Instability of periodic minimals

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Dedicated to Jean Mawhin.

Abstract

We consider second-order Euler-Lagrange systems which are periodic in time. Their periodic solutions may be characterized as the stationary points of an associated action functional, and we study the dynamical implications of minimizing the action. Examples are well-known of stable periodic minimizers, but instability always holds for periodic solutions which are minimal in the sense of Aubry-Mather.

Key words: Periodic minimizers, minimals, quasi-asymptotic solutions, instability.

1 Introduction

Consider the Euler-Lagrange equations

$$\frac{d}{dt}\Big[\nabla_{\dot{x}} L\big(t, x(t), \dot{x}(t)\big)\Big] = \nabla_x L\big(t, x(t), \dot{x}(t)\big)\,. \tag{1}$$

Here the lagrangian $L = L(t, x, \dot{x})$ is given, and may depend on the variables $t \in \mathbb{R}$ (time), $x \in \mathbb{R}^d$ (position) and $\dot{x} \in \mathbb{R}^d$ (velocity). Assume that the dependence on time is T-periodic; then the T-periodic solutions of the system may be seen as the critical points of an associated action functional. In this paper we consider the following question:

> What can be said about periodic solutions which *minimize* the action? More precisely, what can be said about the *dynamics* of such solutions?

This problem is an old one, having being studied by Carathéodory. In a result attributed to Poincaré ([2], Chapter 17, Theorem 1) he was interested in characterizing *periodic minimizers*, i.e., solutions of the periodic variational problem

$$\min \int_0^T L(t, x(t), \dot{x}(t))dt\,, \qquad\qquad x \in C^1(\mathbb{R}/T\mathbb{Z}, \mathbb{R}^d)\,.$$

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Without using these names, Carathéodory studied the connections between the mentioned periodic minimizers and *periodic minimals*, i.e., T-periodic solutions which have lower action than if modified on a compact interval (see Section 2 for a more precise formulation of these notions). Carathéodory observed that periodic minimals are always periodic minimizers and also that the converse holds in the 1-dimensional case $d = 1$. This equivalence has consequences in the linear dynamics: *one dimensional periodic minimizers are never elliptic.*

In the nondegenerate case (when the Hessian quadratic form of the action functional is positive definite) one-dimensional periodic minimizers are hyperbolic, and, in particular, unstable (instability in this paper is understood as the logical negation of Lyapunov stability). In the degenerate case, the periodic minimizer is parabolic and dynamics are not decided at a linear level; nevertheless, instability still holds, as it was shown by Dancer and Ortega [3] and Ortega [8, 9] in the isolated and the analytical cases, and the author [12, 13]. These results cannot be directly generalized to higher dimensions $d \geq 2$ since already Carathéodory ([2], Chapter 17, §411) gave an example of an *elliptic periodic minimizer* in two dimensions (see also [7], Chapter 2, p. 78).

Apparently unrelated, the classical Lagrange-Dirichlet Theorem states the stability of the strict minima of the potential for autonomous natural mechanical systems $L = K - U$. Motivated by the search of a converse, Hagedorn [4] showed that the local maxima of the potential are unstable. This result was subsequently extended in many works; we just mention two of them which are specially related with this paper. Bolotin and Kozlov [1] considered general lagrangians and showed the instability of the equilibrium q_0 assuming that it minimizes the lagrangian in the sense that $L(t, x, \dot{x}) \geq L(t, q_0, 0)$ for any $(t, x, \dot{x})$. Hagedorn and Mawhin [5] used variational techniques to give a simple proof to Hagedorn's Instability Theorem under minimal regularity assumptions.

The results by Hagedorn are related with those by Carathéodory because the maxima of the potential are *periodic minimals*. Also the assumptions of the generalizations above imply that the solution under consideration is a periodic minimal. Our initial question may now be formulated more precisely:

Is it true that periodic minimals are always unstable?

On the other hand, Rabinowitz [10, 11] has established the existence of heteroclinic orbits connecting periodic minimizers of a class of Hamiltonian systems which are reversible in time. The link to the problem which occupies us here comes from the fact that the reversibility assumption implies that every periodic minimizer is actually a periodic minimal[1]. In some sense, Rabinowitz's results are stronger than ours because they do not only show instability, but the existence of heteroclinic orbits; on the other hand, they require some form of isolatedness assumptions.

Using also a variational approach Mather [6] showed the existence of solutions connecting Aubry-Mather sets inside a Birkhoff region of instability of the two-dimensional cylinder. He subsequently generalized these facts to higher dimensions and developed a corresponding theory of action-minimizing measures. But once again, when one tries to apply these results to prove the instability of a particular periodic orbit one needs it to be isolated.

[1] see Lemma 2.2.

In the next section we present our main results. Theorem 2.1 answers affirmatively to the question formulated two paragraphs above. Theorem 2.4 is a global version of the result in which we show the existence of quasi-asymptotic solutions starting from the periodic minimal and arriving to any chosen position

Theorem 2.1 was suggested to me by R. Ortega. I am also indebted to him for his encouragement to write this paper.

2 The main results

Let the real-valued lagrangian $L = L(t, x, \dot{x})$ be continuously defined on some open set $\Omega \subset (\mathbb{R}/T\mathbb{Z}) \times \mathbb{R}^d \times \mathbb{R}^d$. We recall the meaning of this notation: a point $(t, x, \dot{x})$ belongs to Ω if and only if the same happens with $(t + T, x, \dot{x})$; moreover L is T-periodic in time. Throughout this paper we shall further assume

(a) ***Smoothness***. $L : \Omega \to \mathbb{R}$ is twice continuously differentiable with respect to $(x, \dot{x})$.

(b) ***Legendre convexity condition***. The Hessian matrix $\big(\partial^2 L/\partial \dot{x}^2\big)(t, x, \dot{x})$ is positive definite at every point.

