Las desigualdades de Kurdyka-Łojasiewicz-Simon para flujos gradiente en espacios métricos

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Introduction

S. Łojasiewicz. Une propriété topologique des sous-ensembles analytiques réels. In *Les Équations aux Dérivées Partielles (Paris, 1962)*, pages 87–89. Éditions du Centre National de la Recherche Scientifique, Paris, 1963.

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Every real-analytic function $f : \mathcal{U} \to \mathbb{R}$ defined on a neighborhood $\mathcal{U} \subseteq \mathbb{R}^N$ of a, with $f(a) = 0$ satifies **Łojasiewicz (Ł-)inequality**

$$\|\nabla f(x)\| \geq |f(x)|^{\alpha} \quad \text{for all } x \in \mathcal{U} \tag{1}$$

for some exponent $0 < \alpha \leq 1$.

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If the function $f : \mathcal{U} \subset \mathbb{R}^N \to \mathbb{R}$ satisfies inequality (1), then the solutions of the *gradient system*

$$\begin{cases} v'(t) + \nabla f(v(t)) = 0 & \text{for } t > 0, \\ v(0) = v_0, \end{cases} \tag{2}$$

trend to equilibrium.

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$(\mathfrak{M}, d)$ denotes a complete metric space. $1 \leq p < \infty$ and $p' := \frac{p}{p-1}$ be the *Hölder conjugate*

Gradient flows in metric spaces

Definition (Strong upper gradient)

For a proper functional $\mathcal{E} : \mathfrak{M} \to (-\infty, \infty]$, a proper functional $g : \mathfrak{M} \to [0, +\infty]$ is called a strong upper gradient of $\mathcal{E}$ if for every curve $v \in AC(0, +\infty; \mathfrak{M})$, the composition function $g \circ v : (0, +\infty) \to [0, \infty]$ is Borel-measurable and

$$|\mathcal{E}(v(t)) - \mathcal{E}(v(s))| \leq \int_s^t g(v(r))\, |v'|(r)\, dr \qquad \text{for all } a < s \leq t < b$$

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$$|v'|(t) := \lim_{s \to t} \frac{d(v(s), v(t))}{|s - t|} \quad \text{the metric derivative}$$

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Definition

For a given functional $\mathcal{E} : \mathfrak{M} \to (-\infty, \infty]$, the descending slope $|D^- \mathcal{E}| : \mathfrak{M} \to [0, +\infty]$ of $\mathcal{E}$ is given by

$$|D^- \mathcal{E}|(u) := \begin{cases} \limsup_{v \to u} \dfrac{[\mathcal{E}(v) - \mathcal{E}(u)]^-}{d(v, u)} & \text{if } u \in D(\mathcal{E}), \\ +\infty & \text{if otherwise.} \end{cases}$$

Definition

Let $\mathcal{E} : \mathfrak{M} \to (-\infty, \infty]$ be proper functional with strong upper gradient g, and $v \in AC_{loc}(0, +\infty; \mathfrak{M})$. Then v is a p-gradient flow of $\mathcal{E}$ if and only if $\mathcal{E} \circ v : (0, +\infty) \to \mathbb{R}$ is non-increasing and energy dissipation equality

$$\mathcal{E}(v(s)) - \mathcal{E}(v(t)) = \frac{1}{p} \int_s^t |v'|^p(r)\,\mathrm{d}r + \frac{1}{p'} \int_s^t g^{p'}(v(r))\,dr$$

holds for all $0 < s < t < +\infty$.

Definition (constant speed geodesics and λ-geodesic convexity)

A curve $\gamma : [0,1] \to \mathfrak{M}$ is said to be a constant speed geodesic connecting two points $v_0, v_1 \in \mathfrak{M}$ if $\gamma(0) = v_0$, $\gamma(1) = v_1$ and

$$d(\gamma(s), \gamma(t)) = (t-s)d(v_0, v_1) \quad \text{for all } s,\ t \in [0,1] \text{ with } s \leq t.$$

A metric space $(\mathfrak{M}, d)$ with the property that for every two elements v_0, $v_1 \in \mathfrak{M}$, there is at least one constant speed geodesic $\gamma \subseteq \mathfrak{M}$ connecting v_0 and v_1 is called a length space. Given $\lambda \in \mathbb{R}$, a functional $\mathcal{E} : \mathfrak{M} \to (-\infty, \infty]$ is called λ-geodesically convex if for every v_0, $v_1 \in D(\mathcal{E})$, there is a constant speed geodesic $\gamma \subseteq \mathfrak{M}$ connecting v_0 and v_1 such that $\mathcal{E}$ is λ-convex along γ, tha is,

$$\mathcal{E}(\gamma(t)) \leq (1-t)\mathcal{E}(\gamma(0)) + t\mathcal{E}(\gamma(1)) - \frac{\lambda}{2} t(1-t) d^2(\gamma(0), \gamma(1))$$

for all $t \in [0,1]$,

Gradient flows in metric spaces

Proposition

For $\lambda \in \mathbb{R}$, let $\mathcal{E} : \mathfrak{M} \to (-\infty, \infty]$ be a proper λ-geodesically convex functional on a length space $\mathfrak{M}$. Then, if $\mathcal{E}$ is lower semicontinuous, then the descending slope $|D^{-}\mathcal{E}|$ of $\mathcal{E}$ is a strong upper gradient of $\mathcal{E}$, and $|D^{-}\mathcal{E}|$ is lower semicontinuous.

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Definition

An element $\varphi \in \mathfrak{M}$ is called an equilibrium point (or also *critical point*) of a proper functional $\mathcal{E} : \mathfrak{M} \to (-\infty, \infty]$ with strong upper gradient g if $\varphi \in D(g)$ and $g(\varphi) = 0$. We denote by $\mathbb{E}_g = g^{-1}(\{0\})$ the set of all equilibrium points of $\mathcal{E}$ with respect to strong upper gradient g.

