3. A short Review on Critical Point Theory

Let E be a Hilbert space. A functional J is a map from E to $\mathbb{R}$. Suppose $J \in C^1(E, \mathbb{R})$. Then the Frechet derivative $dJ(u)$ is a linear continuous map from E to $\mathbb{R}$ and hence we can define, by the Riesz theorem, the gradient $J'(u)$ of J at u by setting

$$(J'(u) \mid v) = dJ(u)[v], \qquad \forall\, v \in E.$$

Example: Ω bounded domain in $\mathbb{R}^n$, $E = W_0^{1,2}(\Omega)$ with scalar product $(u \mid v) = \int \nabla u \cdot \nabla v$. Let

$$J_1(u) = \frac{1}{2} \int |\nabla u|^2 = \frac{1}{2} \|u\|^2$$

Clearly, $dJ_1(u)[v] = \int \nabla u \cdot \nabla v$. Hence $J_1'(u)$ is the element $w \in E$ such that $(w \mid v) = dJ_1(u)[v]$. Then

$$\int \nabla w \cdot \nabla v = \int \nabla u \cdot \nabla v \Rightarrow w = u.$$

In other words, $J_1'(u) = u$.

Consider now

$$\Phi(u) = \int F(u).$$

One finds $d\Phi(u)[v] = \int F'(u)v$.

The gradient $\Phi'(u)$ is the element of $\phi \in E$ such that $(\phi \mid v) = \int F'(u)v, \quad \forall\, v \in E$. Since $(\phi \mid v) = \int \nabla\phi \cdot \nabla v dx$ we find that ϕ satisfies

$$\int \nabla\phi \cdot \nabla v = \int F'(u)v, \quad \forall\, v \in E.$$

Thus ϕ is the weak (and, by regularity) strong solutions of

$$-\Delta\phi = F'(u),\ x \in \Omega, \qquad \phi(x) = 0,\ x \in \partial\Omega$$

namely

$$\phi(= \Phi'(u)) = (-\Delta)^{-1} \circ F'(u)$$

For example, if $F(u) = \frac{1}{2}\lambda u^2 \pm \frac{1}{p+1}|u|^{p+1}$, everything works provided $1 < p+1 < 2^*$. Recall that $2^* = 2n/n-2$ if $n > 2$, otherwise we set $2^*0+\infty$

A critical point of J is a $u \in E$ such that $J'(u) = 0$.

In our applications critical points are (weak) solutions of differential equations. For example, in the preceding case, the critical points of

$$J(u) = \frac{1}{2}\|u\|^2 - \frac{\lambda}{2}\int |u|^2 \mp \frac{1}{p+1}\int |u|^{p+1}, \quad u \in W_0^{1,2}(\Omega)$$

are solutions of

$$\begin{cases} -\Delta u &= \lambda u \pm |u|^{p-1}u, & x \in \Omega, \\ u &= 0, & x \in \partial\Omega. \end{cases}$$

Existence of critical points

We will focus on two cases:

- Minima
- Mountain-Pass

We will check the abstract results on the model problem

$$(BVP_{\pm}) \qquad \left\{ \begin{array}{rcll} -\Delta u &=& \lambda u \pm |u|^{p-1}u, & x \in \Omega, \\ u &=& 0, & x \in \partial\Omega. \end{array} \right.$$

We will see that the results depend on the sign of the nonlinear term.

Minima

Theorem. Suppose that $J \in C^1(E, \mathbb{R})$ is:

– coercive, i.e. $\lim_{\|u\| \to \infty} J(u) = +\infty$;

– w.l.s.c., i.e. $u_n \rightharpoonup u \ \Rightarrow J(u) \leq \liminf J(u_n)$.

Then J (is bounded from below and) has a global minimum z.

This Theorem applies to $(BVP)_-$ and $p > 1$. Precisely:

– If $\lambda \leq \lambda_1$ (the first eigenvalue of $-\Delta$ on $W_0^{1,2}(\Omega)$), then the minimum is the trivial solution of $(BVP)_-$;

– If $\lambda > \lambda_1$, then the minimum is the positive solution of $(BVP)_-$.

