6. Bifurcation and Perturbation

Here we deal with perturbed problems, variational in nature, whose solutions are critical points of a functional

$$I_\varepsilon(u) = I_0(u) + \varepsilon G(u)$$

We assume that there is a smooth d-dimensional manifold Z such that $I_0'(z) = 0$, for all $z \in Z$. Z is called the critical manifold of the *unperturbed functional* I_0.

Let $T_z Z$ denote the tangent space to Z at $z \in Z$. Since $I_0'(z) = 0$ for all $z \in Z$, differentiating along Z we get

$$(I_0''(z)[v] \mid \phi) = 0, \qquad \forall\, v \in T_z Z,\ \forall\, \phi \in E.$$

This shows that $T_z Z \subseteq Ker[I_0''(z)]$.

Then $Ker[I_0''(z)]$ is not empty and has dimension $\leq d$.

As a consequence, each $z \in Z$ is degenerate and we cannot apply the implicit function theorem to find solutions of $I'_\varepsilon(u) = 0$.

We will assume that Z has the lowest possible degeneracy, namely

$$\text{(ND)} \qquad T_z Z = Ker[I''_0(z)]$$

We also suppose that $I''_0(z)$ is a $0-$Fredholm map.

A critical manifold Z satisfying the above conditions will be called *Non-Degenerate Critical Manifold* (NDCM).

If we look for critical points of I_ε in the form $u = z + w$, with $z \in Z$ and $w \in (T_z Z)^\perp$, we find the equation

$$I'_\varepsilon(z + w) = I'_0(z + w) + \varepsilon G(z + w) = 0$$

Roughly, we will treat this equation as a bifurcation problem, the trivial solution being $u = z$ and the bifurcation parameter(s) $(\varepsilon, z) \in \mathbb{R} \times Z$.

A finite dimensional reduction.

If $P :\mapsto W$ denotes the orthogonal projection onto W, the equation $I'_\varepsilon(z+w) = 0$ is equivalent to the system

$$(\mathrm{S}) \qquad \begin{cases} PI'_\varepsilon(z+w) = 0 & \text{auxiliary equation;} \\ QI'_\varepsilon(z+w) = 0, & \text{bifurcation equation.} \end{cases}$$

where $Q = I_E - P$ is the conjugate projection of P.

Let $F : \mathbb{R} \times Z \times W \to W$ be defined by setting

$$F(\varepsilon, z, w) = PI'_0(z+w)(z+w) + \varepsilon PG'(z+w).$$

F is of class C^1 and one has $F(0, z, 0) = 0$, for every $z \in Z$. Moreover, letting $D_w F(0, z, 0)$ denote the partial derivative with respect to w evaluated at $(0, z, 0)$, one has:

Lemma 1 *If Z is a NDCM, then $D_w F(0, z, 0)$ is invertible as a map from W into itself.*

PROOF. The map $D_w F(0,z,0)W \mapsto W$ is given by

$$D_w F(0,z,0) : v \mapsto P I_0''(z)[v], \qquad v \in W.$$

Remark that $I_0''(z)$ is orthogonal to $T_z Z$. Actually, if $T_z Z$ =span$\{q_i\}$, $i = 1, 2, \ldots, d$, there holds:

$$(I_0''(z)[v] \mid q_i) = (I_0''(z)[q_i] \mid v) = 0,$$

because $q_i \in T_z Z$.

Hence $P I_0''(z)[v] = I_0''(z)[v]$ and the equation $D_w F(0,z,0)[v] = 0$ becomes $I_0''(z)[v] = 0$.