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We denote by $\mathrm{argmin}(\mathcal{E})$ the set of all global minimisers φ of $\mathcal{E}$.

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If for $\lambda \geq 0$, $\mathcal{E}$ is λ-geodesically convex, then

$$\mathrm{argmin}(\mathcal{E}) = \Big\{ \varphi \in D(|D^-\mathcal{E}|) \;\Big|\; |D^-\mathcal{E}|(\varphi) = 0 \Big\} = \mathbb{E}_{|D^-\mathcal{E}|}.$$

Definition

For a curve $v \in C((0,\infty);\mathfrak{M})$, the set

$$\omega(v) := \left\{ \varphi \in \mathfrak{M} \;\middle|\; \text{there is } t_n \uparrow +\infty \text{ s.t. } \lim_{n\to\infty} v(t_n) = \varphi \text{ in } \mathfrak{M} \right\}$$

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For a proper functional $\mathcal{E} : \mathfrak{M} \to (-\infty,\infty]$ and $\varphi \in D(\mathcal{E})$, we call the functional $\mathcal{E}(\cdot|\varphi) : \mathfrak{M} \to (-\infty,\infty]$ defined by

$$\mathcal{E}(v|\varphi) = \mathcal{E}(v) - \mathcal{E}(\varphi) \qquad \text{for every } v \in \mathfrak{M}$$

the relative entropy or relative energy of $\mathcal{E}$ with respect to φ.

Trend to equilibrium in the entropy sense

Proposition

Let $\mathcal{E} : \mathfrak{M} \to (-\infty, +\infty]$ be a proper functional, g a strong upper gradient of $\mathcal{E}$, and v a p-gradient flow of $\mathcal{E}$. Then, the following statements hold.

(1) $\mathcal{E}$ *is a strict Lyapunov function of* v*.*

(2) **(Trend to equilibrium in the entropy sense)** *If for $t_0 \geq 0$, $\mathcal{E}$ restricted on the set $\overline{\mathcal{I}}_{t_0}(v)$ is lower semicontinuous, then for every $\varphi \in \omega(v)$, one has $\varphi \in D(\mathcal{E})$ and*

$$\lim_{t\to\infty} \mathcal{E}(v(t)) = \mathcal{E}(\varphi) = \inf_{\xi \in \overline{\mathcal{I}}_{t_0}(v)} \mathcal{E}(\xi). \tag{3}$$

(3) **(ω-limit points are equilibrium points of $\mathcal{E}$)** *Suppose for $t_0 \geq 0$, $\mathcal{E}$ restricted on the set $\overline{\mathcal{I}}_{t_0}(v)$ is bounded from below and g restricted on the set $\overline{\mathcal{I}}_{t_0}(v)$ is lower semicontinuous. Then the ω-limit set $\omega(v)$ of v is contained in the set $\mathbb{E}_g$ of equilibrium points of $\mathcal{E}$.*

Kurdyka-Łojasiewicz-Simon inequalities in metric spaces

Definition

A proper functional $\mathcal{E} : \mathfrak{M} \to (-\infty, +\infty]$ with strong upper gradient g and equilibrium point $\varphi \in \mathbb{E}_g$ is said to satisfy a Kurdyka-Łojasiewicz inequality on the set $\mathcal{U} \subseteq [g > 0] \cap [\theta'(\mathcal{E}(\cdot|\varphi)) > 0]$ if there is a strictly increasing function $\theta \in W^{1,1}_{loc}(\mathbb{R})$ satisfying $\theta(0) = 0$ and

$$\theta'(\mathcal{E}(v|\varphi))\, g(v) \geq 1 \qquad \text{for all } v \in \mathcal{U}. \tag{4}$$

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In the case that there are $\alpha \in (0, 1]$ and $c > 0$ such that

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Definition

A proper functional $\mathcal{E} : \mathfrak{M} \to (-\infty, +\infty]$ with strong upper gradient g and equilibrium point $\varphi \in \mathbb{E}_g$ is said to satisfy a Łojasiewicz-Simon inequality with exponent $\alpha \in (0, 1]$ near φ if there are $c > 0$ and a set $\mathcal{U} \subseteq D(\mathcal{E})$ with $\varphi \in \mathcal{U}$ such that

$$|\mathcal{E}(v|\varphi)|^{1-\alpha} \leq c\, g(v) \qquad \text{for every } v \in \mathcal{U}. \tag{5}$$

Trend to equilibrium in the metric sense

Theorem (Trend to equilibrium in the metric sense)

Let $\mathcal{E} : \mathfrak{M} \to (-\infty, +\infty]$ be a proper functional with strong upper gradient g, and v be a p-gradient flow of $\mathcal{E}$ with non-empty ω-limit set $\omega(v)$. Suppose, $\mathcal{E}$ is lower semicontinuous on $\overline{\mathcal{I}}_{\overline{t}}(v)$ for some $\overline{t} \geq 0$ and for $\varphi \in \omega(v) \cap \mathbb{E}_g$, there is an $\varepsilon > 0$ such that

$$B(\varphi, \varepsilon) \cap [\mathcal{E}(\cdot|\varphi) > 0] \subseteq [g > 0].$$

If there is a strictly increasing function $\theta \in W^{1,1}_{loc}(\mathbb{R})$ satisfying $\theta(0) = 0$ and $|[\theta > 0, \theta' = 0]| = 0$ such that $\mathcal{E}$ satisfies the Kurdyka-Łojasiewicz inequality (4) on

$$\mathcal{U}_\varepsilon = B(\varphi, \varepsilon) \cap [\mathcal{E}(\cdot|\varphi) > 0] \cap [\theta'(\mathcal{E}(\cdot|\varphi)) > 0], \tag{6}$$

then v has finite length and

$$\lim_{t \to \infty} v(t) = \varphi \qquad \text{in } \mathfrak{M}. \tag{7}$$

Theorem (Decay estimates and finite time of extinction)

