# Uniform circular motion in General Relativity: Existence and extendibility of the trajectories

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**Abstract**

The concept of uniform circular motion in a general spacetime is introduced as a particular case of a planar motion. The initial value problem of the corresponding differential equation is analysed in detail. Geometrically, an observer that obeys a uniform circular motion is characterized as a Lorentzian helix. The completeness of inextensible trajectories is studied in Generalized Robertson-Walker spacetimes and in a relevant family of pp-wave spacetimes. Under reasonable assumptions, the physical interpretation of such results is that a uniform circular observer lives forever, providing the absence of the singularities defined by these timelike curves.

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## 1 Introduction

Uniform circular motion has been widely studied in Special Relativity (see, for instance, [9]). Relevant physical phenomena and paradoxes, usually related with the Thomas precession, have motivated its study, and its interest is still present (see [23] for an intuitive introduction). The usual approach consists in setting a family of inertial observers, one of them is considered 'the center'. Thus, an observer is said to describe a uniform circular motion with respect to the fixed 'center' if the trajectory measured by that family of inertial observers is circular and its angular velocity is constant for them. Other approaches rely on a convenient use of Frenet equations [11], [12].

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Specific motions that can be seen as very particular cases of uniform circular motions have been previously considered in relevant relativistic models with some rotational symmetry, as Schwarzschild, Reissner-Nordström and Kerr spacetimes [16, Ch. 25]. Each of these spacetimes has a remarkable family of observers with a similar role that the inertial observers in Minkowski spacetime. Consider, for example, the Exterior Schwarzschild spacetime, and set the usual coordinate system $(t, r, \theta, \varphi)$. The metric has the form

$$g = -\left(1 - \frac{r_s}{r}\right) dt^2 + \frac{1}{\left(1 - \frac{r_s}{r}\right)} dr^2 + r^2 \left(d\theta^2 + \sin^2\theta \, d\varphi^2\right),$$

where $r_s = 2M$ denotes the Schwarzschild radius (expressed in suitable units) associated to a spherical star with mass $M$. It is often said that a free falling test particle (with mass $m = 1$) describes a uniform circular orbit around the star -with respect to the Schwarzschild observers-, when its worldline has the form

$$\gamma(\tau) = \left( \sqrt{\frac{1 + \Omega^2 R^2}{1 - \frac{r_s}{R}}} \tau \, , \, R \, , \, \frac{\pi}{2}, \, \Omega\tau \right), \tag{1}$$

with

$$R^3 - \frac{3r_s}{2} R^2 - \frac{r_s}{2\Omega^2} = 0, \quad \Omega > 0. \tag{2}$$

Last condition assures that $\gamma$ is a timelike geodesic. We will show in the next sections that (1) also represents a uniform circular observer, even though (2) is not satisfied (and then, it will not be a free falling observer in general). In fact, (1)-(2) will be a limit case of strict uniform circular motions that we will introduce in this article.

The analysis of this kind of motions has a recognized physical and technological interest because they correspond to the orbits of some artificial satellites, planets or stars (see, for instance, [13]).

Our interest here is to introduce a definition of uniform circular motion in a general spacetime, involving only the physical observable quantities measured by the proper observer. In order to determine the inherent kinematic state of an observer we will focus on its proper acceleration. In other words, we will give an 'intrinsic' definition of uniform circular motion, without considering an external family of distinguished observers for which the motion described has a circular trajectory or not. Note, in addition, that the existence of such a family of observers is not guaranteed in a generic spacetime. Of course, our definition will agree with the standard notion in previously quoted cases.

Intuitively, an observer is able to detect its proper acceleration by using a giroscope or, more generally, an *accelerometer*. An accelerometer may be thought as a sphere in the center of which there is a small ball that is held on by elastic cords to the spherical surface. If a free falling observer carries such an accelerometer, then it will notice that the small ball remains just at the center. Whereas the ball will be displaced if the observer obeys an accelerated motion. For instance, a uniform accelerated motion may be recognised from a constant displacement of the small ball [5]. This intuitive idea has the advantage that may be used independently if the spacetime is relativistic or not. Thus, if an observer checks that the small ball describes a plane uniform rotation, then it will believe that it obeys a uniform circular motion.

The first challenge we have to face is to state a notion of 'planar' motion in an arbitrary spacetime. Classically, a motion is said to be planar when the projection of its space-time trajectory on the absolute Euclidean space is contained in a plane. Equivalently, its proper acceleration is contained in the same plane at any instant. This alternative notion may be extended to any spacetime. Obviously, as it happens in Classical Mechanics, a uniform circular motion should be a planar motion. The subtle problem in Relativity consists in giving sense to the sentence *the same plane forever*. This is done by making use of the Fermi-Walker connection of each observer (Definition 1).

Our procedure lies in the realm of modern Lorentzian geometry and, as far as we know, is new in our context (compare with [12]). Let us recall that a *particle* of mass $m > 0$ in a spacetime $(M, \langle\, , \,\rangle)$ is a curve $\gamma : I \longrightarrow M$, such that its velocity $\gamma'$ satisfies $\langle \gamma', \gamma' \rangle = -m^2$ and it is future pointing. A particle with $m = 1$ is called an *observer* and its parameter the *proper time* of $\gamma$ (see, for instance, [19, p. 41]). The covariant derivative of $\gamma'$, $\dfrac{D\gamma'}{dt}$, is its (proper) acceleration, which may be seen as a mathematical translation of the values measured by the accelerometer.

Assume that the particle $\gamma$ obeys a planar motion. In order to arrive to a suitable notion of uniform circular motion, we will require that the modulus of its acceleration remains unchanged, i.e.,

$$\left| \frac{D\gamma'}{dt} \right|^2 = \text{constant}.$$

On the other hand, we need a *connection along* $\gamma$ that permits to compare spatial directions at different instants of the life of $\gamma$, i.e., we need such a connection to compute how the proper acceleration of $\gamma$ changes. In General Relativity this connection is known as the *Fermi-Walker connection* of $\gamma$ (see Section 2 for more details). Thus, using the corresponding Fermi-Walker covariant derivative $\dfrac{\widehat{D}}{dt}$, if a particle obeys a *uniform circular* (UC) motion, it is also necessary that,

$$\left| \frac{\widehat{D}}{dt} \left( \frac{D\gamma'}{dt} \right) \right|^2 = \text{constant},$$

i.e., the modulus of the change of its acceleration should be constant (Section 2). We will arrive to the main notion of Definition 3 collecting suitably the three previous conditions.

On the other hand, UC motions appear naturally in any spacetime, for instance they arise from a dynamical point of view. Considering an electrically charged particle with nonzero rest mass $(\gamma(t), m, q)$ under the influence of an electromagnetic field $F$, then the dynamics of the particle is completely described by the well-known Lorentz force equation (see, for instance, [19, Def. 3.8.1]),

$$m \, \frac{D\gamma'}{dt} = q \, \widetilde{F}(\gamma'),$$

where $\widetilde{F}$ is the (1,1)-tensor field metrically equivalent to the closed 2-form $F$. Now, let us consider in Minkowski spacetime $\mathbb{L}^4$, i.e., $\mathbb{R}^4$ with coordinates $(t, x, y, z)$ endowed with the Lorentzian metric $-dt^2 + dx^2 + dy^2 + dz^2$, the particular electromagnetic field,

$$F = 2B_0 \, dx \wedge dy,$$

where $B_0 > 0$ is a constant (here we use the convention $dx \wedge dy = \frac{1}{2}(dx \otimes dy - dy \otimes dx)$). The family of inertial observers $\partial/\partial t$ measures a uniform magnetic field with modulus $B_0$ and pointing towards $\partial/\partial z$ (and zero electric field) for $F$. Now, the particle $\gamma$ obeys a UC motion, and its trajectory is expressed as [19, Prop. 3.8.2],

$$\gamma(\tau) \,=\, p + \left(\sqrt{1+R^2w^2}\, m\tau, R\cos(wm\tau+\vartheta), R\sin(wm\tau+\vartheta), 0\right),$$

with $w = \frac{qB_0}{m} \in \mathbb{R}$, $p \in \mathbb{L}^4$, $R > 0$ and $\vartheta \in \mathbb{R}$, whenever the initial velocity of the particle with respect to the family of inertial observers lies in the plane $xy$ [19, p. 88].

Next, the paper is organized as follows. In Section 2 several mathematical preliminaries are introduced to arrive to the notion of UC observer (Definition 3). Section 3 is devoted to expose how a UC observer can be seen as a Lorentzian helix in a general spacetime (Equation 3.1). The corresponding differential system and the associated initial value problem are analysed in detail in Section 4. Equivalently, each UC observer is obtained as a solution from only one fourth-order differential equation (Proposition 4.1). A representation of any UC observer is given in Section 5. Section 6 is dedicated to characterize geometrically UC observers as the projections on the spacetime of the integral curves of a certain vector field $G$ defined on a suitable fiber bundle over the spacetime (Lemma 6.1). Using $G$, the completeness of inextensible UC motions is analysed in the search of geometric assumptions which assure that inextensible UC observers do not disappear in a finite proper time. This technique is applied to spacetime which admit a certain timelike symmetry. In fact, we obtain that any UC observer in a Generalized Robertson-Walker spacetime can be extended whenever its worldline lies in a compact subset of the spacetime (Theorem 6.4). Finally, in Section 7, we prove the completeness of inextensible UC observers in an important class of pp-wave spacetimes. In this case we make use of a different and more analytical approach (Theorem 7.3). In both cases, the absence of singularities of this kind is found.

