Large solutions for a class of nonlinear elliptic equations with gradient terms

Tommaso Leonori
University of Rome “La Sapienza”

San José 19/9/2007
I present a result contained in a joint paper with A. Porretta:

*The boundary behavior of blow-up solutions related to a stochastic control problem with state constraint*

...to appear in SIAM Journal on Mathematical Analysis.
We consider the following equation:

\[-\Delta u + u + |\nabla u|^q = f(x) \quad \text{in} \quad \Omega\]
We consider the following equation:

$$-\Delta u + u + |\nabla u|^q = f(x) \quad \text{in} \quad \Omega$$

equipped with the boundary condition

$$u(x) \to +\infty \quad \text{as} \quad x \to \partial \Omega.$$

Here $\Omega$ is a bounded domain of $\mathbb{R}^N$, $N \geq 2$ and we assume $1 < q \leq 2$ and $f(x)$ smooth (namely $W^{1,\infty}(\Omega)$). Solutions that satisfy the explosive boundary condition are known as large solutions.
We consider the following equation:

\[-\Delta u + u + |\nabla u|^q = f(x) \quad \text{in } \Omega\]

equipped with the boundary condition

\[u(x) \to +\infty \quad \text{as } x \to \partial \Omega.\]

Here \(\Omega\) is a bounded domain of \(\mathbb{R}^N, N \geq 2\) and we assume

\[1 < q \leq 2\] and \(f(x)\) smooth (namely \(W^{1,\infty}(\Omega))\).
We consider the following equation:

\[-\Delta u + u + |\nabla u|^q = f(x) \quad \text{in } \Omega\]

equipped with the boundary condition

\[u(x) \to +\infty \quad \text{as } x \to \partial \Omega.\]

Here \(\Omega\) is a bounded domain of \(\mathbb{R}^N, N \geq 2\) and we assume \(1 < q \leq 2\) and \(f(x)\) smooth (namely \(W^{1,\infty}(\Omega)\)).

Solutions that satisfy the explosive boundary condition are known as large solutions.
Motivations: a stochastic control problem

Let us consider the stochastic differential equation

\[ dX_t = a_t \, dt + dB_t \]

where \( B_t \) is the Brownian motion and \( a_t = a(X_t) \) is a feedback control.

We consider the class \( A \) of all (feedback) controls that keep the process \( X_t \) inside the domain \( \Omega \) for any time \( t > 0 \) a.s..

The criterion for optimality is given by the cost functional

\[
J(x, a) = \mathbb{E} \int_0^\infty \left[ f(X_t) + C_q|a(X_t)|^{q'} \right] e^{-t} \, dt
\]

where \( f \) is the assigned cost, \( C_q > 0 \) and \( 1 + q' = 1 \).
Motivations: a stochastic control problem

Let us consider the stochastic differential equation
\[
\begin{cases}
    dX_t = a_t dt + dB_t \\
    X_0 = x \in \Omega
\end{cases}
\]

where $B_t$ is the Brownian motion and $a_t = a(X_t)$ is a feedback control.
Let us consider the stochastic differential equation

\[
\begin{cases}
    dX_t = a_t \, dt + dB_t \\
    X_0 = x \in \Omega
\end{cases}
\]

where \( B_t \) is the Brownian motion and \( a_t = a(X_t) \) is a feedback control.

We consider the class \( \mathcal{A} \) of all (feedback) controls that keep the process \( X_t \) inside the domain \( \Omega \) for any time \( t > 0 \) a.s.
Motivations: a stochastic control problem

Let us consider the stochastic differential equation

\[ \begin{cases} dX_t = a_t dt + dB_t \\ X_0 = x \in \Omega \end{cases} \]

where \( B_t \) is the Brownian motion and \( a_t = a(X_t) \) is a feedback control.

We consider the class \( \mathcal{A} \) of all (feedback) controls that keep the process \( X_t \) inside the domain \( \Omega \) for any time \( t > 0 \) a.s..

The criterion for optimality is given by the cost functional

\[ J(x, a) = \mathbb{E} \int_0^\infty \left( f(X_t) + C_{q'} \left| a(X_t) \right|^q \right) e^{-r t} dt \]

where \( f \) is the assigned cost and \( C_{q'} \) is the cost of the control.
Let us consider the stochastic differential equation

\[ \begin{align*}
    dX_t &= a_t \, dt + dB_t \\
    X_0 &= x \in \Omega
\end{align*} \]

where $B_t$ is the Brownian motion and $a_t = a(X_t)$ is a feedback control.

We consider the class $\mathcal{A}$ of all (feedback) controls that keep the process $X_t$ inside the domain $\Omega$ for any time $t > 0$ a.s..

The criterion for optimality is given by the cost functional ($E$ is the expected value, $C_q > 0$ and $\frac{1}{q'} + \frac{1}{q} = 1$):

\[
    J(x, a) = E \int_0^\infty \left\{ f(X_t) + C_q |a(X_t)|^{q'} \right\} e^{-t} \, dt
\]
Motivations: a stochastic control problem

Let us consider the stochastic differential equation

\[
\begin{aligned}
\left\{ \begin{array}{l}
dX_t = a_t dt + dB_t \\
X_0 = x \in \Omega
\end{array} \right.
\end{aligned}
\]

where \( B_t \) is the Brownian motion and \( a_t = a(X_t) \) is a feedback control.

We consider the class \( \mathcal{A} \) of all (feedback) controls that keep the process \( X_t \) inside the domain \( \Omega \) for any time \( t > 0 \) a.s.

The criterion for optimality is given by the cost functional

\[
J(x, a) = E \int_0^\infty \left\{ \begin{array}{l}
f(X_t) \\
\text{discount factor}
\end{array} \right. 
\]

\( C_q > 0 \) and \( \frac{1}{q'} + \frac{1}{q} = 1 \):

\[
T.Leonori \quad \text{Boundary behavior of blow-up solutions}
\]
Motivations: a stochastic control problem

Let us consider the stochastic differential equation

\[ \begin{cases} dX_t = a_t dt + dB_t \\ X_0 = x \in \Omega \end{cases} \]

where \( B_t \) is the Brownian motion and \( a_t = a(X_t) \) is a feedback control.

We consider the class \( \mathcal{A} \) of all (feedback) controls that keep the process \( X_t \) inside the domain \( \Omega \) for any time \( t > 0 \) a.s..

The criterion for optimality is given by the cost functional \( (E \) is the expected value, \( C_q > 0 \) and \( \frac{1}{q'} + \frac{1}{q} = 1 ):\)

\[ J(x, a) = E \int_0^\infty \left\{ \frac{f(X_t)}{\text{assigned cost}} + C_q |a(X_t)|^{q'} \right\} e^{-t} dt \]

T.Leonori  Boundary behavior of blow-up solutions
Let us consider the stochastic differential equation

\[
\begin{aligned}
dX_t &= a_t \, dt + dB_t \\
X_0 &= x \in \Omega
\end{aligned}
\]

where \( B_t \) is the Brownian motion and \( a_t = a(X_t) \) is a feedback control.

We consider the class \( \mathcal{A} \) of all (feedback) controls that keep the process \( X_t \) inside the domain \( \Omega \) for any time \( t > 0 \) a.s..

