Selected Topics on Bifurcation

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Plan of the lectures:

1. The Rabinowitz global bifurcation theorem

2. Bifurcation for problems on $\mathbb{R}^n$ in the presence of eigenvalues.

3. A short review on Critical Point theory.

4. Bifurcation for variational operators.

5. Bifurcation from the essential spectrum.

6. Bifurcation and perturbation.

References: • A.A. and A. Malchiodi, *Nonlinear Analysis and Semilinear Elliptic Problems*, Cambridge Studies in Adv. Math. n.104 (2007), C.U.P.

• A.A. and A. Malchiodi, *Perturbation Methods nd Semilinear Elliptic Problems on* $\mathbb{R}^n$, Progress in Math. n.240 (2005), Birkhäuser.

1. The Rabinowitz global bifurcation theorem

Let X be a Banach space, $A \in L(X)$ and $T \in C^1(X, X)$ be compact and such that $T(0) = 0$ and $T'(0) = 0$.
We also set $S_\lambda(u) = u - \lambda Au - T(u)$ and denote by Σ the set

$$\Sigma = \{(\lambda, u) \in \mathbb{R} \times X,\ u \neq 0 : S_\lambda(u) = 0\}.$$

If $(\lambda^*, 0) \in \overline{\Sigma}$ then λ^* is a bifurcation point for $S_\lambda = 0$.

A connected component of $\overline{\Sigma}$ is a closed connected set $\mathcal{C} \subset \overline{\Sigma}$ which is maximal with respect to the inclusion.
Let me recall that the Krasnoselski bifurcation theorem says that:

if λ^* is an odd characteristic value of A, then λ^* is a bifurcation point, namely $(\lambda^*, 0) \in \overline{\Sigma}$.

Let $\mathcal{C}$ be the connected component of $\overline{\Sigma}$ containing $(\lambda^*, 0)$.
We are going to discuss a celebrated paper by P. Rabinowitz, which improves the Krasnoselski result by showing that $\mathcal{C}$ is either unbounded or meets

another bifurcation point of $S_\lambda = 0$. The set of characteristic values of A will be denoted by $r(A)$.

Theorem. Let $A \in L(X)$ be compact and let $T \in C^1(X,X)$ be compact and such that $T(0) = 0$ and $T'(0) = 0$. Suppose that λ^* is an odd characteristic value of A. Let $\mathcal{C}$ be the connected component of $\overline{\Sigma}$ containing $(\lambda^*, 0)$. Then either

(a) $\mathcal{C}$ is unbounded; or

(b) $\exists \hat{\lambda} \in r(A) \setminus \{\lambda^*\}$ such that $(\hat{\lambda}, 0) \in \mathcal{C}$.

Although alternative (b) can arise as well, in many applications to nonlinear eigenvalue problems it is possible to rule out the alternative (b).

A case in which this is possible is when one deals with nonlinear Sturm Liouville problems, modeled by

$$\begin{cases} -u'' &= \lambda u + f(x, u, u'), \quad x \in (0, \pi), \\ u(0) &= u(\pi) = 0, \end{cases} \tag{1}$$

where f is Lipschitz and $f(x,u,\xi) = o(\sqrt{u^2+|\xi|^2})$ as $(u,\xi) \longrightarrow (0,0)$, uniformly with respect to $x \in [0,\pi]$.

The numbers k^2, $k \in \mathbb{N}$, are simple eigenvalues of the linearized problem $-u'' = \lambda u$, $u(0) = u(\pi) = 0$ and hence are bifurcation points for (1). One has:

Theorem. From each k^2, $k \in \mathbb{N}$, bifurcates an unbounded connected components $\mathcal{C}_k \subset \Sigma$ of non-trivial solutions of (1). Moreover $\mathcal{C}_k \cap \mathcal{C}_j = \emptyset$ if $k \neq j$.

PROOF. One works on $E = \{u \in C^1(0,\pi) : u(0) = u(\pi) = 0\}$ endowed with the standard norm.