Let $\mathcal{E} : \mathfrak{M} \to (-\infty, +\infty]$ be a proper functional with strong upper gradient g, and v be a p-gradient flow of $\mathcal{E}$ with non-empty ω-limit set $\omega(v)$. Suppose, $\mathcal{E}$ is lower semicontinuous on $\overline{\mathcal{I}}_{\overline{t}}(v)$ for some $\overline{t} > 0$, and for $\varphi \in \omega(v) \cap \mathbb{E}_g$ there are $\varepsilon > 0$, $c > 0$, and $\alpha \in (0,1]$ such that $\mathcal{E}$ satisfies a Łojasiewicz-Simon inequality (5) *with exponent α on $B(\varphi, \varepsilon) \cap D(\mathcal{E})$. Then*

$$d(v(t), \varphi) \leq \frac{c}{\alpha} \left(\mathcal{E}(v(t)|\varphi)\right)^{\alpha} = \mathcal{O}\left(t^{-\frac{\alpha(p-1)}{1-p\alpha}}\right) \qquad \text{if } 0 < \alpha < \frac{1}{p}$$

$$d(v(t), \varphi) \leq c\, p \left(\mathcal{E}(v(t)|\varphi)\right)^{\frac{1}{p}} \leq c\, p \left(\mathcal{E}(v(t_0)|\varphi)\right)^{\frac{1}{p}} e^{-\frac{t}{pc^{p'}}} \qquad \text{if } \alpha = \frac{1}{p}$$

$$d(v(t), \varphi) \leq \begin{cases} \tilde{c}\,(\hat{t} - t)^{\frac{\alpha(p-1)}{p\alpha - 1}} & \text{if } \quad t_0 \leq t \leq \hat{t}, \\ 0 & \text{if } t > \hat{t}, \end{cases} \qquad \text{if } \frac{1}{p} < \alpha \leq 1,$$

Decay estimates and finite time of extinction

Theorem

where,

$$\tilde{c} := \left[\left[\frac{1}{\alpha^{\alpha-1} c} \right]^{\frac{p'-1}{\alpha}} \frac{p\alpha - 1}{\alpha(p-1)} \right]^{\frac{\alpha(p-1)}{p\alpha-1}},$$

$$\hat{t} := t_0 + \alpha^{\frac{\alpha-1}{\alpha(p-1)}} c^{\frac{1}{\alpha(p-1)}} \frac{\alpha(p-1)}{p\alpha-1} \left(\mathcal{E}(v(t_0)|\varphi)\right)^{\frac{p\alpha-1}{\alpha(p-1)}},$$

and $t_0 \geq 0$ can be chosen to be the "first entry time", that is, $t_0 \geq 0$ is the smallest time $\hat{t}_0 \in [0, +\infty)$ such that $v([\hat{t}_0, +\infty)) \subseteq B(\varphi, \varepsilon)$.

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If $\mathcal{E} : \mathfrak{M} \to (-\infty, \infty]$ be a proper, lower semicontinuous, λ-geodesically convex functional on a length space $(\mathfrak{M}, d)$, with $\lambda > 0$ and is bounde from below, then there is a unique minimiser $\varphi \in D(\mathcal{E})$ of $\mathcal{E}$ and

$$\mathcal{E}(v|\varphi) \leq \frac{1}{2\lambda} \, |D^{-}\mathcal{E}|^2(v) \qquad \text{for all } v \in D(\mathcal{E}).$$

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$$\mathcal{E}(v|\varphi) \leq \frac{1}{2\lambda} \, |D^{-}\mathcal{E}|^2(v) \qquad \text{for all } v \in D(\mathcal{E}).$$

Then, every gradient flow v of $\mathcal{E}$ satisfies

$$d(v(t), \varphi) = \mathcal{O}\left(e^{-\lambda t}\right) \qquad \text{as } t \to \infty.$$

Entropy-transportation inequality and KŁ inequality

Definition

A proper functional $\mathcal{E} : \mathfrak{M} \to (-\infty, +\infty]$ with strong upper gradient g is said to satisfy locally a generalised entropy-transportation (ET-) inequality at a point of equilibrium $\varphi \in \mathbb{E}_g$ if there are $\varepsilon > 0$ and a strictly increasing function $\Psi \in C(\mathbb{R})$ satisfying $\Psi(0) = 0$ and

$$\inf_{\hat{\varphi} \in \mathbb{E}_g \cap B(\varphi, \varepsilon)} d(v, \hat{\varphi}) \leq \Psi(\mathcal{E}(v|\varphi)) \tag{8}$$

for every $v \in B(\varphi, \varepsilon) \cap D(\mathcal{E})$. Further, a functional $\mathcal{E}$ is said to satisfy globally a generalised entropy-transportation inequality at $\varphi \in \mathbb{E}_g$ if $\mathcal{E}$ satisfies

$$\inf_{\hat{\varphi} \in \mathbb{E}_g} d(v, \hat{\varphi}) \leq \Psi(\mathcal{E}(v|\varphi)) \quad \text{for every } v \in D(\mathcal{E}). \tag{9}$$

Assumption (E) Suppose, for the proper energy functional $\mathcal{E} : \mathfrak{M} \to (-\infty, +\infty]$ with strong upper gradient g holds:

for all $v_0 \in D(\mathcal{E})$, there is a p-gradient flow v of $\mathcal{E}$ with $v(0+) = v_0$.

