5. Bifurcation from the essential spectrum

We will consider the specific problem

$$(1) \qquad -\Delta u + V(x)u = \lambda u + h(x)|u|^{p-1}u, \qquad u \in W^{1,2}(\mathbb{R}^n).$$

where $p > 1$, V is bounded and $h > 0$. If we assume

$$(2) \qquad V \in L^\infty, \qquad V(x) \geq 0, \qquad \lim_{|x|\to\infty} V(x) = 0,$$

then the spectrum of the linearized problem

$$-\Delta v + V(x)v = \lambda v, \qquad v \in W^{1,2}(\mathbb{R}^n)$$

is the whole half line $[0, \infty)$ and coincides with its essential spectrum, which is the set of all points of the spectrum that are not isolated, jointly with the eigenvalues of infinite multiplicity.

Clearly, none of the bifurcation results proved so far apply to (1).

In order to have an idea of the results we can expect, let us consider the elementary case in one dimension when $V \equiv 0$ and $h \equiv 1$:

$$-u'' = \lambda u + |u|^{p-1}u, \quad u \in W^{1,2}(\mathbb{R}),$$

which can be studied in a straightforward way by a phase-plane analysis.

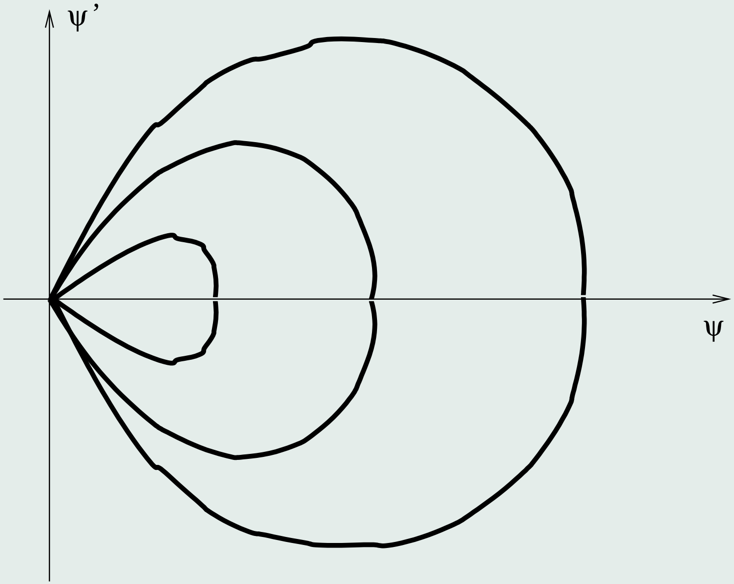
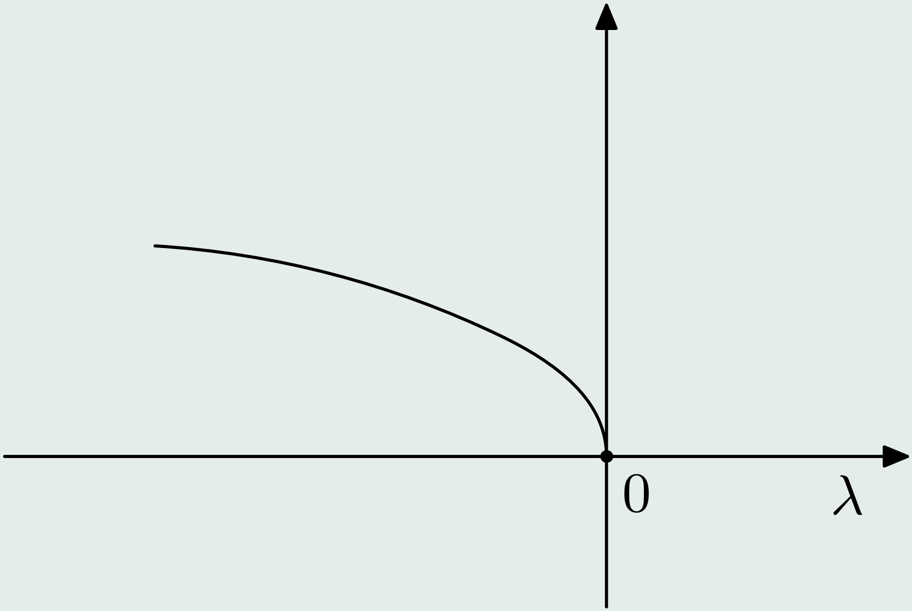


Figure 1: Phase plane portrait of $u'' + \lambda u + u^p = 0$

It follows that from $\lambda = 0$, the bottom of the essential spectrum of $-v'' = \lambda v, \;\; v \in W^{1,2}(\mathbb{R})$, bifurcates a family of solutions (λ, u_λ), $\lambda < 0$, of $-u'' = \lambda u + |u|^{p-1}u$, with $(\lambda, u_\lambda) \to (0,0)$ as $\lambda \uparrow 0$.