Thus $v \in Ker[I_0''(z)] \cap W$ and from (ND) it follows that $v = 0$, namely that $D_w F(0,z,0)$ is injective. Since I_0'' is Fredholm, we deduce that $D_w F(0,z,0) : W \to W$ is invertible. ∎

Lemma 1 allows us to apply the Implicit Function Theorem to $F(\varepsilon, z, w) = 0$, yielding

Lemma 2 *Given any compact subset* Z_c *of* Z *there exists* $\varepsilon_0 > 0$ *such that: for all* $|\varepsilon| < \varepsilon_0$, *for all* $z \in Z_c$, *the auxiliary equation has a unique solution* $w = w_\varepsilon(z)$ *such that:*

(i) $w_\varepsilon(z) \in W = (T_z Z)^\perp$ *and is of class* C^1 *with respect to* $z \in Z_c$ *and* $w_\varepsilon(z) \longrightarrow 0$ *as* $|\varepsilon| \longrightarrow 0$, *uniformly with respect to* $z \in Z_c$, *together with its derivative with respect to* z, w'_ε;

(ii) *more precisely one has that* $\|w_\varepsilon(z)\| = O(\varepsilon)$ *as* $\varepsilon \longrightarrow 0$, *for all* $z \in Z_c$.

We shall now solve the bifurcation equation. In order to do this, let us define the $reduced\ functional$ $\Phi_\varepsilon : Z \longrightarrow \mathbb{R}$ by setting

$$\Phi_\varepsilon(z) = I_\varepsilon(z + w_\varepsilon(z)). \tag{1}$$

Theorem 3 *Let $I_0, G \in C^2(E, \mathbb{R})$ and suppose that I_0 has a smooth critical manifold Z which is non-degenerate, in the sense that (ND) and (Fr) hold. Given a compact subset Z_c of Z, let us assume that Φ_ε has, for $|\varepsilon|$ sufficiently small, a critical point $z_\varepsilon \in Z_c$. Then $u_\varepsilon = z_\varepsilon + w_\varepsilon(z_\varepsilon)$ is a critical point of $I_\varepsilon = I_0 + \varepsilon G$.*

PROOF. (outline) Consider the manifold $Z_\varepsilon = \{z + w_\varepsilon(z)\}$. Since z_ε is a critical point of Φ_ε, it follows that $u_\varepsilon \in Z_\varepsilon$ is a critical point of I_ε constrained on Z_ε and thus u_ε satisfies $I'_\varepsilon(u_\varepsilon) \perp T_{u_\varepsilon} Z_\varepsilon$. Moreover the definition of w_ε implies that $I'_\varepsilon(z + w_\varepsilon(z)) \in T_z Z$. In particular, $I'_\varepsilon(u_\varepsilon) \in T_{z_\varepsilon} Z$. Since, for $|\varepsilon|$ small, $T_{u_\varepsilon} Z_\varepsilon$ and $T_{z_\varepsilon} Z$ are close, see (i) in Lemma 2, it follows that $I'_\varepsilon(u_\varepsilon) = 0$. ∎

In order to use Theorem 3 it is convenient to expand Φ_ε.

Lemma 4 *One has:*

$$\Phi_\varepsilon(z) = c_0 + \varepsilon G(z) + o(\varepsilon), \quad \textit{where } c_0 = I_0(z).$$

PROOF. Recall that

$$\Phi_\varepsilon(z) = I_0(z + w_\varepsilon(z)) + \varepsilon G(z + w_\varepsilon(z)).$$

Let us evaluate separately the two terms above. First we have

$$I_0(z + w_\varepsilon(z)) = I_0(z) + (I_0'(z) \mid w_\varepsilon(z)) + o(\|w_\varepsilon(z)\|).$$

Since $I_0'(z) = 0$ we get

$$I_0(z + w_\varepsilon(z)) = c_0 + o(\|w_\varepsilon(z)\|). \tag{2}$$

Similarly, one has

$$\begin{aligned} G(z + w_\varepsilon(z)) &= G(z) + (G'(z) \mid w_\varepsilon(z)) + o(\|w_\varepsilon(z)\|) \\ &= G(z) + O(\|w_\varepsilon(z)\|). \end{aligned} \tag{3}$$