First one shows that there exists a neighborhood U_k of $(k^2, 0) \in \mathbb{R} \times E$ such that if $(\lambda, u) \in \Sigma \cap U_k$, then u has exactly $k-1$ simple zeroes in $(0,\pi)$.

Moreover, by the uniqueness of the Cauchy problem it follows that the non-trivial solutions of (1) have only simple zeros in $(0,\pi)$.

These two properties, together with the fact that the branch $\mathcal{C}_k \subset \Sigma$ emanating from $(k^2, 0)$ is connected, allow us to rule out the alternative (b) and to show that $\mathcal{C}_k \cap \mathcal{C}_j = \emptyset$ if $k \neq j$, proving the theorem. ■

Concerning the global properties of the bifurcation branches, it is worth mentioning a classical global result by Leray and Schauder.

Theorem. Consider the equation $u = \lambda T(u)$, where $T \in C(X, X)$ is compact, let $\Sigma = \{(\lambda, u) \in \mathbb{R} \times X : u = \lambda T(u)\}$ and let $\mathcal{C}$ denote the connected component of $\overline{\Sigma}$ containing $(0, 0)$. Then $\mathcal{C} = \mathcal{C}^+ \cup \mathcal{C}^-$ where $\mathcal{C}^\pm \subset \mathbb{R}^\pm \times X$ and $\mathcal{C}^+ \cap \mathcal{C}^- = \{(0, 0)\}$.

The global features of the bifurcation set can also be exploited in the case in which we deal with the existence of positive solutions of a class of asymptotically linear elliptic boundary value problems like

$$(2) \qquad \begin{cases} -\Delta u = \lambda f(u), & x \in \Omega, \\ u = 0, & x \in \partial\Omega, \end{cases}$$

where $f \in C(\mathbb{R}^+, \mathbb{R})$ is asymptotically linear.

Let us start with an abstract setting. Let X be a Banach space and consider a map $S(\lambda, u) = u - \lambda T(u)$, with $T \in C(X, X)$ compact. We set $\Sigma = \{(\lambda, u) \in \mathbb{R} \times X \setminus \{0\} : S(\lambda, u) = 0\}$.

To investigate the asymptotic behavior of Σ, it is convenient to introduce the definition of bifurcation from infinity.

Definition. *We say that $\lambda_\infty \in \mathbb{R}$ is a bifurcation from infinity for $S = 0$ if there exist $\lambda_j \to \lambda_\infty$ and $u_j \in X$, such that $\|u_j\| \to \infty$ and $(\lambda_j, u_j) \in \Sigma$.*

Let us now assume that $T = A + G$, with A linear and G bounded. Let us set $z = \|u\|^{-1} u$, and

$$(3) \qquad \Psi(\lambda, z) = \begin{cases} z - \lambda \|z\|^2 T\left(\frac{z}{\|z\|^2}\right), & \text{if } z \neq 0, \\ 0 & \text{if } z = 0. \end{cases}$$

For $z \neq 0$ one has that

$$\Psi(\lambda, z) = z - \lambda A z - \lambda \|z\|^2 G\left(\frac{z}{\|z\|^2}\right).$$

Since G is bounded, then Ψ is continuous at $z = 0$.

Moreover, setting

$$\Gamma = \{(\lambda, z) : z \neq 0,\ \Psi(\lambda, z) = 0\},$$

there holds

$$(\lambda, u) \in \Sigma \iff (\lambda, z) \in \Gamma. \tag{4}$$

In addition, $\|u_j\| \to \infty$ if and only if $\|z_j\| = \|u_j\|^{-1} \to 0$. This and (4) immediately imply

Lemma. λ_∞ is a bifurcation from infinity for $S = 0$ if and only if λ_∞ is a bifurcation from the trivial solution for $\Psi = 0$. In such a case we will say that Σ bifurcates from (λ_∞, ∞).

