From nonlinear electrodynamics to Series of Δ_p's and Regularity theory for non-uniformly elliptic operators

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From joint works with

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1. Maxwell Theory
2. Born-Infeld Theory
3. Link with geometry
4. Direct method of the Calculus of variations and then...
5. Connection to obstacle problems and free boundaries ?
6. Relaxed formulation
7. An approximated model
8. Regularity for non-uniformly elliptic operators

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Maxwell's equations in the vacuum

Maxwell's equations for an electromagnetic field $(\mathbf{E}, \mathbf{B})$ are

$$\begin{aligned} \nabla \times \mathbf{B} - \partial_t \mathbf{E} &= \mathbf{J} && \text{[Ampère law]} \\ \frac{1}{4\pi} \nabla \cdot \mathbf{E} &= \rho && \text{[Gauss's law]} \\ \partial_t B + \nabla \times \mathbf{E} &= 0 && \text{[Faraday's law of induction]} \\ \nabla \cdot \mathbf{B} &= 0 && \text{[Gauss's law for magnetism]} \end{aligned}$$

where ρ and $\mathbf{J}$ are respectively the charge and the current density of an external source.

Maxwell's equations in the vacuum

Choosing gauge potentials $\mathbf{A}, \varphi$, namely assuming

$$\mathbf{B} = \nabla \times \mathbf{A}, \quad \mathbf{E} = -\partial_t \mathbf{A} - \nabla \varphi,$$

we end up with two equations

$$\partial_t \left(\partial_t \mathbf{A} + \nabla \varphi \right) + \nabla \times \left(\nabla \times \mathbf{A} \right) = \mathbf{J}$$

$$\nabla \cdot \left(\partial_t \mathbf{A} + \nabla \varphi \right) = -4\pi \rho.$$

These equations are variational, which means they are the Euler-Lagrange equation of an action deriving from the Lagragian

$$\mathcal{L}_{Maxwell}(\mathbf{A}, \varphi) = \frac{1}{2} \left(|\mathbf{E}|^2 - |\mathbf{B}|^2 \right) + (\mathbf{J} \mid \mathbf{A}) - 4\pi \rho \varphi.$$

Maxwell's equations for an electrostatic field $\mathbf{E} = -\nabla\varphi$ leads to Poisson equation

$$-\Delta\varphi = 4\pi\rho.$$

If $\rho = \delta$ is a point charge, then

$$-(r^2\varphi'(r))' = 0 \text{ for } r > 0,$$

and the unique solution (which vanishes at infinity) is given by

$$\varphi(x) = \frac{1}{|x|}.$$

The energy of the electrostatic field is

$$\frac{1}{8\pi}\int_{\mathbb{R}^3} |\mathbf{E}|^2\, dx = \frac{1}{8\pi}\int_{\mathbb{R}^3} \frac{1}{|x|^2}\, dx = +\infty.$$

The problem of energy divergence

The Feynman
LECTURES ON
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We must conclude that the idea of locating the energy in the field is inconsistent with the assumption of the existence of point charges. One way out of the difficulty would be to say that elementary charges, such as an electron, are not points but are really small distributions of charge. Alternatively, we could say that there is something wrong in our theory of electricity at very small distances, or with the idea of the local conservation of energy. There are difficulties with either point of view. These difficulties have never been overcome; they exist to this day. Sometime later, when we have discussed some additional ideas, such as the momentum in an electromagnetic field, we will give a more complete account of these fundamental difficulties in our understanding of nature.

Assume

$$-\Delta\varphi = 4\pi\rho.$$

The energy of the electrostatic field is

$$\int_{\mathbb{R}^3} |\mathbf{E}|^2\,dx = \int_{\mathbb{R}^3} |\nabla\varphi|^2\,dx = 4\pi\int_{\mathbb{R}^3} \rho\varphi\,dx.$$

Assume

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The energy of the electrostatic field is

$$\int_{\mathbb{R}^3} |\mathbf{E}|^2\,dx = \int_{\mathbb{R}^3} |\nabla\varphi|^2\,dx = 4\pi\int_{\mathbb{R}^3} \rho\varphi\,dx.$$

• If $\rho \in L^{6/5}(\mathbb{R}^3)$, the energy is finite (Sobolev ineq.)

$$\frac{1}{2}\int_{\mathbb{R}^3} \rho\varphi\,dx \leqslant \|\rho\|_{L^{6/5}(\mathbb{R}^3)}\|u\|_{L^6(\mathbb{R}^3)}$$

• if $\rho \in L^1(\mathbb{R}^3)$, which is a relevant physical case, the energy is still infinite in general, namely one can build counter-examples as for instance $\rho(x) = (|x|^{5/2} + |x|^{7/2})^{-1}$ for which the energy is infinite

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Born-Infeld field Theory

Born-Infeld field Theory

Foundations of the New Field Theory

THE new field equations proposed recently[1] can be derived from either of two principles, the first being a rather obvious physical statement, the other an equally obvious mathematical postulate.