The Mountain-Pass Theorem

This Theorem deals with the existence of critical points of a functional $J \in C^1(E, \mathbb{R})$ which satisfies the following two "geometric" assumptions (A):

$A1.$ J has a local strict minimum at, say, $u = 0$: there exist $r, \rho > 0$ such that $J(u) \geq \rho$ for all $u \in E$ with $\|u\| = r$.

$A2.$ $\exists\, v \in E$, $\|v\| > r$, such that $J(v) \leq 0 = J(0)$.

In addition, one assumes the "compactness" condition $(PS)_c$, called Palais-Smale condition at level c

Every sequence u_n such that

(a) $J(u_n) \to c$,

(b) $J'(u_n) \to 0$,

has a converging subsequence.

The sequences satisfying $(a) - (b)$ are called $(PS)_c$ sequences.

For example, if (PS) holds and J is bounded from below, then the steepest descent flow, namely the solutions of the Cauchy problem

$$\begin{cases} \frac{d}{dt}\sigma & = -J'(\sigma) \\ \sigma(0) & = u \end{cases}$$

converges to a critical point of J as $t \to +\infty$. This could be false if (PS) does not hold.

If J is bounded from below and (PS) holds, then J the infimum is attained.

This could be false if (PS) does not hold.

Let $J \in C^1(E, \mathbb{R})$ be a functional that satisfies the assumptions (A1-A2). Without loss of generality, we can also assume (to simplify notation) that $J(0) = 0$.

Consider the class of all paths joining $u = 0$ and $u = v$:

$$\Gamma = \{\gamma \in C([0,1], E) : \gamma(0) = 0, \ \gamma(1) = v\}$$

and set

$$c = \inf_{\gamma \in \Gamma} \max_{t \in [0,1]} J(\gamma(t)).$$

Remark: $c \geq \rho > 0$

Theorem (Mountain-Pass) If $J \in C^1(E, \mathbb{R})$ satisfies (A1-A2) and $(PS)_c$ holds, then c is a positive critical level for J. Precisely, there exists $z \in E$ such that $J(z) = c > 0$ and $J'(z) = 0$. In particular $z \neq 0$ and $z \neq v$.

Remarks. (a) J can be unbounded from above and from below.

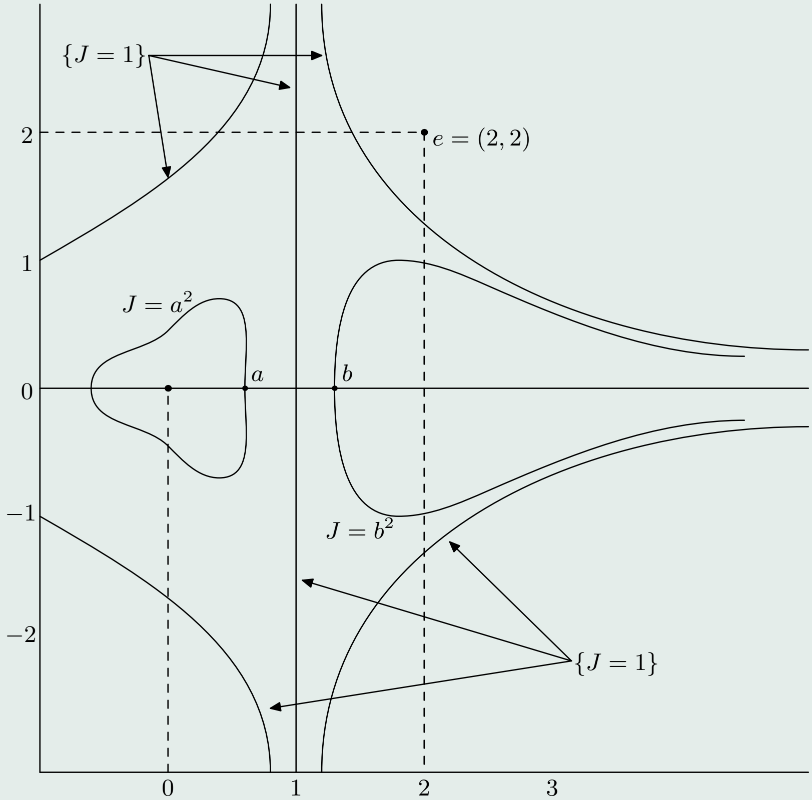
(b) The M-P critical point is a saddle point: if it is non-degenerate, then its Morse index is 1.

