2. Bifurcation for problems on $\mathbb{R}^n$ in the presence of eigenvalues

We will consider the elliptic problem on $\mathbb{R}^n$ of the type

$$\text{(P)} \qquad -\Delta u + q(x)u = \lambda u \pm u^p, \qquad u \in W^{1,2}(\mathbb{R}^n).$$

In the sequel we will always assume that

$$q \in L^2(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n), \qquad 1 < p < 2^* - 1.$$

and

$$\lim_{|x|\to\infty} q(x) = 0.$$

The linearized problem at $u = 0$ is

$$\text{(L)} \qquad -\Delta u + q(x)u = \lambda u, \qquad u \in W^{1,2}(\mathbb{R}^n),$$

The spectrum of (L) depends on the following number

$$(1) \qquad \Lambda := \inf \left\{ \int_{\mathbb{R}^n} [|\nabla u|^2 + qu^2] dx : u \in W^{1,2}(\mathbb{R}^n),\ \|u\|_{L^2} = 1 \right\}.$$

Precisely, it is well known that;

- If $\Lambda < 0$ then the spectrum contains eigenvalues. Indeed, Λ is the lowest eigenvalue of (L) and is simple.

- If $q(x) \geq 0$, then the spectrum is the whole half line $[0, \infty)$ and coincides with the essential spectrum.

The essential spectrum is the set of all points of the spectrum that are not isolated, jointly with the eigenvalues of infinite multiplicity.

General Ref.: C. Stuart

Consider the problem

$$\text{(P')} \qquad -\Delta u + q(x)u = \lambda u - u^p, \qquad u \in W^{1,2}(\mathbb{R}^n)$$

- and assume that $\Lambda < 0$.

Following a joint paper with J. Gamez, we will use an approximation procedure.
Problem (P') will be approximated by problems on balls $B_{R_k} = \{x \in \mathbb{R}^n : |x| < R_k\}$,

$$(P_k) \quad -\Delta u + q(x)u = \lambda u - u^p, \qquad u \in W_0^{1,2}(B_{R_k})) \qquad (R_k \to \infty).$$

The solutions u of (P_k) are extended to all of $\mathbb{R}^n$ by setting $u(x) \equiv 0$ for $|x| > R_k$.

Let $\Sigma^k = \{(\lambda, u) \in \mathbb{R} \times E : \lambda > 0, u > 0,\ -\Delta u + q(x)u = \lambda u - u^p\}$.

Let λ_{R_k} denote the first (lowest) eigenvalues of

$$-\Delta u + q(x)u = \lambda u, \qquad u \in W_0^{1,2}(B_{R_k}),$$

which is given by

$$\lambda_{R_k} = \inf \left\{ \int_{B_{R_k}} [|\nabla u|^2 + qu^2] dx : u \in W_0^{1,2}(B_{R_k}),\ \|u\|_{L^2} = 1 \right\}.$$

Comparing this with the definition (1) of Λ, it follows that

$$\lambda_{R_k} \downarrow \Lambda = \inf_{u \in W^{1,2}(\mathbb{R}^n) : \|u\|_{L^2} = 1} \int_{\mathbb{R}^n} [|\nabla u|^2 + qu^2] dx, \quad (R_k \to \infty).$$

In particular, if $\Lambda < 0$ then $\lambda_{R_k} < 0$ provided $R_k \gg 1$.

Problem (P_k) can be faced by the Rabinowitz global bifurcation theorem:

There exists an unbounded connected component Σ_0^k emanating from $(\lambda_k, 0)$ which lies on the right of λ_k.

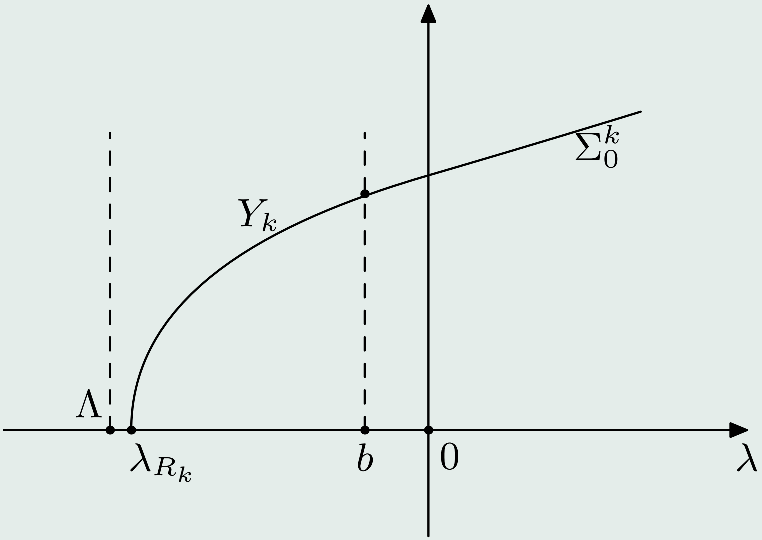
In order to perform a limit as $k \to +\infty$, we will use the following topological result:

Whyburn Lemma. Let Y be a metric space and Y_k a sequence of connected subsets of Y. Suppose that
(i) $\bigcup Y_k$ is precompact,
(ii) $\liminf Y_k \neq \emptyset$
Then $\limsup Y_k$ is precompact and connected.

$\liminf Y_k$ is the set of $y \in Y$ such that every neighborhood of y has nonempty intersection with all but a finite number of Y_k.

$\limsup Y_k$ is the set of $y \in Y$ such that every neighborhood of y has nonempty intersection with infinitely many of the Y_k.

In order to use this lemma, we take $E = W^{1,2}(\mathbb{R}^n)$, endowed with the standard norm. Fixed $b < 0$, let $Y = [\Lambda, b] \times E$ and let Y_k be the connected component of $\{(\lambda, u) \in \overline{\Sigma}_0^k : \lambda \in [\Lambda, b]\}$ such that $(\lambda_{R_k}, 0) \in \overline{\Sigma}_0^k$.



We also let $\Pi : \mathbb{R} \times E$ be defined by setting $\Pi(\lambda, u) = \lambda$.

It is not difficult to check that $\Pi(\overline{\Sigma}_0^k) = [\lambda_{R_k}, +\infty)$. Since $(\lambda_{R_k}, 0) \in \overline{\Sigma}_0^k$ and $\lambda_{R_k} \to \Lambda$, then $(\Lambda, 0) \in \liminf Y_k$ and thus (ii) holds.

Moreover, one has that $b \in \Pi(\overline{\Sigma}_0^k)$ for all $k \gg 1$.

In order to prove that $\bigcup Y_k$ is precompact, we need a preliminary lemma.

Lemma 1. Let $\Lambda < 0$. There exists $\Psi = \Psi_b \in L^2(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n)$, $\Psi > 0$, such that $u < \Psi$ for all $(\lambda, u) \in Y_k$, for all $k \gg 1$.

The proof (sketch) is carried out in 4 steps.

Step 1. Fix a with $b < a < 0$. Since $\lim_{|x|\to\infty} q(x) = 0$ and $a < 0$, the support of $(q(x) - a)^-$ (the negative part of $q - a$) is compact and is contained in the ball B_ρ, for some $\rho > 0$. We define a piecewise linear continuous function $\gamma_\alpha(t)$, $t \in \mathbb{R}$, such that

$$\gamma_\alpha(t) = \begin{cases} -\alpha & t \le \rho, \\ 0 & t \ge \rho + 1. \end{cases}$$

Let

$$\mu_\alpha = \inf \left\{ \int_{\mathbb{R}^n} [|\nabla u|^2 + \gamma_\alpha(|x|)u^2] dx : u \in E, \ \|u\|_{L^2} = 1 \right\}.$$

Since $\gamma_\alpha \leq 0$, it follows that

$$\mu_\alpha \leq \inf\{\int_{\mathbb{R}^n} |\nabla u|^2 dx : u \in E,\ \|u\|_{L^2} = 1\} = 0.$$

It is easy to see that there exists $\alpha^* > 0$ such that $\mu_\alpha < 0$ for all $\alpha > \alpha^*$. Moreover, μ_α is the principal eigenvalue of

$$-\Delta u + \gamma_\alpha(|x|)u = \mu u, \qquad u \in E.$$

We denote by $\varphi_\alpha > 0$ the (normalized) eigenfunction corresponding to $\mu_\alpha < 0$.

In addition, we notice that μ_α depends continuously upon α.

Step 2. From the preceding step it follows that we can find $\alpha_0 > 0$ such that $\mu_0 := \mu_{\alpha_0}$ verifies $b - a < \mu_0 < 0$.

We define a function $\psi \in C^2(\mathbb{R}^n) \cap E$ by setting $\psi(x) = \varphi_{\alpha_0}(x)$ for all $|x| \geq \rho + 1$; in the ball $B_{\rho+1}$ the function ψ is arbitrary, but positive.

One shows that there exists $C > 0$ such that $C\psi$ is a super-solution of (P_k) for all $k \geq 1$ and all $\lambda \geq b$.