(1) Einstein's mechanics is equivalent with the Lagrangian $m_0c^2\{1-(1-v^2/c^2)^{\frac{1}{2}}\}$. Historically, it has been derived from the idea of relativity[2]; but it could just as well have been found from experiments which show that electrons can not be arbitrarily accelerated. From this follows the existence of an upper limit for the velocity c; and the new Lagrangian is the simplest expression which is real only for $v<c$ and gives for the limit of small velocities the classical value $m_0v^2/2$. The problem of finding the exact law of the electromagnetic field can be attacked in a similar way. The classical Lagrangian $L=\frac{1}{2}(H^2-E^2)$ allows infinitely large values for the strengths of the field. But experience leads to the *principle of the finite field*. For the use of the classical function L gives infinite values of self energy and other physical quantities which are, in fact, certainly finite. From this follows the existence of a limit of the field, b (formerly called a^{-1}); and by the same reasoning as in mechanics, one constructs the new Lagrangian

$$L=b^2[1-\{1-b^{-2}(E^2-H^2)\}^{\frac{1}{2}}] \qquad (1)$$

(2) The same result can be obtained by the mathematical postulate of the *invariance of action*.

Using the tensor notation, the classical Lagrangian is $L=\frac{1}{2}f_{kl}f^{kl}$, where $f_{kl}=-f_{kl}$ represents the field (H,E). The integral $\frac{1}{2}\int f_{kl}f^{kl}d\tau$ ($d\tau$ element of space-time) is invariant for linear orthogonal, but not for general, transformations.

If a_{kl} is any tensor and $|a_{kl}|$ its determinant, then $\int\sqrt{|a_{kl}|}\,d\tau$ is an invariant[3]. Now every tensor can be split up into a symmetrical and antisymmetrical part: $a_{kl}=g_{kl}+f_{kl}$; $g_{kl}=g_{lk}$, $f_{lk}=-f_{kl}$. The symmetrical part g_{kl} should be identified with the metrical and f_{kl} with the electromagnetic tensor. If we demand that the actions should be not only invariant, but should also take the form of the well-known expression $\frac{1}{2}\int f_{kl}f^{kl}\,d\tau$ in the case of small electromagnetic fields and cartesian co-ordinate systems, we obtain

$$L=(-|g_{kl}|)^{\frac{1}{2}}-(-|g_{kl}+f_{kl}|)^{\frac{1}{2}} \qquad (2)$$

This expression is entirely equivalent to the expression (1) for a statical field and a cartesian co-ordinate system. In the general (not statical) case an additional term, namely, $b^{-4}(EH)$ in the square root appears. One can get rid of this by choosing another but also invariant expression for L.

M. BORN.
L. INFELD.

[1] NATURE, **132**, 282, Aug. 19, 1933; *Proc. Roy. Soc.*, in the press.
[2] Compare G. Levi-Cività, "Absolute Differential Calculus" (1927), Chap. xi, pp. 286–301.
[3] A. S. Eddington, "The Mathematical Theory of Relativity" (Cambridge, 1923), 107.

Annales de l'I. H. P.

M. Born

Théorie non-linéaire du champ électromagnétique

Annales de l'I. H. P., tome 7, n° 4 (1937), p. 155-265.
<http://www.numdam.org/item?id=AIHP_1937__7_4_155_0>

The classical action

$$\mathcal{L} = \frac{1}{2}mv^2$$

in Newton mechanics is replaced in special relativity by

$$\mathcal{L} = mc^2\left(1 - \sqrt{1 - \frac{v^2}{c^2}}\right),$$

and this provides a maximal admissible velocity of motion.

In Born-Infeld field theory (1933-34), the Lagrangian density

$$\mathcal{L}_B = \frac{b^2}{4\pi}\left(1 - \sqrt{1 - \frac{|\mathbf{E}|^2 - |\mathbf{B}|^2}{b^2}}\right) + \mathbf{J}\cdot\mathbf{A} - \rho\varphi,$$

replace the usual Lagrangian density of Maxwell theory

$$\mathcal{L} = \frac{1}{8\pi}\left(|\mathbf{E}|^2 - |\mathbf{B}|^2\right) + \mathbf{J}\cdot\mathbf{A} - \rho\varphi.$$

We have again chosen a gauge potential $(\varphi, \mathbf{A})$ so that

$$\mathbf{E} = -(\partial_t \mathbf{A} + \nabla\varphi), \quad \mathbf{B} = \nabla \times \mathbf{A}.$$

Yet ρ is the charge density while $\mathbf{J}$ is the current density.