(c) The following example shows that, even on $\mathbb{R}^n$, the geometric assumptions (A1-2) alone, without the (PS) condition, do not suffice for the existence of a M-P critical point.

Let $E = \mathbb{R}^2$ and $J(x, y) = x^2 + (1 - x)^3 y^2$. It is easy to see that $(0, 0)$ is a strict local minimum and that $J(2, 2) = J(0, 0) = 0$.

- The only critical point of J is $(0, 0)$.

- The M-P critical level is $c = 1$ and $(PS)_c$ does not hold for $c = 1$.



$\{J = 1\}$
2
$e = (2, 2)$
1
$J = a^2$
a
b
0
-1
$J = b^2$
-2
$\{J = 1\}$
0
1
2
3

The M-P Theorem applies, for example, to $(BVP)_+$ with $\lambda < \lambda_1$.

If $1 < p < 2^* - 1 = \frac{n+2}{n-2}$ $(n > 2)$ the functional is

$$J(u) = \frac{1}{2}\|u\|^2 - \frac{\lambda}{2}\int u^2 - \frac{1}{p+1}\int |u|^{p+1}, \qquad u \in E = W_0^{1,2}(\Omega).$$

Let us check the assumptions (A1-2):

(A1) The second derivative of $\Phi(u) = \frac{1}{p+1}\int |u|^{p+1}$ is given by $\Phi''(u)[v]^2 = p\int |u|^{p-1}v^2$. Since $p > 1$ we infer $\Phi''(0)[v]^2 = 0$. Then

$$J''(0)[v]^2 = \|v\|^2 - \lambda\int v^2 - \Phi''(0)[v]^2 = \|v\|^2 - \lambda\int v^2.$$

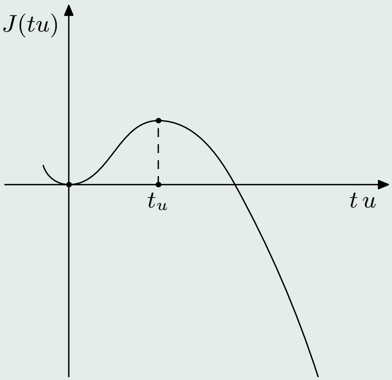
If $\lambda < \lambda_1$ there exists $b > 0$ such that

$$J''(0)[v]^2\|v\|^2 - \lambda\int u^2 \geq b\|v\|^2.$$

(A2) Fix any $\overline{u} \in E$ with $\|\overline{u}\| = 1$, and consider $J(t\overline{u})$, $t > 0$. From

$$J(t\overline{u}) = \frac{1}{2}\,t^2 - \frac{\lambda}{2}\,t^2 \int \overline{u}^2 - \frac{t^{p+1}}{p+1} \int |\overline{u}|^{p+1}$$

it follows that $J(t\overline{u}) \to -\infty$ as $t \to +\infty$.



Finally, for the (PS) condition, let u_n be a $(PS)_c$ sequence.

From $J(u_n) \leq k$ we get

$$(*) \qquad \|u\|^2 \leq 2k + 2\Phi(u_n)$$

From $J'(u_n) \to 0$ we infer

$$\left|\|u\|^2 - (p+1)\Phi(u_n)\right| = |(J'(u_n) \mid u_n)| \leq \|J'(u_n)\|\|u_n\| = o(1)\|u_n\|.$$

Thus

$$\Phi(u_n) \leq \frac{1}{p+1}\|u\|^2 + o(1)\|u_n\|$$

Substituting in (*) we get

$$\|u\|^2 \leq 2k + 2\Phi(u_n) \leq 2k + \frac{2}{p+1}\|u\|^2 + o(1)\|u_n\|$$

and thus

$$\left(1 - \frac{2}{p+1}\right)\|u\|^2 \leq 2k + o(1)\|u_n\| \;\Rightarrow\; \|u_n\| \leq K.$$

Moreover:

(i) Since $\|u_n\| \leq K$, then , up to a subsequence, $u_n \rightharpoonup u^*$.