In order to prove a similar result for (1) with V and h possibly depending on x, we will use variational tools.

Let V satisfy (2) and suppose that h verifies

$$h \in L^\infty, \qquad h(x) > 0, \qquad \lim_{|x|\to\infty} h(x) = 0. \tag{3}$$

Let $1 < p < 2^* - 1$, $l < 0$, and set $E = W^{1,2}(\mathbb{R}^n)$,

$$\|u\|_\lambda^2 = \int_{\mathbb{R}^n} \left[|\nabla u|^2 + V(x)u^2 - \lambda u^2\right] dx, \qquad u \in E,$$

and

$$\Psi(u) = \int_{\mathbb{R}^n} h|u|^{p+1} dx,$$

Let us remark that, for each fixed $\lambda < 0$, $\|\cdot\|_\lambda$ is a norm equivalent to the usual one in $W^{1,2}(\mathbb{R}^n)$.

Consider the functional $J_\lambda : E \mapsto \mathbb{R}$,

$$J_\lambda(u) = \tfrac{1}{2}\|u\|_\lambda^2 - \tfrac{1}{p+1}\Psi(u).$$

Clearly, J_λ is of class C^2 and its critical points give rise to (weak and, by regularity results, strong) solutions of (1) such that $\lim_{|x|\to\infty} u(x) = 0$.

To find critical points of J_λ we can use the Mountain-Pass Theorem. First of all, we show

Lemma 1 *If* (3) *holds, then* Ψ *is weakly continuous and* Ψ' *is compact.*

PROOF. Let $u_k \rightharpoonup u$ in E. Given $\varepsilon > 0$, from (3) it follows that there exists $R > 0$ such that

$$\int_{|x|\geq R} h(x)\left(|u_k|^{p+1} - |u|^{p+1}\right) dx \leq \varepsilon.$$

Since $W^{1,2}(B_R)$ is compactly embedded in $L^{p+1}(B_R)$, we get

$$\int_{|x|<R} h(x)\left(|u_k|^{p+1} - |u|^{p+1}\right) dx \leq \varepsilon,$$

provided $k \gg 1$. Putting together the two preceding inequalities, it follows that Ψ is weakly continuous. The proof that Ψ' is compact is similar. ■

Let $m(\lambda)$ denote the MP critical level of J_λ.
In order to estimate $m(\lambda)$ we strengthen assumptions (2) by requiring

$$|x|^2 V(x) \in L^\infty, \qquad V(x) \geq 0. \tag{4}$$

Moreover, we suppose that h verifies (3) and $\exists\, K > 0,\ C > 0\ \tau \in [0, 2[$ such that

$$h(x) \geq K|x|^{-\tau}, \quad \forall\, |x| \geq C. \tag{5}$$

Lemma 2 *If (3-4-5) hold, then* $m(\lambda) \to 0$ *as* $\lambda \to 0^-$.

PROOF. Fix the function $\phi(x) = |x|e^{-|x|}$ and set $u_\alpha(x) = \phi(\alpha x)$. There holds

$$
\begin{aligned}
\|\nabla u_\alpha\|_{L^2}^2 &= \alpha^{2-n} A_1, \quad A_1 = \int_{\mathbb{R}^n} |\nabla \phi|^2 dx, \\
\|u_\alpha\|_{L^2}^2 &= \alpha^{-n} A_2, \quad A_2 = \int_{\mathbb{R}^n} \phi^2 dx, \\
\int_{\mathbb{R}^n} V u_\alpha^2 dx &\le \alpha^{2-n} A_3, \quad A_3 = c_1 \int_{\mathbb{R}^n} |x|^2 \phi^2 dx,
\end{aligned}
$$

where c_1 is such that $|x|^2 \, |V(x)| \le c_1$. Then one finds

$$
\begin{aligned}
\|u_\alpha\|_\lambda^2 &= \int_{\mathbb{R}^n} \left[|\nabla u_\alpha|^2 + V(x) u_\alpha^2 - \lambda u_\alpha^2 \right] dx \\
&\le A_1 \alpha^{2-n} + A_3 \alpha^{2-n} - \lambda A_2 \alpha^{-n} \\
&\le A_4 \alpha^{2-n}, \qquad (-1 \le \lambda = -\alpha^2 < 0),
\end{aligned}
$$

for some $A_4 > 0$. Moreover, using (5) we deduce

$$
\Psi(u_\alpha) = \int h(x) |u_\alpha|^{p+1} \ge K \int_{|x| \ge C} |x|^{-\tau} |u_\alpha|^{p+1} dx.
$$

Performing the change of variable $y = \alpha x$, we find

$$\begin{aligned}\int_{|x|\geq C} |x|^{-\tau}|\phi(\alpha x)|^{p+1}dx &= \int_{|y|\geq \alpha C} |\frac{y}{\alpha}|^{-\tau}|\phi(y)|^{p+1}\alpha^{-n}dy \\ &= \alpha^{\tau-n}\int_{|y|\geq \alpha C} |y|^{-\tau}|\phi(y)|^{p+1}dy.\end{aligned}$$