If we set

$$I = \frac{1}{2}\left(|\mathbf{E}|^2 - |\mathbf{B}|^2\right),$$

$$\mathbf{D}_{BI} = \frac{1}{4\pi}\frac{\mathbf{E}}{\sqrt{1-\frac{2}{b^2}I}}, \qquad \mathbf{H}_{BI} = \frac{1}{4\pi}\frac{\mathbf{B}}{\sqrt{1-\frac{2}{b^2}I}},$$

we obtain formally the Euler-Lagrange equations

$$\begin{aligned} \nabla\cdot\mathbf{D}_{BI} &= \rho, \\ \nabla\times\mathbf{H}_{BI} - \partial_t\mathbf{D}_{BI} &= \mathbf{J}. \end{aligned}$$

Since the new Lagragian is only invariant for the Lorentz group of transformations, Born and Infeld quickly modified their new Lagrangian as

$$\mathcal{L}_{BI} = \frac{b^2}{4\pi}\left(1 - \sqrt{1 - \frac{|\mathbf{E}|^2 - |\mathbf{B}|^2}{b^2} - \frac{(\mathbf{E}\cdot\mathbf{B})^2}{b^4}}\right) + \mathbf{J}\cdot\mathbf{A} - \rho\varphi$$

$\leadsto$ Born-Infeld, Nature 132 (1933)
$\leadsto$ Born-Infeld, Proc. Roy. Soc. A 144 (1934)
$\leadsto$ Born, ANIHP 7 (1937)

In an electrostatic regime, we formaly have

$$-\mathrm{div}\left(\frac{\nabla\varphi}{\sqrt{1-\frac{|\nabla\varphi|^2}{b^2}}}\right)=4\pi\rho$$

The (large) parameter b is then the *maximal field intensity*.

This gives rise to solution with finite field energy

$$\mathcal{H}_{\mathrm{BI}}(\phi):=\frac{1}{4\pi}\int_{\mathbb{R}^3}\left(\frac{|\nabla\varphi|^2}{\sqrt{1-\frac{|\nabla\varphi|^2}{b^2}}}-b^2\left(1-\sqrt{1-\frac{|\nabla\varphi|^2}{b^2}}\right)\right).$$

Rem : the field energy is the Legendre transform of the action

$$\mathcal{J}_{\mathrm{BI}}(\phi)=\int_{\mathbb{R}^3}\mathcal{L}_{\mathrm{BI}}(\phi)\,dx$$

For a single point charge, we obtain

$$-\mathrm{div}\left(\frac{\nabla\varphi}{\sqrt{1-\frac{|\nabla\varphi|^2}{b^2}}}\right) = 4\pi q\delta$$

and an explicit computation shows

$$\mathbf{E} = \nabla\varphi = \frac{q}{r_0^2\sqrt{1+\frac{|x|^4}{r_0^4}}}\frac{x}{|x|},$$

where $r_0 = q/b$ is interpreted as the radius of the electron.

This **BIon**[1] has finite energy.

[1] Gibbons : A BIon is a finite energy solution of a nonlinear field theory with distributional sources

For an integrable density, if

$$-\mathrm{div}\left(\frac{\nabla\varphi}{\sqrt{1-\frac{|\nabla\varphi|^2}{b^2}}}\right)=\rho$$

holds in a classical or weak sense, then using Morrey-Sobolev ineq. we infer

$$C\|\varphi\|_\infty^N \leqslant \int \frac{|\nabla\varphi|^2}{\sqrt{1-\frac{|\nabla\varphi|^2}{b^2}}} = \int \rho\varphi \leqslant \|\varphi\|_\infty \|\rho\|_{L^1}.$$

BUT, if $\rho \in L^1$, the existence of a weak solution is open...

One can prove the existence of a solution of a relaxed problem.

