Ambrosetti-Prodi Problem for Non-variational Elliptic Systems

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Let $f : \mathbb{R} \to \mathbb{R}$ be a $C^2$-fct s.t.

$(f_1)$ $f(0) = 0$ and $f''(t) > 0$, for all $t$,

$(f_2)$ $\lim_{t \to -\infty} f'(t) = l'$, with $0 < l' < \lambda_1$,

$$\lim_{t \to +\infty} f'(t) = l'', \text{ with } \lambda_1 < l'' < \lambda_2,$$

with $\Omega$ bdd smooth in $\mathbb{R}^N$, consider the Dirichlet problem:
$$\Delta u + f(u) = g \text{ in } \Omega, \quad u = 0 \text{ on } \partial \Omega, \quad (1)$$
Let $f : \mathbb{R} \to \mathbb{R}$ be a $C^2$-fct s.t.

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2. $\lim_{t \to -\infty} f'(t) = l'$, with $0 < l' < \lambda_1$,
   $\lim_{t \to +\infty} f'(t) = l''$, with $\lambda_1 < l'' < \lambda_2$,

with $\Omega$ bdd smooth in $\mathbb{R}^N$, consider the Dirichlet problem:

$$\Delta u + f(u) = g \text{ in } \Omega, \quad u = 0 \text{ on } \partial \Omega, \quad (1)$$

Then $\exists$ in $C^{0,\alpha}(\overline{\Omega})$ a closed connected $C^1$-manifold $M$ s.t.

$C^{0,\alpha}(\overline{\Omega}) \setminus M = A_0 \cup A_2$ (connected components) s.t

1. $g \in A_0 \Rightarrow (1)$ no solution,
2. $g \in M \Rightarrow (1)$ exactly one solution
3. $g \in A_2 \Rightarrow (1)$ exactly 2 solutions
The 1973 paper called much attention to this kind of BVP with nonlinearities of the above type.

As a matter of fact the A-P paper is much more than the PDE problem. It treats the inversion of differentiable mappings with singularities between Banach Spaces.
Earlier Results on Ambrosetti-Prodi

The 1973 paper called much attention to this kind of BVP with nonlinearities of the above type. As a matter of fact the A-P paper is much more than the PDE problem. It treats the inversion of differentiable mappings with singularities between Banach Spaces.

It appears that what we call the A-P phenomena has to do with the crossing of the first eigenvalue. This was soon realized and many papers appeared afterwards. A very partial list of earlier papers includes: Berger-Podolak, Fucik, Kazdan-Warner, Dancer, Hess, Berestycki, Solimini, Adimurthi-Srikanth,...
Let us put the problem in the present framework for the scalar case:

\[-Lu = f(x, u) + t\varphi_1(x) + h(x), \text{ in } \Omega, \ u = 0 \text{ on } \partial \Omega, \quad (2)\]

where \( L = \sum_{i,j=1}^{N} a_{ij}(x) \frac{\partial^2}{\partial x_i \partial x_j} + \sum_{i=1}^{N} b_i(x) \frac{\partial}{\partial x_i} \) is a general strongly elliptic operator.
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From Berestycki-Nirenberg-Varadhan, we know: \( \varphi_1 \) is a first positive eigenfunction of \(-L\varphi_1 = \lambda_1 \varphi_1, \) with \( \varphi_1 = 0 \) on \( \partial \Omega, \) and \( \lambda_1 > 0. \)
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Problem (2) is said of A-P type if

\[
\limsup_{s \to -\infty} \frac{f(x, s)}{s} \leq a' < \lambda_1 < b' \leq \liminf_{s \to +\infty} \frac{f(x, s)}{s}
\]
In the present framework the A-P statement for the case of one equation becomes:

\[(AP) \; \exists t_0 \in R, \text{ s.t. Problem (2) has at least two solutions for } t < t_0, \text{ one solution for } t = t_0 \text{ and no solution for } t < t - 0.\]
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One of the main difficulties in proving the statement on the existence of two solutions comes in the case that \( f \) is superlinear in \( u \). Variational and Topological Methods have been used.
Methods for existence of solutions-1