Roughly, it is easy to check that for $C > 0$ sufficiently large one has that

$$-\Delta(C\psi) + q(C\psi) \geq \lambda(C\psi) - (C\psi)^p, \quad \forall\, |x| \leq \rho + 1.$$

For $|x| > \rho + 1$, one remarks that $\gamma_\alpha \equiv 0$, so $-\Delta\psi = \mu_0\psi$ and one finds $-\Delta\psi + q\psi = (\mu_0 + q)\psi$. The definition of ρ implies that $q > a$ for all $|x| > \rho$ and thus $-\Delta\psi + q\psi \geq (\mu_0 + a)\psi \geq b\psi$. Then for $\lambda \leq b$ we get $-\Delta\psi + q\psi \geq \lambda\psi - \psi^p$ for all $|x| > \rho + 1$, and the claim follows.

Step 3. One proves that $\Psi = C\psi$ is such that $u \leq \Psi$ for all $(\lambda, u) \in Y_k$ with k large. For $\lambda \leq b$, set $f_\lambda(u) := \lambda u - qu - u^p$ and take $M > 0$ such that $f_\lambda + M$ is strictly increasing for $u \in [0, \max \Psi]$. Let v_k be the solution of

$$\begin{cases} -\Delta v_k + M v_k = f_b(\Psi) + M\Psi & |x| < R_k, \\ v_k = 0 & |x| = R_k. \end{cases}$$

We want to show that for all $\lambda \leq b$, v_k is a super-solution of (P_k) but not a solution.

Since $f_b(\Psi) + M\Psi \geq 0$ then $v_k \in \mathcal{P}_k$, where $\mathcal{P}_k$ denotes the $interior$ of the positive cone in $C_0^1(B_{R_k})$.

From the preceding step we know that

$$-\Delta\Psi \geq b\Psi - q\Psi - \Psi^p = f_b(\Psi).$$

From this one easily infers

$$\begin{cases} -\Delta(\Psi - v_k) + M(\Psi - v_k) \geq 0 & |x| < R_k, \\ \Psi - v_k > 0 & |x| = R_k, \end{cases}$$

Then the maximum principle yields

$$(a) \qquad \Psi(x) > v_k(x), \qquad \forall\, |x| < R_k.$$

Since $f_\lambda + M$ is strictly increasing, it follows that

$$f_\lambda(\Psi) + M\Psi > f_\lambda(v_k) + Mv_k.$$

This and the fact that $f_b \geq f_\lambda$ provided $\lambda \leq b$, imply

$$-\Delta v_k = f_b(\Psi) + M\Psi - Mv_k \geq f_\lambda(\Psi) + M\Psi - Mv_k > f_\lambda(v_k), \quad |x| < R_k.$$

This proves our claim.

Step 4. Let us prove that $u < v_k$ for all $(\lambda, u) \in Y_k$. Consider the set $Y_k' = \{(\lambda, v_k - u) : (\lambda, u) \in Y_k\}$. Since $(\lambda_{R_k}, 0) \in Y_k$ then $(\lambda_{R_k}, v_k) \in Y_k'$, and thus $Y_k' \cap ([\Lambda, b] \times \mathcal{P}_k) \neq \emptyset$. Let us check that $Y_k' \subset [\Lambda, b] \times \mathcal{P}_k$. Otherwise, there exists $(\lambda^*, u^*) \in Y_k$ such that $v_k - u^* \in \partial\mathcal{P}_k$. Since v_k is not a solution of (P_k) it follows that $v_k \geq u^*$ but $v_k \not\equiv u^*$ in B_{R_k}. This implies $-\Delta(v_k - u^*) + M(v_k - u^*) \geq f_\lambda(v_k) + Mv_k - f_\lambda(u^*) + Mu^* \geq 0$. By the maximum principle we infer that $v_k > u^*$, namely $v_k - u^* \in \mathcal{P}_k$, while $v_k - u^* \in \partial\mathcal{P}_k$. This proves that $u < v_k$ and thus, using (a) we get $u < v_k < \Psi$ for all $|x| < R_k$, and the proof is completed.

Let us point out that we do not know whether $u < \Psi$ for all $(\lambda, u) \in \Sigma_0^k$, with $\lambda \in [\Lambda, b]$. The proof only works for $(\lambda, u) \in Y_k$.

The preceding lemma allows us to show

Lemma 2. $\bigcup Y_k$ is precompact.