## 2 The notion of uniform circular motion

Consider a spacetime $M$, i.e., an $n(\geq 2)-$dimensional manifold endowed with a time orientable Lorentzian metric $\langle\, ,\, \rangle$ which we agree to have signature $(-,+,...,+)$, and with a fixed time orientation. Points of $M$ are called events and an observer in $M$ is a (smooth) curve $\gamma : I \longrightarrow M$, $I$ being an open interval of $\mathbb{R}$ $(0 \in I)$, such that $\langle \gamma'(t), \gamma'(t)\rangle = -1$ and $\gamma'(t)$ lies in the future time cone in $T_{\gamma(t)}M$ for all $t \in I$. In brief, $\gamma'(t)$ is a future pointing unit timelike vector for any proper time $t$ of $\gamma$.

At each event $\gamma(t)$, the tangent space $T_{\gamma(t)}M$ splits linearly as

$$T_{\gamma(t)}M = T_t \oplus R_t,$$

where $T_t := \mathrm{Span}\{\gamma'(t)\}$ and $R_t := T_t^{\perp}$. Clearly, $T_t$ is a negative definite line in $T_{\gamma(t)}M$ and $R_t$ is a spacelike hyperplane of $T_{\gamma(t)}M$. For $n = 4$, the 3-dimensional subspace $R_t$ may be interpreted as the instantaneous physical space observed by $\gamma$ at the instant $t$ of its clock. Consequently, vectors in $R_t$ represent *observable* quantities for $\gamma$ at $t$. Note that the acceleration vector field $\dfrac{D\gamma'}{dt}$ satisfies $\dfrac{D\gamma'}{dt}(t) \in R_t$, for any $t$. In fact, it is observed by $\gamma$ whereas the velocity vector field $\gamma'$ is not observable by $\gamma$.

To compare $v_1 \in R_{t_1}$ with $v_2 \in R_{t_2}$, for $t_1 < t_2$ and $|v_1| = |v_2|$, on $\gamma$, one could use the parallel transport defined by the Levi-Civita covariant derivative along $\gamma$,

$$P^{\gamma}_{t_1,t_2} : T_{\gamma(t_1)}M \longrightarrow T_{\gamma(t_2)}M.$$

However, this linear isometry does not satisfy $P^{\gamma}_{t_1,t_2}(R_{t_1}) = R_{t_2}$, in general. This is a serious inconvenience. To avoid it, the observer can use a more subtle mathematical tool. In fact, recall that $\gamma$ possesses a (private) connection, called its Fermi-Walker connection, which is defined as follows [19, p. 51]. Consider the Levi-Civita connection $\nabla$ associated to the Lorentzian metric of spacetime. Let $\mathfrak{X}(\gamma) = \{Y : I \longrightarrow TM \ : \ Y_t \in T_{\gamma(t)}M \text{ for all } t\}$ the space of (smooth) vector fields along $\gamma$. The connection $\nabla$ induces a connection along each $\gamma : I \longrightarrow M$ such that the corresponding covariant derivative is the usual covariant derivative of vector field $Y \in \mathfrak{X}(\gamma)$, namely, $\dfrac{DY}{dt}(t) = \nabla_{\gamma'(t)}\widetilde{Y}$, where $\widetilde{Y}$ is a local extension of $Y$ on an open neighbourhood of each event experimented by $\gamma(t)$ in $M$.

For $Y \in \mathfrak{X}(\gamma)$, denote by $Y^T_t$, $Y^R_t$ the orthogonal projections of $Y_t$ on $T_t$ and $R_t$, respectively, i.e.,

$$Y^T_t = -\langle Y_t, \gamma'(t)\rangle\, \gamma'(t) \qquad \text{and} \qquad Y^R_t = Y_t - Y^T_t.$$

Clearly, we have $Y^T, Y^R \in \mathfrak{X}(\gamma)$. According to [19, Prop. 2.2.1], the Fermi-Walker connection of $\gamma$ is the unique connection $\widehat{\nabla}$ along $\gamma$ which satisfies

$$\widehat{\nabla}_X Y = \left(\nabla_X Y^T\right)^T + \left(\nabla_X Y^R\right)^R,$$

for any $X \in \mathfrak{X}(I)$ and $Y \in \mathfrak{X}(\gamma)$, being $\nabla$ the induced connection on $\gamma$ from the Levi-Civita connection of $M$.

Now, let us denote by $\widehat{D}/dt$ the covariant derivative corresponding to $\widehat{\nabla}$. Then, it is not difficult to prove the following relationship with the Levi-Civita covariant derivative [19, Prop. 2.2.2],

$$\frac{\widehat{D}Y}{dt} = \frac{DY}{dt} + \langle \gamma', Y\rangle \frac{D\gamma'}{dt} - \left\langle \frac{D\gamma'}{dt}, Y \right\rangle \gamma', \tag{3}$$

for any $Y \in \mathfrak{X}(\gamma)$. Clearly, we have $\dfrac{\widehat{D}}{dt} = \dfrac{D}{dt}$ if and only if $\gamma$ is a geodesic, i.e., the observer is free falling.

In addition, the following Leibnitz type rule holds,

$$\frac{d}{dt}\langle X\,, Y\rangle = \left\langle \frac{\widehat{D}X}{dt}\,, Y \right\rangle + \left\langle X\,, \frac{\widehat{D}Y}{dt} \right\rangle, \tag{4}$$

for any $X, Y \in \mathfrak{X}(\gamma)$.

Associated to the Fermi-Walker covariant derivative along $\gamma$ there exists a parallel transport

$$\widehat{P}^{\gamma}_{t_1,t_2} : T_{\gamma(t_1)}M \longrightarrow T_{\gamma(t_2)}M,$$

which is a linear isometry and satisfies

$$\widehat{P}^{\gamma}_{t_1,t_2}(R_{t_1}) = R_{t_2}.$$

Therefore, given $v_1 \in R_{t_1}$ and $v_2 \in R_{t_2}$, with $t_1 < t_2$ and $|v_1| = |v_2|$, the observer $\gamma$ may consider $\widehat{P}^{\gamma}_{t_1,t_2}(v_1)$ instead of $v_1$, with the advantage that $\widehat{P}^{\gamma}_{t_1,t_2}(v_1)$ may be compared with $v_2$ (see also [16, Sec. 6.5]).

Now we introduce the crucial concept of 'planar motion' to make precise when an observer considers that it is moving along a plane. Intuitively, an observer will say that its motion is planar when the small ball of its accelerometer moves along a constant plane. In the mathematical translation of this intuitive idea, the main difficulty lies in what is the meaning of a 'constant plane' relative to the observer. For this purpose we will use the Fermi-Walker connection exposed above.

**Definition 1** An observer $\gamma : I \longrightarrow M$ obeys a *planar motion* if for some $t_0 \in I$, there exists an observable plane $\Pi_{t_0} \subset R_{t_0} \subset T_{\gamma(t_0)}M$, such that

$$\widehat{P}^{\gamma}_{t,t_0}\left(\frac{D\gamma'}{dt}(t)\right) \in \Pi_{t_0} \tag{5}$$

for any $t \in I$.

As a direct consequence of the definition, using the equality

$$\frac{\widehat{D}}{dt}\left(\frac{D\gamma'}{dt}\right)(t_0) = \lim_{\varepsilon\to 0}\frac{1}{\varepsilon}\left[\widehat{P}^{\gamma}_{t_0+\varepsilon,t_0}\left(\frac{D\gamma'}{dt}(t_0+\varepsilon)\right) - \frac{D\gamma'}{dt}(t_0)\right],$$

the vector $\frac{\widehat{D}}{dt}\left(\frac{D\gamma'}{dt}\right)(t_0)$ is also in $\Pi_{t_0}$. Indeed, if the motion has not an unchanged direction in the neighbourhood of the instant $t_0$, in the terminology of [6], then the plane $\Pi_{t_0}$ is generated by the proper acceleration of the observer and the variation that it measures, i.e.,

$$\Pi_{t_0} = \operatorname{span}\left\{\frac{D\gamma'}{dt}(t_0)\,,\ \frac{\widehat{D}}{dt}\left(\frac{D\gamma'}{dt}\right)(t_0)\right\}.$$

In this case, we may define the following family of 2-planes along $\gamma$

$$\Pi_t := \operatorname{span}\left\{\widehat{P}^{\gamma}_{t_0,t}\left(\frac{D\gamma'}{dt}(t_0)\right)\ ,\ \widehat{P}^{\gamma}_{t_0,t}\left(\frac{\widehat{D}}{dt}\left(\frac{D\gamma'}{dt}\right)(t_0)\right)\right\} \subset R_t. \tag{6}$$

Observe that this family of planes is Fermi-Walker parallel in the sense of the following definition.