(ii) Since the embedding $W_0^{1,2}(\Omega)$ in $L^{p+1}(\Omega)$ is compact (because $p+1 < 2^*$)) (i) implies that $u_n \to u^*$ strongly in $L^{p+1}(\Omega)$ and we deduce that

$$\Phi(u_n) \to \Phi(u^*).$$

(iii) Recall that $J'(u_n) = u_n - (p+1)\Phi(u_n)$. Hence

$$u_n = J'(u_n) + (p+1)\Phi(u_n)$$

Since $J'(u_n) \to 0$, (ii) and (iii) yield

$$u_n \to (p+1)\Phi(u^*),$$

proving that $(PS)_c$ holds for every c.

The M-P theorem can be extended to cover the case in which $u = 0$ is not a minimum but a saddle.

These results are called "linking theorems" and can be applied to $(BVP)_+$ in the case that $\lambda > \lambda_1$.

4. Bifurcation for Variational Operators

Let E be a Hilbert space and consider the equation

$$(1) \qquad Lu + H(u) = \lambda u, \quad u \in E,$$

where $L : E \to E$ is linear and $H \in C^1(E, E)$ is such that $H(0) = 0$, $H'(0) = 0$. Let $(\cdot \mid \cdot)$ denote the scalar product in E.

Let Σ denote the closure of the set of non-trivial solutions $(\lambda, u) \in \mathbb{R} \times E$ of (1).

- $\mu \in \mathbb{R}$ is a bifurcation point of (1) if $(\mu, 0) \in \Sigma$.

We suppose to be in the variational case, namely:

(A_1) $L \in L(E,E)$ is a symmetric Fredholm operator with index zero.

(A_2) There exists a functional $h \in C^k(E,\mathbb{R})$, for some $k \geq 3$, such that $H(u) = h'(u)$. Moreover $h(0) = h'(0) = h''(0) = 0$.

Let us define $f \in C^k(E,\mathbb{R})$ by setting

$$f(u) = \frac{1}{2}\lambda\|u\|^2 - \frac{1}{2}(Lu \mid u) - h(u), \tag{2}$$

so that $f'(u) = \lambda u - Lu - H(u)$ and Σ is the closure of the set of the critical points u of f on E such that $u \neq 0$.

Since $f'(0)[v] = \lambda v - Lv - H'(0)[v] = \lambda v - Lv - h''(0)[v]$, the linearization of (1) at $u = 0$ is given by

$$\lambda v - Lv = 0.$$

Let $\mu \in \mathbb{R}$ be an eigenvalue of finite multiplicity of L and set $Z = \text{Ker}[\mu I - L]$, where I denotes the identity map in E.

Theorem 1 (Krasnoselski) *Suppose that (A_1) and (A_4) hold and let μ be an isolated eigenvalue of finite multiplicity of L. Then μ is a bifurcation point of (1).*

Other results:

• Marino-Prodi (1968): proof using Morse theory.

• Böhme (1972) who proved that if h is real analytic, then μ is a branching point. An example shows that if h is C^∞, μ can be merely a bifurcation point.

We will prove Theorem 1 under some further assumptions.
Suppose that there is an integer $k \geq 3$ such that $D^j h(0) = 0$, $\forall\ j = 1, \ldots, k-1$, and $D^k h(0) \neq 0$. Let

$$\alpha_k(v) = \frac{1}{k!} D^k h(0)[v]^k, \qquad v \in E.$$

$\alpha_k : Z \to \mathbb{R}$ is homogeneous of degree k and there results

$$h(u) = \alpha_k(u) + o(\|u\|^k) \quad \text{as } \|u\| \to 0.$$

We also assume that

(A_3) $\exists\ \tilde{z} \in Z$ such that $\alpha_k(\tilde{z}) \neq 0$.
(A_4) M and m have the same sign ($M \geq m > 0$ or $m \leq M < 0$).
where

$$M := \max_{\partial B_Z} \alpha_k, \quad m := \min_{\partial B_Z} \alpha_k, \quad B_Z = \{z \in Z\text{"} \|z\| \leq 1\}.$$