Putting together (2) and (3) we infer that

$$\Phi_\varepsilon(z) = c_0 + \varepsilon \Big[G(z) + O(\|w_\varepsilon(z)\|) \Big] + o(\|w_\varepsilon(z)\|). \tag{4}$$

Since $\|w_\varepsilon(z)\| = O(\varepsilon)$, see Lemma 2-$(ii)$, the result follows. ■

The preceding lemma, jointly with Theorem 3 yields

Theorem 5 *Let $I_0, G \in C^2(E, \mathbb{R})$ and suppose that I_0 has a smooth critical manifold Z which is non-degenerate.*
Let $\Gamma := G_{|Z}$ and let $\bar{z} \in Z$ be either

(i) *a strict local maximum or minimum of Γ, or*

(ii) $\bar{z} \in Z$ *is a critical point of Γ satisfying*

(G') $\quad \exists\, r > 0$ *such that the topological degree $d(\Gamma', B_r(\bar{z}), 0) \neq 0$.*

Then for $|\varepsilon|$ small the functional I_ε has a critical point u_ε and if $\bar{z}$ is isolated, then $u_\varepsilon \longrightarrow \bar{z}$ as $\varepsilon \longrightarrow 0$.

From the point of view of bifurcation, we can say that from $\bar{z} \in Z$ branch off solutions of $I'_\varepsilon = 0$.

As a first application of the preceding theorems we are going to prove the result stated in the previous lecture dealing with the existence of a bifurcation from the essential spectrum for the problem

$$(5) \qquad \psi'' + \lambda\psi + h(x)|\psi|^{p-1}\psi = 0, \quad \lim_{|x|\to\infty} \psi(x) = 0,$$

where $p > 1$ and h satisfies

$(h.1)$ $\exists\, \ell > 0 \,:\, h - \ell \in L^1(\mathbb{R})$, and $\int_{\mathbb{R}}(h-\ell)dx \neq 0.$

To frame the problem as a perturbation one, we use the change of variable

$$\psi(x) = \varepsilon^{-\alpha}u(\varepsilon\, x),\ i.e.\ u(x) = \varepsilon^{\alpha}\psi(x/\varepsilon).$$

One finds:

$$\begin{aligned} -u''(x) &= -\varepsilon^{\alpha-2}\psi''(x/\varepsilon) \\ &= \varepsilon^{\alpha-2}\left[\lambda\psi(x/\varepsilon) + h(x/\varepsilon)\psi^p(x/\varepsilon)\right] \\ &= \varepsilon^{\alpha-2}\left[\lambda\varepsilon^{-\alpha}u(x) + h(x/\varepsilon)\varepsilon^{-\alpha p}u^p(x)\right] \\ &= \lambda\varepsilon^{-2}u(x) + h(x/\varepsilon)\varepsilon^{\alpha-2-\alpha p}u^p(x) \end{aligned}$$

Hence, choosing $\lambda = -\varepsilon^2$ and $\alpha = 2/(1-p)$ we find the following equation

$$-u''(x) + u(x) = h(x/\varepsilon)u(x)^p, \quad u \in W^{1,2}(\mathbb{R}), \tag{6}$$

Solutions of (6) are the critical points of

$$I_\varepsilon(u) = \tfrac{1}{2}\|u\|^2 - \frac{1}{p+1}\int_{\mathbb{R}} h(x/\varepsilon)|u|^{p+1}dx$$

on $E = W^{1,2}(\mathbb{R})$.

In order to use the abstract frame, we set $I_\varepsilon(u) = I_0(u) + G(\varepsilon, u)$ where

$$I_0(u) = \tfrac{1}{2}\|u\|^2 - \frac{\ell}{p+1}\int_{\mathbb{R}} |u|^{p+1}dx,$$

and

$$G(\varepsilon, u) = \begin{cases} -\frac{1}{p+1}\int_{\mathbb{R}} \left[h(x/\varepsilon) - \ell\right] |u|^{p+1}dx & \text{if } \varepsilon \neq 0; \\ 0 & \text{if } \varepsilon = 0. \end{cases} \tag{7}$$

Study of the unperturbed problem.