It will be convenient to denote by $\mathcal{F}$ to the family of C^1-smooth functions $x : \mathbb{R} \to \mathbb{R}^d$ traveling inside Ω for all time:

$$\mathcal{F} = \Big\{ x \in C^1(\mathbb{R}, \mathbb{R}^d) : (t, x(t), \dot{x}(t)) \in \Omega \ \forall t \in \mathbb{R} \Big\}.$$

By a *periodic minimal* we mean a function $x_* \in \mathcal{F}$ which is T-periodic and verifies

$$\int_{\mathbb{R}} \Big(L(t, x_* + \phi, \dot{x}_* + \dot{\phi}) - L(t, x_*, \dot{x}_*) \Big) dt \geq 0$$

for any test function ϕ with compact support and such that $x_* + \phi \in \mathcal{F}$. Equivalently (see [2], Chapter 17, Theorem 1), periodic minimals might be defined as a T-periodic functions $x_* \in \mathcal{F}$ such that

$$\int_0^{nT} L(t, x_*(t), \dot{x}_*(t)) dt \leq \int_0^{nT} L(t, x(t), \dot{x}(t)) dt$$

for any nT-periodic $x \in \mathcal{F}$.

The notion of minimality expressed above is *weak* because, depending on the set Ω, it may happen that the action on x_* is compared only to curves which are close in the C^1 topology. At a first glance it may seem a unnecessary generalization of the global periodic minimals of the Aubry-Mather theory; however, this is the concept considered by Carathéodory in [2], Chapter 17, §412 − 413.

It is well-known that periodic minimals must be solutions of the Euler-Lagrange equations (1). The main result of this paper states the instability of these solutions.

Theorem 2.1. *Every periodic minimal is unstable.*

A remark on the meaning on instability. It will be understood as the logical negation of Lyapunov stability. Our system being Hamiltonian, past and future instability are equivalent.

Theorem 2.1 may be criticized on the grounds that it may be not so easy to show the existence of periodic minimals. The classical direct methods of the Calculus of Variations, usually designed to find minima of functionals defined on Banach spaces, are more adapted to show the existence of *periodic minimizers*, i.e., T-periodic functions $x_* \in \mathcal{F}$ such that

$$\int_0^T L(t, x_*(t), \dot{x}_*(t))dt \leq \int_0^T L(t, x(t), \dot{x}(t))dt$$

for any T-periodic $x \in \mathcal{F}$.

As mentioned in the Introduction, Carathéodory showed that in the one-dimensional case $d = 1$ periodic minimizers are periodic minimals. In more dimensions the result is false in general, but continues to hold when the lagrangian is reversible, i.e.

$$(t, x, \dot{x}) \in \Omega \text{ if and only if } (-t, x, -\dot{x}) \in \Omega\,, \qquad L(t, x, \dot{x}) = L(-t, x, -\dot{x})\,.$$

Indeed, the arguments of Proposition 2.2 and Corollary 2.10 of [10] may be repeated in our framework, to show

Lemma 2.2. *Let the lagrangian L be reversible. Then every periodic minimizer is a periodic minimal.*

The combination of Theorem 2.1 and Lemma 2.2 immediately yields the following result:

Corollary 2.3. *Let the lagrangian L be reversible. Then periodic minimizers are unstable.*

In Sections 3 and 4 we will be interested in results of global nature. Correspondingly, we shall further need the following global assumptions:

(c) ***The lagrangian is globally defined***. With the notation used above this means that $\Omega = (\mathbb{R}/T\mathbb{Z}) \times \mathbb{R}^d \times \mathbb{R}^d$.

(d) ***The lagrangian grows quadratically with respect to $\dot{x}$.*** By this we mean that

(d$_1$) There is some constant $M > 0$ such that

$$L(t, x, \dot{x}) \geq \frac{|\dot{x}|^2}{2M} - M\,, \qquad \frac{\partial^2 L}{\partial \dot{x}^2}(t, x, \dot{x}) \geq \frac{1}{M}\,,$$

for any $(t, x, \dot{x}) \in (\mathbb{R}/T\mathbb{Z}) \times \mathbb{R}^d \times \mathbb{R}^d$.

(d$_2$) Given any compact set $K \subset \mathbb{R}^d$ there exists some constant $N = N_K > 0$ such that

$$L(t, x, \dot{x}) \leq N\big(1+|\dot{x}|^2\big)\,, \quad \big|\nabla_x L\big(t, x, \dot{x}\big)\big| \leq N\big(1+|\dot{x}|^2\big)\,, \quad \big|\nabla_{\dot{x}} L\big(t, x, \dot{x}\big)\big| \leq N\big(1+|\dot{x}|\big)\,,$$

for any $(t, x, \dot{x}) \in (\mathbb{R}/T\mathbb{Z}) \times K \times \mathbb{R}^d$.

We point out that the second inequality in (d_1) means

$$v^* \left(\frac{\partial^2 L}{\partial \dot{x}^2}(t, x, \dot{x}) \right) v \geq \frac{|v|^2}{M} \quad \forall v \in \mathbb{R}^d .$$

Some of these assumptions are similar to those employed in Chapter 2 of [7]; the differences are motivated by the facts that we are dealing with several space dimensions instead of just one, that our lagrangian is not periodic on x, and that we do not assume smoothness with respect to time. In Section 4 we will prove the following

Theorem 2.4. *Assume* (a),(b),(c),(d) *above. Assume further that x_* is a periodic minimal. Then for every $x_0 \in \mathbb{R}^d$ and every $\epsilon > 0$ there exists some solution $u = u(t)$ of (1) with*

$$u(0) = x_*(0) , \qquad |\dot{u}(0) - \dot{x}_*(0)| < \epsilon , \qquad u(t_0) = x_0 ,$$

for some $t_0 > 0$.