The criterion for optimality is given by the cost functional (\( E \) is the expected value, \( C_q > 0 \) and \( \frac{1}{q'} + \frac{1}{q} = 1 \)):

\[
J(x, a) = E \int_0^\infty \left\{ \underbrace{f(X_t)}_{\text{assigned cost}} + \underbrace{C_q |a(X_t)|^{q'}}_{\text{cost of the control}} \right\} e^{-t} \, dt
\]
Motivations: a stochastic control problem

Hence

\[ \inf_{a \in A} J(x, a), \]

is achieved and defines the value function

\[ u(x) = \inf_{a \in A} J(x, a), \]

that solves the problem

\[
\begin{align*}
-\Delta u + u + |\nabla u|^q &= f(x) \\
|u(x)| &\to +\infty \quad \text{as} \quad d(x) \to 0,
\end{align*}
\]

where

\[ d(x) = \text{dist}(x, \partial \Omega). \]
Motivations: a stochastic control problem

Hence

\[ \inf_{a \in A} J(x, a), \]

where

\[ A = \{ a \in C^0(\Omega, \mathbb{R}^N) : X_t \in \Omega, \forall t > 0 \text{ a.s.} \}, \]
Motivations: a stochastic control problem

Hence

$$\inf_{a \in A} J(x, a),$$

where

$$A = \{ a \in C^0(\Omega, \mathbb{R}^N) : X_t \in \Omega, \forall t > 0 \text{ a.s.} \} ,$$

is achieved and defines the value function

$$u(x) = \inf_{a \in A} J(x, a),$$
Motivations: a stochastic control problem

Hence

$$\inf_{a \in A} J(x, a),$$

where

$$A = \{ a \in C^0(\Omega, \mathbb{R}^N) : X_t \in \Omega, \forall t > 0 \ a.s. \},$$

is achieved and defines the value function

$$u(x) = \inf_{a \in A} J(x, a),$$

that solves the problem

$$\begin{cases}
-\Delta u + u + |\nabla u|^q = f(x) & \text{in } \Omega \\
u(x) \to +\infty & \text{as } d(x) \to 0,
\end{cases}$$

where $$d(x) = \text{dist}(x, \partial\Omega).$$
Known results

In the paper
it has been proved:
Known results

In the paper
it has been proved:

- existence and uniqueness of the solution \( u \in W^{2,p}_{\text{loc}}(\Omega) \), \( \forall p > 1 \);
Known results

In the paper
it has been proved:

- existence and uniqueness of the solution $u \in W^{2,p}_{\text{loc}}(\Omega)$, $\forall p > 1$;
- asymptotic estimates $u(x)$,
In the paper “J.-M. Lasry, P.-L. Lions, Nonlinear elliptic equations with singular boundary conditions and stochastic control with state constraints. I. The model problem, Math. Ann. 283 (1989), n. 4, 583–630” it has been proved:

1. existence and uniqueness of the solution $u \in W_{\text{loc}}^{2,p}(\Omega)$, $\forall p > 1$;
2. asymptotic estimates $u(x)$, i.e. as $d(x) \to 0$:

$$u(x) \sim C^* d(x)^{-\frac{2-q}{q-1}} \quad \text{if } 1 < q < 2, \quad C^* = \left(\frac{q-1}{2-q}\right)^{-\frac{2-q}{q-1}},$$
Known results

In the paper
it has been proved:

- existence and uniqueness of the solution \( u \in W^{2,p}_{\text{loc}}(\Omega) \), \( \forall p > 1 \);
- asymptotic estimates \( u(x) \), i.e. as \( d(x) \to 0 \):

\[
\begin{align*}
    u(x) & \sim C^* d(x)^{-\frac{2-q}{q-1}} \quad \text{if } 1 < q < 2, \\
    C^* & = \frac{(q-1)^{-\frac{2-q}{q-1}}}{2-q} , \\
    u(x) & \sim -\log(d(x)) \quad \text{if } q = 2;
\end{align*}
\]
Known results

In the paper
it has been proved:

- existence and uniqueness of the solution \( u \in W^{2,p}_{\text{loc}}(\Omega) \), \( \forall p > 1 \);
- asymptotic estimates \( u(x) \), i.e. as \( d(x) \to 0 \):

\[
    u(x) \sim C^* d(x)^{-\frac{2-q}{q-1}} \quad \text{if } 1 < q < 2, \quad C^* = \frac{(q - 1)^{-\frac{2-q}{q-1}}}{2 - q},
\]

\[
    u(x) \sim -\log(d(x)) \quad \text{if } q = 2;
\]

- the unique optimal control is \( a(x) = -q|\nabla u(x)|^{q-2} \nabla u(x) \).
First order estimates on the gradient

More recently in


It has been proved a first order estimate on $\nabla u$ near the boundary,
More recently in


It has been proved a first order estimate on $\nabla u$ near the boundary, namely

$$
\begin{cases}
\lim_{x \to x_0 \in \partial \Omega} d(x) \frac{1}{q-1} \nabla u(x) = (q - 1)^{- \frac{1}{q-1}} \nu(x_0), \\
\frac{\partial u}{\partial \tau} = o \left( \frac{\partial u}{\partial \nu} \right)
\end{cases}
$$

where $\nu(x) = -\nabla d(x)$ is the outer normal.
More recently in


It has been proved a first order estimate on $\nabla u$ near the boundary, namely

$$\lim_{x \to x_0 \in \partial \Omega} d(x)^{\frac{1}{q-1}} \nabla u(x) = (q - 1)^{-\frac{1}{q-1}} \nu(x_0),$$

$$\frac{\partial u}{\partial \tau} = o \left( \frac{\partial u}{\partial \nu} \right)$$

where $\nu(x) = -\nabla d(x)$ is the outer normal.

Such result has been proved via scaling and blow-up.
The results on the first order, in particular, say that the solution and the gradient (and consequently the control) depend only on the distance to the boundary.

\[ u(x) \sim \psi(d(x)) \]
\[ \nabla u \sim -\psi'(d(x)) \nu(x) \]

where \( \psi(s) \) is the solution of the ODE:

\[
\begin{align*}
-\psi''(s) + |\psi'(s)|^q &= 0 \\
\psi(s) &\to +\infty \quad \text{as} \quad s \to 0^+.
\end{align*}
\]

Note that such a solution exists since \( 1 < q \leq 2 \).
The results on the first order, in particular, say that the solution and the gradient (and consequently the control) depend only on the distance to the boundary. Thus

\[ u(x) \sim \psi(d(x)) \quad \text{and} \quad \nabla u \sim -\psi'(d(x)) \nu(x) \]

where \( \psi(s) \) is the solution of the ODE

\[
\begin{aligned}
-\psi''(s) + |\psi'(s)|^q &= 0 \\
\text{for} \quad s &\in (0, 1), \\
\lim_{s \to 0^+} \psi(s) &= +\infty.
\end{aligned}
\]
The results on the first order, in particular, say that the solution and the gradient (and consequently the control) depend only on the distance to the boundary. Thus

\[ u(x) \sim \psi(d(x)) \quad \text{and} \quad \nabla u \sim -\psi'(d(x)) \nu(x) \]

where \( \psi(s) \) is the solution of the ODE

\[
\begin{cases} 
-\psi''(s) + |\psi'(s)|^q = 0 & s \in (0, 1), \\
\lim_{s \to 0^+} \psi(s) = +\infty.
\end{cases}
\]
The results on the first order, in particular, say that the solution and the gradient (and consequently the control) depend only on the distance to the boundary. Thus

\[ u(x) \sim \psi(d(x)) \quad \text{and} \quad \nabla u \sim -\psi'(d(x)) \nu(x) \]

where \( \psi(s) \) is the solution of the ODE

\[
\begin{cases}
-\psi''(s) + |\psi'(s)|^q = 0 & s \in (0, 1), \\
\lim_{{s \to 0^+}} \psi(s) = +\infty.
\end{cases}
\]

Note that such solution exists since \( 1 < q \leq 2 \).
The aim of our work is:
The aim of our work is:

- to give a **more precise picture** of the behavior of the gradient (and consequently of the control) near $\partial \Omega$;
The aim of our work is:
- to give a more precise picture of the behavior of the gradient (and consequently of the control) near $\partial \Omega$;
- to study second order effects;
The aim of our work is:

- to give a **more precise picture** of the behavior of the gradient (and consequently of the control) near \( \partial \Omega \);
- to study **second order effects**;
- look at the role played by the **geometry of the domain**.
Theorem (L.-Porretta)