Entropy-transportation inequality and KŁ inequality

Theorem (Global KŁ- and ET-inequality)

For $\lambda \geq 0$, let $\mathcal{E} : \mathfrak{M} \to (-\infty, +\infty]$ be a proper, lower semicontinuous, λ-geodesically convex functional on a length space $(\mathfrak{M}, d)$. Suppose, $\mathcal{E}$ and the descending slope $|D^-\mathcal{E}|$ satisfying Assumption (E) and for $\varphi \in \mathbb{E}_{|D^-\mathcal{E}|}$, the set $[\mathcal{E}(\cdot|\varphi) \neq 0] \subset [|D^-\mathcal{E}| > 0]$. Then, the following statements are equivalent.

(1) (**KŁ-inequality**) *There is a strictly increasing function $\theta \in W^{1,1}_{loc}(\mathbb{R})$ satisfying $\theta(0) = 0$ and $|[\theta \neq 0, \theta' = 0]| = 0$, and $\mathcal{E}$ satisfies a Kurdyka-Łojasiewicz inequality on $\mathcal{U} := [\mathcal{E}(\cdot|\varphi) > 0] \cap [\theta'(\mathcal{E}(\cdot|\varphi)) > 0]$.*

(2) (**ET-inequality**)*There is a strictly increasing function $\Psi \in C(\mathbb{R})$ satisfying $\Psi(0) = 0$ and $s \mapsto \Psi(s)/s$ belongs to $L^1_{loc}(\mathbb{R})$ such that $\mathcal{E}$ satisfies the generalised entropy-transportation inequality*

$$\inf_{\tilde{\varphi} \in argmin(\mathcal{E})} d(v, \tilde{\varphi}) \leq \Psi(\mathcal{E}(v|\varphi)) \qquad \text{for all } v \in D(\mathcal{E}).$$

Corollary (**Global ŁS- and ET-inequality**)

For $\lambda \geq 0$, let $\mathcal{E} : \mathfrak{M} \to (-\infty, +\infty]$ be a proper, lower semicontinuous, λ-geodesically convex functional on a length space $(\mathfrak{M}, d)$. Suppose, $\mathcal{E}$ and the descending slope $|D^{-}\mathcal{E}|$ satisfy Assumption (E) and for $\varphi \in \mathbb{E}_{|D^{-}\mathcal{E}|}$, the set $[\mathcal{E}(\cdot|\varphi) > 0] \subset [g > 0]$. Then, for $\alpha \in (0, 1]$, the following statements hold.

(1) (**ŁS-inequality implies ET-inequality**) *If there is a $c > 0$*

$$(\mathcal{E}(v|\varphi))^{1-\alpha} \leq c|D^{-}\mathcal{E}|(v) \qquad \text{for all } v \in D(\mathcal{E}) \tag{10}$$

then $\mathcal{E}$ satisfies

$$\inf_{\tilde{\varphi} \in argmin(\mathcal{E})} d(v, \tilde{\varphi}) \leq \frac{c}{\alpha}(\mathcal{E}(v|\varphi))^{\alpha} \qquad \text{for all } v \in D(\mathcal{E}). \tag{11}$$

(2) (**ET-inequality implies ŁS-inequality**) *If there is a $c > 0$ such that $\mathcal{E}$ satisfies (11), then $\mathcal{E}$ satisfies*

$$(\mathcal{E}(v|\varphi))^{1-\alpha} \leq \frac{c}{\alpha}|D^{-}\mathcal{E}|(v) \qquad \text{for all } v \in D(\mathcal{E}). \tag{12}$$

Aplications. The classical Hilbert space case

In the case $(H, (.,.)_H)$ is a real Hilbert space and $\mathcal{E} : H \to (-\infty, +\infty]$ a proper, lower semicontinuous and semi-convex functional. Then, the following well-known generation theorem holds

Theorem

For every $v_0 \in \overline{D(\mathcal{E})}$, there is a unique strong solution v of

$$\begin{cases} v'(t) + \partial\mathcal{E}(v(t)) \ni 0, & t \in (0, \infty), \\ v(0) = v_0. \end{cases} \tag{13}$$

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The sub-differential $\partial\mathcal{E}$ of $\mathcal{E}$ is given by

$$\partial\mathcal{E} = \left\{ (v,u) \in H \times H \,\middle|\, \liminf_{t\downarrow 0} \frac{\mathcal{E}(v+tw) - \mathcal{E}(v)}{t} \geq (u,w) \ \text{ for all } \ w \in H \right\}.$$

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For $v \in D(|D^-\mathcal{E}|)$, the descending slope

$$|D^-\mathcal{E}|(v) = \min\left\{ \|u\|_H \,\middle|\, u \in \partial\mathcal{E}(v) \right\}$$

and $|D^-\mathcal{E}|$ is a strong upper gradient of $\mathcal{E}$.

Applications. The classical Hilbert space case

Corollary

Let $\mathcal{E} : H \to (-\infty, +\infty]$ be a proper, lower semicontinuous and semi-convex functional on a Hilbert space H. Suppose v is solution of (13) and there are c, $\varepsilon > 0$ and an equilibrium point $\varphi \in \omega(v)$ such that $\mathcal{E}$ satisfies a Łojasiewicz-Simon inequality

$$|\mathcal{E}(v|\varphi)|^{1-\alpha} \leq c\,\|u\| \qquad \text{for every } v \in B(\varphi, \varepsilon) \text{ and } u \in \partial\mathcal{E}(v).$$