The unperturbed problem $I_0'(u) = 0$ is the equation

$$(8) \qquad -u''(x) + u(x) = \ell |u(x)|^{p-1} u(x), \quad u \in W^{1,2}(\mathbb{R})$$

which has a unique even positive solution $U(x)$ such that

$$U'(0) = 0, \qquad \lim_{|x| \to \infty} U(x) = 0.$$

Then I_0 has a one dimensional critical manifold given by

$$Z = \{z_\xi(x) := U(x+\xi) : \xi \in \mathbb{R}\}.$$

Moreover, every z_ξ is a Mountain-Pass critical point of I_0.

In order to show that Z is non-degenerate we will make use of the following lemma.

Lemma 6 *Let $y(x)$ be a solution of*

$$-y''(x) + Q(x)y(x) = 0,$$

where $Q(x)$ is continuous and there exist $a, R > 0$ such that $Q(x) \geq a > 0$, for all $|x| > R$. Then either $\lim_{|x|\to\infty} y(x) = 0$ or $\lim_{|x|\to\infty} y(x) = \infty$. Moreover, the solutions y satisfying the first alternative are unique, up to a constant.

Lemma 7 *Z is non-degenerate.*

PROOF. Let $v \in Ker[I_0''(z_\xi)]$, namely a solution of the linearized equation $I_0''(z_\xi)[v] = 0$,

$$-v''(x) + v(x) = \ell\, p\, z_\xi^{p-1}(x)v(x), \quad v \in W^{1,2}(\mathbb{R}). \tag{9}$$

A solution of (9) is given by $z'_\xi(x) = U'(x+\xi)$, spanning the tangent space $T_{z_\xi}Z$.
Set $Q = 1 - \ell\, p z_\xi^{p-1}$. Since $\lim_{|x|\to\infty} z_\xi(x) = 0$ then $\lim_{|x|\to\infty} Q(x) = 1$ and we can apply Lemma 6 yielding that all the solutions $v \in W^{1,2}(\mathbb{R})$ of (9) are given by $c\, z'_\xi$, for some constant $c \in \mathbb{R}$.

This shows that $Ker[I_0''(z_\xi)] \subseteq T_{z_\xi} Z$ and implies that Z is ND. ■

We now apply the abstract existence results. Actually, here the perturbation term is not of the form $\varepsilon G(u)$ but $G(\varepsilon, u)$.
However, it is possible to overcome this difficulty, showing that
the map $u \mapsto G(\varepsilon, u)$ is of class C^2 and there holds that $G(0, u) = 0$, $D_u G(0, u) = 0$, $D_u^2 G(0, u) = 0$.

Moreover, one has:

$$G(\varepsilon, z_\xi) = -\frac{\varepsilon}{p+1} \int_{\mathbb{R}} [h(y) - 1] U^{p+1}(\varepsilon y + \xi) dy.$$

By the Dominated Convergence Theorem we infer

$$\lim_{|\varepsilon| \to 0} \frac{G(\varepsilon, z_\xi)}{\varepsilon} = -\frac{1}{p+1} \left(\int_{\mathbb{R}} [h(y) - 1] dy \right) U^{p+1}(\xi) = -\frac{1}{p+1} \gamma\, U^{p+1}(\xi),$$

and this proves that

$$G(\varepsilon, z_\xi) = -\varepsilon \frac{1}{p+1} \gamma\, U^{p+1}(\xi) + o(\varepsilon)$$

as $|\varepsilon| \to 0$, uniformly for $|\xi|$ bounded.