In view of Lemma 2.2, Theorem 2.4 also holds for periodic minimizers of reversible systems. We close this Section with an example in which this result is applied to systems of the type considered by Rabinowitz in [10, 11]. Consider the Hamiltonian system

$$\ddot{q} + W_q(t, q) = f(t) , \tag{HS}$$

where $q \in \mathbb{R}^d$ and the functions W and f satisfy

(W) $W \in C^2(\mathbb{R} \times \mathbb{R}^d, \mathbb{R})$ and is T-periodic in t and T_i periodic in q_i, $1 \leq i \leq d$,

(f_1) $f \in C(\mathbb{R}, \mathbb{R}^d)$ and is T-periodic in t,

(f_2) $\frac{1}{T} \int_0^T f(t) dt = 0$,

(V_1) $V(t, q) = W(t, q) - f(t)q$ is even in t.

Well known arguments show that under these assumptions the set $\mathcal{K}_1$ of periodic minimizers is nonempty. Assuming further that $\mathcal{K}_1$ consists of isolated points it was observed in [10], p. 818 that its elements are unstable. The isolatedness assumption was replaced by a weaker ones in [11]. Now we can forget it completely; the price is that, instead of heteroclinic connections we just obtain quasi-asymptotic solutions.

Corollary 2.5. *Assume $(W), (f_1), (f_2), (V_1)$. Then, every T-periodic solution $q_* \in \mathcal{K}_1$ has the following property: for any $q_0 \in \mathbb{R}^d$ and every $\epsilon > 0$ there exists some solution $q = q(t)$ of (HS) with*

$$q(0) = q_*(0) , \qquad |\dot{q}(0) - \dot{q}_*(0)| < \epsilon , \qquad q(t_0) = q_0 ,$$

for some $t_0 > 0$.

3 Minimals on a compact interval

In this Section we assume (a),(b),(c),(d). The notion of minimals, borrowed from Carathéodory ([2], Chapter 17, §412 − 413) as well as the Aubry-Mather theory, will be a key ingredient in our discussion. Let us fix real numbers $a < b$ and denote by $H^1([a,b], \mathbb{R}^d)$ to the usual Sobolev space of absolutely continuous functions with square-integrable derivative. A function $x = x(t)$ in this space is called *minimal between a and b* (or simply *minimal* when the interval is clear from the context) provided that

$$\int_a^b L(t, x(t), \dot{x}(t))\, dt \le \int_a^b L(t, y(t), \dot{y}(t))\, dt\,,$$

for any $y \in H^1([a,b], \mathbb{R}^d)$ with $y(a) = x(a)$ and $y(b) = x(b)$. The set of all functions $x : [a,b] \to \mathbb{R}^d$ which are minimal between a and b will be denoted by $\mathcal{M}([a,b])$.

The remaining of this section is devoted to point out certain well-known properties of this class of functions which will be needed in the sequel. We start with a regularity result.

Lemma 3.1. *Every function $x \in \mathcal{M}([a,b])$ is a C^1 solution of the Euler-Lagrange equations (1).*

Observe that, since the lagrangian L is not assumed smooth with respect to time, this does not imply x to be C^2; however, the function $t \mapsto \nabla_{\dot{x}} L(t, x(t), \dot{x}(t))$ must be C^1 on $[a,b]$ and the equality (1) must hold pointwise. We do not include a proof of Lemma 3.1 since, under slightly different assumptions, it may be found, for instance, in [7], pp. 39-40.

There is a second fact which will play an important role in the next section. This is an elementary version of the more general *Tonelli Theorem*, and provides the existence of minimals connecting any two given points.

Lemma 3.2. *For any $x_a, x_b \in \mathbb{R}^d$ there exists some $x \in \mathcal{M}([a,b])$ with $x(a) = x_a$ and $x(b) = x_b$.*

Again this result is well known, and its proof can be transcribed, step by step, from the proof of Theorem 2.2.1 of [7], which states the same result in a related framework. Thus, we do not include it in this paper.

Before closing this section we present a related statement which asserts the compactness of certain subsets of $\mathcal{M}([a,b])$ under quite mild assumptions.

Lemma 3.3. *Let the sequence $\{x_n\}_n \subset \mathcal{M}([a,b])$ be such that*

$$\big\{x_n(a)\big\}_n, \quad \big\{x_n(b)\big\}_n$$

are both bounded. Then there is a partial subsequence of $\{x_n\}$ which converges to some function $x \in \mathcal{M}([a,b])$ in the C^1 topology.

To keep the pace of the exposition, we accept this result now and postpone its proof to the end of the paper.

4 A sequence of quasi-asymptotic solutions

In this Section we prove Theorem 2.4. With this aim we assume (a),(b),(c),(d) and the existence of a periodic minimal x_*.

Before going into the details of the proof we observe that there is no loss of generality in assuming that

$$x_* \equiv 0\,, \qquad\qquad L(t,0,0) \equiv 0\,, \tag{2}$$

as otherwise we would replace our lagrangian L by $\mathcal{L}(t,x,\dot{x}) = L(t,x+x_*(t),\dot{x}+\dot{x}_*(t)) - L(t,x_*(t),\dot{x}_*(t))$. It will help us to simplify the notation.