Then,

$$\|v(t) - \varphi\|_H \leq \tfrac{c}{\alpha}\,(\mathcal{E}(v(t)|\varphi))^{\alpha} = \mathcal{O}\left(t^{-\frac{\alpha}{1-2\alpha}}\right) \qquad \text{if } 0 < \alpha < \tfrac{1}{2}$$

$$\|v(t) - \varphi\|_H \leq c\,2\,(\mathcal{E}(v(t)|\varphi))^{\frac{1}{2}} \leq c\,2\,(\mathcal{E}(v(t_0)|\varphi))^{\frac{1}{2}}\; e^{-\frac{t}{2c^2}} \qquad \text{if } \alpha = \tfrac{1}{2}$$

$$\|v(t) - \varphi\|_H \leq \begin{cases} \tilde{c}\,(\hat{t} - t)^{\frac{\alpha}{2\alpha-1}} & \text{if } \; t_0 \leq t \leq \hat{t}, \\ 0 & \text{if } t > \hat{t}, \end{cases} \qquad \text{if } \tfrac{1}{2} < \alpha \leq 1,$$

where, $t_0 \geq 0$ can be chosen to be the "first entry time", that is, $t_0 \geq 0$ is the smallest time $\hat{t}_0 \in [0, +\infty)$ such that $v([\hat{t}_0, +\infty)) \subseteq B(\varphi, \varepsilon)$.

Applications. Finite Extinction time of the Dirichlet-Total Variational Flow

$$\begin{cases} v_t = \mathrm{div}\left(\frac{Dv}{|Dv|}\right) & \text{in } \Omega \times (0, +\infty), \\ v = 0 & \text{on } \partial\Omega \times (0, +\infty), \\ v(0) = v_0 & \text{on } \Omega, \end{cases} \tag{14}$$

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Problem (14) can be rewritten as an abstract initial value problem (13) in the Hilbert space $H = L^2(\Omega)$ for the energy functional $\mathcal{E} : L^2(\Omega) \to (-\infty, +\infty]$ given by

$$\mathcal{E}(v) := \begin{cases} \int_\Omega |Dv| + \int_{\partial\Omega} |v| & \text{if } v \in BV(\Omega) \cap L^2(\Omega), \\ +\infty & \text{if otherwise.} \end{cases} \tag{15}$$

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Extinction time

$$T^*(v_0) := \inf\left\{ t > 0 \,\middle|\, v(s) = 0 \quad \text{for all } s \geq t \right\}.$$

Applications. Finite Extinction time of the Dirichlet-Total Variational Flow

Theorem

Suppose $N \leq 2$ and for $v_0 \in L^2(\Omega)$, let v be the unique strong solution of problem (14). Then,

$$T^*(v_0) \leq \begin{cases} s + S_1\,|\Omega|^{1/2}\,\mathcal{E}(v(s)) & \text{if } N = 1, \\ s + S_2\,\mathcal{E}(v(s)) & \text{if } N = 2, \end{cases}$$

for arbitrarily small $s > 0$, and

$$\|v(t)\|_{L^2(\Omega)} \leq \begin{cases} \tilde{c}\,(T^*(v_0) - t) & \text{if } 0 \leq t \leq T^*(v_0), \\ 0 & \text{if } t > T^*(v_0), \end{cases} \tag{16}$$

where S_N is the best constant in Sobolev inequality and $\tilde{c} > 0$.

Skech of proof By Sobolev's inequality in $BV(\Omega)$, we have

$$\|v\|_{L^{1^*}(\Omega)} \leq S_N \left(\int_\Omega |Dv| + \int_{\partial\Omega} |v| \right) \quad \text{for all } v \in BV(\Omega) \tag{17}$$

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(18) is a entropy-transportation inequality for $\Psi(s) = C\, s$, $(s \in \mathbb{R})$, which by Corollary 14, is equivalent to the Łojasiewicz-Simon inequality

$$1 \leq C\, |D^- \mathcal{E}|(v), \qquad (v \in [\mathcal{E} > 0]).$$

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In dimension $N = 2$, the extinction time

$$T^*(v_0) \leq \frac{1}{\sqrt{2\pi}} \int_\Omega |Dv_0|.$$

Gradient flows in spaces of probability measures.

Let $(X, \mathcal{B}, d)$ be a *Polish space* equipped with their Borel σ-algebra $\mathcal{B}$.

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For $s \in \{0, 1\}$, let $\pi_s : X \times X \to X$ be defined by $\pi_s(x, y) := (1-s)x + sy$. For given measures μ_0, $\mu_1 \in \mathcal{P}(X)$, the set of transport plans with marginals μ_0 and μ_1 is denoted by

$$\Pi(\mu_0, \mu_1) := \Big\{ \gamma \in \mathcal{P}(X \times X) \,\Big|\, \pi_{0\#}\gamma = \mu_0,\ \pi_{1\#}\gamma = \mu_1 \Big\}.$$

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Fixed $x_0 \in X$, the space of finite p-moment

$$\mathcal{P}_{p,d}(X) := \left\{\mu \in \mathcal{P}(X) \,\middle|\, \int_X d(x_0,x)^p \, \mathrm{d}\mu(x) < +\infty\right\},$$

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The pair $(\mathcal{P}_{p,d}(X), W_{p,d})$ is called the p-Wasserstein space.

Gradient flows in spaces of probability measures.