Let $\Omega$ be a regular open subset of $\mathbb{R}^N$, and let $H(\varsigma)$ be the mean curvature of $\partial \Omega$ computed at $\varsigma$ and $\bar{x}$ the projection of $x \in \Omega$ on $\partial \Omega$. Then

$$\partial u(x) \partial \nu = (q - 1)^{-1} q^{-1} d(x) 1^{q-1} + (N - 1) H(x) 2 d(x) + o(d(x))$$

and

$$\begin{cases} \partial u(x) \partial \tau \in L^\infty(\Omega) & \text{if } \frac{3}{2} < q \leq 2, \\ \partial u(x) \partial \tau = O(|\log d(x)|) & \text{if } q = \frac{3}{2}, \\ \partial u(x) \partial \tau = O(d(x)^{q - 3/2}) & \text{if } 1 < q < \frac{3}{2}. \end{cases}$$
Main result

Theorem (L.-Porretta)

Let $\Omega$ be a regular open subset of $\mathbb{R}^N$, and let $H(\varsigma)$ be the mean curvature of $\partial \Omega$ computed at $\varsigma$ and $\bar{x}$ the projection of $x \in \Omega$ on $\partial \Omega$. Then $\forall 1 < q \leq 2$, as $d(x) \to 0$, 
\[
\partial u(x) / \partial \nu = \left( q - \frac{1}{2} \right) \frac{1}{q - 1} d(x) + \left( q - 1 \right) \left( N - 1 \right) H(x) d(x) + o(d(x)) \]

and 
\[
\begin{cases} 
\partial u(x) / \partial \tau \in L^\infty(\Omega) & \text{if } \frac{3}{2} < q \leq 2 \\
\partial u(x) / \partial \tau = O(|\log d(x)|) & \text{if } q = \frac{3}{2} \\
\partial u(x) / \partial \tau = O\left( d(x)^{\frac{2}{q} - 3q - 1} \right) & \text{if } 1 < q < \frac{3}{2}.
\end{cases}
\]
Theorem (L.-Porretta)

Let $\Omega$ be a regular open subset of $\mathbb{R}^N$, and let $H(\varsigma)$ be the mean curvature of $\partial \Omega$ computed at $\varsigma$ and $\bar{x}$ the projection of $x \in \Omega$ on $\partial \Omega$. Then $\forall 1 < q \leq 2$, as $d(x) \to 0$,

$$\frac{\partial u(x)}{\partial \nu} = (q - 1)^{-\frac{1}{q-1}} \frac{1}{d(x)^{\frac{1}{q-1}}}$$
**Main result**

**Theorem (L.-Porretta)**

Let \( \Omega \) be a regular open subset of \( \mathbb{R}^N \), and let \( H(\varsigma) \) be the mean curvature of \( \partial \Omega \) computed at \( \varsigma \) and \( \overline{x} \) the projection of \( x \in \Omega \) on \( \partial \Omega \). Then \( \forall 1 < q \leq 2 \), as \( d(x) \to 0 \),

\[
\frac{\partial u(x)}{\partial \nu} = \frac{(q - 1)^{-\frac{1}{q-1}}}{d(x)^{\frac{1}{q-1}}} \left[ 1 + \left( \frac{N - 1)H(\overline{x})}{2} \right) d(x) \right]
\]
Theorem (L.-Porretta)

Let $\Omega$ be a regular open subset of $\mathbb{R}^N$, and let $H(\varsigma)$ be the mean curvature of $\partial \Omega$ computed at $\varsigma$ and $\bar{x}$ the projection of $x \in \Omega$ on $\partial \Omega$. Then $\forall 1 < q \leq 2$, as $d(x) \to 0$,

$$\frac{\partial u(x)}{\partial \nu} = \frac{(q - 1)^{-\frac{1}{q-1}}}{d(x)^{\frac{1}{q-1}}} \left[ 1 + \frac{(N - 1)H(\bar{x})}{2} d(x) + o(d(x)) \right]$$
Main result

Theorem (L.-Porretta)

Let $\Omega$ be a regular open subset of $\mathbb{R}^N$, and let $H(\varsigma)$ be the mean curvature of $\partial \Omega$ computed at $\varsigma$ and $\bar{x}$ the projection of $x \in \Omega$ on $\partial \Omega$. Then $\forall 1 < q \leq 2$, as $d(x) \to 0$,

$$\frac{\partial u(x)}{\partial \nu} = (q-1)^{-\frac{1}{q-1}}\left[1 + \frac{(N-1)H(\bar{x})}{2}d(x) + o(d(x))\right]$$

and

$$\begin{cases}
\frac{\partial u(x)}{\partial \tau} \in L^\infty(\Omega) & \text{if } \frac{3}{2} < q \leq 2, \\
\frac{\partial u(x)}{\partial \tau} = O(|\log d|) & \text{if } q = \frac{3}{2}, \\
\frac{\partial u(x)}{\partial \tau} = O\left(d^{\frac{2q-3}{q-1}}\right) & \text{if } 1 < q < \frac{3}{2}.
\end{cases}$$
Recalling that, by [LL], $a = -q|\nabla u|^{q-2}\nabla u$, we deduce, as $d(x) \to 0$:

- If $1 < q < 2$, $a(x) = -q'|\nabla u|^{q-2}\nabla u - q'(N-1)\frac{H(x)}{\nu(x)} + o(1)$;
- If $q = 2$, $a(x) = -2d(x)\nu(x) - (N-1)\frac{H(x)}{\nu(x)} + \psi(x)\tau(x)$,

where $\tau(x) \in \mathbb{R}^N$, $|\tau| = 1$, $\tau \cdot \nu = 0$, $\psi \in L^\infty(\Omega)$. 
Recalling that, by [LL], \( a = -q|\nabla u|^{q-2}\nabla u \), we deduce, as \( d(x) \to 0 \):
Recalling that, by [LL], $a = -q|\nabla u|^{q-2}\nabla u$, we deduce, as $d(x) \to 0$:

- if $1 < q < 2$

$$a(x) = -\frac{q'}{d(x)}\nu(x)$$
Recalling that, by [LL], \( a = -q|\nabla u|^{q-2}\nabla u \), we deduce, as \( d(x) \to 0 \):

- if \( 1 < q < 2 \)

\[
a(x) = -\frac{q'}{d(x)} \nu(x) - \frac{q'(N-1)}{2} H(\bar{x}) \nu(x)
\]
Recalling that, by [LL], \( a = -q|\nabla u|^{q-2}\nabla u \), we deduce, as \( \delta(x) \to 0 \):

- if \( 1 < q < 2 \)

\[
a(x) = -\frac{q'}{d(x)}\nu(x) - \frac{q'(N-1)}{2}H(x)\nu(x) + o(1);
\]
Recalling that, by [LL], \( a = -q|\nabla u|^{q-2}\nabla u \), we deduce, as \( d(x) \to 0 \):

- **if** \( 1 < q < 2 \)

\[
a(x) = -\frac{q'}{d(x)} \nu(x) - \frac{q'(N - 1)}{2} H(x) \nu(x) + o(1);
\]

- **if** \( q = 2 \)

\[
a(x) = -\frac{2}{d(x)} \nu(x)
\]
Recalling that, by [LL], $a = -q|\nabla u|^{q-2}\nabla u$, we deduce, as $d(x) \to 0$:

- if $1 < q < 2$

  $$a(x) = -\frac{q'}{d(x)}\nu(x) - \frac{q'(N-1)}{2}H(x)\nu(x) + o(1);$$

- if $q = 2$

  $$a(x) = -\frac{2}{d(x)}\nu(x) - (N-1)[H(x) + o(1)]\nu(x).$$
Recalling that, by [LL], $a = -q|\nabla u|^{q-2}\nabla u$, we deduce, as $d(x) \to 0$:

- if $1 < q < 2$

$$a(x) = -\frac{q'}{d(x)}\nu(x) - \frac{q'(N-1)}{2}H(x)\nu(x) + o(1);$$

- if $q = 2$

$$a(x) = -\frac{2}{d(x)}\nu(x) - (N-1)[H(x) + o(1)]\nu(x) + \psi(x)\tau(x),$$

where $\tau(x) \in \mathbb{R}^N$, $|\tau| = 1$, $\tau \cdot \nu = 0$, $\psi \in L^\infty(\Omega)$. 
Remark

The (unique) **optimal control**:

1. it is **singular** at the boundary;
2. it is mainly directed in the normal direction, pointing inside;
3. in the tangential directions, it vanishes as $d(x) \to 0$, if $1 < \varrho < 2$, while it is bounded if $\varrho = 2$;
4. it has maximum intensity in those points close to the boundary where the boundary is more “curved” (i.e. on the hypersurfaces parallel to $\partial \Omega$, it achieves its maximum in those points in which the mean curvature is maximum).
Remark

The (unique) optimal control:

1. it is singular at the boundary;
2. it is mainly directed in the normal direction, pointing inside;

3. in the tangential directions, it vanishes as $d(x) \to 0$, if $1 < q < 2$ while it is bounded if $q = 2$;

4. it has maximum intensity in those points close to the boundary where the boundary is more "curved" (i.e. on the hypersurfaces parallel to $\partial \Omega$, it achieves its maximum in those points in which the mean curvature is maximum).
Remark

The (unique) **optimal control**: 

1. it is **singular** at the boundary;
2. it is mainly directed in the **normal direction**, pointing inside;
3. in the tangential directions, it **vanishes** as \( d(x) \to 0 \), if \( 1 < q < 2 \)
Remark

The (unique) **optimal control**:

1. it is **singular** at the boundary;
2. it is mainly directed in the **normal direction**, pointing inside;
3. in the tangential directions, it **vanishes** as $d(x) \to 0$, if $1 < q < 2$ while it is **bounded** if $q = 2$. 
Remark

The (unique) **optimal control**: 

1. it is **singular** at the boundary;
2. it is mainly directed in the **normal direction**, pointing inside;
3. in the tangential directions, it **vanishes** as $d(x) \to 0$, if $1 < q < 2$ while it is **bounded** if $q = 2$;
4. it has **maximum** intensity in those points close to the boundary where the boundary is more “curved”
Remark

The (unique) optimal control:

1. it is singular at the boundary;
2. it is mainly directed in the normal direction, pointing inside;
3. in the tangential directions, it vanishes as $d(x) \to 0$, if $1 < q < 2$ while it is bounded if $q = 2$;
4. it has maximum intensity in those points close to the boundary where the boundary is more “curved” (i.e. on the hypersurfaces parallel to $\partial \Omega$, it achieves its maximum in those points in which the mean curvature is maximum).
Idea of the proof.

Let us assume $1 < q < 2$, the case $q = 2$ is a bit different (easier).

We introduce a corrector term, 

$$S = d - q^{-1} \sum_{k=0}^{m} \sigma_k (x) d_k(x),$$

where $m > 0$, $\sigma_0 = C_* = (q - 1)^{-q}$. Then we define

$$z = u - S$$

and we look at the equation solved by $z$, i.e.

$$-\Delta z + z + |\nabla z + \nabla S|_q^{q - 1} = f(x) + g(x),$$

where $g(x) = \Delta S - S - |\nabla S|_q^{q - 1}$.

We observe that from the result of Porretta and Veron we deduce that $|\nabla z + \nabla S|_q^{q - 1} \sim -q^{-1} \nabla z \cdot \nabla d + O(d^2 - q^{-1} |\nabla z|_2^2)$. 
Idea of the proof.

Let us assume $1 < q < 2$, then we introduce a corrector term, $(a formal expansion of u) S = d - 2 - q q - 1(x)m \sum_{k=0}^{m} \sigma_k(x) d_k(x)$, where $m > 0$, $\sigma_0(x) = C^* = (q - 1)^{-2} - q q - 1 2^{-q}$. Then we define $z = u - S$ and we look at the equation solved by $z$, i.e.

$$-\Delta z + z + |\nabla z + \nabla S|^q - |\nabla S|^q = f(x) + g(x)$$

where $g(x) = \Delta S - S - |\nabla S|^q$. We observe that from the result of Porretta and Veron we deduce that $|\nabla z + \nabla S|^q - |\nabla S|^q \sim -q q - 1 \nabla z \cdot \nabla d d + O(d^2 - q q - 1 |\nabla z|^2)$.
Idea of the proof.

Let us assume $1 < q < 2$, the case $q = 2$ is a bit different (easier).

We introduce a corrector term, $(a formal expansion of $u$)

$$S = d - 2 - q - 1 \sum_{k = 0}^{m} \sigma_k(x)d_k(x),$$

where $m > 0$, $\sigma_0 = C^* = (q - 1)^{-2 - q}$.

Then we define $z = u - S$ and we look at the equation solved by $z$, i.e.

$$-\Delta z + z + |\nabla z + \nabla S|^{q - 2} = f(x) + g(x),$$

where $g(x) = \Delta S - S - |\nabla S|^{q - 2}$.

We observe that from the result of Porretta and Veron we deduce that $|\nabla z + \nabla S|^{q - 2} \sim -q^{-2 - 1} \nabla z \cdot \nabla d + O(d^2 - q^{-2 - 1} |\nabla z|^{2})$. 

T.Leonori 
Boundary behavior of blow-up solutions
Idea of the proof.

Let us assume $1 < q < 2$, the case $q = 2$ is a bit different (easier).
We introduce a corrector term, (a formal expansion of $u$)

$$S = d^{-\frac{2-q}{q-1}}(x) \sum_{k=0}^{m} \sigma_k(x)d^k(x), \quad m > 0, \quad \sigma_0 = C^* = \frac{(q - 1)^{\frac{2-q}{q-1}}}{2 - q}.$$
Idea of the proof.

Let us assume $1 < q < 2$, the case $q = 2$ is a bit different (easier).
We introduce a corrector term, (a formal expansion of $u$)

$$S = d^{-\frac{2-q}{q-1}}(x) \sum_{k=0}^{m} \sigma_k(x) d^k(x), \quad m > 0, \quad \sigma_0 = C^* = \frac{(q - 1)^{\frac{2-q}{q-1}}}{2 - q}.$$ 

Then we define $z = u - S$
Idea of the proof.

Let us assume $1 < q < 2$, the case $q = 2$ is a bit different (easier). We introduce a corrector term, (a formal expansion of $u$)

$$S = d^{-\frac{2-q}{q-1}}(x) \sum_{k=0}^{m} \sigma_k(x)d^k(x), m > 0, \quad \sigma_0 = C^* = \left(\frac{q - 1}{2 - q}\right)^{-\frac{2-q}{q-1}}.$$

Then we define $z = u - S$ and we look at the equation solved by $z$, i.e.

$$-\Delta z + z + |\nabla z + \nabla S|^q - |\nabla S|^q = f(x) + g(x)$$

where $g(x) = \Delta S - S - |\nabla S|^q$. 
Idea of the proof.

Let us assume \( 1 < q < 2 \), the case \( q = 2 \) is a bit different (easier).

We introduce a corrector term, (a formal expansion of \( u \))

\[
S = d^{-\frac{2-q}{q-1}}(x) \sum_{k=0}^{m} \sigma_k(x) d^k(x), \ m > 0, \quad \sigma_0 = C^* = \frac{(q-1)^{\frac{2-q}{q-1}}}{2 - q}.
\]

Then we define \( z = u - S \) and we look at the equation solved by \( z \), i.e.

\[
-\Delta z + z + |\nabla z + \nabla S|^q - |\nabla S|^q = f(x) + g(x)
\]

where \( g(x) = \Delta S - S - |\nabla S|^q \).