**Definition 2** Given an observer $\gamma : I \longrightarrow M$ in the spacetime $M$, a family of planes along $\gamma$, $\{\Pi_t\}_{t\in I}$, is said to be Fermi-Walker parallel if for any $t_1, t_2 \in I$ and for any vector $v \in \Pi_{t_1}$, the following relation holds

$$\widehat{P}^{\gamma}_{t_1,t_2}(v) \in \Pi_{t_2}\,.$$

In addition, the previous family of planes (6) satisfies the following property.

**Lemma 2.1** *For any* $t, t_1 \in I$, *we have*

$$\widehat{P}^{\gamma}_{t,t_1}\left(\frac{D\gamma'}{dt}(t)\right) \in \Pi_{t_1}\,.$$

*Proof.* Taking the inverse mapping of $\widehat{P}^{\gamma}_{t,t_0}$ in (5), we have that there exist $a, b \in \mathbb{R}$ such that

$$\frac{D\gamma'}{dt}(t) = a\,\widehat{P}^{\gamma}_{t_0,t}\left(\frac{D\gamma'}{dt}(t_0)\right) + b\,\widehat{P}^{\gamma}_{t_0,t}\left(\frac{\widehat{D}}{dt}\left(\frac{D\gamma'}{dt}\right)(t_0)\right).$$

Now, the desired relation follows by taking $\widehat{P}^{\gamma}_{t,t_1}$ in both members of the previous equality. $\square$

It should be pointed out that the family $\{\Pi_t\}_{t\in I}$ satisfies the previous property, but it is not unique in general (a generically planar motion may be a free falling motion from some instant). However, if the observer $\gamma$ is not an unchanged direction observer [6], i.e., if $\left\{\frac{D\gamma'}{dt}(t), \frac{\widehat{D}}{dt}\left(\frac{D\gamma'}{dt}\right)(t)\right\}$ are linearly independent for any $t \in I$, then the only family of 2-planes satisfying Lemma 2.1 is $\{\Pi_t\}_{t\in I}$.

**Example 2.2** Consider an observer in the Exterior Schwarzschild spacetime with the general expression

$$\gamma(\tau) = \left(t(\tau)\,,\, r(\tau)\,,\, \frac{\pi}{2}\,,\, \varphi(\tau)\right), \quad \tau \in I.$$

We claim that observers of this form always obey a planar motion.
Let $\{\Pi_\tau\}_{\tau\in I}$ be the following family of spacelike 2-planes along $\gamma$,

$$\Pi_\tau := R_\tau \cap (\partial_\theta)^{\perp}, \quad R_\tau = \left(\gamma'(\tau)\right)^{\perp}.$$

Since the Christoffel symbols $\Gamma^{\theta}_{ij} = 0$ for $i, j \in \{t, r, \varphi\}$, it is immediate that $\frac{D\gamma'}{d\tau}(\tau) \in \Pi_\tau$. We only have to prove that $\{\Pi_\tau\}_{\tau\in I}$ is a Fermi-Walker parallel family of 2-planes along $\gamma$. A direct computation gives $\frac{\widehat{D}\,\partial_\theta}{d\tau}(\tau) = \frac{r'(\tau)}{r(\tau)}\,\partial_\theta(\tau)$. Therefore, $\widehat{P}^{\gamma}_{\tau_1,\tau_2}\left(\mathrm{span}(\partial_\theta)\right) = \mathrm{span}(\partial_\theta)$, and then, since $\widehat{P}^{\gamma}_{\tau_1,\tau_2}$ is an isometry, we have

$$\widehat{P}^{\gamma}_{\tau_1,\tau_2}\left(\left(\partial_\theta(\tau_1)\right)^{\perp}\right) = \left(\partial_\theta(\tau_1)\right)^{\perp}.$$

We conclude the proof taking into account that $\widehat{P}^{\gamma}_{\tau_1,\tau_2}\left(R_{\tau_1}\right) = R_{\tau_2}$.

Now, we will introduce a uniform circular (UC) motion as a very particular case of planar motion. Intuitively, a UC observer will see that the small ball of its accelerometer is rotating with constant angular velocity. Thus, we introduce the following notion.

**Definition 3** An observer $\gamma : I \longrightarrow M$ which satisfies a planar motion is said to obey a UC motion if

$$\left|\frac{D\gamma'}{dt}\right|^2 = a^2 \qquad \text{and} \qquad \left|\frac{\widehat{D}}{dt}\left(\frac{D\gamma'}{dt}\right)\right|^2 = a^2 w^2, \tag{7}$$

where the constants $a, w$ satisfy $a, w > 0$ and $a < w$.

Here $a$ is the modulus of the acceleration, and $w$ corresponds to the angular velocity that the observer perceives. Therefore, motivated from the classical relation between the radius $R$, the angular velocity $w$ and the centripetal acceleration $a$ on a circular motion,

$$a = w^2 R,$$

a UC observer will measure a uniform rotation with frequency $\frac{w}{2\pi}$ and 'radius' equal to $R := a/w^2$. We emphasize that this quantity $R$ does not represent a real observable distance in general. It is only the radius of the trajectory which the UC observer assumes, using the classical intuition, from the evolution of its acceleration. The assumption $a < w$ is imposed to exclude other different kinds of motion, as we will discuss it at the end of Section 5.

**Remark 2.3** Note that if $a = 0$ were permitted, we would recover the definition of a free falling observer. Moreover, when $a > 0$, if $w = 0$ is permited, we would obtain the definition of a uniform accelerated observer [5]. Observe that, in this case, the trajectory measured by the observer may be thought as having infinite radius, i.e., the observer obeys a rectilinear motion [6].

**Example 2.4** Let us consider the Exterior Schwarzschild spacetime and the observer (1) presented in the introduction. By one hand, from Example 2.2, we know that $\gamma$ describes a planar motion. By the other hand, making some computations, we get

$$\left|\frac{D\gamma'}{d\tau}\right|^2 = a^2, \qquad a = \left(1 - \frac{r_s}{R}\right)^{-1/2}\left(\Omega^2\left(R - \frac{3r_s}{2}\right) - \frac{r_s}{2R^2}\right),$$

and,

$$\left|\frac{\widehat{D}}{d\tau}\left(\frac{D\gamma'}{d\tau}\right)\right|^2 = a^2w^2,$$

$$w^2 = \frac{\Omega^2}{1 - \frac{r_s}{R}}\left(1 + \Omega^2R^2\right)\left[1 - \frac{3r_s}{2R} - \Omega^2\left(R - \frac{3r_s}{2}\right)^2\right].$$

Therefore, if $R$ and $\Omega$ satisfy that $w(R, \Omega) > a(R, \Omega)$, and $a(R, \omega) > 0$ (if $a = 0$ we recover the free falling observer (1)-(2)), the observer $\gamma$ obeys a UC motion.

**Remark 2.5** There is a link between our definition of UC motion and the notion of helical symmetry in stationary axisymmetric spacetimes, see [22, Chap.V]. In fact, with the terminology of the previous example, the vector field $\xi = \partial_t + \Omega\partial_\varphi$ is helical Killing. The UC observers built in Example 2.4 are integral curves of $\xi$. A relevant difference between both definitions is that we consider individual curves obeying a UC motion instead of a family of integral curves of certain Killing vector field [22, p. 93]. Besides, we deal with spacetimes without any symmetry assumption.

Naturally, in order to determine a UC observer trajectory, it is necessary to know the initial observable 2-plane, the initial spin sense and the initial values of the position, 4-velocity and proper acceleration. In an $n(\geq 3)$-dimensional spacetime, the initial 2-plane can be determined by means of $n-3$ observable directions $u_4, \cdots, u_n \in \gamma'(0)^\perp$, orthogonal to the initial acceleration $\frac{D\gamma'}{dt}(0)$. So, the vector $\frac{\widehat{D}}{dt}\left(\frac{D\gamma'}{dt}\right)(0)$ will point towards the unique observable direction that is orthogonal to $\frac{D\gamma'}{dt}(0)$ in $R_{t_0}$ and $u_4, \cdots, u_n$. From equation (7), the modulus of the vector $\frac{\widehat{D}}{dt}\left(\frac{D\gamma'}{dt}\right)(0)$ is also known, and it is equal to $aw$. However, the initial spin sense is needed to determine the sense of that vector.