Proof. Let $(\lambda_j, u_j) \in \bigcup Y_k$. We can assume that $\lambda_j \to \lambda$, for some $\lambda \in [\Lambda, b]$. From Lemma 1 it follows there is $c_1 > 0$ such that

$$\|u_j\|_{L^2} \leq c_1, \qquad \forall\, j.$$

From (P_k) we also get

$$(2) \qquad \int_{\mathbb{R}^n} |\nabla u_j|^2 dx + \int_{\mathbb{R}^n} q u_j^2 dx = \lambda \int_{\mathbb{R}^n} u_j^2 dx - \int_{\mathbb{R}^n} u_j^{p+1} dx.$$

From (2) it follows that $\exists\, c_2 > 0$ such that $\|u_j\| \leq c_2$ and hence, up to a subsequence, $u_j \rightharpoonup u$ in E.

Since u_j verifies

$$\int \nabla u_j \cdot \nabla \phi + \int q\, u_j\, \phi = \lambda \int u_j \phi - \int u_j^p \phi, \quad \forall\, \phi \in C_0^\infty(\mathbb{R}^n)$$

then u satisfies

$$\int \nabla u \cdot \nabla \phi + \int q u \phi = \lambda \int u \phi - \int u^p \phi,\ \forall\, \phi \in C_0^\infty(\mathbb{R}^n). \tag{3}$$

Set $G_\lambda(u) = \lambda u - qu - u^p$. Equation (2) can be written as

$$\|u_j\|^2 = \int u_j^2 + \int G_{\lambda_j}(u_j) u_j. \tag{4}$$

Moreover, by density, we can set $\phi = u_j$ in (3) yielding

$$\int \nabla u_j \cdot \nabla u = \int G_\lambda(u) u_j \tag{5}$$

Similarly, letting $\phi = u$, we get $\int |\nabla u|^2 = \int G_\lambda(u) u$ and hence

$$\|u\|^2 = \int G_\lambda(u) u + \int_{\mathbb{R}^n} u^2. \tag{6}$$

Using (4), (5) and (6), we infer

$$
\begin{aligned}
\|u_j - u\|^2 &= \|u_j\|^2 + \|u\|^2 - 2\int \nabla u_j \cdot \nabla u - 2\int u_j u \\
&= \int u_j^2 + \int G_{\lambda_j}(u_j)u_j + \int G_\lambda(u)u + \int u^2 \\
&\quad -2\int G_\lambda(u)u_j - 2\int u_j u \\
&= \int \left[G_{\lambda_j}(u_j) - G_\lambda(u)\right] u_j + \int G_\lambda(u)[u - u_j] \\
&\quad + \int u\,[u - u_j] + \int u_j[u_j - u].
\end{aligned}
$$

Since $u_j < \Psi \in L^2(\mathbb{R}^n)$ we find

$$
\begin{aligned}
\|u_j - u\|^2 &\leq \int \left|G_{\lambda_j}(u_j) - G_\lambda(u)\right| \Psi + \int |G_\lambda(u)|\,|u - u_j| \\
&\quad + \int |u|\,|u - u_j| + \int \Psi|u_j - u|.
\end{aligned}
$$

Since

$$|G_{\lambda_j}(u_j) - G_\lambda(u)| \le |\lambda_j - \lambda|\,|u_j - u| + |q|\,|u_j - u| + |u_j^p - u^p|,$$

also taking into account that $u_j \rightharpoonup u$ in E, it readily follows that all the integrals in the right hand side of the preceding equation tend to zero. Thus $\|u_j - u\|^2 \to 0$, proving that $u_j \to u$ strongly in E.

We are now ready to prove our main result

Theorem. (A.A - J.L. Gamez) If (1) holds, then there exists a connected set $\Sigma_0 = \{(\lambda, u) \in \mathbb{R} \times E\}$ such that

(a) u is a positive solution of (P');

(b) $(\Lambda, 0) \in \overline{\Sigma}_0$ and $\Pi\overline{\Sigma}_0 \supset [\Lambda, 0)$.

Proof. We set $\Sigma_0 = \limsup Y_k \setminus \{(\Lambda, 0)\}$.

We use the Whyburn Lemma: Let Y be a metric space and Y_k a sequence of connected subsets of Y such that
(i) $\bigcup Y_k$ is precompact,
(ii) $\liminf Y_k \neq \emptyset$
Then $\limsup Y_k$ is precompact and connected.

Therefore Σ_0 is connected and it is easy to check that any $(\lambda, u) \in \Sigma_0$ is a non-negative solution of (P). To prove (a) we need to show that $u > 0$.