Using the abstract results we deduce that the reduced functional is given by

$$\Phi_\varepsilon(\xi) = I_\varepsilon(z_\xi + w_\varepsilon(z_\xi)) = c_0 - \varepsilon \frac{1}{p+1} \gamma\, U^{p+1}(\xi) + o(\varepsilon), \quad \text{as } \varepsilon \to 0.$$

Then we can apply the abstract existence Theorem with

$$\Gamma(\xi) = -\frac{1}{p+1} \gamma\, U^{p+1}(\xi).$$

Γ has a maximum at $\xi = 0$ and hence, for all $|\varepsilon| > 0$ small, equation (6) has a solution $u_\varepsilon \simeq z_{\xi_\varepsilon} = U(x + \xi_\varepsilon)$, with $\xi_\varepsilon \to 0$.

These u_ε correspond to a family (λ, ψ_λ) of solutions to (5) given by

$$\lambda = -\varepsilon^2, \qquad\qquad \psi_\lambda(x) = (-\lambda)^{1/(p-1)} u_\varepsilon(\varepsilon\, x).$$

Since $\lambda \to 0^-$, $p > 1$ and $u_\varepsilon \simeq U$, it follows that $\|\psi_\lambda\|_\infty \to 0$ as $\lambda \to 0^-$.

This shows that (λ, ψ_λ) gives rise to a bifurcation (on the left) from $\lambda = 0$, the infimum of the essential spectrum.

Moreover, one has

$$\|\psi_\lambda\|^2_{L^2(\mathbb{R})} = \varepsilon^{4/(p-1)} \int_{\mathbb{R}} u_\varepsilon^2(\varepsilon x)dx = (-\lambda)^{(5-p)/2(p-1)} \|u_\varepsilon\|^2_{L^2(\mathbb{R})}.$$

This proves that

$$\lim_{\lambda \to 0^-} \|\psi_\lambda\|^2_{L^2(\mathbb{R})} = \begin{cases} 0 & \text{if} \quad 1 < p < 5 \\ const > 0 & \text{if} \quad p = 5 \\ +\infty & \text{if} \quad p > 5 \end{cases}$$

and completes the proof of the bifurcation Theorem.

Other Applications.

The abstract setting applies to many other variational problems, perturbative in nature. We will discuss in the sequel two of them.
A common feature is that the corresponding Euler functionals do not satisfy the (PS) compactness condition.

1. Equations on $\mathbb{R}^n$. We follow the paper
A.A - J. Garcia Azorero - I. Peral, Adv. Nonlin. Studies,2001

Let us consider the equation

$$-\Delta u + u = (1 + \varepsilon h(x))u^p, \qquad u \in W^{1,2}(\mathbb{R}^n), \tag{10}$$

where $1 < p < (n+2)/(n-2)$.

Here $E = W^{1,2}(\mathbb{R}^n)$, and

$$I_\varepsilon(u) = I_0(u) + \varepsilon G(u)$$

where

$$I_0(u) = \frac{1}{2}\|u\|^2 - \int |u|^{p+1}, \qquad G(u) = -\frac{1}{p+1}\int h(x)|u|^{p+1}.$$

It is understood that $h \in L^\infty(\mathbb{R}^n)$ in such a way $h(x)|u|^{p+1} \in L^1(\mathbb{R}^n)$ for all $u \in E$.

The Critical manifold Z is given by $Z = \{U_\xi(x) = U(x-\xi) : \xi \in \mathbb{R}^n\} \simeq \mathbb{R}^n$, where U is the (unique) radial positive function satisfying

$$-\Delta U + U = U^p.$$

Since $I_0''(U_\xi)[v] = 0$ is equivalent to

$$-\Delta v + v = pU_\xi^{p-1}v, \qquad v \in W^{1,2}(\mathbb{R}^n),$$

then, to study the Kernel of $I_0''(U_\xi)$ we have to study the preceding linear equation.
As in the ODE case, it is possible to show that Z is a NDCM. Actually,