We observe that from the result of Porretta and Veron we deduce that

\[
|\nabla z + \nabla S|^q - |\nabla S|^q \sim -\frac{q}{q-1} \frac{\nabla z \cdot \nabla d}{d} + O(d^{\frac{2-q}{q-1}}|\nabla z|^2).
\]
Idea of the proof

The nonlinear term generate a transport term, singular at the boundary, that has a regularizing effect.
Idea of the proof

The nonlinear term generate a transport term, singular at the boundary, that has a regularizing effect.

Thus we deal with an equation of the type

$$-\Delta z + z - \frac{q}{q-1} \frac{\nabla z \cdot \nabla d}{d} + d^{\frac{2-q}{q-1}} |\nabla z|^2 (1 + o(1)) = f(x) + g(x).$$

We would like to prove (via scaling) $\nabla u \rightarrow \nabla S$. However we do not know the behavior of $z = u - S$ on $\partial \Omega$ so that this approach fails.

In fact, for our aim, it is enough to prove $|\nabla u - \nabla S| \in L^\infty(\Omega)$, i.e. $z \in W^{1, \infty}(\Omega)$. 
Idea of the proof

The nonlinear term generate a transport term, singular at the boundary, that has a regularizing effect.

Thus we deal with an equation of the type

$$\begin{align*}
-\Delta z + z - \frac{q}{q-1} \frac{\nabla z \cdot \nabla d}{d} + d^{\frac{2-q}{q-1}} |\nabla z|^2 (1 + o(1)) &= f(x) + g(x).
\end{align*}$$

We would like to prove (via scaling) $\nabla u \rightarrow \nabla S$. 
Idea of the proof

The nonlinear term generate a transport term, singular at the boundary, that has a regularizing effect.

Thus we deal with an equation of the type

$$-\Delta z + z - \frac{q}{q-1} \frac{\nabla z \cdot \nabla d}{d} + d^{\frac{2-q}{q-1}} |\nabla z|^2 (1 + o(1)) = f(x) + g(x).$$

We would like to prove (via scaling) $\nabla u \to \nabla S$. However we do not know the behavior of $z = u - S$ on $\partial \Omega$ so that this approach fails.
Idea of the proof

The nonlinear term generate a transport term, singular at the boundary, that has a regularizing effect.

Thus we deal with an equation of the type

\[-\Delta z + z - \frac{q}{q-1} \frac{\nabla z \cdot \nabla d}{d} + d^{\frac{2-q}{q-1}} |\nabla z|^2 (1 + o(1)) = f(x) + g(x).\]

We would like to prove (via scaling) $\nabla u \to \nabla S$. However we do not know the behavior of $z = u - S$ on $\partial \Omega$ so that this approach fails.

In fact, for our aim, it is enough to prove $|\nabla u - \nabla S| \in L^\infty(\Omega)$
The nonlinear term generate a transport term, singular at the boundary, that has a regularizing effect.

Thus we deal with an equation of the type

$$-\Delta z + z - \frac{q}{q-1} \frac{\nabla z \cdot \nabla d}{d} + d^{\frac{2-q}{q-1}} |\nabla z|^2 (1 + o(1)) = f(x) + g(x).$$

We would like to prove (via scaling) $\nabla u \to \nabla S$.
However we do not know the behavior of $z = u - S$ on $\partial \Omega$ so that this approach fails.

In fact, for our aim, it is enough to prove $|\nabla u - \nabla S| \in L^\infty(\Omega)$, i.e. $z \in W^{1,\infty}(\Omega)$. 
Idea of the proof

Our approach uses a weighted version of

Bernstein Method 1910, Serrin '60, P.-L. Lions '80. It is a technique to obtain Lipschitz estimates: let $v$ be a solution of an elliptic equation you show that $|\nabla v|^2$ is a subsolution of an equation of the same type, you prove that it is bounded using the strong maximum principle (SMP).
Idea of the proof

Our approach uses a weighted version of the Bernstein Method.
Idea of the proof

Our approach uses a weighted version of

Bernstein Method

Bernstein 1910,
Idea of the proof

Our approach uses a weighted version of

**Bernstein Method**

Bernstein 1910, Serrin ’60,
Idea of the proof

Our approach uses a weighted version of

Bernstein Method

Bernstein 1910, Serrin ’60, P.-L. Lions ’80.
Idea of the proof

Our approach uses a weighted version of

Bernstein Method

Bernstein 1910, Serrin ’60, P.-L. Lions ’80.

It is a technique to obtain Lipschitz estimates:
Idea of the proof

Our approach uses a weighted version of

Bernstein Method

Bernstein 1910, Serrin ’60, P.-L. Lions ’80.

It is a technique to obtain Lipschitz estimates:

let \( v \) be a solution of an elliptic equation
Idea of the proof

Our approach uses a weighted version of

**Bernstein Method**

Bernstein 1910, Serrin ’60, P.-L. Lions ’80.

It is a technique to obtain Lipschitz estimates:

let \( v \) be a solution of an elliptic equation
you show that \( |\nabla v|^2 \) is a subsolution of an equation of the same type,
Idea of the proof

Our approach uses a weighted version of

**Bernstein Method**

Bernstein 1910, Serrin ’60, P.-L. Lions ’80.

It is a technique to obtain Lipschitz estimates:

let $v$ be a solution of an elliptic equation
you show that $|\nabla v|^2$ is a subsolution of an equation of the same type,
you prove that it is bounded using the strong maximum principle (SMP).
Idea of the proof

Several versions of this method are known.

In order to prove global Lipschitz estimates we need to approximate the problem with a sequence that satisfies a Neumann boundary condition.

We set

\[-\Delta u_n + u_n + |\nabla u_n|^q = f(x) \text{ in } \Omega\]

\[\partial u_n / \partial \nu = \partial S_n / \partial \nu \text{ on } \partial \Omega\]

where

\[\begin{align*}
S_n &= d^{-2} - q^{-1} n(x) \\
m \sum_{k=0} d_k n(x) \\
m > 0 \\
d_n &= d(x) + n
\end{align*}\]

and we prove first order estimates, i.e.:

\[u_n \sim C^* d^{-2} - q^{-1} n \text{ (via sub and supersolutions)}\]

\[\nabla u_n \sim 2^{-q} C^* d^{-1} q^{-1} n \nu \text{ (via scaling and blow-up)}\]
Idea of the proof

Several version of this method are known. In order to prove global Lipschitz estimates we need to approximate the problem with a sequence that satisfies a Neumann boundary condition.
Idea of the proof

Several versions of this method are known. In order to prove global Lipschitz estimates we need to approximate the problem with a sequence that satisfies a Neumann boundary condition. We set

\[
\begin{cases}
-\Delta u_n + u_n + |\nabla u_n|^q = f(x) \quad &\text{in} \quad \Omega \\
\frac{\partial u_n}{\partial \nu} = \frac{\partial S_n}{\partial \nu} \quad &\text{on} \quad \partial \Omega
\end{cases}
\]

where

\[
S_n = d_n^{-\frac{2-q}{q-1}}(x) \sum_{k=0}^{m} \sigma_k(x)d_n^k(x), \quad m > 0, \quad d_n = d(x) + \frac{1}{n}
\]
Idea of the proof

Several versions of this method are known. In order to prove global Lipschitz estimates we need to approximate the problem with a sequence that satisfies a Neumann boundary condition. We set

\[
\begin{cases}
-\Delta u_n + u_n + |\nabla u_n|^q = f(x) & \text{in } \Omega \\
\frac{\partial u_n}{\partial \nu} = \frac{\partial S_n}{\partial \nu} & \text{on } \partial \Omega
\end{cases}
\]

where

\[
S_n = d_n^{-\frac{2-q}{q-1}} (x) \sum_{k=0}^{m} \sigma_k(x) d_n^k(x), \; m > 0, \quad d_n = d(x) + \frac{1}{n}
\]

and we prove first order estimates, i.e.:

- \( u_n \sim C d_n^{-\frac{2-q}{q-1}} \)
Idea of the proof

Several version of this method are known. In order to prove global Lipschitz estimates we need to approximate the problem with a sequence that satisfies a Neumann boundary condition. We set

\[
\begin{cases}
-\Delta u_n + u_n + |\nabla u_n|^q = f(x) & \text{in } \Omega \\
\frac{\partial u_n}{\partial \nu} = \frac{\partial S_n}{\partial \nu} & \text{on } \partial \Omega
\end{cases}
\]

where

\[
S_n = d_n^{-\frac{2-q}{q-1}}(x) \sum_{k=0}^{m} \sigma_k(x) d_n^k(x), \quad m > 0, \quad d_n = d(x) + \frac{1}{n}
\]

and we prove first order estimates, i.e.:

- \( u_n \sim C^* d_n^{-\frac{2-q}{q-1}} \) (via sub and supersolutions)
Idea of the proof

Several versions of this method are known. In order to prove global Lipschitz estimates we need to approximate the problem with a sequence that satisfies a Neumann boundary condition. We set

\[
\begin{align*}
-\Delta u_n + u_n + |\nabla u_n|^q &= f(x) & \text{in} & & \Omega \\
\frac{\partial u_n}{\partial \nu} &= \frac{\partial S_n}{\partial \nu} & \text{on} & & \partial \Omega
\end{align*}
\]

where

\[
S_n = d_n^{\frac{2-q}{q-1}}(x) \sum_{k=0}^{m} \sigma_k(x) d_n^k(x), \quad m > 0, \quad d_n = d(x) + \frac{1}{n}
\]

and we prove first order estimates, i.e.:

- \( u_n \sim C^* d_n^{-\frac{2-q}{q-1}} \) (via sub and supersolutions)
- \( \nabla u_n \sim \frac{2-q}{q-1} C^* d_n^{-\frac{1}{q-1}} \nu \)
Idea of the proof

Several version of this method are known. In order to prove global Lipschitz estimates we need to approximate the problem with a sequence that satisfies a Neumann boundary condition. We set

\[
\begin{cases}
-\Delta u_n + u_n + |\nabla u_n|^q = f(x) & \text{in } \Omega \\
\frac{\partial u_n}{\partial \nu} = \frac{\partial S_n}{\partial \nu} & \text{on } \partial \Omega
\end{cases}
\]

where

\[
S_n = d_n^{\frac{2-q}{q-1}}(x) \sum_{k=0}^{m} \sigma_k(x) d_k(x), \quad m > 0, \quad d_n = d(x) + \frac{1}{n}
\]

and we prove first order estimates, i.e.:

- \( u_n \sim C^* d_n^{\frac{2-q}{q-1}} \) (via sub and supersolutions)

- \( \nabla u_n \sim \frac{2-q}{q-1} C^* d_n^{\frac{1}{q-1}} \nu \) (via scaling and blow-up)
Idea of the proof

Then we consider the equation solved by $z_n = u_n - S_n$

$$\begin{cases} 
-\Delta z_n + z_n + |\nabla z_n + \nabla S_n|^q - |\nabla S_n|^q = f(x) + g_n(x) & \text{in } \Omega \\
\frac{\partial z_n}{\partial \nu} = 0 & \text{on } \partial \Omega 
\end{cases}$$

with

$$g_n(x) = \Delta S_n - S_n - |\nabla S_n|^q,$$
Idea of the proof

Then we consider the equation solved by $z_n = u_n - S_n$

$$
\begin{cases}
-\Delta z_n + z_n + |\nabla z_n + \nabla S_n|^q - |\nabla S_n|^q = f(x) + g_n(x) & \text{in } \Omega \\
\frac{\partial z_n}{\partial \nu} = 0 & \text{on } \partial \Omega
\end{cases}
$$

with

$$g_n(x) = \Delta S_n - S_n - |\nabla S_n|^q,$$

Now, fix the coefficients $\sigma_k$ such that the right hand side is smooth.
Idea of the proof

Then we consider the equation solved by $z_n = u_n - S_n$

$$
\begin{cases}
-\Delta z_n + z_n + |\nabla z_n + \nabla S_n|^q - |\nabla S_n|^q = f(x) + g_n(x) & \text{in } \Omega \\
\frac{\partial z_n}{\partial \nu} = 0 & \text{on } \partial \Omega
\end{cases}
$$

with

$$g_n(x) = \Delta S_n - S_n - |\nabla S_n|^q,$$

Now, fix the coefficients $\sigma_k$ such that the right hand side is smooth.

As before, the equation solved by $z_n$ is similar to

$$-\Delta z_n + z_n - \frac{q}{q-1} \frac{\nabla z_n \cdot \nabla d}{d_n} + d_n^{2-q} |\nabla z_n|^2 (1 + o(1)) = f(x) + g_n(x).$$
Idea of the proof

Then we consider the equation solved by $z_n = u_n - S_n$

\[
\begin{aligned}
-\Delta z_n + z_n + |\nabla z_n + \nabla S_n|^q - |\nabla S_n|^q &= f(x) + g_n(x) \quad \text{in } \Omega \\
\frac{\partial z_n}{\partial \nu} &= 0 \quad \text{on } \partial \Omega
\end{aligned}
\]

with

\[
g_n(x) = \Delta S_n - S_n - |\nabla S_n|^q,
\]

Now, fix the coefficients $\sigma_k$ such that the right hand side is smooth.

As before, the equation solved by $z_n$ is similar to

\[
-\Delta z_n + z_n - \frac{q}{q-1} \frac{\nabla z_n \cdot \nabla d}{d_n} + d_n^{\frac{2-q}{q-1}} |\nabla z_n|^2 (1 + o(1)) = \underbrace{f(x) + g_n(x)}_{\text{smooth}}.
\]

Now we have a boundary condition!
Idea of the proof

Then, in order to apply a “weighted ” version of the Bernstein’s method,
Idea of the proof

Then, in order to apply a “weighted” version of the Bernstein’s method, we look at the equation solved by \( w_n = \Phi(d_n)|\nabla z_n|^2 \).
Idea of the proof

Then, in order to apply a “weighted” version of the Bernstein’s method, we look at the equation solved by $w_n = \Phi(d_n)|\nabla z_n|^2$.

On $\partial \Omega$, we have:

$$\frac{\partial z_n}{\partial \nu} = 0$$
Idea of the proof

Then, in order to apply a “weighted” version of the Bernstein’s method, we look at the equation solved by $w_n = \Phi(d_n)|\nabla z_n|^2$.

On $\partial \Omega$, we have:

$$\frac{\partial z_n}{\partial \nu} = 0 \Rightarrow \frac{\partial \Phi(d_n)|\nabla z_n|^2}{\partial \nu}$$
Idea of the proof

Then, in order to apply a “weighted” version of the Bernstein’s method, we look at the equation solved by $w_n = \Phi(d_n)|\nabla z_n|^2$.

On $\partial\Omega$, we have:

$$\frac{\partial z_n}{\partial \nu} = 0 \Rightarrow \frac{\partial \Phi(d_n)|\nabla z_n|^2}{\partial \nu} \leq \left[ C_\Omega - \frac{\Phi'(d_n)}{\Phi(d_n)} \right] \Phi(d_n)|\nabla z_n|^2$$

Thus the maximum of $w_n$ is not achieved on the boundary.

**Step 1.** $\Phi(t) = t^{2\beta}, 0 < \beta < 1$, Bernstein + SMP $w_n$ bounded $\Rightarrow$ estimate for $z_n$ in $C_0^{1-\beta}(\Omega)$

**Step 2.** $\Phi(t) = e^{\lambda t}, \lambda \gg 1$, Bernstein + SMP $w_n$ bounded $\Rightarrow$ estimate for $z_n$ in $W^{1,\infty}(\Omega)$.