Of particular interest in mathematical physics is also the superposition of point charges:

$$\begin{cases} -\mathrm{div}\left(\dfrac{\nabla\phi}{\sqrt{1-|\nabla\phi|^2}}\right) = 4\pi\displaystyle\sum_{\mathrm{k}=1}^{\ell} \mathrm{a_k}\delta_{\mathrm{x_k}}, \quad x \in \mathbb{R}^3, \\ \phi(x) \to 0, \quad \text{as } |x| \to \infty. \end{cases}$$

One would like
-the existence of a weak solution of finite energy
-analyze the behavior around the point charges

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Electrostatic field without external source

Without source, the equation for the electric field is

$$-\mathrm{div}\left(\frac{\nabla\varphi}{\sqrt{1-\frac{|\nabla\varphi|^2}{b^2}}}\right)=0.$$

If we can integrate by part then $\nabla\varphi = 0$ so that $\mathbf{E} = 0$.

The finiteness of the energy justifies the integration by part.

Let $\mathbb{L}^{3+1} := \{(x,t) \in \mathbb{R}^3 \times \mathbb{R}\}$ with the flat metric $+++-$.

Let $\Omega \subset \mathbb{R}^3$ be a bounded convex domain.

If M is the graph of a function $u \in C^{0,1}(\Omega)$, we say that M is

- weakly spacelike if $\|\nabla u\| \leqslant 1$ a.e.
- spacelike if $|u(x) - u(y)| < \|x - y\|$ whenever $x \neq y$
- strictly spacelike if $u \in C^1(\Omega)$ and $\|\nabla u\| < 1$ in Ω

We then define the area integral

$$\int_\Omega \left(\sqrt{1 - |\nabla u(x)|^2} \right) dx.$$

An important problem in classical Relativity is that of determining existence and regularity properties of maximal and constant mean curvature hypersurfaces. These are spacelike submanifolds of codimension one in the spacetime manifold, with the property that the trace of the extrinsic curvature is respectively zero, constant. Such surfaces are important because they provide Riemannian submanifolds with properties which reflect those of the spacetime.

Robert Bartnik and Leon Simon, Commun. Math. Phys. 87, 131-152 (1982)

This amounts to maximize

$$E(u) = \int_\Omega \left(\sqrt{1 - |\nabla u(x)|^2} + \int_0^{u(x)} H(x,t)dt \right) dx,$$

amongst

$$\mathcal{C}(\varphi, \Omega) = \{u \in C^{0,1}(\Omega) : \mathsf{Lip}(u) \leqslant 1 \ \& \ u(x) = \varphi(x), \ x \in \partial\Omega\}.$$

Bartnik-Simon

If H and φ are given bounded functions, the variational problem can be solved iff $\mathcal{C}(\varphi, \Omega)$ is non-empty.

Mean curvature in Minkowski space

The maximal spacelike hypersurfaces have zero Lorentz mean curvature

$$-\mathrm{div}\left(\frac{\nabla \mathrm{u}}{\sqrt{1-|\nabla \mathrm{u}|^2}}\right) = 0.$$

Bernstein's problem in Minkowski space

Calabi (1968 for $n \leqslant 4$) and Cheng-Yau (1976) proved that any entire maximal spacelike hypersurface must be affine.

Remarks:

- in opposition to the euclidean case ($n \leqslant 7$), there is no restriction on the dimension.

- the only non-strictly spacelike entire area maximizing hypersurfaces are hyperplanes of slope 1 (Bartnik)

The equation for the magnetic field is then

$$-\nabla \times \left(\frac{\nabla \times \mathbf{A}}{\sqrt{1 + \frac{|\nabla \times \mathbf{A}|^2}{b^2}}} \right) = 0.$$

Bernstein problem for vector fields

The Calabi-Cheng-Yau result applies to the scalar field ψ defined by

$$\nabla \psi = \frac{\nabla \times \mathbf{A}}{\sqrt{1 + \frac{|\nabla \times \mathbf{A}|^2}{b^2}}}.$$

Bartnik-Simon

Let Ω be a bounded and $C^{2,\alpha}$ for some $\alpha > 0$. Suppose
(i) φ is bounded and has an extension $\bar{\varphi} \in C^{2,\alpha}(\bar{\Omega})$ satisfying

$$|\nabla\bar{\varphi}(x)| \leqslant 1 - \theta_0, \quad x \in \bar{\Omega} \text{ for some } \theta_0 > 0;$$

(ii) $H \in C^{0,\alpha}(\Omega \times \mathbb{R})$ is bounded, with $\sup |H| \leqslant \Lambda$.

Then the variational problem has a $C^{2,\alpha}(\bar{\Omega})$ solution and there is $\theta = \theta(\Lambda, \Omega, \theta_0, \varphi) > 0$ such that

$$|\nabla u(x)| \leqslant 1 - \theta, \quad x \in \bar{\Omega}.$$

Remark: this was improved (and precised) by Corsato-Obersnel-Omari-Rivetti in $W^{2,r}$ for an L^{∞} curvature.