Consider the free energy $\mathcal{E} : \mathcal{P}_p(\mathbb{R}^N) \to (-\infty, +\infty]$ composed by

$$\mathcal{E} = \mathcal{H}_F + \mathcal{H}_V + \mathcal{H}_W \tag{19}$$

of the internal energy

$$\mathcal{H}_F(\mu) := \begin{cases} \displaystyle\int_{\mathbb{R}^N} F(\rho)\,\mathrm{d}x & \text{if } \mu = \rho\,\mathcal{L}^N, \\ +\infty & \text{if } \mu \in \mathcal{P}_p(\mathbb{R}^N) \setminus \mathcal{P}_p^{ac}(\mathbb{R}^N), \end{cases}$$

the potential energy

$$\mathcal{H}_V(\mu) := \begin{cases} \displaystyle\int_{\mathbb{R}^N} V\,\mathrm{d}\mu & \text{if } \mu = \rho\,\mathcal{L}^N, \\ +\infty & \text{if } \mu \in \mathcal{P}_p(\mathbb{R}^N) \setminus \mathcal{P}_p^{ac}(\mathbb{R}^N), \end{cases}$$

and the interaction energy

$$\mathcal{H}_W(\mu) := \begin{cases} \displaystyle\frac{1}{2}\int_{\mathbb{R}^N\times\mathbb{R}^N} W(x-y)\,\mathrm{d}(\mu\otimes\mu)(x,y) & \text{if } \mu = \rho\,\mathcal{L}^N, \\ +\infty & \text{if } \mu \in \mathcal{P}_p(\mathbb{R}^N) \setminus \mathcal{P}_p^{ac}(\mathbb{R}^N), \end{cases}$$

We assume that the function

(F) : $F : [0, +\infty) \to \mathbb{R}$ is a convex differential function satisfying

$$F(0) = 0, \quad \liminf_{s \downarrow 0} \frac{F(s)}{s^\alpha} > -\infty \quad \text{for some } \alpha > N/(N+p), \quad (20)$$

$$\text{the map } s \mapsto s^N F(s^{-N}) \text{ is convex and non increasing in } (0, +\infty), \quad (21)$$

there is a $C_F > 0$ such that

$$F(s + \hat{s}) \leq C_F\,(1 + F(s) + F(\hat{s})) \quad \text{for all } s,\ \hat{s} \geq 0, \text{ and} \quad (22)$$

$$\lim_{s \to +\infty} \frac{F(s)}{s} = +\infty \qquad \text{(super-linear growth at infinity);} \quad (23)$$

(V) : $V : \mathbb{R}^N \to (-\infty, +\infty]$ is a proper, lower semicontinuous, λ-convex for some $\lambda \in \mathbb{R}$, and the effective domain $D(V)$ has a convex, nonempty interior $\Omega := \text{int}\, D(V) \subset \mathbb{R}^N$.

(W) : $W : \mathbb{R}^N \to [0, +\infty)$ is a convex, differentiable, and even function and there is a $C_W > 0$ such that

$$W(x + \hat{x}) \leq C_W\,(1 + W(x) + W(\hat{x})) \quad \text{for all } x,\ \hat{x} \in \mathbb{R}^N. \quad (24)$$

Proposition

Suppose, the functions F, V and W satisfy the hypotheses **(F)**, **(V)** *and* **(W)**, *and $\mathcal{E} : \mathcal{P}_p(\mathbb{R}^N) \to (-\infty, +\infty]$ is the functional given by* (19)*. Then, for $\mu = \rho\mathcal{L}^N \in D(\mathcal{E})$, one has $\mu \in D(|D^-\mathcal{E}|)$ if and only if*

$$P_F(\rho) \in W^{1,1}_{loc}(\Omega), \quad \rho\,\xi_\rho = \nabla P_F(\rho) + \rho\nabla V + \rho(\nabla W) * \rho \tag{25}$$

for some $\xi_\rho \in L^{p'}(\mathbb{R}^N, \mathbb{R}^N; \mathrm{d}\mu)$, where $P_F(x) := xF'(x) - F(x)$ is the associated "pressure function" of F. Moreover, the vector field ξ_ρ satisfies

$$|D^-\mathcal{E}|(\mu) = \left(\int_{\mathbb{R}^N} |\xi_\rho(x)|^{p'}\,\mathrm{d}\mu\right)^{\frac{1}{p'}}. \tag{26}$$

Gradient flows in spaces of probability measures.

For every $\mu_0 \in D(\mathcal{E})$, there is a p-gradient flow $\mu : [0,+\infty) \to \mathcal{P}_p(\mathbb{R}^N)$ of $\mathcal{E}$ with initial value $\lim_{t\downarrow 0} \mu(t) = \mu_0$. Moreover, for every $t > 0$, $\mu(t) = \rho(t)\,\mathcal{L}^N$ with $\mathrm{supp}(\rho(t)) \subseteq \overline{\Omega}$, and ρ is a distributional solution of the following quasilinear parabolic-elliptic boundary-value problem

$$\begin{cases} \rho_t + \mathrm{div}(\rho\, \boldsymbol{U}_\rho) = 0 & \text{in } (0,+\infty)\times\Omega, \\ \boldsymbol{U}_\rho = -|\xi_\rho|^{p'-2}\xi_\rho & \text{in } (0,+\infty)\times\Omega, \\ \boldsymbol{U}_\rho \cdot \boldsymbol{n} = 0 & \text{in } (0,+\infty)\times\partial\Omega, \end{cases} \tag{27}$$

with $P_F(\rho) \in L^1_{loc}((0,+\infty); W^{1,1}_{loc}(\Omega))$ and

$$\xi_\rho = \frac{\nabla P_F(\rho)}{\rho} + \nabla V + (\nabla W) * \rho \in L^\infty_{loc}((0,+\infty); L^{p'}(\Omega, \mathbb{R}^N; \mathrm{d}\mu(\cdot))),$$

where, $\boldsymbol{n}$ in (27) denotes the outward unit normal to the boundary $\partial\Omega$ which in the case $\Omega = \mathbb{R}^N$ needs to be neglected.
If the function $F \in C^2(0,+\infty)$, then one has that

$$-\boldsymbol{U}_\rho = |F''(\rho)\nabla\rho + \nabla V + (\nabla W)*\rho|^{p'-2}\left(F''(\rho)\nabla\rho + \nabla V + (\nabla W)*\rho\right).$$

Problem (27) includes the

doubly nonlinear diffusion equation

$$\rho_t - \mathrm{div}(|\nabla \rho^m|^{p'-2} \nabla \rho^m) = 0$$

$(V = W = 0,\ F(s) = \frac{m\, s^q}{q(q-1)}$ for $q = m + 1 - \frac{1}{p'-1}$, $\frac{1}{p'-1} \neq m \geq \frac{N-(p'-1)}{N(p'-1)})$

Fokker-Planck equation with interaction term through porous medium

$$\rho_t = \Delta \rho^m + \mathrm{div}\left(\rho(\nabla V + (\nabla W) * \rho)\right)$$

$(p = 2,\ F(s) = \frac{s^m}{(m-1)}$ for $1 \neq m \geq 1 - \frac{1}{N})$.