The initial plane $\Pi_0$ is given by

$$\Pi_0 = \operatorname{span}\left\{ \frac{D\gamma'}{dt}(0), \frac{\widehat{D}}{dt}\left(\frac{D\gamma'}{dt}\right)(0) \right\}.$$

The following vectors,

$$u_1 = \gamma'(0), \quad u_2 = \frac{1}{a}\frac{D\gamma'}{dt}(0), \quad u_3 = \frac{1}{aw}\frac{\widehat{D}}{dt}\left(\frac{D\gamma'}{dt}\right)(0),$$

are orthonormal from Definition 3. Consider the following Fermi-Walker parallel vector fields along $\gamma$,

$$e_4(t), \cdots, e_n(t)$$

satisfying the initial conditions $e_i(0) = u_i$, for each $4 \leq i \leq n$. Now, the family of Fermi-Walker parallel planes (6), corresponding to the UC observer $\gamma$ is given by

$$\Pi_t = \operatorname{span}\{\widehat{P}^{\gamma}_{0,t}(u_2), \widehat{P}^{\gamma}_{0,t}(u_3)\} = \left(\operatorname{span}\{\widehat{P}^{\gamma}_{0,t}(u_1), e_4(t), \cdots, e_n(t)\}\right)^{\perp} \subset R_t.$$

**Remark 2.6** Note that, in the physically relevant case $n = 4$, we have

$$e_4(t) = \frac{1}{a^2 w}\frac{D\gamma'}{dt} \times \frac{\widehat{D}}{dt}\left(\frac{D\gamma'}{dt}\right),$$

where $\times$ denotes the natural cross product defined in $R_t$.

From the previous discussion we can state the initial value equations for a UC observer with 'frequency' $\frac{w}{2\pi}$ and 'radius' $R = \frac{a}{w^2}$. Such an observer obeys the following system of equations:

$$\langle \gamma', \gamma' \rangle = -1, \tag{8}$$

$$\left| \frac{D\gamma'}{dt} \right|^2 = a^2, \tag{9}$$

$$\left| \frac{\widehat{D}}{dt}\left(\frac{D\gamma'}{dt}\right) \right|^2 = a^2 w^2, \tag{10}$$

$$\frac{\widehat{D}e_i}{dt} = 0 \quad \text{for} \quad 4 \leq i \leq n, \tag{11}$$

$$\left\langle \frac{D\gamma'}{dt}, e_i \right\rangle = 0 \quad \text{for} \quad 4 \leq i \leq n, \tag{12}$$

under the initial conditions

$$\gamma(0) = p, \quad \gamma'(0) = u_1, \quad \frac{D\gamma'}{dt}(0) = a\, u_2, \quad \frac{\widehat{D}}{dt}\left(\frac{D\gamma'}{dt}\right)(0) = aw\, u_3, \tag{13}$$

$$e_i(0) = u_i \quad \text{for} \quad 4 \leq i \leq n,$$

where $p$ is an event in the $n$-dimensional spacetime $M$.

Note that (12) automatically implies that

$$\left\langle \frac{\widehat{D}}{dt}\left(\frac{D\gamma'}{dt}\right)(t)\,, e_i(t)\right\rangle = 0 \quad \text{for} \quad 4 \leq i \leq n, \quad t \in I.$$

The local existence and uniqueness of this initial problem is not yet guaranteed because it is not written it in the normal form (therefore the classical Picard-Lindeloff Theorem can not be applied). On the other hand, an initial condition is imposed to the third derivative, in spite of the system being of third order. However, in Section 4 we will prove that system (8)-(13) has a unique inextensible solution.

## 3 UC motion as a Lorentzian helix

In this section we analyse the UC motion from a more geometric viewpoint. In order to do that, we will consider the Frenet equations associated with an observer. These equations showed to have an important role when dealing with different kinematics aspects of relativistic motions [1], [15]. First, we proceed to find the Frenet equations of each UC observer.

Let $\gamma : I \longrightarrow M$ be a UC observer with angular velocity $w$ and radius $R = \frac{a}{w^2}$. We define the following three vector fields along $\gamma$, which are orthonormal from equations (4) and (7),

$$e_1(t) = \gamma'(t),$$

$$e_2(t) = \frac{1}{a}\frac{D\gamma'}{dt}(t),$$

$$e_3(t) = \frac{1}{aw}\frac{\widehat{D}}{dt}\left(\frac{D\gamma'}{dt}\right)(t).$$

Let $\{u_4, \cdots, u_n\}$ be $n-3$ orthonormal vectors in $T_{\gamma(0)}M$, such that,

$$\{e_1(0), e_2(0), e_3(0), u_4, \ldots, u_n\}$$

is an orthogonal basis of $T_{\gamma(0)}M$. Consider the Fermi-Walker parallel vector fields along $\gamma$ starting at $u_i$,

$$e_i(t) = \widehat{P}^{\gamma}_{0,t}(u_i), \quad \text{for} \quad 4 \leq i \leq n.$$

Since a UC motion is a planar motion, the 2-plane $\Pi_t$ is orthogonal to the subspace generated by $\{\widehat{P}^{\gamma}_{0,t}(u_i)\}_{4\leq i\leq n}$, and the vector fields $\{e_j(t)\}_{1\leq j\leq n}$ are orthonormal at every instant $t \in I$.

Now, we are in a position to obtain the Frenet equations. A direct computation give us

$$\frac{De_1}{dt} = a\, e_2.$$

On the other hand,

$$\frac{De_2}{dt} = \frac{1}{a}\left[\frac{\widehat{D}}{dt}\left(\frac{D\gamma'}{dt}\right) + a^2\gamma'\right] = a\, e_1 + w\, e_3.$$

Taking into account that $\gamma$ is a UC observer, we obtain

$$\Big\langle \frac{De_3}{dt}\,, e_1 \Big\rangle = \frac{1}{aw}\Big\langle \frac{\widehat{D}}{dt}\left(\frac{\widehat{D}}{dt}\left(\frac{D\gamma\,'}{dt}\right)\right)\,, \gamma\,' \Big\rangle = 0,$$

and

$$\Big\langle \frac{De_3}{dt}\,, e_2 \Big\rangle = \frac{1}{a^2 w}\left[\frac{d}{dt}\langle \frac{\widehat{D}}{dt}\left(\frac{D\gamma\,'}{dt}\right)\,, \frac{D\gamma\,'}{dt}\rangle - \Big|\frac{\widehat{D}}{dt}\left(\frac{D\gamma\,'}{dt}\right)\Big|^2\right] = -w.$$

Hence, we get

$$\frac{De_3}{dt} = \frac{1}{aw}\frac{\widehat{D}}{dt}\left(\frac{\widehat{D}}{dt}\left(\frac{D\gamma\,'}{dt}\right)\right) = -w\, e_2.$$

Finally, for $4 \leq i \leq n$,

$$\frac{De_i}{dt} = \frac{\widehat{D}e_i}{dt} - \langle \gamma\,', e_i\rangle \frac{D\gamma\,'}{dt} + \langle \frac{D\gamma\,'}{dt}\,, e_i\rangle \gamma\,' = 0. \tag{14}$$

Summarizing, the Frenet equations corresponding to a UC observer are

$$\frac{De_1}{dt} = a\, e_2, \tag{15}$$

$$\frac{De_2}{dt} = a\, e_1 + w\, e_3, \tag{16}$$

$$\frac{De_3}{dt} = -w\, e_2, \tag{17}$$

$$\frac{De_i}{dt} = 0 \qquad \text{for} \quad 4 \leq i \leq n. \tag{18}$$

Therefore, Frenet equations gives that a UC observer is a Lorentzian helix, i.e., a unit timelike curve with constant curvature and torsion, and with vanishing higher order curvatures (see, for instance, [14]).

Conversely, assume Frenet system of equations $(15) - (18)$ holds true for $\gamma$ with the initial conditions (13). Note that no information is obtained from (15). On the other hand, Equation (16) can be written as

$$\frac{\widehat{D}}{dt}\left(\frac{D\gamma\,'}{dt}\right) = \frac{D^2\gamma\,'}{dt^2} - a^2\,\gamma\,'.$$

The relation between Fermi-Walker and Levi-Civita covariant derivatives given in (3), allows us conclude

$$\left(\Big|\frac{D\gamma\,'}{dt}\Big|^2 - a^2\right)\gamma\,' = \Big\langle \gamma\,', \frac{D\gamma\,'}{dt}\Big\rangle \frac{D\gamma\,'}{dt}.$$

Multiplying this expression by $\dfrac{D\gamma\,'}{dt}$ we get,

$$|\gamma\,'|^2 = -1 \qquad \text{and} \qquad \Big|\frac{D\gamma\,'}{dt}\Big|^2 = a^2,$$

which are the first two equations on the system (8)-(12). On the other hand, from (17) we obtain the fourth-order equation,

$$\frac{D}{dt}\left[\frac{D^2\gamma\,'}{dt^2} + (w^2 - a^2)\gamma\,'\right] = 0.$$

Since we know that $\langle \dfrac{D\gamma'}{dt}, \gamma' \rangle = 0$, from second Frenet equation, we conclude that $\left| \dfrac{\widehat{D}}{dt}\left(\dfrac{D\gamma'}{dt}\right) \right|^2$ is constant along $\gamma$. Now, from initial conditions (13), we get the second equation of (7).