Proof

Let W denote the orthogonal complement of Z in E: $E = Z \oplus W$, and let P denote the orthogonal projection on W, parallel to Z. Setting $u = z+w$, $z \in Z$, $w \in W$ and $\lambda = \mu + \epsilon$, equation (1) becomes

$$F(\epsilon, z, w) := (\mu I - L)w + \epsilon z + \epsilon w - H(z+w) = 0.$$

Lemma A. There exists $w = w(\epsilon, z)$ defined in a neighborhood $\mathcal{O}$ of $(0,0)$ in $\mathbb{R} \times Z$ such that $PF(\epsilon, z, w) = 0$. Moreover $w \in C^k(\mathcal{O}, W)$ and one has that $w(\epsilon, 0) \equiv 0$, $D_z^j w(0,0) = 0 \; \forall\, j = 1, \ldots, k-2$. In particular, $\exists\, a > 0$ such that $\|w(\epsilon, z)\| \leq \|z\|$, for all $(\varepsilon, z) \in calO$. uniformly for $|\epsilon|$ small.

PROOF. One has that $PF(0,0,0) = 0$ as well as

$$PD_wF(0,0,0)[v] = \mu v - Lv, \quad (v \in W).$$

Then $PD_wF(0,0,0)$ is injective and hence invertible, because L is Fredholm. Then the result follows from the Implicit Function Theorem. ∎

Let us define $\Phi_\epsilon : Z \longrightarrow \mathbb{R}$ by setting

$$\Phi_\epsilon(z) = f(z + w(\epsilon, z)).$$

Lemma B. If $z_\epsilon \in Z$ is a critical point of Φ_ϵ then $u_\epsilon = z_\epsilon + w(\epsilon, z_\epsilon)$ is a solution of (1) with $\lambda = \mu + \epsilon$. Furthermore, if $z_\epsilon \neq 0$ and $\|z_\epsilon\| \longrightarrow 0$ as $|\epsilon| \longrightarrow 0$, then $u_\epsilon \neq 0$ and $\|u_\epsilon\| \longrightarrow 0$.

PROOF. If $z_\epsilon \in Z$ is a critical point of Φ_ϵ there results

$$(f'(u_\epsilon) \mid \zeta + D_z w(\epsilon, z_\epsilon)[\zeta]) = 0, \quad \forall\, \zeta \in Z.$$

Recall that $Pf'(z + w(\epsilon, z)) = 0$ for all $z \in Z$. In particular, $Pf'(u_\epsilon) = 0$, namely $f'(u_\epsilon) \in Z$. Since $D_z w(\epsilon, z_\epsilon)[\zeta] \in W$ we infer

$$(f'(u_\epsilon) \mid D_z w(\epsilon, z_\epsilon)[\zeta]) = 0, \quad \forall\, \zeta \in Z.$$

Thus $(f'(u_\epsilon) \mid \zeta) = 0, \quad \forall\, \zeta \in Z$. Using again the fact that $Pf'(u_\epsilon) = 0$ we conclude that $f'(u_\epsilon) = 0$. ■

Let $M \geq m > 0$ (if $m \leq M < 0$, we simply consider $\varepsilon < 0$ or $-\Phi_\varepsilon$ with $\varepsilon > 0$) and let us prove that Φ_ε has a Mountain-Pass critical point for $\varepsilon > 0$ small.

Let us evaluate $\Phi_\varepsilon(z)$. One has (for brevity we write w instead of $w(\epsilon, z)$)

$$\Phi_\epsilon(z) = \frac{\epsilon}{2}\|z\|^2 + \frac{1}{2}(\mu + \epsilon)\|w\|^2 - \frac{1}{2}(Lw \mid w) - h(z + w).$$

Since w satisfies $(\mu I - L)w + \epsilon(z + w) = H(z + w)$ it follows that

$$(\mu + \epsilon)\|w\|^2 - (Lw \mid w) = (H(z + w) \mid w)$$

thus

$$\Phi_\epsilon(z) = \frac{\epsilon}{2}\|z\|^2 + \frac{1}{2}(H(z + w) \mid w) - h(z + w).$$

Moreover, for some $s \in (0, 1)$

$$h(z + w) = h(z) + (H(z + sw) \mid w).$$

Hence we find

$$\Phi_\epsilon(z) = \frac{\epsilon}{2}\|z\|^2 - h(z) + \frac{1}{2}(H(z+w) \mid w) - (H(z+sw) \mid w). \tag{3}$$

Next, let us take $\mu < m/(1+2^k)$.