Therefore there exists $A_5 > 0$ such that

$$\Psi(u_\alpha) \geq \alpha^{\tau-n}A_5.$$

Putting together all the preceding estimates, we get

$$J_\lambda(u_\alpha) \leq a\,\alpha^{2-\alpha} - b\,\alpha^{\tau-n}.$$

Recall that

$$m(\lambda) = \inf_{\gamma\in\Gamma}\max_t J_\lambda(\gamma(t)),$$

where Γ is the class of all paths joining 0 and v, $J_\lambda(v) \leq 0$.
Since $t \mapsto tu_\alpha$ is a path in Γ, then

$$m(\lambda) \leq \max_t J_\lambda(tu_\alpha), \qquad (\lambda = -\alpha^2).$$

We now use the above estimate to evaluate $\max_t J_\lambda(tu_\alpha)$.

There holds (up to un-influent constants)

$$J_\lambda(tu_\alpha) \leq \beta(t) := \frac{1}{2}\alpha^{2-\alpha}t^2 - \frac{1}{p+1}\alpha^{\tau-n}t^{p+1}.$$

The maximum of β is achieved at $t_\alpha = \alpha^{\frac{2-\tau}{p-1}}$ and

$$\beta(t_\alpha) = \alpha^{\left(2\,\frac{2-\tau}{p-1}-n\right)}\left[\frac{1}{2}\alpha^2 - \frac{1}{p+1}\alpha^{p+\tau-1}\right].$$

Therefore

$$m(\lambda) \leq \alpha^{\left(2\,\frac{2-\tau}{p-1}-n\right)}\left[\frac{1}{2}\alpha^2 - \frac{1}{p+1}\alpha^{p+\tau-1}\right], \quad (\lambda = -\alpha^2).$$

Since $p > 1$, the quantity $[\cdots]$ tends to zero as $\alpha \to 0$.

Thus, if $1 < p < 1 + \frac{2(2-\tau)}{n}$, we find that $m(\lambda) \to 0$ as $\lambda = -\alpha^2 \to 0^-$. ■

We have proved that the MP critical point u_λ satisfies $J_\lambda(u_\lambda) = m(\lambda) \to 0$ as $\lambda \to 0^-$.

Since $J_\lambda(u_\lambda) = \frac{1}{2}\|u_\lambda\|_\lambda^2 - \frac{1}{p+1}\Psi(u_\lambda)$ and u_λ is a critical point, we have

$$(J_\lambda(u_\lambda) \mid u_\lambda) = 0 \Rightarrow \|u_\lambda\|_\lambda^2 = \Psi(u_\lambda),$$

and hence

$$J_\lambda(u_\lambda) = \left(\frac{1}{2} - \frac{1}{p+1}\right) \|u_\lambda\|_\lambda^2$$

and thus $\|u_\lambda\|_\lambda^2 \to 0$ as $\lambda \to 0^-$.

In conclusion, we can state the following theorem due to C. Stuart:

Theorem 3 *Suppose that (3-4-5) hold. If* $1 < p < 1 + \frac{2(2-\tau)}{n}$*, then the bottom of the essential spectrum,* $\lambda = 0$*, is a bifurcation point for* (1)*. Precisely, for all* $\lambda < 0$ *there is a family of nontrivial solutions* u_λ *of* (1) *such that* $\|u_\lambda\|_\lambda \to 0$ *as* $\lambda \to 0^-$.

We anticipate that, using some perturbation result we will discuss in the sequel, one can obtain other bifurcation results. For example one can prove

Theorem 4 *Consider the equation*

$$\psi'' + \lambda\psi + h(x)|\psi|^{p-1}\psi = 0, \quad \lim_{|x|\to\infty} \psi(x) = 0, \tag{6}$$

where $p > 1$ *and* h *satisfies*

$(h.1)$ $\exists\, \ell > 0 \,:\, h - \ell \in L^1(\mathbb{R}),$ *and* $\int_{\mathbb{R}}(h-\ell)dx \neq 0.$

Then (6) *has a family of solutions* (λ, ψ_λ) *such that* $\lambda \longrightarrow 0^-$ *and* $\psi_\lambda \longrightarrow 0$ *as* $\lambda \longrightarrow 0^-$ *in the* $C(\mathbb{R})$ *topology. Moreover, one has:*

$$\lim_{\lambda\to 0^-} \|\psi_\lambda\|^2_{L^2(\mathbb{R})} = \begin{cases} 0 & \textit{if} \quad 1 < p < 5 \\ const > 0 & \textit{if} \quad p = 5 \\ +\infty & \textit{if} \quad p > 5 \end{cases} \tag{7}$$

Finally, if $p \geq 2$*, the family* (λ, ψ_λ) *is a curve.*