Let us start by fixing some point $x_0 \in \mathbb{R}^d \backslash \{0\}$. Using Lemma 3.2 we may find, for each $n \in \mathbb{N}$, some function $u_n \in \mathcal{M}([0,nT])$ with

$$u_n(0) = 0\,, \qquad\qquad u_n(nT) = x_0\,,$$

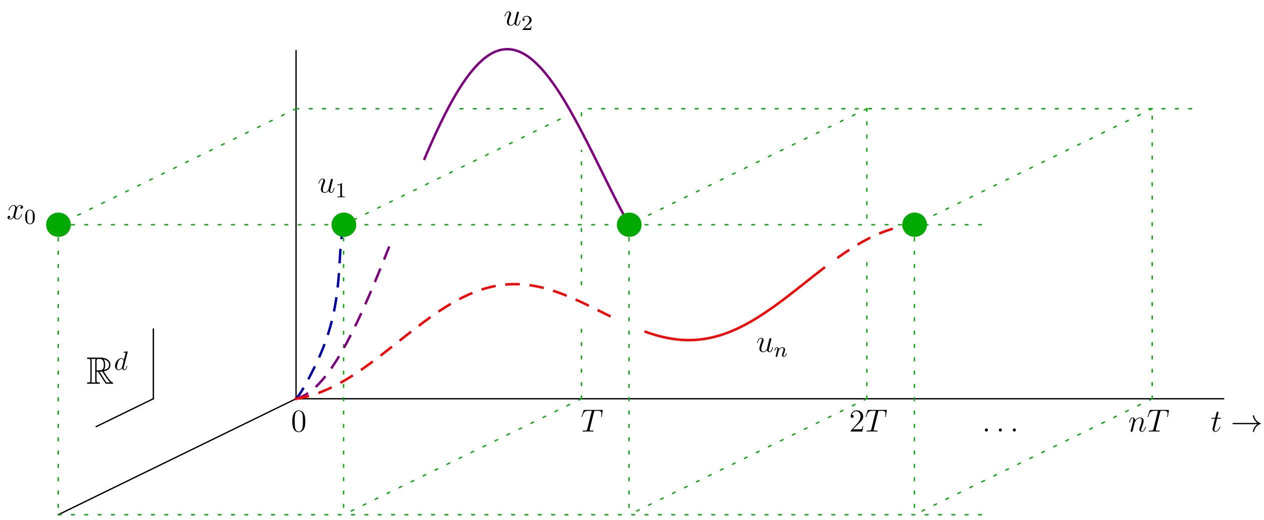

Figure 1: u_n joins the positions 0 and x_0 on the time interval $[0,T]$.

(see Fig. 1). Our proof will consist in showing that

$$\dot{u}_n(0) \to 0 \text{ as } n \to +\infty\,;$$

with this aim an important role will be played by the sequence of real numbers

$$c_n := \int_0^{nT} L(t,u_n(t),\dot{u}_n(t))\,dt\,.$$

First Step: $\{c_n\}_n$ is decreasing.

To check this statement we choose some $n \in \mathbb{N}$ and consider the piecewise C^1 function $w : [0,(n+1)T] \to \mathbb{R}^d$ defined by

$$w(t) := \begin{cases} 0 & \text{if } t \in [0,T]\,, \\ u_n(t-T) & \text{if } t \in]T,(n+1)T]\,. \end{cases}$$

Then, w and u_{n+1} coincide at $t=0$ and $t=(n+1)T$. The minimality of u_{n+1} implies that

$$c_{n+1} = \int_0^{(n+1)T} L(t,u_{n+1}(t),\dot u_{n+1}(t))dt \le \int_0^{(n+1)T} L(t,w(t),\dot w(t))dt = c_n\,,$$

(see Fig. 2(a)). It shows this step.

Second Step: $\{c_n\}_n$ is bounded from below.

To show this fact we fix some C^1 function $\varphi : [0,T] \to \mathbb{R}^d$ with $\varphi(0) = x_0$ and $\varphi(T) = 0$. Choose some $n \in \mathbb{N}$ and define $w : [0,(n+1)T] \to \mathbb{R}^d$ by

$$w(t) := \begin{cases} u_n(t) & \text{if } t \in [0,nT]\,, \\ \varphi(t-nT) & \text{if } t \in]nT,(n+1)T]\,. \end{cases}$$

It vanishes at times $t = 0,(n+1)T$ and, x_* being a periodic minimal, we deduce

$$0 \le \int_0^{(n+1)T} L(t,w(t),\dot w(t))dt = \int_0^{nT} L(t,u_n(t),\dot u_n(t))dt + \int_0^{T} L(t,\varphi(t),\varphi(t))dt\,,$$

so that $c_n \ge -\int_0^T L(t,\varphi(t),\varphi(t))dt$, see Fig. 2(b). The step is proved.

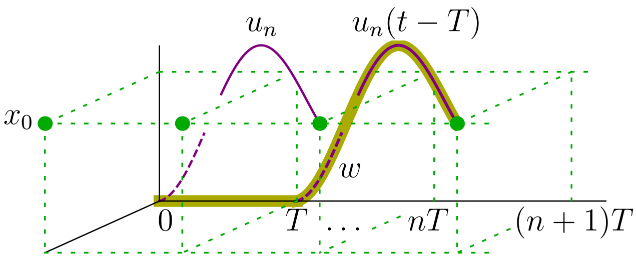

Figure 2(a): The action on w is c_n.

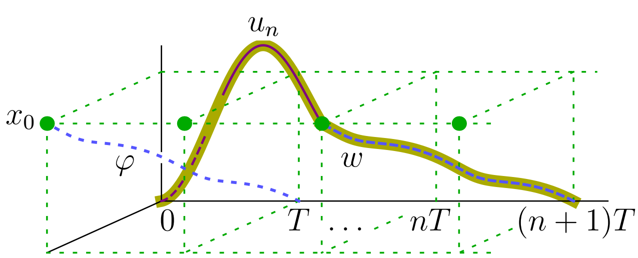

Figure 2(b): The action on w is nonnegative.

Third Step: There exists some $0 < t_* < T$ such that $\{u_n(t_*)\}_n$ is bounded.