Since $h'(u) = H(u)$ and $D^j h(0) = 0,\ \forall\, j \le k-1,\ \exists\, \rho = \rho_\mu > 0$ s.t.

$$\|H(u)\| \le \mu \|u\|^{k-1}, \qquad \forall\, \|u\| < \rho,$$

and

$$h(z) = \alpha_k(z) + \beta(z), \qquad |\beta(z)| \le \mu \|z\|^k,\ \forall\, \|z\| < \rho.$$

Lemma A implies that for all $r < \rho/2$ there exists $\varepsilon_0 > 0$ such that

$$\|w(\varepsilon, z)\| \leq \|z\|, \qquad \forall\ \|z\| < r,\ \ \forall\ \varepsilon < \varepsilon_0$$

and hence, if $\|z\| < r$ and $\varepsilon < \varepsilon_0$, one has that

$$\|z + w(\varepsilon, z)\| \leq 2\|z\| < 2r < \rho$$

and this yields

$$\|H(z + w(\varepsilon, z))\| \leq \mu 2^{k-1}\, \|z\|^{k-1}, \qquad \forall\ \|z\| < r,\ \ \forall\ \varepsilon < \varepsilon_0.$$

Then

$$\Phi_\epsilon(z) = \frac{\epsilon}{2}\|z\|^2 - h(z) + \frac{1}{2}(H(z+w) \mid w) - (H(z+sw) \mid w)$$

where

$$|(H(z+w) \mid w)| \leq \|H(z+w)\| \times \|w\| \leq \mu 2^{k-1} \|z\|^k, \qquad \forall \|z\| < r, \ \forall \varepsilon < \varepsilon_0.$$

and

$$h(z) = \alpha_k(z) + \beta(z), \qquad |\beta(z)| \leq \mu \|z\|^k, \ \forall \|z\| < \rho.$$

In conclusion, we have found that

$$\Phi_\epsilon(z) = \frac{\epsilon}{2}\|z\|^2 - \alpha_k(z) + R(\varepsilon, z)$$

where $R(\varepsilon, z) = \frac{1}{2}(H(z+w) \mid w) - (H(z+sw) \mid w) + \beta(z)$ satisfies

$$|R(\varepsilon, z)| \le \mu 2^k \|z\|^k + \mu \|z\|^k, \qquad \forall \|z\| < r, \ \ \forall \varepsilon < \varepsilon_0.$$

- From $\Phi_\varepsilon(z) > 0$ we find for $\|z\| < r$ and $\varepsilon < \varepsilon_0$:

$$\frac{\epsilon}{2}\|z\|^2 > \alpha_k(z) - R(\varepsilon, z) \ge m\|z\|^k - \mu(1+2^k)\|z\|^k = [m - \mu(1+2^k)]\|z\|^k$$

Since $m > \mu(1+2^k)$ and $k \ge 3$ it follows that the set $\{\Phi_\varepsilon(z) > 0\}$ is bounded and contained, for ε small, in the ball $\{z \in Z : \|z\| < \rho\}$.

- Φ_ε has a local strict minimum at $z = 0$.

- Furthermore, using (A_3) one has (for $\varepsilon > 0$ small)

$$\Phi_\varepsilon(t\tilde{z}) = \frac{1}{2}t^2 - t^k\alpha(\tilde{z}) + R(\varepsilon, t^k) \to -\infty, \qquad (t \to +\infty).$$

- Since the set $\{\Phi_\varepsilon(z) > 0\}$ is bounded, it follows that (PS) holds.

Applying the Mountain-Pass theorem to Φ_e we find a critical point z_ε. This completes the proof in the case that (A_4) holds.