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Derivation of the Euler Lagrange equation

Consider the basic problem of minimizing

$$\int_\Omega L(\nabla v(x))\,dx$$

where L is smooth and convex. Since for a minimizer u

$$\frac{1}{\varepsilon}\int_\Omega \left(L(\nabla(u(x)+\varepsilon\eta)) - L(\nabla u(x))\right)\,dx \geqslant 0,$$

we expect that (whenever we can justify the convergence)

$$\int_\Omega \left(\nabla L(\nabla u(x)) \cdot \nabla \eta(x)\right)\,dx = 0.$$

The convergence is somehow automatic on the set

$$\nabla L(\nabla u(x)) \cdot \nabla \eta(x) \geqslant 0$$

but not on its complementary because we usually miss the information that

$$\int_\Omega |\nabla L(\nabla u(x)) \cdot \nabla \eta(x)|\,dx < \infty.$$

Derivation of the Euler Lagrange equation

Theorem [Degiovanni and Marzocchi]

Assume that L is convex, differentiable and defined on $\mathbb{R}^N$, but without any upper growth conditions. Then the Euler Lagrange equations holds, namely

$$\int_\Omega (\nabla L(\nabla u(x)) \cdot \nabla \eta(x))\, dx = 0.$$

for arbitrary compactly supported smooth η.

Remarks:

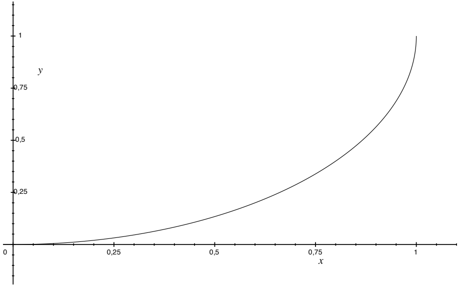
- this result was extended by Cellina to variations which are not necessarily regular.
- in the scalar case, Marcellini proved $W^{2,2}_{loc}$ regularity

Derivation of the Euler Lagrange equation

The volume integral

$$I(u) = \int \left(1 - \sqrt{1 - |\nabla u|^2}\right)\, dx$$

is weakly lower semi-continuous by convexity but not C^1



It cannot be extended in a convex way.

Derivation of the Euler Lagrange equation

We have a scalar dependence on the gradient but we have to deal with the restriction

$$|\nabla u(x)| \leqslant 1$$

which is not enough to controle

$$\int_\Omega \left(\frac{|\nabla u(x)|^2}{\sqrt{1 - |\nabla u(x)|^2}} \right) dx.$$

Also, we can only take the variations η satisfying (for small $t > 0$)

$$|\nabla(u(x) + t\eta(x))| \leqslant 1$$

i.e.

$$\nabla u(x) \cdot \nabla \eta(x) < 0$$

whenever $|\nabla u(x)| = 1$.

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Connection to obstacle problems and free boundaries ?

In the problem of elastoplasticity, the convex constraint on the gradient

$$|\nabla u(x)| \leqslant 1$$

is replaced by adequate obstacles

$$u_- \leqslant u(x) \leqslant u_+$$

See e.g. Brezis-Sibony and further developments.

Can we do that in this framework ?

Then the Euler-Lagrange equation is satisfied outside the free boundary, see e.g. Caffarelli-Friedman.

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Consider the Maxwell-Born-Infeld equation with source

$$\begin{cases} -\operatorname{div}\left(\dfrac{\nabla\phi}{\sqrt{1-|\nabla\phi|^2}}\right) = \rho, \quad x \in \mathbb{R}^N, \\ \lim\limits_{|x|\to\infty} \phi(x) = 0. \end{cases} \tag{$\mathcal{BI}$}$$

where $N \geqslant 3$.

We look for a solution in weak sense in the space

$$\mathcal{X} := \left\{ u \in \mathcal{D}^{1,2}(\mathbb{R}^N) : \nabla u \in L^\infty(\mathbb{R}^N) \text{ and } \|\nabla u\|_{L^\infty} \leqslant 1 \right\}$$

assuming $\rho \in \mathcal{X}^\star$ (which includes Radon measures).