To check this fact we define, for each $n \in \mathbb{N}$,

$$t_n := \min\left\{t > 0 : |x_n(t)| = |x_0|\right\}.$$

This step will be proved by showing that $\{t_n\}_n$ is bounded away from zero. It will be helpful to establish firstly the following

Claim. *The sequence* $\int_{t_n}^{nT} L(t,u_n(t),\dot u_n(t))dt$ *is bounded from below.*

Proof of the claim. Fix $\varphi : [0,T] \to \mathbb{R}$ as in the second step, choose some $n \in \mathbb{N}$ and define $w : [-T,(n+1)T] \to \mathbb{R}^d$ by

$$w(t) := \begin{cases} \left(\frac{t+T}{t_n+T}\right) u_n(t_n) & \text{if } -T \le t \le t_n\,, \\ u_n(t) & \text{if } t_n \le t \le nT\,, \\ \varphi(t-nT) & \text{if } nT \le t \le (n+1)T\,. \end{cases}$$

Since $w(-T) = w((n+1)T) = 0$, the minimality property of $x_* \equiv 0$ gives

$$\int_{-T}^{(n+1)T} L(t,w(t),\dot w(t))dt \ge 0\,,$$

from where the claim follows.

□

Since from the first step we also know that $\{c_n\}_n$ is bounded from above, we conclude that

$$\int_0^{t_n} L(t,u_n(t),\dot u_n(t))\,dt = c_n - \int_{t_n}^{nT} L(t,u_n(t),\dot u_n(t))\,dt$$

is also bounded from above. Remembering the first part of assumption (d_1) we see that $(1/M)\int_0^{t_n} |\dot u_n(t)|^2 dt - Mt_n$ is also bounded from above, and, combining Cauchy-Schwarz inequality with the fact that $\int_0^{t_n} |\dot u_n(t)|dt \ge x_0$ for all n, we obtain that the sequence $\{|x_0|^2/(Mt_n) - Mt_n\}_n$ is bounded from above. Thus, $\{t_n\}_n$ is bounded away from zero, concluding the step.

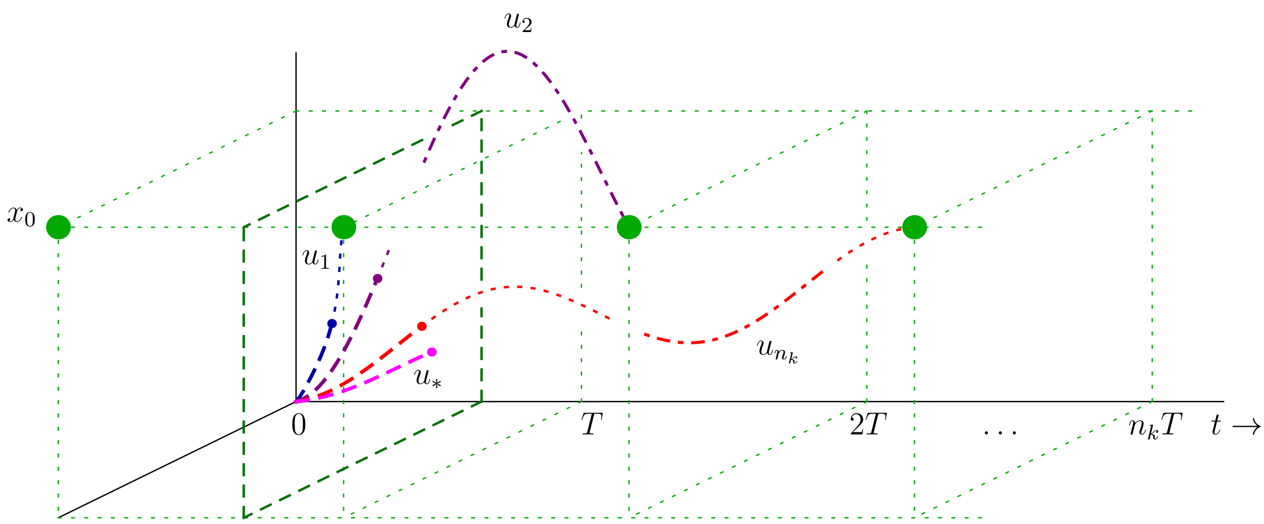

Figure 3: The partial subsequence $\{u_{n_k}\}_k$ converges to u_* on $[0,t_*]$.

Final Step: The end of the proof.

It follows from the third step and Lemma 3.3 that there is a partial subsequence $\{u_{n_k}\}_k$ which converges, in the $C^1([0,t_*],\mathbb{R}^d)$ sense, to some $u_*:[0,t_*]\to\mathbb{R}^d$, see Fig. 3.

All what remains to see now is that $\dot u_*(0)=0$. We argue by a contradiction argument and assume instead that $\dot u_*(0)\neq 0$. The function

$$w(t) := \begin{cases} 0 & \text{if } 0\le t\le T\,,\\ u_*(t-T) & \text{if } T\le t\le T+t_* \end{cases}$$

is only piecewise C^1, and by Lemma 3.1, it cannot be minimal. Thus, Lemma 3.2 states the existence of some minimal function $v:[0,T+t_*]\to\mathbb{R}^d$ satisfying

$$v(0)=0\,,\qquad v(T+t_*)=u_*(t_*)\,,\qquad \int_0^{T+t_*} L(t,v(t),\dot v(t))dt < \int_0^{t_*} L(t,u_*(t),\dot u_*(t))dt\,.$$

Let us choose some $\epsilon>0$ such that

$$\int_0^{T+t_*} L(t,v(t),\dot v(t))dt < \int_0^{t_*} L(t,u_*(t),\dot u_*(t))dt - 2\epsilon\,,$$

and define, for every $k\ge 1$, the piecewise C^1 function

$$w_k(t) = \begin{cases} v(t) + \big(t/(T+t_*)\big)\big(u_{n_k}(t_*)-u_*(t_*)\big) & \text{if } t\in[0,T+t_*]\,,\\ u_{n_k}(t-T) & \text{if } t\in[T+t_*,(n_k+1)T]\,. \end{cases}$$