(A_4) can be substituted by a different assumption.

Let $\xi \in \partial B_Z$, resp. $\eta \in \partial B_Z$, be such that $\alpha_k(\xi) = M$, resp. $\alpha_k(\eta) = m$. We assume

(A_5) kM and km are not eigenvalues of the matrix $D^2\alpha_k(\xi)$, resp. $D^2\alpha_k(\eta)$.

To use (A_5) we consider again the auxiliary functional

$$\Phi_\epsilon(z) = \frac{\epsilon}{2}\|z\|^2 - \alpha_k(z) + R(\varepsilon, z).$$

Let

$$\Gamma_\epsilon(z) = \frac{1}{2}\epsilon\,\|z\|^2 - \alpha_k(z), \qquad z \in Z.$$

Since $\alpha_k \not\equiv 0$, either $M := \max_T \alpha_k > 0$ or $\min_T \alpha_k < 0$. Assume the former: in the other case it suffices to consider $-\varepsilon$ instead of ε.

The functional Γ_ε has the Mountain-Pass geometry.

Let $\xi \in T$ be a point where M is achieved. By homogeneity it immediately follows that $\alpha_k'(\xi) = k\alpha(\xi)\xi = kM\xi$.

Moreover, $p_\varepsilon = t_\varepsilon \xi$ is a critical point of Γ_ε whenever t_ε satisfies the equation

$$t^{k-2} = \frac{\varepsilon}{kM} \qquad (\varepsilon > 0).$$

It is easy to check that p_ε is the Mountain-Pass critical point of Γ_ϵ we were seeking. Let us explicitely point out that one has $p_\epsilon \longrightarrow 0$ as $\epsilon \longrightarrow 0^+$.

Lemma C. p_ϵ is a non-degenerate mountain-pass critical point of Γ_ϵ and there results

$$i(\Gamma'_\epsilon, p_\epsilon) = -1. \tag{4}$$

PROOF. Let I_Z denote the identity in Z. There results

$$D^2\Gamma_\epsilon(p_\epsilon) = \epsilon I_Z - D^2\alpha_k(p_\epsilon).$$

Since $p_\epsilon = t_\epsilon \xi$ one finds

$$D^2\Gamma_\epsilon(p_\epsilon) = \epsilon I_Z - t_\epsilon^{k-2} D^2\alpha_k(\xi) = \epsilon I_Z - \frac{\epsilon}{kM} D^2\alpha_k(\xi).$$

By (A_5) kM is not an eigenvalue of $D^2\alpha_k(\xi)$. Hence $D^2\Gamma_\epsilon(p_\epsilon)$ is invertible and p_ϵ is a non degenerate critical point of Γ_ϵ.

As an non degenerate mountain-pass critical point, it is well known that (4) holds. ■

- Lemma C
- $\Phi_\varepsilon(z) = \Gamma_\varepsilon(z) + R(\varepsilon, z)$, and
- the properties of the topological degree

imply that for $\epsilon > 0$ sufficiently small one also has

$$deg(\Phi'_\epsilon, B(p_\epsilon, \delta), 0) = -1, \quad \delta > 0 \text{ small.}$$

where $B(p_\epsilon, \delta)$ denote a ball in Z centered in p_ϵ with radius δ.

In particular Φ_ϵ has a critical point $z_\epsilon \in E$ in $B(p_\epsilon, \delta)$. ■

In fact, if (A_5) holds, we can sharpen Theorem 1.

• If Σ contains a connected set S such that $(\mu, 0) \in S$ and $S \setminus \{(\mu, 0)\} \neq \emptyset$, we will say that μ is a branching point.

Theorem 2 *Suppose that (A_1, A_2, A_3) and (A_5) hold and let μ be an isolated eigenvalue of finite multiplicity of L. Then μ is a branching point of (1).*

Assumption (A_3) rules out a counterexample of Böhme where $h \not\equiv 0$ is C^∞ with all the derivatives at $u = 0$ equal to zero and μ is not a branching point.

(A_5) rules out, e.g. α_k such that $\alpha_k(z) \equiv c\|z\|^k$ on Z. If this is violated there are examples showing that μ can be a bifurcation point but not a branching point.

