

How to approximate the heat equation with Neumann boundary conditions by nonlocal diffusion problems

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Non-local diffusion.

The function J . Let $J : \mathbf{R}^N \rightarrow \mathbf{R}$, nonnegative, smooth with

$$\int_{\mathbf{R}^N} J(r) dr = 1.$$

Assume that is compactly supported and radially symmetric.

Non-local diffusion equation

$$u_t(x, t) = J * u - u(x, t) = \int_{\mathbf{R}^N} J(x - y) u(y, t) dy - u(x, t).$$

Non-local diffusion.

In this model, $u(x, t)$ is the density of individuals in x at time t and $J(x - y)$ is the probability distribution of jumping from y to x . Then

$$(J * u)(x, t) = \int_{\mathbf{R}^N} J(x - y)u(y, t)dy$$

is the rate at which the individuals are arriving to x from other places

$$-u(x, t) = - \int_{\mathbf{R}^N} J(y - x)u(x, t)dy$$

is the rate at which they are leaving from x to other places.

References

- P. Bates, P- Fife, X. Ren, X. Wang. Arch. Rat. Mech. Anal. (1997).
- P. Fife. Trends in nonlinear analysis. Springer, 2003.

Non-local diffusion.

The non-local equation shares some properties with the classical heat equation

$$u_t = \Delta u.$$

Properties

- Existence, uniqueness and continuous dependence on the initial data.
- Maximum and comparison principles.
- Perturbations propagate with infinite speed. If u is a nonnegative and nontrivial solution, then $u(x, t) > 0$ for every $x \in \mathbf{R}^N$ and every $t > 0$.

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There is no regularizing effect for the non-local model.

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Newmann boundary conditions.

One of the boundary conditions that has been imposed to the heat equation is the *Neumann boundary condition*,
 $\partial u / \partial \eta(x, t) = g(x, t)$, $x \in \partial \Omega$.

Non-local Neumann model

$$u_t(x, t) = \int_{\Omega} J(x-y)(u(y, t) - u(x, t)) dy + \int_{\mathbf{R}^N \setminus \Omega} G(x-y)g(y, t) dy$$

for $x \in \Omega$.

Since we are integrating in Ω , we are imposing that diffusion takes place only in Ω . The last term takes into account the prescribed flux (given by the data $g(x, t)$) of individuals from outside.

Existence, uniqueness and a comparison principle

Theorem

For every $u_0 \in L^1(\Omega)$ and $g \in L_{loc}^\infty((0, \infty); L^1(\mathbf{R}^N \setminus \Omega))$ there exists a unique solution u such that $u \in C([0, \infty); L^1(\Omega))$ and $u(x, 0) = u_0(x)$.

Moreover the solutions satisfy the following comparison property:

if $u_0(x) \leq v_0(x)$ in Ω , then $u(x, t) \leq v(x, t)$ in $\Omega \times [0, \infty)$.

In addition the total mass in Ω satisfies

$$\int_{\Omega} u(y, t) dy = \int_{\Omega} u_0(y) dy + \int_0^t \int_{\Omega} \int_{\mathbf{R}^N \setminus \Omega} G(x-y) g(y, s) dy dx ds.$$

Asymptotic behavior

Theorem

Let $g(x, t) = h(x)$ such that

$$0 = \int_{\Omega} \int_{\mathbf{R}^N \setminus \Omega} G(x - y) h(y) dy dx.$$

Then there exists a unique solution φ of the problem

$$0 = \int_{\Omega} J(x - y)(\varphi(y) - \varphi(x)) dy + \int_{\mathbf{R}^N \setminus \Omega} G(x - y) h(y) dy$$

that verifies $\int_{\Omega} u_0 = \int_{\Omega} \varphi$ and there exists $\beta = \beta(J, \Omega) > 0$ such that

$$\|u(t) - \varphi\|_{L^2(\Omega)} \leq e^{-\beta t} \|u_0 - \varphi\|_{L^2(\Omega)}.$$

Asymptotic behavior

If the compatibility conditions does not hold then solutions are unbounded.

Here β_1 is given by

$$\beta_1 = \inf_{u \in L^2(\Omega), \int_{\Omega} u = 0} \frac{\frac{1}{2} \int_{\Omega} \int_{\Omega} J(x-y)(u(y) - u(x))^2 dy dx}{\int_{\Omega} (u(x))^2 dx}.$$

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F. Andreu, J. M. Mazón, J. D. R., J. Toledo. Preprint.