Finally, directly from definition of $e_i(t)$, we deduce equations (11). As a consequence,using equations (18),

$$0 = \frac{\widehat{D}e_i}{dt} = \langle e_i, \gamma' \rangle \frac{D\gamma'}{dt} - \langle \frac{D\gamma'}{dt}, e_i \rangle \gamma',$$

and we obtain equations (12).

The previous arguments gives rise to the following result.

**Proposition 3.1** *An observer $\gamma : I \longrightarrow M$ is a UC observer, with angular velocity $w$ and radius $R = \frac{a}{w^2}$, if and only if $\gamma$ is a unit timelike Lorentzian helix, i.e., a curve with constant curvature $a$ and torsion $w$, while the rest of higher order curvatures are identically zero.*

**Remark 3.2** We note that, if the spacetime has constant sectional curvature, the reduction of codimension Erbacher Theorem (see [7]) implies that the UC observers are contained in a 3-dimensional totally geodesic Lorentzian submanifold.

# 4 The associated Cauchy problem

Observe that Frenet equations (15)-(18) constitute a fourth order differential system for $\gamma$. The associated initial value problem is

$$\frac{\widehat{D}}{dt}\left(\frac{D\gamma'}{dt}\right) = \frac{D^2\gamma'}{dt^2} - a^2\gamma', \tag{19}$$

$$\frac{D}{dt}\left[\frac{D^2\gamma'}{dt^2} + (w^2 - a^2)\gamma'\right] = 0, \tag{20}$$

$$\gamma(0) = p, \quad \gamma'(0) = u_1, \quad \frac{D\gamma'}{dt}(0) = a\, u_2, \quad \frac{\widehat{D}}{dt}\left(\frac{D\gamma'}{dt}\right)(0) = aw\, u_3, \tag{21}$$

where $u_1, u_2, u_3 \in T_pM$ are orthogonal vectors satisfying $|u_1|^2 = -1$ and $|u_2|^2 = |u_3|^2 = 1$.

Note that equations (18) are encoded in this new system of equations. Let $u_i$, $i = 4, \ldots n$ be a set of orthonormal vectors such that $\{u_i\}_{1\leq i\leq n}$ is a base of $T_pM$. Let $e_i(t)$, $4 \leq i \leq n$ be the corresponding Fermi-Walker parallel vector fields along a solution $\gamma$ of (19)-(20). Thus,

$$\frac{De_i}{dt} = \left\langle \frac{D\gamma'}{dt}, e_i \right\rangle \gamma' \qquad 4 \leq i \leq n.$$

Multiplying by $\dfrac{D^2\gamma'}{dt^2}$ and using (19) and (20), we get after some computations,

$$\frac{d^2}{dt^2}\left\langle \frac{D\gamma'}{dt}, e_i \right\rangle = -w^2 \left\langle \frac{D\gamma'}{dt}, e_i \right\rangle.$$

From the initial conditions (21), we conclude that $\gamma$ satisfies (18).

Since there are two different equations the existence and uniqueness of the solution is not immediate. Now, we are going to prove that this initial problem is equivalent to the following one, which clearly has a unique inextensible solution.

$$\frac{D}{dt}\left[\frac{D^2\gamma\,'}{dt^2}+\langle\gamma\,',\frac{D\gamma\,'}{dt}\rangle\frac{D\gamma\,'}{dt}+\left(w^2-\left|\frac{D\gamma\,'}{dt}\right|^2\right)\gamma\,'\right]=0, \tag{22}$$

$$\gamma(0)=p,\quad \gamma\,'(0)=u_1,\quad \frac{D\gamma\,'}{dt}(0)=a\,u_2,\quad \frac{\widehat{D}}{dt}\left(\frac{D\gamma\,'}{dt}\right)(0)=aw\,u_3, \tag{23}$$

where $u_1,u_2,u_3\in T_pM$ satisfy the above conditions.

Taking into account (3), it is clear that a solution of (19)-(21) is a solution of (22)-(23). For the converse, we only have to prove that $|\gamma\,'|^2$ and $\left|\frac{D\gamma\,'}{dt}\right|^2$ are constant along $\gamma$.

Denote by $x(t):=|\gamma\,'|^2$ and $y(t):=\left|\frac{D\gamma\,'}{dt}\right|^2$. Then, the initial values are

$$x(0)=-1,\quad x'(0)=x''(0)=0,\quad y(0)=a^2,\quad y'(0)=y''(0)=0. \tag{24}$$

Multiplying equation (22) by $\gamma\,'$ and $\frac{D\gamma\,'}{dt}$, we obtain respectively,

$$x'''-3y'+x'x''+(w^2-y)x'-2xy'-x'y=0, \tag{25}$$

and

$$\left|\frac{D^2\gamma\,'}{dt^2}\right|^2=\frac{1}{2}\,y''+\frac{1}{2}x''y-\frac{1}{4}x'y'+w^2y-y^2=\frac{1}{2}y''+f(x',x'',y,y'), \tag{26}$$

where $f(x',x'',y,y')$ denotes the corresponding terms.

On the other hand, multiplying (22) by $\frac{D^2\gamma\,'}{dt^2}$ and using (26), we get

$$\frac{1}{4}\,y'''+\frac{1}{2}\left(f(x',x'',y,y')\right)'-x''y'+x'y''+\frac{1}{2}\,yy'+x'y''+x'f(x',x'',y,y')-\frac{w^2}{2}\,y'=0. \tag{27}$$

Equations (25) and (27), under the initial conditions (24), have a unique solution by the classical Picard-Lindeloff Theorem. Since $x(t)=-1$ and $y(t)=a^2$ satisfy this initial value problem, we get the announced conclusion.

Previous results are picked up in the following proposition.

**Proposition 4.1** *The three following assertions are equivalent*

(*a*) *A curve $\gamma$ in $M$ is solution of* (8)-(13).

(*b*) *A curve $\gamma$ in $M$ is solution of* (19)-(21).

(*c*) *A curve $\gamma$ in $M$ is* (*the unique*) *solution of* (22)-(23).

We can rewrite the equivalences as follows.

**Corollary 4.2** *There exists a unique inextensible UC observer in $M$ for each initial data* (13).

# 5 A representation of the solutions

At this point, we remark that although Definition 3 has a clear physical meaning, (based on how an observer obeying a uniform circular motion measures its proper acceleration), we will use strongly the following representation of the solutions to study the completeness of the inextensible circular trajectories. This representation of the UC observers is also useful to obtain explicit solutions of the equations of motion in certain cases.

**Proposition 5.1** *Let $M$ be an $n(\geq 3)$-dimensional spacetime, $n \geq 3$, and let $a$, $w$ be two positive constants, $a < w$. Let us consider $u_1$, $u_2$, $u_3 \in T_pM$ three orthogonal vectors such that $|u_1|^2 = -1$ and $|u_2|^2 = |u_3|^2 = 1$, and consider $n-3$ vectors $\{u_4, \cdots, u_n\} \subset T_pM$ completing the previous three ones to an orthonormal basis of $T_pM$. Then, the velocity of the only UC observer $\gamma$ satisfying the initial conditions* (21) *has the form*

$$\gamma'(t) = \frac{w}{\sqrt{w^2-a^2}}\, L(t) + \frac{a}{\sqrt{w^2-a^2}} \left[\cos\left(\sqrt{w^2-a^2}\, t\right) M(t) + \sin\left(\sqrt{w^2-a^2}\, t\right) N(t)\right], \quad (28)$$

*being $L, M, N$, three unit* (*Levi-Civita*) *parallel vector fields along $\gamma$ satisfying*

$$L(0) = \tfrac{1}{\sqrt{w^2-a^2}}(wu_1 + au_3),$$

$$M(0) = \tfrac{-1}{\sqrt{w^2-a^2}}(au_1 + wu_3),$$

$$N(0) = u_2.$$

*Proof.* Let $\gamma$ be the unique UC observer satisfying the initial conditions (23). Consider $n-3$ vectors $\{u_4, \cdots, u_n\} \subset T_pM$ completing the previous $\{u_1, u_2, u_3\}$ to an orthonormal basis of $T_pM$. We may construct the Fermi-Walker parallel vector fields along $\gamma$, $\{e_4, \cdots, e_n\}$, as before, and the Levi-Civita parallel vector fields $L(t), M(t), N(t)$ verifying the initial conditions given above. We intend to show that the vector field $Z(t)$,

$$Z(t) = \frac{w}{\sqrt{w^2-a^2}}\, L(t) + \frac{a}{\sqrt{w^2-a^2}} \left[\cos\left(\sqrt{w^2-a^2}\, t\right) M(t) + \sin\left(\sqrt{w^2-a^2}\, t\right) N(t)\right],$$

corresponds with the velocity $\gamma'$ and satisfies the initial conditions (23).