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If $L$ is of *divergence form*, one can use the Calculus of Variations to get the second solution. In general a first solution, for small $t$, can be obtained by the Method of Lower and Upper Solutions. To use Critical Point Theory, as usual, one needs some compactness which it is obtained by an appropriate growth on $f$ with respect to $u$. 
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Already in the 1980's this technique was used by deF-Solimini and K.C.Chang to obtain multiplicity for $f$ subcritical, namely $f(x, s) \sim s^p$, for $1 < p < \frac{N+2}{N-2}$. 
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For superlinear problems this is not easy matter. The first results in this area treated $f(x, s)$ with linear growth in $s$, or polynomial growth in $s$ with a power at most $\frac{N+1}{N-1}$. This restriction comes from the use of Hardy inequality as initiated by Brézis-Turner for existence of positive solutions for superlinear problems.
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Our results next improve this growth up to $\frac{N+2}{N-2}$, and also take care of the case of systems.
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Let us write the system in the form:

\[-L_1 u_1 = f_1(x, u_1, u_2) + t_1 \varphi_1 + h_1(x)\]
\[-L_2 u_2 = f_2(x, u_1, u_2) + t_2 \varphi_2 + h_2(x) \ (PS)_t\]
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\[f(x, .) = (f_1(x, .), f_2(x, .)) : \mathbb{R}^2 \to \mathbb{R}^2 \text{ is quasi-monotone},\]

that is, \(f_i(x, s)\) is non-decreasing in \(s_j, i \neq j\). This is for Max. Principle.
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\[f(x, \cdot) = (f_1(x, \cdot), f_2(x, \cdot)) : R^2 \to R^2 \text{ is quasi-monotone, that is, } f_i(x, s) \text{ is non-decreasing in } s_j, i \neq j.\] This is for Max. Principle.

An A-P result for system should state:

\[(APS): \exists \Gamma \subset R^2, \text{ a Lipschitz curve that splits } R^2 \text{ into two parts } A_0 \text{ and } A_2 \text{ s.t. problem } (PS)_t \text{ has at least two solutions if } t = (t_1, t_2) \in A_2, \text{ at least one solution if } t \in \Gamma, \text{ and no solution if } t \in A_0.\]
The condition of crossing the first eigenvalue in the scalar case can be restated as: \( \exists \) const’s \( a, b, C \) s.t. 
\[ \lambda_1(\Delta + a) > 0, \ \text{and} \ \lambda_1(\Delta + b) < 0, \ \text{where} \]
\[ f(x, s) \geq as - C \ \text{for} \ s \leq 0, \ \quad f(x, s) \geq bs - C \ \text{for} \ s \geq 0, \ \forall x \in \Omega \]
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For the system we have:
\( \exists \) cooperative matrices \( A_1(x), A_2(x), \), and constants \( b_1, b_2 \) s.t.\[ \lambda_1(L + A_1) > 0, \quad \lambda_1(L + A_2) < 0, \text{ where} \]
\[ f(x, s) \geq A_1(x)s - b_1e \text{ in } \{s \in \mathbb{R}^2, s \leq 0\} \]
\[ f(x, s) \geq A_2(x)s - b_2e \text{ in } \{s \in \mathbb{R}^2, s \geq 0\} \]
Here \( f(x, s) = (f_1(x, s), f_2(x, s)) \) and \( e = (e_1, e_2) \)
Let $A$ be a cooperative matrix. Busca-Sirakov extendend Berestycki-Nirenberg-Varadhan to systems. So the \emph{principal eigenvalue} of $L + A$ is defined by

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\lambda_1 = \lambda_1(L + A) = \sup \{ \lambda \in \mathbb{R} : \exists \psi \in W^{2,N}_{loc}(\Omega, \mathbb{R}^2), \text{ s.t. } \psi > 0, (L + A + \lambda I)\psi \leq 0 \text{ in } \Omega \}.
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the following statements are equivalent:

(i) $\lambda_1(L + A) > 0$

(ii) $\exists \psi \in W^{2,N}(\Omega, \mathbb{R}^2) \cap C(\overline{\Omega})$ s.t. $\psi \geq e, (L + A)\psi \leq 0$ in $\Omega$

(iii) $(L + A)$ satisfies the Max Principle:

If $(L + A)u \leq 0$ in $\Omega$ and $u \geq 0$ on $\partial \Omega$, then $u \geq 0$ in $\Omega$
The a priori bounds here are proved using the Blow-up Method, as introduced by Gidas-Spruck to treat the case of positive solutions for superlinear problems.
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For that matter we shall write the \( f_i \) as:

\[
\begin{align*}
    f_1(x, u_1, u_2) &= a(x)u_1^{\alpha_{11}} + b(x)u_2^{\alpha_{12}} + g_1(x, u_1, u_2) \\
    f_2(x, u_1, u_2) &= c(x)u_1^{\alpha_{21}} + d(x)u_2^{\alpha_{22}} + g_2(x, u_1, u_2),
\end{align*}
\]

where the \( g_i \) are the lower order terms. We assume that the \( \alpha_{ij} > 1 \) and the coefficients \( \in C(\bar{\Omega}) \) and \( \geq 0 \).
Construction of Blow-up pairs

In the Blow-up procedure the following lines appear naturally (here $\vec{\beta} = (\beta_1, \beta_2) \in R^2$):

$l_1 = \{\vec{\beta} | \beta_1 + 2 - \beta_1 \alpha_{11} = 0\}$,  $l_3 = \{\vec{\beta} | \beta_1 + 2 - \beta_2 \alpha_{12} = 0\}$

$l_4 = \{\vec{\beta} | \beta_2 + 2 - \beta_1 \alpha_{21} = 0\}$,  $l_2 = \{\vec{\beta} | \beta_2 + 2 - \beta_2 \alpha_{22} = 0\}$
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\[\Pi = \left(\frac{2}{\alpha_{22} - 1}, \frac{2}{\alpha_{21}}\right)\]
Blow-up pairs $\vec{\beta}$ are defined as the points that are in the intersection of at least two lines and lie to the left of or on $l_1$, below or on $l_2$, below or on $l_3$, and above or on $l_4$. 
A-P result for systems

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There are three cases:

(i) Case A. The intersection of $l_1$ and $l_2$ is a blow-up pair. Here $a(x) \geq c > 0$ and $d(x) \geq c > 0$. Take $(\beta^0_1, \beta^0_2) = l_1 \cap l_2$.

(ii) Case B. The intersection of $l_3$ and $l_4$ is a blow-up pair. Here $b(x) \geq c > 0$ and $c(x) \geq c > 0$. Take $(\beta^0_1, \beta^0_2) = l_3 \cap l_4$.

(iii) Case C. Neither $l_1 \cap l_2$ nor $l_3 \cap l_4$ is a blow-up pair. So either $l_1 \cap l_3$ or $l_2 \cap l_4$ is a blow-up pair. $b(x), c(x) \geq c > 0$. 

A-P result for systems
THEOREM (deF-Sirakov). Suppose there is a crossing of the first eigenvalue and a blow-up pair $\vec{\beta} = (\beta_1^0, \beta_2^0)$ can be chosen satisfying

$$\min\{\beta_1^0, \beta_2^0\} > \frac{N - 2}{2}, \quad \max\{\beta_1^0, \beta_2^0\} > N - 2.$$ 

Then (APS) holds.
LEMMA 1. (Subsolution) For any $t \in \mathbb{R}^2$ there is a subsolution $u \leq 0$ of the system.
Existence of a first solution

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(Proof Lemma 1). Take $K = 2 \max\{|h_i| + |t_i|\} + b_1$. By the Max Principle this subsolution is just a solution of the Dirichlet problem for:

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LEMMA 2. (Supersolution) \( \exists t_0 \in R \) s.t. for \( t \leq t_0 \) the system has a supersolution.
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(Proof Lemma 2). Choose \( p_1, p_2 \) s.t.

\[ f(x, s) \leq C_1 (1 + s_1^{p_1} + s_2^{p_2}) e. \]

Let \( \overline{u} \) be the solution of

\[ Lu + h^+ + C_1 e \text{ in } \Omega, \ u = 0 \text{ on } \partial \Omega. \]

Then the \( t_0 \in \mathbb{R} \) is chosen in such a way that \( -t_0 \varphi_1 \geq (\overline{u}_1^{p_1} + \overline{u}_2^{p_2}) \), which is possible by Hopf.

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Bound on negative part of solution

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**LEMMA 3.** For each \(C_0 \in R_+\), \(\exists \text{ const } M\), s.t. for \(t \geq -C_0 e\) and any solution \(u\) of \((PS_t)\) with this \(t\) we have \(\|u^-\| \leq M\).
One difficulty in applying the Blow-up Method comes from the fact that solutions of \((PS)_t\) change sign. So the process of passing to the limit in order to get a Liouville type of result involves a control on the negative part of solutions.