$$Ker[I_0''(U_\xi)] = T^{U_\xi}Z(= span\{D_{x_i}U_\xi : i = 1, \cdots, n\})$$

Lemma 8 *Suppose that $h \in L^\infty$ holds and that*

$$(h_0) \qquad \lim_{|x|\to\infty} h(x) = 0.$$

Then

$$\lim_{|\xi|\to\infty} \Gamma(\xi) = 0.$$

PROOF. Given $\rho > 0$ we set

$$\Gamma_\rho(\xi) := \int_{|x|<\rho} h(x) U^{p+1}(x-\xi)dx, \quad \Gamma^*_\rho(\xi) = \int_{|x|>\rho} h(x) U^{p+1}(x-\xi)dx,$$

in such a way that

$$\Gamma(\xi) = -\frac{1}{p+1}\left[\Gamma_\rho(\xi) + \Gamma^*_\rho(\xi)\right].$$

Since U tends to zero at infinity, it follows immediately that $\Gamma_\rho(\xi)$ tends to zero as $|\xi|$ tends to infinity.

Furthermore, since also h tends to zero at infinity, we have that $\Gamma^*_\rho(\xi) = o_\rho(1)$, where $o_\rho(1)$ tends to zero as ρ tends to infinity. By the arbitrarity of ρ we obtain immediately the conclusion. ■

The previous lemma allows us to prove the existence of solutions of (10), provided $\Gamma(\xi) \not\equiv 0$. Actually, we can show

Theorem 9 *Let $h \in L^\infty$ satisfy (h_0). Moreover, suppose that*

(h_1) $\quad \int_{\mathbb{R}^n} h(x) U^{p+1}(x) \neq 0$;

Then (10) has a positive solution provided $|\varepsilon|$ is small enough.

PROOF. By Lemma 8, $\Gamma(\xi)$ tends to zero as $|\xi| \to \infty$.

If (h_1) holds then $\Gamma(0) = -\frac{1}{p+1}\int_{\mathbb{R}^n} h(x) U^{p+1}(x) \neq 0$. Then Γ is not identically zero and it follows that Γ has a maximum or a minimum on $\mathbb{R}^n$, and the existence of a solution follows from Theorem 5.

Positivity follows from the fact that the solution is found near U_ξ. ■

Assumptions (h_1) can be eliminated by studying directly the reduced functional

$$\Phi_\varepsilon(\xi) = I_\varepsilon(U_\xi + w(\varepsilon,\xi)).$$

One proves that $\lim_{|\xi|\to\infty} w(\varepsilon,\xi) = 0$ (strongly in E) for all $|\varepsilon| \ll 1$ and this leads to show that

$$\lim_{|\xi|\to\infty} \Phi_\varepsilon(\xi)) = 0, \qquad (|\varepsilon| \ll 1).$$

Therefore $\Phi_\varepsilon(\xi)$ has at least a critical point (a maximum or a minimum).

By Theorem 3 we know that critical points of Φ_ε give rise to solutions of $I'_\varepsilon = 0$. So we obtain

Theorem 10 *Suppose that $h \in L^\infty$ satisfy (h_0).*

Then for all $|\varepsilon|$ small, problem (10) has a positive solution.

2. Standing waves of NLS equations.

The second application deals with a singular perturbation problem:

$$(NLS_\varepsilon) \qquad \begin{cases} -\varepsilon^2\Delta u + V(x)u = u^p, & \text{in } \mathbb{R}^n \\ u > 0, \quad u \in W^{1,2}(\mathbb{R}^n), \end{cases}$$

where $p > 1$ is subcritical and V is a smooth bounded function.