A direct computation gives $|Z(t)|^2 = -1$ and

$$\frac{DZ}{dt}(t) = a\left(-\sin\left(\sqrt{w^2-a^2}\, t\right) M(t) + \cos\left(\sqrt{w^2-a^2}\, t\right) N(t)\right).$$

Then, by using the orthogonality of $M$ and $N$, equation (9) is automatically satisfied. Analogously, after easy computations we get

$$\frac{\widehat{D}}{dt}\left(\frac{Z}{dt}\right) = -\frac{a^2 w}{\sqrt{w^2-a^2}}\, L - \frac{a w^2}{\sqrt{w^2-a^2}} \left(\cos\left(\sqrt{w^2-a^2}\, t\right) M + \sin\left(\sqrt{w^2-a^2}\, t\right) N\right),$$

and equation (10) holds.

Next, from the choice of $u_i$, $i = 4, ..., n$, we get

$$\langle L(0)\,, u_i\rangle = \langle M(0)\,, u_i\rangle = \langle N(0)\,, u_i\rangle = 0 \quad \text{for} \quad 4 \leq i \leq n.$$

Since $e_i(t)$, $i=4,...,n$, are Fermi-Walker parallel, the same argument used in (14) leads to the orthogonality relations

$$\langle L(t)\,, e_i(t)\rangle = \langle M(t)\,, e_i(t)\rangle = \langle N(t)\,, e_i(t)\rangle = 0 \quad \text{for} \quad 4 \le i \le n.$$

Therefore, $\langle Z(t)\,, e_i(t)\rangle = 0 \quad \text{for} \quad 4 \le i \le n$, and (12) are satisfied.

The equations (11) and (12) are satisfied because of assumptions on $L$, $M$ and $N$ and their initial relations with the vectors $u_4, \cdots, u_n$. Moreover, the initial conditions are straightforward satisfied.

The uniqueness of inextensible solutions of initial value problem (8)-(13) leads to $\gamma'(t) = Z(t)$ for $t \in I$.

$\square$

**Remark 5.2** By using the Levi-Civita parallel transport, we can express (28) as the following first order integro-differential equation,

$$\gamma'(t) = \frac{1}{\sqrt{w^2-a^2}}\left[w\, P^{\gamma}_{0,t}(v_1) + a\,\cos\left(\sqrt{w^2-a^2}\,t\right) P^{\gamma}_{0,t}(v_2) + a\,\sin\left(\sqrt{w^2-a^2}\,t\right) P^{\gamma}_{0,t}(u_2)\right],$$
$$|u_1|^2 = -1, \qquad |u_2|^2 = |u_3|^2 = 1, \qquad \langle u_1,\, u_2\rangle = \langle u_1,\, u_3\rangle = \langle u_2,\, u_3\rangle = 0.$$

with $v_1 = \dfrac{wu_1 + au_3}{\sqrt{w^2-a^2}}$ and $v_2 = -\dfrac{au_1 + wu_3}{\sqrt{w^2-a^2}}$.

Note that expression (28) is only a representation of the solutions of (22). In fact, in order to compute the parallel vector fields $L(t)$, $M(t)$ and $N(t)$ for $t > 0$, if is necessary to know the solution curve $\gamma$. However, formula (28) is interesting and it will be used in the analytical study of the completeness of inextensible trajectories made in Section 6.

**Example 5.3** Consider the 3-dimensional Minkowski spacetime $M = \mathbb{L}^3$ endowed with the standard coordinate system $(t, x, y)$. In this spacetime, every motion is obviously planar. A UC observer, with frequency $\frac{w}{2\pi}$ and 'radius' $\frac{a}{w^2}$, satisfying the initial conditions

$$\gamma(0) = (0,\, \frac{a}{w^2-a^2},\, 0), \quad \gamma'(0) = (\frac{w}{\sqrt{w^2-a^2}},\, 0,\, \frac{a}{\sqrt{w^2-a^2}}),$$
$$\frac{D\gamma'}{d\tau}(0) = (0,\, -a,\, 0), \quad \frac{\widehat{D}}{d\tau}\Big(\frac{D\gamma'}{d\tau}\Big)(0) = (\frac{a^2 w}{\sqrt{w^2-a^2}},\, 0,\, \frac{aw^2}{\sqrt{w^2-a^2}}),$$

is given by,

$$\gamma(\tau) = \big(t(\tau),\, x(\tau),\, y(\tau)\big),$$

where

$$t(\tau) = \frac{w\tau}{\sqrt{w^2-a^2}}, \quad x(\tau) = \frac{a}{w^2-a^2}\cos\left(\sqrt{w^2-a^2}\,\tau\right), \quad y(\tau) = \frac{a}{w^2-a^2}\sin\left(\sqrt{w^2-a^2}\,\tau\right).$$

The physical interpretation may be as follows. The UC observer measures acceleration with constant modulus equal to $a$ and angular velocity $w$. Thus, he induces that its radius is

$R = \frac{a}{\sqrt{w^2-a^2}}$. However, the family of inertial observers which measure an initial velocity of the UC observer equal to $\frac{a}{w}$, detect that it describes an uniform circular motion with acceleration $a$ but with angular velocity, $\Omega = \sqrt{w^2 - a^2} < w$. The radius of the observed trajectory by these inertial observers is $\frac{a}{w^2-a^2}$, which increases as $w$ goes to $a$.

We end this section discussing the condition $a < w$ (or $wR/c < 1$, where $c = 1$ the speed of light) imposed in Definition 3. A similar procedure to the above one leads to the following representation of the solution of (22) whenever $a > w$ holds.

$$\gamma'(t) = \frac{-w}{\sqrt{a^2 - w^2}}\, L(t) + \frac{a}{\sqrt{a^2 - w^2}} \left[\cosh\left(\sqrt{a^2 - w^2}\, t\right) M(t) + \sinh\left(\sqrt{a^2 - w^2}\, t\right) N(t)\right],$$

being $L, M, N \in \mathfrak{X}(\gamma)$, unit (Levi-Civita) parallel vector fields satisfying

$$L(0) = \tfrac{1}{\sqrt{a^2-w^2}}(wu_1 + au_3),$$
$$M(0) = \tfrac{-1}{\sqrt{a^2-w^2}}(au_1 + wu_3),$$
$$N(0) = u_2.$$

Observe that, for long values of proper time $t$, this motion approaches to a uniformly accelerated motion [6, Th. 2.1].

By the other hand, if $a = w$ holds, we have

$$\gamma'(t) = \left(1 + \frac{a^2}{2}\, t^2\right) L(t) + \frac{a^2}{2}\, t^2\, M(t) + at\, N(t),$$

being $L, M, N \in \mathfrak{X}(\gamma)$ as before and now satisfying

$$L(0) = u_1, \quad M(0) = u_3, \quad N(0) = u_2.$$

In this case, for long times, such a observer approaches to a lightlike trajectory in an accelerated way. In particular, in $\mathbb{L}^3$, we have that $\gamma'(\tau) \approx \frac{a^2\tau^2}{2}\,(\partial_t + \partial_x)$ when $\tau$ is big enough, which is far to any possible circular motion.

Both cases show that no suitable relativistic definition of uniform circular motion is possible with $a \geq w$.

# 6 Completeness of the inextensible UC trajectories in spacetimes with certain timelike symmetries

This section is devoted to the study of the completeness of the inextensible UC observers. First of all, we are going to relate the solutions of equation (22) with the integral curves of a certain vector field on a Stiefel type bundle on $M$ (compare with [8, p. 6]).

Given a Lorentzian linear space $E$ and $a, w \in \mathbb{R}$, $w > a > 0$, denote by $V_{n,3}^{a,w}(E)$ the $(n, 3)$-Stiefel manifold over $E$, defined by

$$V_{n,3}^{a,w}(E) = \left\{(v_1, v_2, v_3) \in E^3 \ : \ |v_1|^2 = -1,\ |v_2|^2 = a^2,\ |v_3|^2 = a^2w^2,\ \ \langle v_i, v_j\rangle = 0,\ i \neq j\right\}.$$

The (n,3)-Stiefel bundle over the spacetime $M$ is then defined as follows,

$$V_{n,3}^{a,w}(M) = \bigcup_{p\in M} \{p\} \times V_{n,3}^{a,w}(T_pM).$$

Now we construct a vector field $G \in \mathfrak{X}\big(V_{n,3}^{a,w}(M)\big)$ which is the key tool in the study of completeness. Let $(p, u_1, au_2, awu_3)$ be a point of $V_{n,3}^{a,w}(M)$ and $f \in C^\infty\big(V_{n,3}^{a,w}(M)\big)$. Let $\sigma$ be the unique inextensible curve solution of (22) satisfying the initial conditions

$$\sigma(0) = p, \quad \sigma'(0) = u_1, \quad \frac{D\sigma'}{dt}(0) = au_2, \quad \frac{\widehat{D}}{dt}\Big(\frac{D\sigma'}{dt}\Big)(0) = awu_3.$$

We define

$$G_{(p,u_1,au_2,awu_3)}(f) := \frac{d}{dt}\Big|_{t=0} f\Big(\sigma(t), \sigma'(t), \frac{D\sigma'}{dt}(t), \frac{\widehat{D}}{dt}\Big(\frac{D\sigma'}{dt}\Big)(t)\Big).$$

From results of Section 4, we have

$$\Big(\sigma(t), \sigma'(t), \frac{D\sigma'}{dt}(t), \frac{\widehat{D}}{dt}\Big(\frac{D\sigma'}{dt}\Big)(t)\Big) \in V_{n,3}^{a,w}(M),$$

and $G$ is well defined.