Since $w_k\to v$ in the C^1 sense on $[0,T+t_*]$ and $u_{n_k}\to u_*$ in the C^1 sense on $[0,t_*]$, we may find some k_0 such that

$$\int_0^{T+t_*} L(t,w_k(t),\dot w_k(t))dt < \int_0^{t_*} L(t,u_{n_k}(t),\dot u_{n_k}(t))dt - \epsilon\,, \qquad\qquad k\ge k_0\,.$$

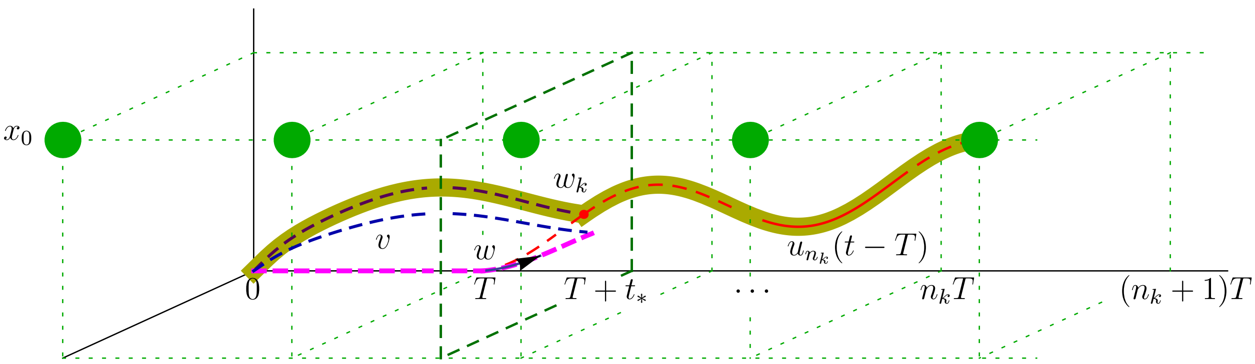

Figure 4: The action on w_k decreases a fixed quantity the action on u_{n_k}.

Consequently,

$$\int_0^{(n_k+1)T} L(t,w_k(t),\dot w_k(t))dt < c_{n_k} - \epsilon\,,$$

and, in particular,

$$c_{n_k+1} < c_{n_k} - \epsilon\,, \qquad\qquad k \geq k_0\,,$$

(see Fig. 4). It contradicts the fact that, as stated by the first and second steps the decreasing sequence $\{c_n\}$ is convergent, and concludes the proof.

5 From the global theorem to a local one

Even though both results deal with instability, Theorem 2.4 is a global result, while Theorem 2.1 is local. The path to arrive to the latter from the first is given by the following extension result.

Proposition 5.1. *Let the lagrangian $L : \Omega \to \mathbb{R}$ verify assumptions* (a),(b) *and let x_* be an associated periodic minimal. Then there exists a modified lagrangian $\tilde{L}$ verifying* (a),(b),(c),(d) *which coincides with L on some neighborhood of x_* and for which x_* remains a periodic minimal.*

As we already observed in the proof of Theorem 2.4, there is no loss of generality in assuming (2), and this will be done throughout this section. This will help us to simplify the notations. For instance, the neighborhood of x_* mentioned above must contain some set of the form

$$\bar{\mathcal{N}}_R := \Big\{(t, x, \dot{x}) : t \in \mathbb{R}/\mathbb{Z},\ |x| \leq R,\ |\dot{x}| \leq R\Big\}$$

which should be contained in Ω. Another example: $x_* \equiv 0$ is a periodic minimal for $\tilde{L}$ provided that

$$\int_{-\infty}^{+\infty} \tilde{L}(t, \phi(t), \dot{\phi}(t))dt \geq 0 \ \text{ for any } \phi \in C^1(\mathbb{R}, \mathbb{R}^d) \text{ with compact support.}$$

This property is perhaps the least direct to check in particular cases and we shall obtain it next from the following feature of $\tilde{L}$: there are numbers $\tau > S > R$ such that $\bar{\mathcal{N}}_\tau \subset \Omega$ and

($\tilde{L}_1$) $\tilde{L}(t, x, \dot{x}) = L(t, x, \dot{x})$ on $\bar{\mathcal{N}}_R$.

($\tilde{L}_2$) $\tilde{L}(t, x, \dot{x}) \geq L(t, x, \dot{x})$ on $\bar{\mathcal{N}}_\tau$.

($\tilde{L}_3$) $\tilde{L}(t, x, \dot{x}) = \Psi(|\dot{x}|)$ on $\Big((\mathbb{R}/T\mathbb{Z}) \times \mathbb{R}^d \times \mathbb{R}^d\Big) \backslash \bar{\mathcal{N}}_\tau$. Here $\Psi : [0, +\infty[\to \mathbb{R}$ is some C^2 function with

$$\Psi(v),\ \Psi''(v) > 0 \text{ on } [0, +\infty[, \qquad \Psi'(0) = 0, \qquad \Psi(v) = v^2/2 + C \text{ if } v \geq S\,, \qquad (3)$$

for some constant $C \in \mathbb{R}$.

The assertions in (3) may be roughly described by saying that Ψ mimics the behavior of a positive quadratic polynomial. Correspondingly, ($\tilde{L}_3$) states that outside $\bar{\mathcal{N}}_\tau$ the modified lagrangian $\tilde{L}(t, x, \dot{x})$ depends only on $\dot{x}$, looks like $|\dot{x}|^2/2 + C$ and is always positive.