Gradient flows in spaces of probability measures.

Every equilibrium point $\nu = \rho_\infty \mathcal{L}^N \in \mathbb{E}_{|D^- \mathcal{E}|}$ of $\mathcal{E}$ can be characterised by

$$\begin{cases} P_F(\rho_\infty) \in W^{1,1}_{loc}(\Omega) \qquad \text{with} \\ \xi_{\rho_\infty} = \dfrac{\nabla P_F(\rho_\infty)}{\rho_\infty} + \nabla V + (\nabla W) * \rho_\infty = 0 \quad \text{a.e. on } \Omega. \end{cases}$$

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Further, for every p-gradient flow μ of $\mathcal{E}$ and equilibrium point $\nu \in \mathbb{E}_{|D^-\mathcal{E}|}$, we have

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{E}(\mu(t)) = -|D^-\mathcal{E}|^{p'}(\mu(t)) = -\mathcal{I}_{p'}(\mu(t)|\nu),$$

where the generalised relative Fischer information of μ with respect to ν is given by

$$\mathcal{I}_{p'}(\mu|\nu) = \int_\Omega -\boldsymbol{U}_\rho \cdot \xi_\rho \, \mathrm{d}\mu.$$

Gradient flows in spaces of probability measures.

Definition

We call a functional $f : \mathbb{R}^N \to (-\infty, +\infty]$ uniformly λ-p-convex for some $\lambda \in \mathbb{R}$ if the interior $\Omega = \mathrm{int}(D(f))$ of f is nonempty, f is differentiable on Ω and for every $x \in \Omega$,

$$f(y) - f(x) \geq \nabla f(x) \cdot (y - x) + \lambda \, |y - x|^p \qquad \text{for all } y \in \mathbb{R}^N.$$

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(V*) $V : \mathbb{R}^N \to (-\infty, +\infty]$ is proper, lower semicontinuous function, the effective domain $D(V)$ of V has nonempty interior $\Omega := \mathrm{int}\, D(V) \subseteq \mathbb{R}^N$, and V is uniformly λ_V-p-convex for some $\lambda_V \in \mathbb{R}$;

Gradient flows in spaces of probability measures.

Theorem

Suppose that the functions F, V and W satisfy the hypotheses **(F)**, **(V*)** *with $\lambda_V \in \mathbb{R}$ and* **(W)**. *Further, suppose $F \in C^2(0,\infty) \cap C[0,+\infty)$ and let $\mathcal{E} : \mathcal{P}_p(\mathbb{R}^N) \to (-\infty,+\infty]$ be the functional given by* (19). *Then, the following statements hold.*

(ET-inequality) *For an equilibrium point $\nu = \rho_\infty \mathcal{L}^N \in \mathbb{E}_{|D^-\mathcal{E}|}$ of $\mathcal{E}$ with $\rho_\infty \in W^{1,\infty}(\Omega)$ and $\inf \rho_\infty > 0$, and every $\mu = \rho \mathcal{L}^N \in D(\mathcal{E})$,*

$$\lambda_V \, W_p^p(\mu,\nu) \leq \mathcal{E}(\mu|\nu). \tag{28}$$

(p-Talagrand transportation inequality) *If $\lambda_V > 0$, then entropy-transportation inequality* (28) *is equivalent to the p-Talagrand inequality*

$$W_p(\mu,\nu) \leq \frac{1}{\lambda_V^{1/p}} \sqrt[p]{\mathcal{E}(\mu|\nu)} \tag{29}$$

holding for an equilibrium point $\nu = \rho_\infty \mathcal{L}^N \in \mathbb{E}_{|D^-\mathcal{E}|}$ of $\mathcal{E}$ with $\rho_\infty \in W^{1,\infty}(\Omega)$ and all $\mu = \rho \mathcal{L}^N \in D(\mathcal{E})$.

Gradient flows in spaces of probability measures.

Theorem

(generalised ŁS-inequality) *If* $\hat{\lambda} > 0$ *then for every probability measures* $\mu_1 = \rho_1 \mathcal{L}^N$, $\mu_2 = \rho_2 \mathcal{L}^N \in \mathcal{P}_p^{ac}(\Omega)$ *with* $\rho_2 \in W^{1,\infty}(\Omega)$ *and* $\inf \rho_2 > 0$, *one has that*

$$\mathcal{E}(\mu_2|\mu_1) + (\lambda_V - \hat{\lambda})\, W_p^p(\mu_1,\mu_2) \leq \frac{p-1}{p^{p'}} \frac{1}{\hat{\lambda}^{1/(p-1)}} \, |D^-\mathcal{E}|^{p'}(\mu_2).$$

