phi \Big) dx \geq \lambda \int \frac{v\,\phi}{|x|^2} dx + \int f \phi dx,$$

for all $\phi \in \mathcal{C}_0^\infty(\Omega)$.

Outline of the proof. By contradiction suppose that problem (PH) has a positive solution v for some $\lambda > \Lambda_N$

Then by iteration we could construct $u \in W_0^{1,p}(B_\eta(0))$ for all $p < \frac{N}{N-1}$ and $u \in L^m(B_\eta(0))$ for all $m < \frac{N}{N-2}$. We will choose $\eta > 0$ below

For $\phi \in \mathcal{C}_0^\infty(B_\eta(0))$ consider $\dfrac{\phi^2}{u}$ as test function in (PH), then

$$-\int_{B_\eta(0)} \frac{|\nabla u|^2 \phi^2}{u^2} dx + 2\int_{B_\eta(0)} \frac{\phi \nabla \phi}{u} \nabla u \, dx + \int_{B_\eta(0)} \frac{|\nabla u|^q \phi^2}{u} dx \geq \lambda \int_{B_\eta(0)} \frac{\phi^2}{|x|^2} dx.$$

Direct computation provides

$$\int_{B_\eta(0)} \frac{|\nabla u|^q \phi^2}{u} dx \leq \frac{q}{2} \epsilon_0^{\frac{2}{q}} \int_{B_\eta(0)} \frac{|\nabla u|^2}{u^2} \phi^2 \, dx + \frac{2-q}{2} \epsilon_0^{-\frac{2}{2-q}} \int_{B_\eta(0)} u^{\frac{2(q-1)}{2-q}} \phi^2 \, dx$$

ϵ_0 is a positive number to be chosen later.

Optimality of the results: Hardy Potential.

On the other hand we have

$$2\int_{B_\eta(0)} \frac{\phi\nabla\phi}{u}\nabla u\,dx \le \epsilon_1^2\int_{B_\eta(0)} \frac{\phi^2|\nabla u|^2}{u^2}\,dx + \epsilon_1^{-2}\int_{B_\eta(0)} |\nabla\phi|^2\,dx.$$

Hence it follows that fixed $\epsilon_1^2\lambda > \Lambda_N$ and $\epsilon_0 > 0$ small enough such that $(1-\epsilon_1^2-\frac{q}{2}\epsilon_0^{\frac{2}{q}}) \ge 0$,

$$\epsilon_1^2\lambda\int_{B_\eta(0)} \frac{\phi^2}{|x|^2}\,dx \le \epsilon_1^2\frac{2-q}{2}\epsilon_0^{-\frac{2}{2-q}}\int_{B_\eta(0)} u^{\frac{2(q-1)}{2-q}}\phi^2\,dx + \int_{B_\eta(0)} |\nabla\phi|^2\,dx.$$

Now,

$$\int_{B_\eta(0)} u^{\frac{2(q-1)}{2-q}}\phi^2\,dx \le S^{-1}\Big(\int_{B_\eta(0)} u^{\frac{N(q-1)}{2-q}}\,dx\Big)^{\frac{2}{N}}\int_{B_\eta(0)} |\nabla\phi|^2\,dx.$$

where S is the classical Sobolev constant. Since $q < \frac{N}{N-1}$, $\frac{N(q-1)}{2-q} < \frac{N}{N-2}$ hence we conclude that

$$\int_{B_\eta(0)} u^{\frac{N(q-1)}{2-q}}\,dx \to 0 \text{ as } \eta \to 0.$$

Then we can fix $\eta > 0,\ \epsilon_0,\ \epsilon_1 > 1$ such that

$$\epsilon_1^2\lambda\Big\{1+\epsilon_1^2\frac{2-q}{2}\epsilon_0^{-\frac{2}{2-q}}S^{-1}\Big(\int_{B_\eta(0)} u^{\frac{N(q-1)}{2-q}}\,dx\Big)^{\frac{2}{N}}\Big\}^{-1} \equiv \lambda_1 > \Lambda_N.$$

Therefore we conclude that

$$\lambda_1\int_{B_\eta(0)} \frac{\phi^2}{|x|^2}\,dx \le \int_{B_\eta(0)} |\nabla\phi|^2\,dx,$$

a contradiction with Hardy inequality.