Theorem [B., D'Avenia, Pomponio, CMP 2016]

There is a unique solution φ in weak sense, namely for all $\psi \in \mathcal{X} \cap C_c^\infty(\mathbb{R}^N)$, we have

$$\int_{\mathbb{R}^N} \frac{|\nabla \varphi|^2}{\sqrt{1-|\nabla \varphi|^2}}\, dx - \int_{\mathbb{R}^N} \frac{\nabla \varphi \cdot \nabla \psi}{\sqrt{1-|\nabla \varphi|^2}}\, dx \leqslant \langle \rho, \varphi - \psi \rangle.$$

which implies

$$\frac{|\nabla \varphi|^2}{\sqrt{1-|\nabla \varphi|^2}} \in L^1(\mathbb{R}^N) \text{ and } \mu_{\mathcal{L}}(\{x \in \mathbb{R}^N \mid |\nabla \varphi| = 1\}) = 0.$$

BE AWARE :

$\mu_{\mathcal{L}}(\{x \in \mathbb{R}^N \mid |\nabla \varphi| = 1\}) = 0 \not\Rightarrow$ φ is a weak solution of the PDE !!!

$$I(u) = \int_{\mathbb{R}^N} \left(1 - \sqrt{1 - |\nabla u|^2}\right) dx - \langle \rho, u \rangle$$

✓ Cases completely solved:

Theorem [B., d'Avenia, Pomponio]

If ρ is *radially symmetric* or *locally bounded*, then the minimizer u_ρ of I, is a weak solution of the PDE.

In the radial situation, we have enough test functions (one-dimensionality greatly helps).

In the locally bounded case, we use Bartnik-Simon local regularity.

$$\rho = \sum_{k=1}^{n} a_k \delta_{x_k}, \quad I(u) = \int_{\mathbb{R}^N} \left(1 - \sqrt{1 - |\nabla u|^2}\right) dx - \sum_{k=1}^{n} a_k u(x_k)$$

Theorem [Kiessling],[B.-D'Avenia-Pomponio]

- u_ρ is a distributional solution of (P) in $\mathbb{R}^N \setminus \{x_1, \ldots, x_n\}$, i.e.

$$\int_{\mathbb{R}^3} \frac{\nabla u_\rho \cdot \nabla v}{\sqrt{1 - |\nabla u_\rho|^2}} dx = 0 \quad \text{for all } v \in C_c^\infty(\mathbb{R}^N \setminus \{x_1, \ldots, x_n\});$$

- $u_\rho \in C^\infty(\mathbb{R}^N \setminus \Gamma) \cap C(\mathbb{R}^N)$, $\Gamma := \bigcup_{k \neq j} \overline{x_k x_j}$;
- $|\nabla u_\rho| < 1$ in $\mathbb{R}^N \setminus \Gamma$ and u_ρ is classical solution of (P) in $\mathbb{R}^N \setminus \Gamma$;
- for $k \neq j$ either u_ρ is classical solution on $\overline{x_k x_j}$ or

$$u_\rho(tx_k + (1-t)x_j) = tu_\rho(x_k) + (1-t)u_\rho(x_j) \quad \text{for all } t \in (0,1).$$

Assymptotic behaviour around the charges

For the Born-Infeld equation, an isolated singularity is removable or

Theorem [Ecker]

For every $k = 1, \dots, n$,

(i) there exists $\lim_{h \to 0^+} \dfrac{u_\varrho(hx + x_k) - u_\varrho(x_k)}{h} =: \ell_x^+(x_k)$ for every direction x and $|\ell_x^+(x_k)| = 1$;

(ii) x_k is a relative strict minimizer (resp. maximizer) of u_ϱ if $a_k < 0$ (resp. $a_k > 0$).

THEOREM [KIESSLING],[B.-D'AVENIA-POMPONIO]

- If $a_k \cdot a_j > 0$, u_ρ is a classical solution on $\mathrm{int}(\overline{x_k x_j})$;
- $\exists\ \sigma = \sigma(x_1, \dots, x_n) > 0$ s.t. if

$$\max_{k=1,\dots,n} |a_k| < \sigma,$$

  u_ρ is a classical solution in $\mathbb{R}^N \setminus \{x_1, \dots, x_n\}$;
- $\exists\ \tau = \tau(a_1, \dots, a_n) > 0$ s.t. if

$$\min_{1 \leqslant k \neq j \leqslant n} |x_k - x_j| > \tau,$$

  u_ρ is a classical solution in $\mathbb{R}^N \setminus \{x_1, \dots, x_n\}$.

In all these cases, $u_\rho \in C^\infty(\mathbb{R}^N \setminus \{x_1, \dots, x_n\})$, $|\nabla u_\rho| < 1$ in $\mathbb{R}^N \setminus \{x_1, \dots, x_n\}$, and $\lim_{x \to x_k} |\nabla u_\rho(x)| = 1$.