(generalised Log-Sobolev inequality) *If* $\lambda_V > 0$, *then for every probability measures* $\mu_1 = \rho_1 \mathcal{L}^N$, $\mu_2 = \rho_2 \mathcal{L}^N \in \mathcal{P}_p^{ac}(\Omega)$ *with* $\rho_2 \in W^{1,\infty}(\Omega)$ *and* $\inf \rho_2 > 0$ *and* $\nu \in \mathbb{E}_{|D^-\mathcal{E}|}$, *one has that*

$$\mathcal{E}(\mu_2|\mu_1) \leq \frac{p-1}{p^{p'}} \frac{1}{\lambda_V^{1/(p-1)}} \mathcal{I}_{p'}(\mu_2|\nu).$$

(p-HWI inequality) *For every probability measures* $\mu_1 = \rho_1 \mathcal{L}^N$, $\mu_2 = \rho_2 \mathcal{L}^N \in \mathcal{P}_p^{ac}(\Omega)$ *with* $\rho_2 \in W^{1,\infty}(\Omega)$ *and* $\inf \rho_2 > 0$, *one has*

$$\mathcal{E}(\mu_2|\mu_1) + \lambda_V\, W_p^p(\mu_1,\mu_2) \leq \mathcal{I}_{p'}^{1/p'}(\mu_2|\nu)\, W_p(\mu_1,\mu_2).$$

Corollary (**Equivalence between global ET-, ŁS- and Log-Sobolev inequality**)

Suppose that the functions F, V and W satisfy the hypotheses **(F)**, **(V)** *and* **(W)**. *Further, suppose $F \in C^2(0,\infty) \cap C[0,+\infty)$ and let $\mathcal{E} : \mathcal{P}_p(\mathbb{R}^N) \to (-\infty,+\infty]$ be the functional given by* (19). *Then, the following statements hold.*

(1) If for $\nu = \rho_\infty \mathcal{L}^N \in \mathbb{E}_{|D^-\mathcal{E}|}$, there is some $\hat{\lambda} > 0$ such that $\mathcal{E}$ satisfies entropy transportation inequality

$$W_p(\mu,\nu) \leq \hat{\lambda}\,(\mathcal{E}(\mu|\nu))^{\frac{1}{p}} \qquad \text{for all } \mu \in D(\mathcal{E}), \tag{30}$$

then $\mathcal{E}$ satisfies the Łojasiewicz-Simon inequality

$$\mathcal{E}(\mu|\mu_\infty)^{1-\frac{1}{p}} \leq \hat{\lambda}\,|D^-\mathcal{E}|(\mu) \qquad \text{for all } \mu \in D(|D^-\mathcal{E}|), \tag{31}$$

or equivalently, $\mathcal{E}$ satisfies the Log-Sobolev inequality

$$\mathcal{E}(\mu|\mu_\infty)^{1-\frac{1}{p}} \leq \hat{\lambda}^{\frac{1}{1-\frac{1}{p}}}\,\mathcal{I}_{p'}(\mu|\nu) \qquad \text{for all } \mu \in D(|D^-\mathcal{E}|), \tag{32}$$

Corollary (**Equivalence between global ET-, ŁS- and Log-Sobolev inequality**)

(2) If for $\nu = \rho_\infty \mathcal{L}^N \in \mathbb{E}_{|D^-\mathcal{E}|}$*, there is some* $\hat{\lambda} > 0$ *such that* $\mathcal{E}$ *satisfies Log-Sobolev inequality* (32)*, then* $\mathcal{E}$ *satisfies entropy transportation inequality*

$$W_p(\mu,\nu) \leq \hat{\lambda} p \left(\mathcal{E}(\mu|\nu)\right)^{\frac{1}{p}} \qquad \text{for all } \mu \in D(\mathcal{E}),$$

Corollary (**Equivalence between global ET-, ŁS- and Log-Sobolev inequality**)

(2) *If for* $\nu = \rho_\infty \mathcal{L}^N \in \mathbb{E}_{|D^-\mathcal{E}|}$, *there is some* $\hat{\lambda} > 0$ *such that* $\mathcal{E}$ *satisfies Log-Sobolev inequality* (32), *then* $\mathcal{E}$ *satisfies entropy transportation inequality*

$$W_p(\mu,\nu) \leq \hat{\lambda} p \left(\mathcal{E}(\mu|\nu)\right)^{\frac{1}{p}} \qquad \text{for all } \mu \in D(\mathcal{E}),$$

Corollary (**Trend to equilibrium and exponential decay rates)**)

Suppose that the functions F, V *and* W *satisfy the hypotheses* **(F)**, **(V*)** *with* $\lambda_V > 0$ *and* **(W)**. *Further, suppose* $F \in C^2(0,\infty) \cap C[0,+\infty)$ *and let* $\mathcal{E} : \mathcal{P}_p(\mathbb{R}^N) \to (-\infty,+\infty]$ *be the functional given by* $\mathcal{E} = \mathcal{H}_F + \mathcal{H}_V + \mathcal{H}_W$. *Then, there is a unique minimiser* $\nu = \rho_\infty \mathcal{L}^N \in \mathbb{E}_{|D^-\mathcal{E}|}$ *of* $\mathcal{E}$ *and for every initial value* $\mu_0 \in D(\mathcal{E})$, *the p-gradient flow* μ *of* $\mathcal{E}$ *trends to* ν *in* $\mathcal{P}_p(\Omega)$ *as* $t \to +\infty$ *and for all* $t \geq 0$,

$$W_p(\mu(t),\nu) \leq \frac{(p-1)^{1/p'}}{\lambda_V^{1/p}} \left(\mathcal{E}(\mu(t)|\nu)\right)^{\frac{1}{p}} \leq \frac{(p-1)^{1/p'}}{\lambda_V^{1/p}} \left(\mathcal{E}(\mu_0|\nu)\right)^{\frac{1}{p}} e^{-\frac{tp^{\frac{1}{p}}}{p-1}\lambda_V^{\frac{1}{p-1}}}.$$

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