Lemma 5.2. *Let the lagrangian $L : \Omega \to \mathbb{R}$ verify* (a),(b) *and let $x_* \equiv 0$ be an associated periodic minimal. Let the lagrangian $\tilde{L}$ verify* (a),(b),(c),(d) *and assume that there are numbers $0 < R < S < \tau$ such that $\bar{\mathcal{N}}_\tau \subset \Omega$ and $(\tilde{L}_1), (\tilde{L}_2), (\tilde{L}_3)$. Then $x_* \equiv 0$ is a periodic minimal for $\tilde{L}$.*

Proof. Using a contradiction argument, we assume instead that there exists some $x \in H^1(\mathbb{R}, \mathbb{R}^d)$ with compact support, say, contained in $[a, b]$, such that

$$\int_a^b \tilde{L}(t, x(t), \dot{x}(t))dt < 0 \,.$$

Remembering Lemma 3.2 we see that there is some function in $\mathcal{M}([a, b])$ vanishing at $t = a, b$. The action on this function cannot be greater than the action on x, and then it is not restrictive to assume that x solves the Euler-Lagrange equations associated to $\tilde{L}$ on the time interval $[a, b]$.

On the other hand, $(\tilde{L}_1) - (\tilde{L}_2)$ imply that there must exist some time $t_0 \in [a, b]$ with either $|x(t_0)| > \tau$ or $|\dot{x}(t_0)| > \tau$. But in view of $(\tilde{L}_3)$, in a neighborhood of $(t_0, x(t_0), \dot{x}(t_0))$ our lagrangian has the form $\tilde{L}(t, x, \dot{x}) = \Psi(|\dot{x}|)$, whose associated Euler-Lagrange equations are

$$(d/dt)F(\dot{x}(t)) = 0 \,, \tag{4}$$

where $F : \mathbb{R}^d \to \mathbb{R}^d$ is given by

$$F(\dot{x}) := \begin{cases} \dfrac{\Psi'(|\dot{x}|)}{|\dot{x}|}\,\dot{x} & \text{if } \dot{x} \neq 0 \,, \\ 0 & \text{if } \dot{x} = 0 \,. \end{cases}$$

This mapping is a diffeomorphism from $\mathbb{R}^d$ to itself, and hence, the Euler-Lagrange equations (4) are equivalent to $\ddot{x} \equiv 0$. From here and the boundary conditions $x(a) = 0 = x(b)$ one easily arrives to a contradiction. □

In view of this result, Proposition 5.1 and hence Theorem 2.1 will be proved if we are show the following

Lemma 5.3. *For any lagrangian $L : \Omega \to \mathbb{R}$ verifying* (a),(b) *and having $x_* \equiv 0$ as a periodic minimal there exists some $\tilde{L}$ under the assumptions of Lemma 5.2.*

This will be our goal in the remaining of this section. We start with the construction of a function of two real variables satisfying certain properties.

Lemma 5.4. *Given numbers $0 < R < S$ there exists a C^2 function $g : [0, +\infty[\times [0, +\infty[\to \mathbb{R}$ with*

(g_1) $g(u, v) \geq 0$, $\frac{\partial^2 g}{\partial v^2}(u, v) \geq 0$ *on* $[0, +\infty[\times [0, +\infty[$,

(g_2) $\frac{\partial g}{\partial v}(u, 0) = 0 = \frac{\partial g}{\partial u}(v, 0)$ *for any* $u, v \geq 0$,

(g_3) $g \equiv 0$ *on* $[0, R] \times [0, R]$.

(g_4) *$g(u,v) = \Psi(v)$ on $\big([0,+\infty[\times[0,+\infty[\big)\setminus\big(]0,S[\times]0,S[\big)$. Here $\Psi : [0,+\infty[\to \mathbb{R}$ is some C^2 function verifying (3).*

Proof. Choose C^1 functions $\phi, \psi : [0,+\infty[\to \mathbb{R}$ with

- $\phi(v) = 0$ if $0 \le v \le R$; $\quad \phi'(v) > 0$ for any $v \in]R,S[$; $\quad \phi(v) = v$ if $v \ge S$.
- $\psi(0) = 0$; $\quad \psi'(v) > 0$ on $[0,S]$; $\quad \int_0^S \psi(v)dv < \int_0^S \phi(v)dv$; $\quad \psi(v) = v$ if $v \ge S$.

Let now $m : [0,+\infty[\to \mathbb{R}$ be a C^2 cut-off function with

$$m(u) = 0 \text{ on } [0,R], \quad 0 \le m(u) \le 1, \quad m(u) = 1 \text{ on } [S,+\infty[\,,$$

and define

$$g(u,v) := \big(1 - m(u)\big)\Phi(v) + m(u)\Psi(v)\,, \qquad (u,v) \in [0,+\infty[\times[0,+\infty[\,,$$

the functions Φ and Ψ being, respectively, the primitives of ϕ and ψ verifying

$$\Phi(0) = 0, \qquad \Psi(0) = \int_0^S \phi(v)dv - \int_0^S \psi(v)dv > 0\,.$$

One immediately checks that Ψ satisfies (3), while g satisfies (g_i), $1 \le i \le 4$. It concludes the proof. $\square$

Let g be given by the previous Lemma and define

$$h(x,\dot{x}) := g(|x|,|\dot{x}|), \qquad (x,\dot{x}) \in \mathbb{R}^d \times \mathbb{R}^d\,.$$

Then, (g_2) implies that h is a C^2 function on $\mathbb{R}^d \times \mathbb{R}^d$, while (g_1) and (g_2) together imply that the Hessian matrix $\partial^2 h/\partial \dot{x}^2$ is positive semidefinite everywhere. We are led to the following multidimensional extension of Lemma 5.4:

Corollary 5.5. *Given $0 < R < S$ there exists a C^2 function $h : \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}$ such that*