Theorem [B., Colasuonno, Földes, preprint 2017]

Let $\mathcal{K}_+ := \{k : a_k > 0\}$ and $\mathcal{K}_- := \{k : a_k < 0\}$. If

$$(\bullet) \quad C_N \left[\Big(\sum_{k \in \mathcal{K}_+} a_k \Big)^{\frac{1}{N-1}} + \Big(- \sum_{k \in \mathcal{K}_-} a_k \Big)^{\frac{1}{N-1}} \right] < \min_{1 \leqslant j \neq \ell \leqslant n} |x_j - x_\ell|,$$

$C_N := \left(\frac{N}{\omega_{N-1}}\right)^{\frac{1}{N-1}} \frac{N-1}{N-2}$, then $|\nabla u_\rho| < 1$ in $\mathbb{R}^N \setminus \{x_1, \ldots, x_n\}$, $u_\rho \in C(\mathbb{R}^N) \cap C^\infty(\mathbb{R}^N \setminus \{x_1, \ldots, x_n\})$, and u_ρ is classical solution in $\mathbb{R}^N \setminus \{x_1, \ldots, x_n\}$.

For two opposite-sign charges we have a better result by using comparison principle and the radial symmetry of the solution with one charge.

Condition $(\bullet)$ is not sharp, we can prove a more precise (but less explicit) sufficient condition where C_N is given by an *ugly* formula.

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• Maxwell's equation for E in the vacuum is formally a first-order approximation of Born-Infeld equation

• [D.Fortunato, L.Orsina, L.Pisani, 2002] introduced a second-order approximation to obtain a finite energy solution in the case $\rho \in L^1(\mathbb{R}^3)$

• [Kiessling],[B.-D'Avenia-Pomponio] study higher-order approximations : the Lagrangian density can be written as the following series

$$1-\sqrt{1-|\nabla u|^2} = \sum_{h=1}^{\infty} \frac{\alpha_h}{2h} |\nabla u|^{2h}, \quad \alpha_h = \frac{(2h-3)!!}{(2h-2)!!} > 0, \ |\nabla u| \leqslant 1$$

and consequently the operator can be seen as

$$-Q(u) = -\sum_{h=1}^{\infty} \alpha_h \Delta_{2h} u$$

Let $m \in \mathbb{N}$. We consider the problem

$$(P_m) \begin{cases} -\sum_{h=1}^m \alpha_h \Delta_{2h} u = \sum_{k=1}^n a_k \delta_{x_k} & \text{in } \mathbb{R}^N, \\ \lim_{|x|\to\infty} u(x) = 0 \end{cases}$$

in the space $\mathcal{X}_{2m} := \overline{C_c^\infty(\mathbb{R}^N)}^{\|\cdot\|_{\mathcal{X}_{2m}}}$ endowed with

$$\|u\|_{\mathcal{X}_{2m}} := \left[\int_{\mathbb{R}^N} |\nabla u|^2 dx + \left(\int_{\mathbb{R}^N} |\nabla u|^{2m} dx \right)^{1/m} \right]^{1/2}$$

- For $2m > \max\{N, 2^*\}$, $2^* = \frac{2N}{N-2}$: $\mathcal{X}_{2m} \hookrightarrow C_0^{0,\beta_m}(\mathbb{R}^N)$, where $\beta_m := 1 - \frac{N}{2m}$ and

$$C_0^{0,\beta_m}(\mathbb{R}^N) := \{v \in C^{0,\beta_m} \ : \ \lim_{|x|\to\infty} u(x) = 0\} \subset C_0(\mathbb{R}^N)$$

$$\Rightarrow \rho = \sum_{k=1}^n a_k \delta_{x_k} \in (C_0(\mathbb{R}^N))^* \subset (C_0^{0,\beta_m}(\mathbb{R}^N))^* \subset (\mathcal{X}_{2m})^*$$

Big advantage: the operator in (P_m) is not singular at finite value of the gradient

Is (P_m) a good approximation of (P)?

Associated functional

$$I_m(u) := \sum_{h=1}^{m} \frac{\alpha_h}{2h} \int_{\mathbb{R}^N} |\nabla u|^{2h} dx - \sum_{k=1}^{n} a_k u(x_k) \quad \text{for all } u \in \mathcal{X}_{2m}.$$

Now $u \in \mathcal{X}_{2m}$ weak solution of $(P_m) \Leftrightarrow u_m$ is a critical point of I_m

[Kiessling],[B.-D'Avenia-Pomponio]

Let $2m > \max\{N, 2^*\}$. Then,

- I_m has one and only one critical point u_m, a minimizer;
- $u_m \rightharpoonup u_\rho$ in $\mathcal{X}_{2m}$ and the convergence is uniform in compact sets of $\mathbb{R}^N$.