The following result follows easily,

**Lemma 6.1** *There exists a unique vector field $G$ on $V_{n,3}^{a,w}(M)$ such that the curves $t \longmapsto \Big(\gamma(t), \gamma'(t), \frac{D\gamma'}{dt}(t), \frac{\widehat{D}}{dt}\big(\frac{D\gamma'}{dt}\big)(t)\Big)$ are the integral curves of $G$, for any solution $\gamma$ of equation* (22).

Once defined $G$, we will look for assumptions which assert its completeness.

Recall that an integral curve $\alpha$ of a vector field defined on some interval $[0, b)$, $b < +\infty$, can be extended to $b$ (as an integral curve) if and only if there exists an increasing sequence $\{t_m\}_m$, $t_m \to b$, such that $\{\alpha(t_m)\}_m$ converges (see for instance [17, Lemma 1.56]). The following technical result directly follows from this fact and Lemma 6.1.

**Lemma 6.2** *Let $\gamma : [0, b) \longrightarrow M$ be a solution of equation* (22) *with $0 < b < \infty$. The curve $\gamma$ can be extended to $b$ as a solution of* (22) *if and only if there exists a sequence $\Big\{\gamma(t_m), \gamma'(t_m), \frac{D\gamma'}{dt}(t_m), \frac{\widehat{D}}{dt}\big(\frac{D\gamma'}{dt}\big)(t_m)\Big\}_m$ which is convergent in $V_{n,3}^{a,w}(M)$ when $t_m \to b$.*

Although we know that $|\gamma'(t)|^2 = -1$, this is not enough to apply Lemma 6.2 even in the geometrically relevant case of $M$ compact. The reason is similar to the possible geodesic incompleteness of a compact Lorentzian manifold (see for instance [17, Ex. 7.16],[20]).

However, it is relevant that if a compact Lorentzian manifold admits a timelike conformal vector field, then it must be geodesically complete [20]. Therefore, from a geometric viewpoint, it is natural to assume the existence of such infinitesimal conformal symmetry to deal with the extendibility of the solutions of (22)-(23).

Recall that a vector field $K$ on $M$ is called *conformal* if the Lie derivative of the metric with respect to $K$ satisfies

$$L_K\langle\, ,\rangle = 2h\langle\, ,\rangle, \tag{29}$$

for some $h \in C^\infty(M)$, equivalently, the local flows of $K$ are conformal maps. In particular, if (29) holds with $h = 0$, $K$ is called *Killing* vector field.

On the other hand, if a vector field $K$ satisfies

$$\nabla_X K = hX \qquad \text{for} \quad \text{all} \qquad X \in \mathfrak{X}(M), \tag{30}$$

then clearly we get (29). Moreover, for the 1-form $K^b$ metrically equivalent to $K$, we have

$$dK^b(X,Y) = \langle \nabla_X K, Y\rangle - \langle \nabla_Y K, X\rangle = 0,$$

for all $X, Y \in \mathfrak{X}(M)$, i.e., $K^b$ is closed. We will call a $K$ which satisfies (30) a *conformal and closed* vector field. A Lorentzian manifold which admits a timelike conformal and closed vector field is locally a Generalized Robertson-Walker spacetime [4], [21].

The following result, inspired from [2, Lemma 9], will be decisive to assure that the image of the curve in $V^{a,w}_{n,3}(M)$, associated to a UC observer $\gamma$, is contained in a compact subset.

**Lemma 6.3** *Let $M$ be a spacetime and let $Q$ be a unit timelike vector field. If $\gamma : I \longrightarrow M$ is a solution of (22)-(23) such that $\gamma(I)$ lies in a compact subset of $M$ and $\langle Q, \gamma'\rangle$ is bounded on $I$, then the image of $t \longmapsto \Big(\gamma(t), \gamma'(t), \frac{D\gamma'}{dt}(t), \frac{\widehat{D}}{dt}\big(\frac{D\sigma'}{dt}\big)(t)\Big)$ is contained in a compact subset of $V^{a,w}_{n,3}(M)$.*

*Proof.* Consider the 1-form $Q^b$ metrically equivalent to $Q$ and the auxiliary Riemannian metric $g_R := \langle\, ,\, \rangle + 2\, Q^b \otimes Q^b$. We have,

$$g_R(\gamma', \gamma') = \langle \gamma', \gamma'\rangle + 2\,\langle Q, \gamma'\rangle^2,$$

which, by hypothesis, is bounded on $I$. Hence, there exists a constant $c > 0$ such that

$$\Big(\gamma(I), \gamma'(I), \frac{D\gamma'}{dt}(I), \frac{\widehat{D}}{dt}\big(\frac{D\gamma'}{dt}\big)(I)\Big) \subset C,$$

$$C := \big\{(p, u_1, au_2, awu_3) \in V^{a,w}_{n,3}(M) \,:\, p \in C_1, \quad g_R(u_1, u_1) \leq c\big\},$$

where $C_1$ is a compact set on $M$ such that $\gamma(I) \subset C_1$. Hence, $C$ is a compact in $V^{a,w}_{n,3}(M)$. □

Now, we are in a position to state the following completeness result (compare with [2, Th. 1] and [3, Th. 1]),

**Theorem 6.4** *Let $M$ be a spacetime which admits a timelike conformal and closed vector field $K$. If $\mathrm{Inf}_M \sqrt{-\langle K, K\rangle} > 0$ then, each solution $\gamma : I \longrightarrow M$ of (22)-(23) such that $\gamma(I)$ lies in a compact subset of $M$ can be extended.*

*Proof.* Let $I = [0, b)$, $0 < b < +\infty$, be the domain of a solution $\gamma$ of equation (22)-(23). Multiplying $\gamma'$ by the vector field $K$ and making use of the representation (28), we obtain,

$$\langle K\,, \gamma'\rangle = \frac{w}{\sqrt{w^2 - a^2}}\langle K\,, L\rangle + \frac{a}{\sqrt{w^2 - a^2}}\Big[\cos\big(\sqrt{w^2 - a^2}\,t\big)\,\langle K\,, M\rangle + \sin\big(\sqrt{w^2 - a^2}\,t\big)\,\langle K\,, N\rangle\Big].$$

On the other hand, taking into account that $L$ is Levi-Civita parallel and (30),

$$\frac{d}{dt}\langle K\,,L\rangle = \left\langle \frac{DK}{dt}\,,L\right\rangle = \langle \gamma'\,,L\rangle\,(h\circ\gamma) = -\frac{w(h\circ\gamma)}{\sqrt{w^2-a^2}}.$$

Analogously,

$$\frac{d}{dt}\langle K\,,M\rangle = \frac{a(h\circ\gamma)}{\sqrt{w^2-a^2}}\cos\left(\sqrt{w^2-a^2}\,t\right),$$

and

$$\frac{d}{dt}\langle K\,,N\rangle = \frac{a(h\circ\gamma)}{\sqrt{w^2-a^2}}\sin\left(\sqrt{w^2-a^2}\,t\right).$$

Using now that $\gamma(I)$ is contained in a compact of $M$, the function $h\circ\gamma$ is bounded on $I$. Therefore, since $I$ is assumed bounded, the functions $\langle K\,,L\rangle$, $\langle K\,,M\rangle$ and $\langle K\,,N\rangle$ are bounded on $I$ and, as consequence, there exists a constant $c_1>0$ such that

$$|\langle K,\gamma'\rangle| < c_1. \tag{31}$$

Now, if we put $Q := \dfrac{K}{\sqrt{-\langle K,\,K\rangle}}$, then $Q$ is a unit timelike vector field such that, by (31),

$$|\langle Q,\gamma'\rangle| \leq m\,c_1 \qquad \text{on} \quad I,$$

where $m = \mathrm{Sup}_M\left(-\langle K,\,K\rangle\right)^{-1/2} < \infty$. The proof ends making use of Lemmas 6.2 and 6.3. $\square$

**Remark 6.5** Note that the previous theorem implies the following result of mathematical interest: *Let $M$ be a compact spacetime which admits a timelike conformal and closed vector field $K$. Then, each inextensible solution of* (22)-(23) *must be complete.* Note that the Lorentzian universal covering of $M$ inherits the completeness of inextensible UC observers form the same fact on $M$.

# 7 Completeness of UC trajectories in a Plane Wave spacetime

In this section, we study the completeness of the inextensible UC trajectories with positive prescribed accelerations, but working in a more analytical way.