Examples

Consider the bvp

$$(5) \qquad \begin{cases} -\lambda \, \Delta u &= u + G'(u), \quad \text{in } \Omega, \\ u &= 0, \quad \text{on } \partial\Omega, \end{cases}$$

where Ω is a bounded domain in $\mathbb{R}^n$ with smooth boundary $\partial\Omega$ and G satisfies, for some integer $k \geq 3$,
(G_1) $G \in C^k(\mathbb{R})$,
(G_2) $G(u) = \frac{1}{k}u^k + o(|u|^k)$, as $u \to 0$.
Let $E = H_0^1(\Omega)$ be the usual Sobolev space endowed with scalar product

$$(u \mid v) = \int_\Omega \nabla u \cdot \nabla v \, dx.$$

Define L and h by

$$(Lu \mid v) = \int_\Omega u(x)v(x)\, dx, \qquad h(u) = \int_\Omega G(u(x))\, dx.$$

Let us point out that the bifurcating solutions of (5) have norm which is small in E and, by regularity, in $C(\Omega)$. Thus, without loss of generality, we can assume that G is, say, quadratic at infinity so that h is well defined and smooth.

Setting $f(u) = \frac{1}{2}\lambda\|u\|^2 - \frac{1}{2}(Lu \mid u) - h(u)$ we get

$$f'(u) = \lambda u - Lu - h'(u)$$

Hence $(f'(u)|v) = 0$ is equivalent to $\lambda(u|v) - (Lu|v) - (h'(u)|v) = 0$ for all $v \in E$, namely

$$\lambda \int_\Omega \nabla u \cdot \nabla v \, dx = \int_\Omega u(x)v(x)\, dx + \int_\Omega G'(u(x))v(x)\, dx$$

Thus critical points of f are weak (and, by regularity, strong) solutions of (5).

Moreover, let μ be an eigenvalue of L with eigenfunction ϕ: $L\phi = \mu\phi$. From

$$(L\phi \mid v) = \mu(\phi \mid v), \qquad \forall v \in E$$

it follows that

$$\int_\Omega \phi v \, dx = \mu \int_\Omega \nabla\phi \cdot \nabla v \, dx \;\Rightarrow\; \phi = -\mu\Delta\phi.$$

Thus the μ are nothing but the characteristic value of $-\Delta$ on $H_0^1(\Omega)$.

It is immediate to verify that the assumptions (A_1) and (A_2) hold true.

Let $dimZ = dimKer[L - \mu I]$ be spanned by φ_1, φ_2.
Any $z \in Z = Ker[L - \mu I]$ has the form $z = z_1\varphi_1 + z_2\varphi_2$. Then we find

$$\alpha_k(z) = \frac{1}{k} \int_\Omega (z_1\varphi_1 + z_2\varphi_2))^k \, dx.$$

Then (A_3) holds if $\exists \, (z_1, z_2) \in \mathbb{R}^2$ such that

$$\int_\Omega (z_1\varphi_1 + z_2\varphi_2))^k \, dx \neq 0.$$

In particular, (A_3) is always satisfied if k is even.
If k is odd, say $k = 3$, (A_3) holds provided e.g. at least one of the following integrals

$$\int_\Omega \varphi_1^3, \quad \int_\Omega \varphi_1^2\varphi_2, \quad \int_\Omega \varphi_1\varphi_2^2, \quad \int_\Omega \varphi_2^3$$

is different from zero.

As for (A_5), a straight calculation shows:

1) let $k = 3$ and let

$$\int_\Omega \varphi_1^3 = \int_\Omega \varphi_2^3 = 1, \int_\Omega \varphi_1^2\varphi_2 = \int_\Omega \varphi_1\varphi_2^2 = 0.$$

Then (A_5) holds.

2) let $k = 4$ and let

$$\int_\Omega \varphi_1^4 = \int_\Omega \varphi_2^4 = 1, \int_\Omega \varphi_1^2\varphi_2^2 = a, \int_\Omega \varphi_1^3\varphi_2 = \int_\Omega \varphi_1\varphi_2^3 = 0.$$

Then (A_5) holds for all a but $a = 1$.