**Lemma 3.** For each \(C_0 \in \mathbb{R}^+\), \(\exists\) const \(M\), s.t. for \(t \geq -C_0 e\) and any solution \(u\) of \((PS)_t\) with this \(t\) we have 
\[\|u^-\| \leq M.\]

**Lemma 4.** For each \(C_0 \in \mathbb{R}^+\), \(\exists\) const \(C_1\), s.t. for \(t \geq -C_0 e\) and any solution \(u\) of \((PS)_t\) with this \(t\) we have 
\[t_i^+ \leq C_1(1 + \|u_i^+\|) \leq C_1(1 + \|u\|), \text{ for } i = 1, 2\]
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LEMMA 5. $\exists$ a const. $C$ s.t. $\forall t \geq e$ and every solution $u = (u_1, u_2)$ of $(PS)_t$ corresponding to this $t$, we have

$$\|u_1\|^{1+\frac{2}{\beta_1}} \leq Ct_1 \quad \text{and} \quad \|u_2\|^{1+\frac{2}{\beta_2}} \leq Ct_2.$$
LEMMA 4. For each $C_0 \in R_+$, $\exists$ const $C_1$, s.t. for $t \geq -C_0 e$ and any solution $u$ of $(PS)_t$ with this $t$ we have $t^+_i \leq C_1 (1 + \|u^+_i\|) \leq C_1 (1 + \|u\|)$, for $i = 1, 2$.

LEMMA 5. $\exists$ a const. $C$ s.t. $\forall t \geq e$ and every solution $u = (u_1, u_2)$ of $(PS)_t$ corresponding to this $t$, we have $\|u_1\|^{1+\frac{2}{\beta_1}} \leq Ct_1$ and $\|u_2\|^{1+\frac{2}{\beta_2}} \leq Ct_2$.

Lemmas 4 and 5 prove that $(PS)_t$ has no solution for large $t$. 
Proof of the (APS)

We have the following:

(i) If $C$ is sufficiently large, $(PS)_t$ has a minimal solution for $t \leq -Ce$. 
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(iii) A priori bound: given $t_0 \in \mathbb{R}^2$, the (eventual) solutions of $(PS)_t$ for all $t \geq t_0$ are bounded by the same constant.
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(iii) A priori bound: given $t_0 \in \mathbb{R}^2$, the (eventual) solutions of $(PS)_t$ for all $t \geq t_0$ are bounded by the same constant.

The curve $\Gamma$ is defined by parametrization with respect to the line $H = \{t \in \mathbb{R}^2 | t_1 + t_2 = 0\}$. For $t_0 \in H$, let $A(t_0) = \{k \in \mathbb{R} | (PS)_{t_0 + ke}$ has a solution$\}$. 
Proof of the (APS), cont.

We have

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For each \( t_0 \in H \) \( \exists k_0 \in \mathbb{R} \) s.t. \((PS)_{t_0+ke}\) does not a solution for \( k \geq k_0\).
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So \( K : H \to R \) is defined by \( K(t) = \sup A(t) \).
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\( A(t_0) \) is an interval. Indeed, let \( k \in A(t_0) \) and \( k' \leq k \).

Since a solution of \( (PS)_{t_0 + ke} \) is a supersolution of \( (PS)_{t_0 + k'e} \), \( k' \in A(t_0) \).
Existence of 2 solutions

Define $S_t : C^{1,\alpha}(\Omega)^2 \to C^{1,\alpha}(\Omega)^2$ by $u = (S_t)v$, where

$$-Lu = f(x, v) + t\varphi_1(x) + h(x) \text{ in } \Omega, \ u = 0 \text{ on } \partial\Omega.$$
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Fix \( t_0 \in H \) and \( k_0 < K(t_0) \). We know \((PS)_{t_0+k_0}e\) has a minimal solution.
Existence of 2 solutions

Define $S_t : C^{1,\alpha}(\Omega)^2 \to C^{1,\alpha}(\Omega)^2$ by $u = (S_t)v$, where $-Lu = f(x,v) + t\varphi_1(x) + h(x)$ in $\Omega$, $u = 0$ on $\partial \Omega$.

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$\exists k_1 \in R$, s.t. $(PS)_{t_0+k_e}$ has no solution for $k \geq k_1$. So

$\deg(I - S_{t_0+k_1e}, B_R, 0) = 0$. for large ball $B_R \subset C^{1,\alpha}(\Omega)^2$. 
PARABENS

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