We have already remarked that for each $k \geq 1$, $(\lambda, u) \in \Sigma_k$ implies that $\lambda > \lambda_{R_k}$, and this yields that $(\lambda, u) \in \Sigma_0 \ \Rightarrow \ \lambda > \Lambda$. Suppose that there exist $(\lambda_j, u_j) \in Y_{k_j}$ such that $(\lambda_j, u_j) \to (\lambda, 0)$ as $k_j \to \infty$.
Recall that u_j satisfies:

$$-\Delta u_j + q u_j = \lambda_j u_j - u_j^p, \qquad u \in W_0^{1,2}(B_{R_j}).$$

Since $\lambda_{R_j} \downarrow \Lambda$ and $\lambda_j \to \lambda > \Lambda$, then given $\delta > 0$ there exists $\ell \in \mathbb{N}$ such that $\lambda_{R_j} < \Lambda + \delta < \lambda_j$, for all $k_j \geq \ell$. Then

$$-\Delta u_j + q u_j > (\Lambda + \delta) u_j - u_j^p, \qquad u \in W_0^{1,2}(B_{R_j}).$$

Therefore u_j is a super-solution of

$$(7) \qquad -\Delta u + qu = (\Lambda + \delta)u - u^p, \qquad u \in W_0^{1,2}(B_{R_\ell}).$$

One can also find $\varepsilon_j \ll 1$ such that $\varepsilon_j \varphi_1$ is a sub-solution of (7) such that $\varepsilon_j \varphi_1 \le u_j$ in B_{R_ℓ} and thus there exists a positive solution $\widetilde{u}_j$ of (7).

Since $u_j \to 0$, then also $\widetilde{u}_j \to 0$ and therefore $\Lambda + \delta$ is a bifurcation point of positive solutions of (7). This is not possible, since the unique bifurcation point of positive solutions of (7) is $\lambda_{R_\ell} < \Lambda + \delta$. This contradiction proves that $u > 0$.

Since $(\Lambda, 0) \in \limsup Y_k$ it follows immediately that $(\Lambda, 0) \in \overline{\Sigma}_0$.

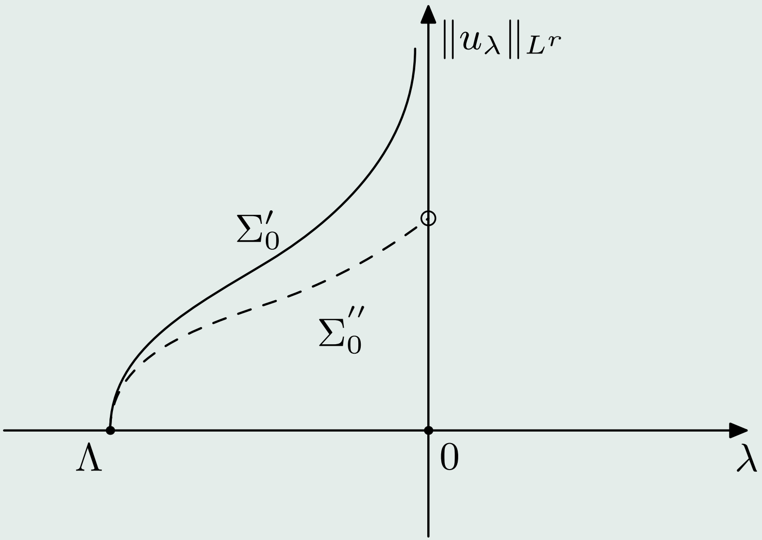
As already remarked before, $b \in \Pi(\overline{\Sigma}_0^k)$ for all $k \gg 1$ and all $b \in (\Lambda, 0)$. Repeating the arguments carried out in Lemma 2, it follows that $b \in \Pi(\overline{\Sigma}_0)$.

Finally, from the fact that Σ_0 is connected one deduces that $[\Lambda, 0) \subset \Pi(\overline{\Sigma}_0)$.

It is possible to complete the statement of the previous Theorem by showing that as $\lambda \uparrow 0$ the solutions u_λ such that $(\lambda, u_\lambda) \in \Sigma_0$ satisfy:

(i) $\|u_\lambda\|_{L^r} \leq$ const. if $r > n/(n-2)$;

(ii) $\|u_\lambda\|_{L^r} \to \infty$ if $r \leq n/(n-2)$.



By similar arguments one can handle sublinear problems on $\mathbb{R}^n$, see Brezis and Kamin.

Theorem. Let $\rho \in L^\infty$, and suppose that $\exists U \in L^\infty \cap L^2$ such that $-\Delta U = \rho$ in $\mathbb{R}^n$. Then, for all $0 < q < 1$ the problem

$$-\Delta u = \lambda \rho(x) u^q, \qquad u \in W^{1,2}(\mathbb{R}^n),$$

possesses a branch Σ of positive solutions bifurcating from $(0,0)$ and such that $\Pi(\overline{\Sigma})$.