Problem (NLS_ε) arises in the study of the Nolinear Schrödinger Equation

$$i\hbar\frac{\partial\psi}{\partial t} = -\hbar^2\Delta\psi + \tilde{V}(x)\psi - |\psi|^{p-1}\psi \qquad \text{in } \mathbb{R}^n,$$

where $\psi : \mathbb{R} \times \mathbb{R}^n \to \mathbb{C}$ is the *wave function*, $\tilde{V} : \mathbb{R}^n \to \mathbb{R}$ is the potential and $\hbar$ is the *Planck constant*.

Looking for *standing wave solutions*, namely solutions of the form $\psi(t,x) = e^{-\frac{i}{\hbar}\omega t}u(x)$, the function u is easily seen to satisfy (NLS_ε), with $V = \tilde{V} - \omega$ and $\varepsilon = \hbar$. Since $\varepsilon = \hbar$ is very small, one is interested is the asymptotic behavior of solutions in the limit $\varepsilon \to 0$, the so-called $semiclassical\ limit$.

We assume the following conditions on the potential V

(V1) $V \in C^2(\mathbb{R}^n)$, and $\|V\|_{C^2(\mathbb{R}^n)} < +\infty$;

(V2) $\lambda_0^2 = \inf_{\mathbb{R}^n} V > 0$.

We say that a solution v_ε of (NLS_ε) *concentrates* at x_0 (as $\varepsilon \longrightarrow 0$) provided

$$(11)\ \ \forall\, \delta > 0, \quad \exists\, \varepsilon_0 > 0,\ R > 0\ : v_\varepsilon(x) \leq \delta,\ \forall\, |x - x_0| \geq \varepsilon\, R,\ \varepsilon < \varepsilon_0.$$

We will show the following typical result.

Theorem 11 *Let $(V1)$ and $(V2)$ hold, and suppose x_0 is a non-degenerate critical point of V, namely for which $V''(x_0)$ is non-singular. Then there exists a solution $\bar{v}_\varepsilon$ of (NLS_ε) which concentrates at x_0 as $\varepsilon \longrightarrow 0$.*

To simplify the notation (and without losing generality) we will suppose that $x_0 = 0$ and that $V(0) = 1$.

To frame (NLS_ε) in the abstract setting, we first make the change of variable $x \mapsto \varepsilon x$ and rewrite equation(NLS_ε) as

$$\begin{cases} -\Delta u + V(\varepsilon x)u = u^p & \text{in } \mathbb{R}^n \\ u > 0, \quad u \in W^{1,2}(\mathbb{R}^n). \end{cases} \tag{12}$$

If $u_\varepsilon(x)$ is a solution of (12) then $v_\varepsilon(x) := u_\varepsilon(x/\varepsilon)$ solves (NLS_ε).

We set again $E = W^{1,2}(\mathbb{R}^n)$ and consider the functional $I_\varepsilon \in C^2(E, \mathbb{R})$,

$$I_\varepsilon(u) = \frac{1}{2}\int_{\mathbb{R}^n} \left(|\nabla u|^2 + V(\varepsilon x)u^2\right) - \frac{1}{p+1}\int_{\mathbb{R}^n} |u|^{p+1}. \tag{13}$$

Setting

$$I_0(u) = \frac{1}{2}\|u\|^2 - \frac{1}{p+1}\int_{\mathbb{R}^n} |u|^{p+1}, \quad \|u\|^2 = \int_{\mathbb{R}^n} \left(|\nabla u|^2 + u^2\right) dx$$

the functional I_ε takes the form

$$I_\varepsilon(u) = I_0(u) + \frac{1}{2}\int_{\mathbb{R}^n} (V(\varepsilon x) - 1)\, u^2 dx \equiv I_0(u) + G(\varepsilon, u).$$

Obviously, for any fixed $u \in E$, we have $G(\varepsilon, u) \longrightarrow 0$ as $\varepsilon \longrightarrow 0$ and hence we can still view I_ε as a perturbation of I_0.
We define U and Z as before. However, the abstract method requires suitable modifications because one cannot apply the Implicit Function Theorem.