Approximations

Now, our goal is to show that the Neumann problem for the heat equation, can be approximated by suitable nonlocal Neumann problems.

More precisely, for given J we consider the rescaled kernels

$$J_\varepsilon(\xi) = C_1 \frac{1}{\varepsilon^N} J\left(\frac{\xi}{\varepsilon}\right), \quad G_\varepsilon(\xi) = C_1 \frac{1}{\varepsilon^N} G\left(\frac{\xi}{\varepsilon}\right)$$

with

$$C_1^{-1} = \frac{1}{2} \int_{B(0,d)} J(z) z_N^2 dz,$$

which is a normalizing constant in order to obtain the Laplacian in the limit instead of a multiple of it.

Approximations

Then, we consider the solution $u_\varepsilon(x, t)$ to

$$\left\{ \begin{array}{l} (u_\varepsilon)_t(x, t) = \frac{1}{\varepsilon^2} \int_{\Omega} J_\varepsilon(x - y)(u_\varepsilon(y, t) - u_\varepsilon(x, t)) dy \\ \quad + \frac{1}{\varepsilon} \int_{\mathbf{R}^N \setminus \Omega} G_\varepsilon(x - y)g(y, t) dy, \\ u_\varepsilon(x, 0) = u_0(x). \end{array} \right.$$

Note that the scaling of the diffusion, $1/\varepsilon^2$, is different from the scaling of the boundary flux, $1/\varepsilon$, as happens for the heat equation with Neumann boundary conditions.

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Approximations

Our first result deals with homogeneous boundary conditions, this is, $g \equiv 0$.

Theorem

Let $g = 0$ and let $u \in C^{2+\alpha, 1+\alpha/2}(\overline{\Omega} \times [0, T])$ be the solution to the heat equation with Neumann boundary conditions $\partial u / \partial \eta = 0$ and u_ε be the solution to the nonlocal model. Then,

$$\sup_{t \in [0, T]} \|u_\varepsilon(\cdot, t) - u(\cdot, t)\|_{L^\infty(\Omega)} \rightarrow 0$$

as $\varepsilon \rightarrow 0$.

Note that this result holds for every G since $g \equiv 0$, and that the assumed regularity in u is guaranteed if $u_0 \in C^{2+\alpha}(\overline{\Omega})$ and $\partial u_0 / \partial \eta = 0$.

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Approximations

Now let the “Neumann” kernel be given by

$$G(\xi) = C_2 J(\xi),$$

where C_2 is such that

$$\int_0^d \int_{\{z_N > s\}} J(z) (C_2 - z_N) dz ds = 0.$$

This choice of G is natural since we are considering a flux with a jumping probability that is a scalar multiple of the same jumping probability that moves things in the interior of the domain, J .

Approximations

In this case we can prove convergence with $g \neq 0$ but in a weaker sense.

Theorem

Let

$$g \in C^{1+\alpha, (1+\alpha)/2}(\overline{(\mathbf{R}^N \setminus \Omega)} \times [0, T]),$$

$$u \in C^{2+\alpha, 1+\alpha/2}(\overline{\Omega} \times [0, T])$$

the solution to the heat equation with Neumann boundary condition, $\partial u / \partial \eta = 0$ and u_ε be the solution to the nonlocal model. Then, for each $t \in [0, T]$

$$u_\varepsilon(x, t) \rightharpoonup u(x, t) \quad * - \text{weakly in } L^\infty(\Omega)$$

as $\varepsilon \rightarrow 0$.

Approximations

Idea of why the involved scaling is correct

Let us give an heuristic idea in one space dimension, with $\Omega = (0, 1)$, of why the scaling involved is the right one. We assume that

$$\int_1^\infty G(1 - y) dy = - \int_{-\infty}^0 G(-y) dy = \int_0^1 J(y) y dy.$$

Approximations

We have

$$\begin{aligned} u_t(x, t) = & \frac{1}{\varepsilon^2} \int_0^1 J_\varepsilon(x - y) (u(y, t) - u(x, t)) dy \\ & + \frac{1}{\varepsilon} \int_{-\infty}^0 G_\varepsilon(x - y) g(y, t) dy \\ & + \frac{1}{\varepsilon} \int_1^{+\infty} G_\varepsilon(x - y) g(y, t) dy := A_\varepsilon u(x, t). \end{aligned}$$