(h_1) *$h(x,\dot{x}) \ge 0$, $\big(\partial^2 h/\partial \dot{x}^2\big)(x,\dot{x}) \ge 0$ on $\mathbb{R}^d \times \mathbb{R}^d$.*

(h_2) *$h(x,\dot{x}) = 0$ if $|x|, |\dot{x}| \le R$.*

(h_3) *$h(x,\dot{x}) = \Psi(|\dot{x}|)$ if $\max(|x|,|\dot{x}|) \ge S$. Here $\Psi : [0,+\infty[\to \mathbb{R}$ is some C^2 function verifying (3).*

We point out that the second inequality of (h_1) is just another way of saying that the matrix $\big(\partial^2 h/\partial \dot{x}^2\big)(x,\dot{x})$ is positive semidefinite at each point. On the other hand, straightforward computations show that, if $\Psi : [0,+\infty[\to \mathbb{R}$ has class C^2 and verifies (3), then the Hessian matrix of the C^2 function $\dot{x} \in \mathbb{R}^d \mapsto \Psi(|\dot{x}|)$ is positive definite everywhere. Thus, the function h of Corollary 5.5 verifies

(h_4) $h(x,\dot{x}) > 0, \quad \dfrac{\partial^2 h}{\partial \dot{x}^2}(x,\dot{x}) > 0 \qquad$ if $\max(|x|,|\dot{x}|) \ge S$, (the first inequality follows immediately from (3)). We are now prepared to complete the (almost) last step which remained in our way to Theorem 2.1.

Proof of Lemma 5.3. Choose numbers $R < S < \tau$ such that $\bar{\mathcal{N}_\tau} \subset \Omega$ and pick some cut-off function $k : \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}$ of class C^2 such that

$$k(x, \dot{x}) = 1 \text{ if } |x|, |\dot{x}| \leq S\,, \qquad k(x, \dot{x}) = 0 \text{ if } \max\{|x|, |\dot{x}|\} \geq \tau\,.$$

Choose now some function $h : \mathbb{R}^d \to \mathbb{R}^d$ as in Corollary 5.5. In view of (h_4) we may find some number $\gamma > 0$ such that $\tilde{L}(t, x, \dot{x}) := k(x, \dot{x})L(t, x, \dot{x}) + \gamma h(x, \dot{x})$ verifies

$$\tilde{L}(t, x, \dot{x}) \geq L(t, x, \dot{x})\,, \quad \frac{\partial^2 \tilde{L}}{\partial \dot{x}^2}(t, x, \dot{x}) > 0 \qquad \text{on } \bar{\mathcal{N}_\tau} \tag{5}$$

One easily checks that $\tilde{L}$ verifies all the assumptions of Lemma 5.2. The proof is complete. □

The last task remaining is the proof of Lemma 3.3, which was skipped in Section 3.

Proof of Lemma 3.3. For any $n \in \mathbb{N}$ we consider the linear curve

$$y_n(t) := \left(\frac{b - t}{b - a}\right) x_n(a) + \left(\frac{t - a}{b - a}\right) x_n(b)\,, \qquad t \in [a, b]\,.$$

It joins the points $x_n(a)$ and $x_n(b)$ on the time interval $[a, b]$, and the minimality property of x_n implies

$$\int_a^b L(t, x_n(t), \dot{x}_n(t))dt \leq \int_a^b L(t, y_n(t), \dot{y}_n(t))dt\,,$$

and since $\{y_n\}_n$ is bounded on the C^1 topology, the sequence of integrals in the left side of the inequality must be bounded from above. Remembering the first part of assumption (d_1) we see that $\{x_n\}_n$ must be bounded in the H^1 topology and the inclusion $H^1([a, b], \mathbb{R}^d) \subset C([a, b], \mathbb{R}^d)$ being compact, after possibly passing to a subsequence we may assume that $x_n \to x$ uniformly on $[a, b]$.

We consider the sequence $\{\eta_n\}_n$ of functions defined by

$$\eta_n(t) := \nabla_{\dot{x}} L(t, x_n(t), \dot{x}_n(t))\,, \qquad t \in [a, b]\,. \tag{6}$$

The last statement of assumption (d_2) implies that this sequence is bounded in the L^2 topology. On the other hand, all the the η_n's have class C^1 and verify

$$\dot{\eta}_n(t) = \nabla_x L(t, x_n(t), \dot{x}_n(t))\,, \qquad t \in [a, b]\,, \tag{7}$$

and the second part of assumption (d_2) gives that $\{\dot{\eta}_n\}_n$ is bounded in the L^1 topology. It follows that $\{\eta_n\}_n$ is uniformly bounded.

The combination of the middle part of assumption (d_1) with the Implicit Function Theorem implies the existence of a continuous function $S : (\mathbb{R}/T\mathbb{Z}) \times \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}^d$ such that

$$\nabla_{\dot{x}} L(t, x, \dot{x}) = y \;\text{ if and only if }\; \dot{x} = S(t, x, y)\,.$$

In this way, (6) may be rewritten as

$$\dot{x}_n(t) = S(t, x_n(t), \eta_n(t)), \qquad t \in [a, b]. \tag{8}$$

It follows that $\{\dot{x}_n\}_n$ is uniformly bounded, and (7) implies that $\{\dot{\eta}_n\}_n$ is uniformly bounded. The Ascoli-Arzelà Lemma then gives that, after possibly passing to a subsequence, $\eta_n \to \eta_*$ uniformly on $[a, b]$, and in view of (8), also $\{\dot{x}_n\}_n$ converges uniformly on $[a, b]$. Thus, x is a C^1 function and $x_n \to x$ in the C^1 topology. Since $x_n \in \mathcal{M}([a, b])$ for any n we see that $x \in \mathcal{M}([a, b])$. The proof is complete. □

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