Hence:

$$|\nabla u_n - \nabla S_n|$$

uniformly bounded in $L^\infty(\Omega)$. 

T.Leonori

Boundary behavior of blow-up solutions
Idea of the proof

Then, in order to apply a “weighted” version of the Bernstein’s method, we look at the equation solved by $w_n = \Phi(d_n)|\nabla z_n|^2$.

On $\partial \Omega$, we have:

$$\frac{\partial z_n}{\partial \nu} = 0 \Rightarrow \frac{\partial \Phi(d_n)|\nabla z_n|^2}{\partial \nu} \leq \left[ C_\Omega - \frac{\Phi'(d_n)}{\Phi(d_n)} \right] \Phi(d_n)|\nabla z_n|^2 \leq 0 \cdot w_n$$

Thus the maximum of $w_n$ is not achieved on the boundary.
Idea of the proof

Then, in order to apply a “weighted” version of the Bernstein’s method, we look at the equation solved by \( w_n = \Phi(d_n)|\nabla z_n|^2 \).

On \( \partial \Omega \), we have:

\[
\frac{\partial z_n}{\partial \nu} = 0 \Rightarrow \frac{\partial \Phi(d_n)|\nabla z_n|^2}{\partial \nu} \leq \left[ C_{\Omega} - \frac{\Phi'(d_n)}{\Phi(d_n)} \right] \Phi(d_n)|\nabla z_n|^2 \leq 0
\]

Thus the maximum of \( w_n \) is not achieved on the boundary.

Step 1.
Then, in order to apply a “weighted” version of the Bernstein’s method, we look at the equation solved by \( w_n = \Phi(d_n)|\nabla z_n|^2 \).

On \( \partial\Omega \), we have:

\[
\frac{\partial z_n}{\partial \nu} = 0 \Rightarrow \frac{\partial \Phi(d_n)|\nabla z_n|^2}{\partial \nu} \leq \left[ C_\Omega - \frac{\Phi'(d_n)}{\Phi(d_n)} \right] \Phi(d_n)|\nabla z_n|^2 \leq 0.
\]

Thus the maximum of \( w_n \) is not achieved on the boundary.

**Step 1.** \( \Phi(t) = t^{2\beta} \), \( 0 < \beta < 1 \),
Idea of the proof

Then, in order to apply a “weighted” version of the Bernstein’s method, we look at the equation solved by \( w_n = \Phi(d_n)|\nabla z_n|^2 \).

On \( \partial\Omega \), we have:

\[
\frac{\partial z_n}{\partial \nu} = 0 \Rightarrow \frac{\partial \Phi(d_n)|\nabla z_n|^2}{\partial \nu} \leq \left[ C_\Omega - \frac{\Phi'(d_n)}{\Phi(d_n)} \right] \Phi(d_n)|\nabla z_n|^2 \leq 0 \cdot w_n.
\]

Thus the maximum of \( w_n \) is not achieved on the boundary.

Step 1. \( \Phi(t) = t^{2\beta}, \ 0 < \beta < 1 \), Bernstein + SMP
Idea of the proof

Then, in order to apply a “weighted ” version of the Bernstein’s method, we look at the equation solved by $w_n = \Phi(d_n)|\nabla z_n|^2$.

On $\partial \Omega$, we have:

$$\frac{\partial z_n}{\partial \nu} = 0 \Rightarrow \frac{\partial \Phi(d_n)|\nabla z_n|^2}{\partial \nu} \leq \left[ C_\Omega - \frac{\Phi'(d_n)}{\Phi(d_n)} \right] \Phi(d_n)|\nabla z_n|^2 \leq 0.$$

Thus the maximum of $w_n$ is not achieved on the boundary.

**Step 1.** $\Phi(t) = t^{2\beta}$, $0 < \beta < 1$, Bernstein + SMP

$w_n$ bounded
Idea of the proof

Then, in order to apply a “weighted” version of the Bernstein’s method, we look at the equation solved by \( w_n = \Phi(d_n)|\nabla z_n|^2 \).

On \( \partial \Omega \), we have:

\[
\frac{\partial z_n}{\partial \nu} = 0 \Rightarrow \frac{\partial \Phi(d_n)|\nabla z_n|^2}{\partial \nu} \leq \left[ C_\Omega - \frac{\Phi'(d_n)}{\Phi(d_n)} \right] \frac{\Phi(d_n)|\nabla z_n|^2}{w_n} \leq 0
\]

Thus the maximum of \( w_n \) is not achieved on the boundary.

**Step 1.** \( \Phi(t) = t^{2\beta} \), \( 0 < \beta < 1 \), Bernstein + SMP

\( w_n \) bounded \( \Rightarrow \) estimate for \( z_n \) in \( C^{0,1-\beta}(\overline{\Omega}) \)
Idea of the proof

Then, in order to apply a “weighted” version of the Bernstein’s method, we look at the equation solved by \( w_n = \Phi(d_n)|\nabla z_n|^2 \).

On \( \partial \Omega \), we have:

\[
\frac{\partial z_n}{\partial \nu} = 0 \Rightarrow \frac{\partial \Phi(d_n)|\nabla z_n|^2}{\partial \nu} \leq \left[ C_\Omega - \frac{\Phi'(d_n)}{\Phi(d_n)} \right] \frac{\Phi(d_n)|\nabla z_n|^2}{w_n}. \]

Thus the maximum of \( w_n \) is not achieved on the boundary.

**Step 1.** \( \Phi(t) = t^{2\beta} \), \( 0 < \beta < 1 \), Bernstein + SMP

\( w_n \) bounded \( \Rightarrow \) estimate for \( z_n \) in \( C^{0,1-\beta}(\Omega) \)

**Step 2.**
Idea of the proof

Then, in order to apply a “weighted” version of the Bernstein’s method, we look at the equation solved by \( w_n = \Phi(d_n)|\nabla z_n|^2 \).

On \( \partial \Omega \), we have:

\[
\frac{\partial z_n}{\partial \nu} = 0 \Rightarrow \frac{\partial \Phi(d_n)|\nabla z_n|^2}{\partial \nu} \leq \left[ C_\Omega - \frac{\Phi'(d_n)}{\Phi(d_n)} \right] \Phi(d_n)|\nabla z_n|^2 \leq 0 \cdot w_n.
\]

Thus the maximum of \( w_n \) is not achieved on the boundary.

**Step 1.** \( \Phi(t) = t^{2\beta}, \quad 0 < \beta < 1 \), Bernstein + SMP \( w_n \) bounded \( \Rightarrow \) estimate for \( z_n \) in \( C^{0,1-\beta}(\overline{\Omega}) \)

**Step 2.** \( \Phi(t) = e^{\lambda t}, \quad \lambda >> 1 \),
Idea of the proof

Then, in order to apply a “weighted” version of the Bernstein’s method, we look at the equation solved by \( w_n = \Phi(d_n)|\nabla z_n|^2 \).

On \( \partial \Omega \), we have:

\[
\frac{\partial z_n}{\partial \nu} = 0 \Rightarrow \frac{\partial \Phi(d_n)|\nabla z_n|^2}{\partial \nu} \leq \left[ C_\Omega - \frac{\Phi'(d_n)}{\Phi(d_n)} \right] \Phi(d_n)|\nabla z_n|^2 \cdot \frac{w_n}{w_n} \leq 0
\]

Thus the maximum of \( w_n \) is not achieved on the boundary.