Theorem [B., Colasuonno, Földes]

Let $2m > \max\{N, 2^*\}$. Then,

$$u_m \in C_0^{0,\beta_m}(\mathbb{R}^N) \cap C^\infty(\mathbb{R}^N \setminus \{x_1, \dots, x_n\}).$$

* [Lieberman, 1988] + linearization + bootstrap;
* Regularity results on inhomogeneous operators of the form $\Delta_p + \Delta_q$ by Marcellini, Acerbi, Mingione do not apply: they have p and q close enough, while we need to let $m \to \infty$. We strongly use the fact that $\alpha_h > 0$ for all h and that in the operator sum there is also the Laplacian;
* $\lim_{m\to\infty} \beta_m = 1$ in accordance with the aim that the solutions of (P_m) should approximate solutions of (P).

Theorem [B., Colasuonno, Földes]

Let $2m > \max\{N, 2^*\}$, and $k = 1, \ldots, n$. Then

$$\lim_{x \to x_k} \frac{u_m(x) - u_m(x_k)}{|x - x_k|^{\frac{2m-N}{2m-1}}} = K_m$$

for some $K_m = K_m(a_k, \alpha_m, N) \in \mathbb{R}$ such that $K_m \cdot a_k < 0$.

- blow up argument (for the gradient) + Riesz potential estimates [Baroni, 2015]
- u_m behaves near the singularities like the fundamental solution of the Δ_{2m} [Serrin - improvement of Veron & Kichenassamy]
- $K_m = -\mathrm{sign}(a_k) \frac{2m-1}{2m-N} \left(\frac{|a_k|}{N|B_1|\alpha_m} \right)^{\frac{1}{2m-1}}$ and so $\lim_{m \to \infty} K_m = -\mathrm{sign}(a_k)$.

Consequences

(I) $\lim_{x\to x_k} \frac{|\nabla u_m(x)|}{|x-x_k|^{\frac{1-N}{2m-1}}} = K'_m$, with $K'_m := \frac{2m-N}{2m-1}|K_m|$;

(II) x_k is a relative strict maximizer (resp. minimizer) of u_m if $a_k > 0$ (resp. $a_k < 0$).

- (II) is an easy consequence of the fact that $K_m \cdot a_k < 0$. We find the same nature of the singularities of u_ρ ;
- $\lim_{m\to\infty} K'_m = 1$, and so

$$\lim_{m\to\infty} |\nabla u_m(x)| \sim 1 \qquad \text{for } x \text{ close to } x_k,$$

in accordance with $|\nabla u_\rho| \to 1$ as $x \to x_k$.

1. Maxwell Theory
2. Born-Infeld Theory
3. Link with geometry
4. Direct method of the Calculus of variations and then...
5. Connection to obstacle problems and free boundaries ?
6. Relaxed formulation
7. An approximated model
8. Regularity for non-uniformly elliptic operators

Regularity of the minimizer

It is well known that the solution of Poisson's equation

$$-\Delta u = \rho,$$

is $C^{1,\alpha}_{loc}$ as soon as ρ is an admissible data (say $\rho \in L^{2_*}(\mathbb{R}^N)$ with $2_* := \frac{2N}{N+2}$) and $\rho \in L^p$ with $p > N$.

What about the BI model ?

When $\rho = 1/|x|^\alpha$, we recover the same threshold $p > N$ so the result cannot be true with $p \leqslant N$.

Regularity of the minimizer

THEOREM [B., IACOPETTI]

Assume that $p > 2N$.

- If $\rho \in L^p(\mathbb{R}^N) \cap L^{2_*}(\mathbb{R}^N)$, then $u_\rho \in W^{2,2}_{\mathrm{loc}}(\mathbb{R}^N)$.
- There exists a constant $c = c(N, p)$ such that for any $\rho \in L^p(\mathbb{R}^N) \cap L^{2_*}(\mathbb{R}^N)$ satisfying $|\rho|_p + |\rho|_{2_*} \leq c$ then u_ρ is a weak solution of the PDE, it is strictly spacelike and $u \in C^{1,\gamma}_{\mathrm{loc}}(\mathbb{R}^N)$, for some $\gamma \in (0, 1)$.

Difficulties :

- The "linearly frozen" operator is not uniformly elliptic
- No regularity theory available except if $\rho \in L^\infty$
- We are even not sure that the minimizer is a weak solution
- We combine
  - revisited gradient estimates of Bartnik-Simon
  - Mingione's regularity results (using Riesz potential)