Let $g \in C^{0,\alpha}(\mathbb{R}^+, \mathbb{R})$ be such that

$$(5) \qquad f(u) = mu + g(u), \quad m > 0, \quad |g(u)| \leq \text{Const.}, \quad g(0) \geq 0.$$

Theorem. Let (5) hold. Then $\lambda_\infty := \lambda_1/m$ is a bifurcation from infinity for S, and the only one. More precisely, there exists a connected component Σ_∞ of Σ bifurcating from (λ_∞, ∞) which corresponds to an unbounded connected component $\Gamma_\infty \subset \Gamma$ bifurcating from the trivial solution of $\Psi_\lambda(u) = 0$ at $(\lambda_\infty, 0)$.

Using similar arguments one can study the bifurcation from the trivial solution for $S_\lambda = 0$, yielding

Theorem. Let (5) hold. Then

(a) If $f(0) > 0$ there exists an unbounded connected component $\Sigma_0 \subset \Sigma$, with $\Sigma_0 \subset]0, \infty) \times X$, such that $(0, 0) \in \overline{\Sigma}_0$. Moreover, $(\lambda, 0) \in \overline{\Sigma}_0 \Rightarrow \lambda = 0$.

(b) If $f(0) = 0$ and the right-derivative $f'_+(0)$ exists and is positive, then letting

$$\lambda_0 := \frac{\lambda_1}{f'_+(0)},$$

there exists an unbounded connected component $\Sigma_0 \subset \Sigma$ such that $(\lambda_0, 0) \in \overline{\Sigma}_0$ and $(\lambda, 0) \in \overline{\Sigma}_0 \Rightarrow \lambda = \lambda_0$.

The next theorem studies the relationships between Σ_∞ and Σ_0.

Theorem. Suppose that the same assumptions made in the previous Theorems hold. Then

(a) If $\exists\, \alpha > 0$ such that $f(u) \geq \alpha u$, $\forall\, u \geq 0$, then setting $\Lambda = \lambda_1/\alpha$ one has that $\Sigma_0 \subset]0, \Lambda]$. As a consequence, $\Sigma_0 = \Sigma_\infty$.

(b) If $\exists\, s_0 > 0$ such that $f(s_0) \leq 0$, then $S_\lambda(u) \neq 0$ for all $u \in X$ with $\|u\|_\infty = s_0$. As a consequence, $\Sigma_0 \cap \Sigma_\infty = \emptyset$.

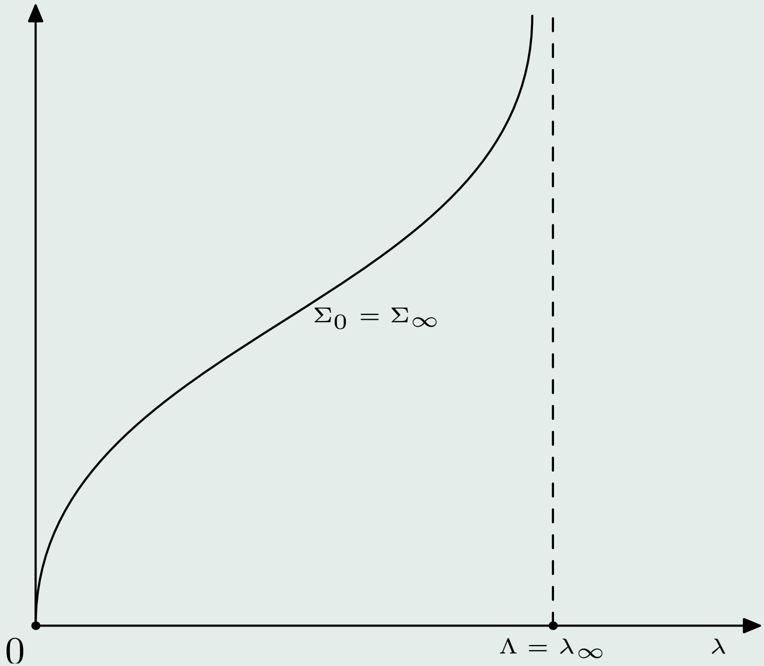

Figure 1: Bifurcation diagram in case (a), with $f(0) > 0$ and $\Lambda = \lambda_\infty$.