**Step 1.** \( \Phi(t) = t^{2\beta} \), \( 0 < \beta < 1 \), Bernstein + SMP

\( w_n \) bounded \( \Rightarrow \) estimate for \( z_n \) in \( \mathcal{C}^{0,1-\beta}(\overline{\Omega}) \)

**Step 2.** \( \Phi(t) = e^{\lambda t} \), \( \lambda >> 1 \), Bernstein + SMP
Idea of the proof

Then, in order to apply a “weighted ” version of the Bernstein’s method, we look at the equation solved by $w_n = \Phi(d_n)|\nabla z_n|^2$.

On $\partial \Omega$, we have:

$$\frac{\partial z_n}{\partial \nu} = 0 \Rightarrow \frac{\partial \Phi(d_n)|\nabla z_n|^2}{\partial \nu} \leq \left[ C_\Omega - \frac{\Phi'(d_n)}{\Phi(d_n)} \right] \Phi(d_n)|\nabla z_n|^2 w_n \leq 0$$

Thus the maximum of $w_n$ is not achieved on the boundary.

**Step 1.** $\Phi(t) = t^{2\beta}$, $0 < \beta < 1$, Bernstein + SMP

$w_n$ bounded $\Rightarrow$ estimate for $z_n$ in $C^{0,1-\beta}(\Omega)$

**Step 2.** $\Phi(t) = e^{\lambda t}$, $\lambda >> 1$, Bernstein + SMP

$w_n$ bounded
Idea of the proof

Then, in order to apply a “weighted ” version of the Bernstein’s method, we we look at the equation solved by \( w_n = \Phi(d_n)|\nabla z_n|^2 \).

On \( \partial \Omega \), we have:

\[
\frac{\partial z_n}{\partial \nu} = 0 \Rightarrow \frac{\partial \Phi(d_n)|\nabla z_n|^2}{\partial \nu} \leq \left[ C_\Omega - \frac{\Phi'(d_n)}{\Phi(d_n)} \right] \Phi(d_n)|\nabla z_n|^2 \leq 0
\]

Thus the maximum of \( w_n \) is not achieved on the boundary.

**Step 1.** \( \Phi(t) = t^{2\beta}, \ 0 < \beta < 1 \), Bernstein + SMP
\( w_n \) bounded \( \Rightarrow \) estimate for \( z_n \) in \( C^{0,1-\beta}(\Omega) \)

**Step 2.** \( \Phi(t) = e^{\lambda t}, \ \lambda >> 1 \), Bernstein + SMP
\( w_n \) bounded \( \Rightarrow \) estimate for \( z_n \) in \( W^{1,\infty}(\Omega) \).
Idea of the proof

Then, in order to apply a “weighted” version of the Bernstein’s method, we look at the equation solved by $w_n = \Phi(d_n)|\nabla z_n|^2$.

On $\partial \Omega$, we have:

$$\frac{\partial z_n}{\partial \nu} = 0 \Rightarrow \frac{\partial \Phi(d_n)|\nabla z_n|^2}{\partial \nu} \leq \left[C_\Omega - \frac{\Phi'(d_n)}{\Phi(d_n)}\right]\Phi(d_n)|\nabla z_n|^2 w_n \leq 0.$$

Thus the maximum of $w_n$ is not achieved on the boundary.

**Step 1.** $\Phi(t) = t^{2\beta}$, $0 < \beta < 1$, Bernstein + SMP

$w_n$ bounded $\Rightarrow$ estimate for $z_n$ in $C^{0,1-\beta}(\Omega)$

**Step 2.** $\Phi(t) = e^{\lambda t}$, $\lambda >> 1$, Bernstein + SMP

$w_n$ bounded $\Rightarrow$ estimate for $z_n$ in $W^{1,\infty}(\Omega)$.

Hence:

$$|\nabla u_n - \nabla S_n| \text{ uniformly bounded in } L^\infty(\Omega).$$
Actually we have characterized any singular term of $\nabla u$, 

$$\alpha = 2 - q - \frac{1}{q} - \frac{1}{\partial u(x)} - \alpha C^* d\alpha + 1 + \sum_{k=1}^{\alpha} \left[ (k - \alpha) \sigma_k(x) d\alpha - k + 1 \right] \in L^\infty(\Omega)$$

and

$$\partial u(x) - \sum_{k=1}^{\alpha} \nabla \sigma_k(x) \cdot \tau d\alpha - k \in L^\infty(\Omega)$$

that is a stronger result than the one stated.

By computations we have that

$$\sigma_1 = (q - 1) - 2 - q - \frac{1}{3} - 2 d(x)$$

and noting that $\Delta d(x) \bigg|_{\partial \Omega} = (N - 1) H(x)$ we deduce the thesis.
Actually we have characterized any singular term of $\nabla u$, i.e.

$$\alpha = \frac{2-q}{q-1}$$

$$\frac{\partial u(x)}{\partial \nu} - \frac{\alpha C^*}{d^{\alpha+1}} + \sum_{k=1}^{[\alpha]+1} \left[ \frac{(k-\alpha)\sigma_k(x)}{d^{\alpha-k+1}(x)} - \frac{\nabla \sigma_{k-1}(x) \cdot \nu}{d^{\alpha-k+1}(x)} \right] \in L^\infty(\Omega)$$
Idea of the proof

Actually we have characterized any singular term of $\nabla u$, i.e.

$$\alpha = \frac{2-q}{q-1}$$

$$\frac{\partial u(x)}{\partial \nu} - \frac{\alpha C^*}{d^{\alpha+1}} + \sum_{k=1}^{[\alpha]+1} \left[ \frac{(k - \alpha)\sigma_k(x)}{d^{\alpha-k+1}(x)} - \frac{\nabla \sigma_{k-1}(x) \cdot \nu}{d^{\alpha-k+1}(x)} \right] \in L^\infty(\Omega)$$

and

$$\frac{\partial u(x)}{\partial \tau} - \sum_{k=1}^{[\alpha]} \frac{\nabla \sigma_k(x) \cdot \tau}{d^{\alpha-k}} \in L^\infty(\Omega)$$
Idea of the proof

Actually we have characterized any singular term of $\nabla u$, i.e.

$$\alpha = \frac{2-q}{q-1}$$

$$\frac{\partial u(x)}{\partial \nu} - \frac{\alpha C^*}{d^{\alpha+1}} + \sum_{k=1}^{[\alpha]+1} \left[ \frac{(k - \alpha)\sigma_k(x)}{d^{\alpha-k+1}(x)} - \frac{\nabla \sigma_{k-1}(x) \cdot \nu}{d^{\alpha-k+1}(x)} \right] \in L^\infty(\Omega)$$

and

$$\frac{\partial u(x)}{\partial \tau} - \sum_{k=1}^{[\alpha]} \frac{\nabla \sigma_k(x) \cdot \tau}{d^{\alpha-k}} \in L^\infty(\Omega)$$

that is a stronger result than the one stated.
Idea of the proof

Actually we have characterized any singular term of $\nabla u$, i.e.

$$\alpha = \frac{2-q}{q-1}$$

$$\frac{\partial u(x)}{\partial \nu} - \frac{\alpha C^*}{d^{\alpha+1}} + \sum_{k=1}^{[\alpha]+1} \left[ \frac{(k - \alpha)\sigma_k(x)}{d^{\alpha-k+1}(x)} - \frac{\nabla \sigma_{k-1}(x) \cdot \nu}{d^{\alpha-k+1}(x)} \right] \in L^\infty(\Omega)$$

and

$$\frac{\partial u(x)}{\partial \tau} - \sum_{k=1}^{[\alpha]} \frac{\nabla \sigma_k(x) \cdot \tau}{d^{\alpha-k}(x)} \in L^\infty(\Omega)$$

that is a stronger result than the one stated.

By computations we have that

$$\sigma_1 = \frac{(q - 1)^{-\frac{2-q}{q-1}} \Delta d(x)}{3 - 2q} \frac{\Delta d(x)}{2}$$

and noting that $\Delta d(x) \mid_{\partial \Omega} = (N - 1)H(x)$ we deduce the thesis.
GRACIAS !