Approximations

If $x \in (0, 1)$ a Taylor expansion gives that for any fixed smooth u and ε small enough, the right hand side $A_\varepsilon u$ becomes

$$A_\varepsilon u(x) = \frac{1}{\varepsilon^2} \int_0^1 J_\varepsilon(x-y) (u(y) - u(x)) dy$$

$$= \frac{1}{\varepsilon^3} \int_0^1 J\left(\frac{x-y}{\varepsilon}\right) (u(y) - u(x)) dy$$

$$= \frac{1}{\varepsilon^2} \int_{\mathbf{R}} J(w) (u(x - \varepsilon w) - u(x)) dw$$

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Approximations

$$= \frac{u_x(x)}{\varepsilon} \int_{\mathbf{R}} J(w) w \, dw + \frac{u_{xx}(x)}{2} \int_{\mathbf{R}} J(w) w^2 \, dw + O(\varepsilon)$$

As J is even

$$\int_{\mathbf{R}} J(w) w \, dw = 0$$

and hence,

$$A_\varepsilon u(x) \approx u_{xx}(x),$$

and we recover the Laplacian for $x \in (0, 1)$.

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Approximations

If $x = 0$ and ε small,

$$A_\varepsilon u(0) = \frac{1}{\varepsilon^2} \int_0^1 J_\varepsilon(-y) (u(y) - u(0)) dy + \frac{1}{\varepsilon} \int_{-\infty}^0 G_\varepsilon(-y) g(y) dy$$

$$= \frac{1}{\varepsilon^3} \int_0^1 J\left(\frac{-y}{\varepsilon}\right) (u(y) - u(0)) dy + \frac{1}{\varepsilon^2} \int_{-\infty}^0 G\left(\frac{-y}{\varepsilon}\right) g(y) dy$$

$$= \frac{1}{\varepsilon^2} \int_{-\infty}^0 J(w) (-u(-\varepsilon w) + u(0)) dw + \frac{1}{\varepsilon} \int_0^{+\infty} G(w) g(-\varepsilon w) dw$$

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Approximations

$$= -\frac{u_x(0)}{\varepsilon} \int_{-\infty}^0 J(w) w \, dw + \frac{g(0)}{\varepsilon} \int_0^{+\infty} G(w) \, dw + O(1)$$

$$\approx \frac{C_2}{\varepsilon} (u_x(0) + g(0)).$$

then

$$-u_x(0) = g(0)$$

and we recover the boundary condition

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and we recover the boundary condition

Idea of the proof. General case

We set

$$w_\varepsilon = u_\varepsilon - u$$

and let $\tilde{u}$ be a $C^{2+\alpha, 1+\alpha/2}$ extension of u to $\mathbf{R}^N \times [0, T]$. We define

$$L_\varepsilon(v) = \frac{1}{\varepsilon^2} \int_{\Omega} J_\varepsilon(x - y)(v(y, t) - v(x, t)) dy$$

and

$$\tilde{L}_\varepsilon(v) = \frac{1}{\varepsilon^2} \int_{\mathbf{R}^N} J_\varepsilon(x - y)(v(y, t) - v(x, t)) dy.$$

Idea of the proof. General case

Then

$$\begin{aligned}(w_\varepsilon)_t &= L_\varepsilon(u_\varepsilon) - \Delta u + \frac{1}{\varepsilon} \int_{\mathbf{R}^N \setminus \Omega} G_\varepsilon(x, x-y) g(y, t) dy \\&= L_\varepsilon(w_\varepsilon) + \tilde{L}_\varepsilon(\tilde{u}) - \Delta u + \frac{1}{\varepsilon} \int_{\mathbf{R}^N \setminus \Omega} G_\varepsilon(x, x-y) g(y, t) dy \\&\quad - \frac{1}{\varepsilon^2} \int_{\mathbf{R}^N \setminus \Omega} J_\varepsilon(x-y) (\tilde{u}(y, t) - \tilde{u}(x, t)) dy.\end{aligned}$$

Or

$$(w_\varepsilon)_t - L_\varepsilon(w_\varepsilon) = F_\varepsilon(x, t),$$

Our main task in order to prove the uniform convergence result is to get bounds on F_ε .

Idea of the proof. General case

However, the proofs of our results are much more involved than simple Taylor expansions.

This is due to the fact that for each $\varepsilon > 0$ there are points $x \in \Omega$ for which the ball in which integration takes place, $B(x, \varepsilon)$, is not contained in Ω and moreover, when working in several space dimensions, one has to take into account the geometry of the domain.

THANKS !!!