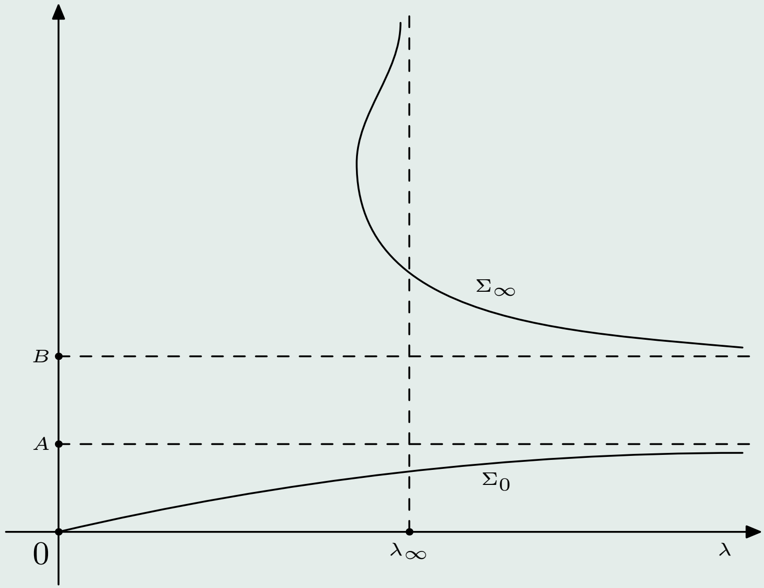

Figure 2: Bifurcation diagram in case (b), with $f(0) > 0$. The interval $[A, B]$ is such that $f(u) \leq 0$ if and only if $u \in [A, B]$.

Finally, it is possible to give conditions that allow us to describe in a precise way the behavior of the branch bifurcating from infinity. They are the counterparts of the conditions that provide a sub-critical or a super-critical bifurcation from the trivial solution.
Precisely, suppose that either

$$\gamma' := \liminf_{u \to +\infty} g(u) > 0, \tag{6}$$

or

$$\gamma'' := \limsup_{u \to +\infty} g(u) < 0. \tag{7}$$

Then Σ_∞ bifurcates to the left, respectively to the right, of (λ_∞, ∞).

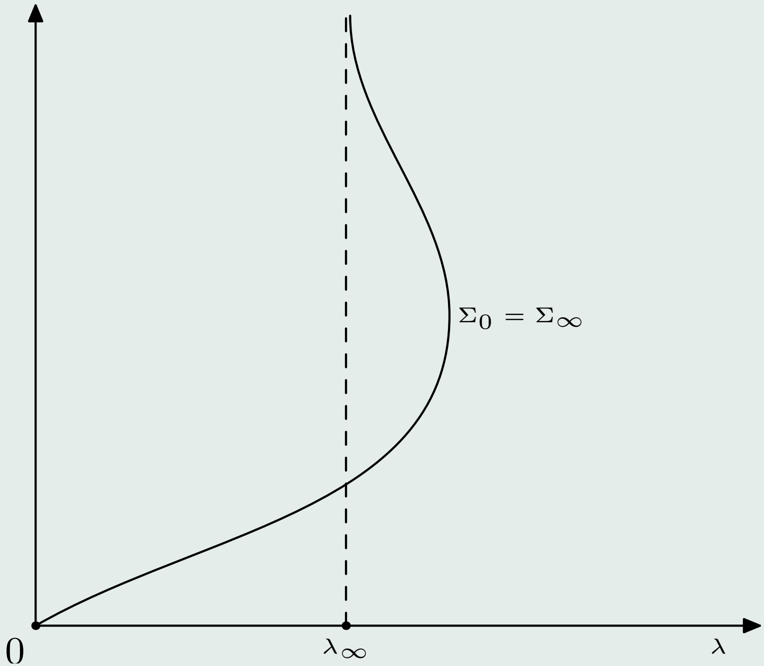

Figure 3: Bifurcation diagram when $f(0) > 0$ and $\gamma" < 0$