We look again for solutions $u = U_\xi + w$, with $w \in W$, and consider the system

$$\begin{cases} PI'_\varepsilon(U_\xi + w) & = 0, \\ QI'_\varepsilon(U_\xi + w) & = 0 \end{cases}$$

which is equivalent to $I'_\varepsilon(U_\xi + w) = 0$.

At this point, instead of using the Implicit Function Theorem, we write

$$PI'_\varepsilon(U_\xi + w) = PI'_\varepsilon(U_\xi) + PI''_\varepsilon(U_\xi)[w] + R(\xi, w),$$

where $R(\xi, w) = o(\|w\|)$, uniformly with respect to bounded $|\xi|$.

Next, using the non-degeneracy of Z, one can show that $PI''_\varepsilon(U_\xi)$ is uniformly invertible for ξ belonging to a fixed bounded set of $\mathbb{R}^n$.

Setting $A_{\varepsilon,\xi} = -(PI''_\varepsilon(U_\xi))^{-1}$, the equation $PI'_\varepsilon(z_\xi + w) = 0$, namely

$$PI'_\varepsilon(U_\xi) + PI''_\varepsilon(U_\xi)[w] + R(\xi, w) = 0$$

can be written in the form

$$w = A_{\varepsilon,\xi}\left(PI'_\varepsilon(U_\xi) + R(\xi, w)\right) := N_{\varepsilon,\xi}(w).$$

It is also possible to show that $N_{\varepsilon,\xi}$ is a contraction in some ball of W provided ε is sufficiently small.

This allows us to solve the auxiliary equation finding a solution $w_\varepsilon(\xi)$ which is of class C^1 with respect to ξ. Furthermore, since $V'(0) = 0$, one finds that $w_\varepsilon(\xi) = O(\varepsilon^2)$, uniformly with respect to ξ in a bounded set.

At this point we can repeat the expansion of Φ_ε obtaining again

$$\Phi_\varepsilon(\xi) = c_0 + \varepsilon^2\,\Gamma(\xi) + o(\varepsilon^2),$$

where $c_0 = I_0(U)$ and

$$\Gamma(\xi) = \tfrac{1}{2}\int_{\mathbb{R}^n} \langle V''(0)x, x\rangle U^2(x-\xi)dx.$$

A straight calculation yields

$$\begin{aligned}\Gamma(\xi) &= \tfrac{1}{2}\int_{\mathbb{R}^n} \langle V''(0)(y+\xi), (y+\xi)\rangle U^2(y)dy \\ &= \tfrac{1}{2}\int_{\mathbb{R}^n} \langle V''(0)y, y\rangle U^2(y)dy + \tfrac{1}{2}\int_{\mathbb{R}^n} \langle V''(0)\xi, \xi\rangle U^2(y)dy \\ &= c_1 + c_2\langle V''(0)\xi, \xi\rangle,\end{aligned}$$

where

$$c_1 = \tfrac{1}{2}\int_{\mathbb{R}^n} \langle V''(0)y, y\rangle U^2(y)dy, \qquad c_2 = \tfrac{1}{2}\int_{\mathbb{R}^n} U^2(x)dx.$$

In other words, up to un-influent positive constants,

$$\Gamma(\xi) = \langle V''(0)\xi, \xi\rangle$$

Then $\xi = 0$ is a non-degenerate critical point of Γ and therefore, from the general theory, it follows that for $\varepsilon \ll 1$, I_ε has a critical point $u_\varepsilon = U_{\xi_\varepsilon} + w_\varepsilon(\xi_\varepsilon)$, with $\xi_\varepsilon \longrightarrow 0$ as $\varepsilon \longrightarrow 0$.

In conclusion, coming back to the solutions v_ε of (NLS_ε) , we find that this equation has a solution $\bar{v}_\varepsilon(x) \sim U(\frac{x-\xi_\varepsilon}{\varepsilon})$ that concentrates at $x = 0$, proving Theorem 11.

References:

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