Let us consider a spacetime $M$ admitting a global chart $\left(x_1,x_2,\cdots,x_n\right)$. In these coordinates, we can write Equation (28) as follows

$$\begin{aligned}
\gamma'_k(t) =& \frac{w}{\sqrt{w^2-a^2}}\,L_k(t) + \frac{a}{\sqrt{w^2-a^2}}\left[\cos\left(\sqrt{w^2-a^2}\,t\right)M_k(t)+\sin\left(\sqrt{w^2-a^2}\,t\right)N_k(t)\right],\\
L'_k(t) =& \textstyle\sum_{i,j}\dfrac{-\Gamma^k_{ij}}{\sqrt{w^2-a^2}}\left[wL_iL_j + a\cos\left(\sqrt{w^2-a^2}t\right)L_i\,M_j + a\sin\left(\sqrt{w^2-a^2}t\right)L_i\,N_j\right],\\
M'_k(t) =& \textstyle\sum_{i,j}\dfrac{-\Gamma^k_{ij}}{\sqrt{w^2-a^2}}\left[wM_iL_j + a\cos\left(\sqrt{w^2-a^2}t\right)M_i\,M_j + a\sin\left(\sqrt{w^2-a^2}t\right)M_i\,N_j\right],\\
N'_k(t) =& \textstyle\sum_{i,j}\dfrac{-\Gamma^k_{ij}}{\sqrt{w^2-a^2}}\left[wN_iL_j + a\cos\left(\sqrt{w^2-a^2}t\right)N_i\,M_j + a\sin\left(\sqrt{w^2-a^2}t\right)N_i\,N_j\right],
\end{aligned} \tag{32}$$

$$\gamma_k(0)=p_k,\ L_k(0)=\tfrac{1}{\sqrt{w^2-a^2}}(wu_{1k}+au_{3k}),\ M_k(0)=\tfrac{-1}{\sqrt{w^2-a^2}}(au_{1k}+wu_{3k}),\ N_k(0)=u_{2k}.$$

Here, $u_{1k}$, $u_{2k}$ and $u_{3k}$ are the coordinates of the vectors $u_1$, $u_2$ and $u_3$ respectively, they satisfy

$$\sum_{i,j} u_{1i}\, u_{1j}\, g_{ij}(0) = -1, \quad \sum_{i,j} u_{2i}\, u_{2j}\, g_{ij}(0) = \sum_{i,j} u_{3i}\, u_{3j}\, g_{ij}(0) = 1,$$
$$\sum_{i,j} u_{1i}\, u_{2j}\, g_{ij}(0) = \sum_{i,j} u_{1i}\, u_{3j}\, g_{ij}(0) = \sum_{i,j} u_{2i}\, u_{3j}\, g_{ij}(0) = 0,$$

being $g_{ij}(0)$ the coefficients of the metric at the point $\gamma(0)$ in these coordinates. Moreover, all the Christoffel symbols are evaluated on $\gamma$, i.e., $\Gamma^k_{ij}(t) := \Gamma^k_{ij}\big(\gamma_1(t), \cdots, \gamma_n(t)\big)$.

A (four dimensional) *Plane Wave* is a spacetime $(M, g)$ which admits a Brinkmann coordinate system [10], i.e., a coordinate system in which the metric has the form

$$g = H(u,x,y)du^2 + 2dudv + dx^2 + dy^2,$$

where $H(u,x,y)$ is a quadratic function of the coordinates $x$ and $y$ with coefficients depending on $u$, that is,

$$H(u,x,y) = A(u)x^2 + B(u)y^2 + C(u)xy + D(u)x + E(u)y + F(u). \tag{33}$$

From now on, it is assumed that $M$ admits a global Brinkmann coordinate system, which we will denote by $(u,v,x,y)\big(= (x_1,x_2,x_3,x_4)\big)$. We also identify $M$ with $\mathbb{R}^4$.

In these coordinates, the Christoffel symbols of $g$ are easily computed as follows

$$\Gamma^1_{ij} = 0, \qquad \text{for all } \; i,j = 1,\ldots,4, \tag{34}$$
$$\Gamma^2_{11} = \frac{1}{2}\frac{\partial H}{\partial u}, \;\; \Gamma^2_{13} = \Gamma^2_{31} = \frac{1}{2}\frac{\partial H}{\partial x}, \;\; \Gamma^2_{14} = \Gamma^2_{41} = \frac{1}{2}\frac{\partial H}{\partial y}, \tag{35}$$
$$\Gamma^3_{11} = -\frac{1}{2}\frac{\partial H}{\partial x}, \;\; \Gamma^4_{11} = -\frac{1}{2}\frac{\partial H}{\partial y}, \tag{36}$$

and the remaining symbols are zero.

Now, let us consider a UC observer $\gamma : I \to \mathbb{R}^4$ satisfying the initial conditions

$$\gamma(0) = p, \quad \gamma'(0) = u_1, \quad \frac{D\gamma'}{dt}(0) = a\,u_2, \quad \frac{\widehat{D}}{dt}\left(\frac{D\gamma'}{dt}\right)(0) = aw\,u_3,$$

with $p, u_1, u_2, u_3 \in \mathbb{R}^4$. Our final objective is to prove that such trajectory is extensible to the whole real line, i.e., that the maximal interval of definition of $\gamma$ is $I = \mathbb{R}$.

By Proposition 5.1, we can write

$$\gamma'(t) = \frac{w}{\sqrt{w^2 - a^2}}\, L(t) + \frac{a}{\sqrt{w^2 - a^2}} \left[\cos\left(\sqrt{w^2 - a^2}\, t\right) M(t) + \sin\left(\sqrt{w^2 - a^2}\, t\right) N(t)\right]$$

where $L, M, N : I \to \mathbb{R}^4$ are solutions of system (32) with initial conditions

$$L(0) = \frac{1}{\sqrt{w^2 - a^2}}(wu_1 + au_3), \quad M(0) = \frac{-1}{\sqrt{w^2 - a^2}}(au_1 + wu_3), \quad N(0) = u_2.$$

Writing in coordinates $L = (L_1, L_2, L_3, L_4)$, $M = (M_1, M_2, M_3, M_4)$, $N = (N_1, N_2, N_3, N_4)$, we have a simple but important fact.

**Lemma 7.1** *The first components of $L, M, N$ are constant with value*

$$L_1 = \frac{1}{\sqrt{w^2 - a^2}}(wu_{11} + au_{31}),\ M_1 = \frac{-1}{\sqrt{w^2 - a^2}}(au_{11} + wu_{31}),\ N_1 = u_{21}$$

*Proof.* It follows trivially from (34) and (32) that $L_1' = M_1' = N_1' = 0$, then $L_1, M_1, N_1$ are constants and equal to the respective initial conditions.

□

A direct consequence of the latter lemma is that

$$\gamma_1'(t) = \frac{w}{\sqrt{w^2 - a^2}}\, L_1 + \frac{a}{\sqrt{w^2 - a^2}}\left[\cos\left(\sqrt{w^2 - a^2}\, t\right) M_1 + \sin\left(\sqrt{w^2 - a^2}\, t\right) N_1\right]$$

with $L_1, M_1, N_1 =$ constant, and we have an explicit expression for $\gamma_1(t)$ as

$$\gamma_1(t) = p_1 + \frac{wt}{\sqrt{w^2 - a^2}}\, L_1 + \tag{37}$$

$$+\frac{a}{\sqrt{w^2 - a^2}}\left[M_1 \int_0^t \cos\left(\sqrt{w^2 - a^2}\, t\right) dt + N_1 \int_0^t \sin\left(\sqrt{w^2 - a^2}\, t\right) dt\right].$$

**Lemma 7.2** *As solutions of system* (32)*, the functions $L_3, M_3, N_3, L_4, M_4, N_4$ are extensible to the whole real line.*

*Proof.* The equations from (32) for $k = 3, 4$ are

$$\begin{aligned}
L_k'(t) &= \frac{-\Gamma_{11}^k}{\sqrt{w^2 - a^2}}\left[wL_1^2 + a\cos\left(\sqrt{w^2 - a^2} t\right) L_1\, M_1 + a\sin\left(\sqrt{w^2 - a^2} t\right) L_1\, N_1\right], \\
M_k'(t) &= \frac{-\Gamma_{11}^k}{\sqrt{w^2 - a^2}}\left[wM_1 L_1 + a\cos\left(\sqrt{w^2 - a^2} t\right) M_1^2 + a\sin\left(\sqrt{w^2 - a^2} t\right) M_1\, N_1\right], \\
N_k'(t) &= \frac{-\Gamma_{11}^k}{\sqrt{w^2 - a^2}}\left[wN_1 L_1 + a\cos\left(\sqrt{w^2 - a^2} t\right) N_1\, M_1 + a\sin\left(\sqrt{w^2 - a^2} t\right) N_1^2\right].
\end{aligned}$$

The expressions between brackets are trigonometric functions. We define

$$\begin{aligned}
f(t) &= \frac{1}{\sqrt{w^2 - a^2}}\left[wL_1^2 + a\cos\left(\sqrt{w^2 - a^2} t\right) L_1\, M_1 + a\sin\left(\sqrt{w^2 - a^2} t\right) L_1\, N_1\right], \\
g(t) &= \frac{1}{\sqrt{w^2 - a^2}}\left[wM_1 L_1 + a\cos\left(\sqrt{w^2 - a^2} t\right) M_1^2 + a\sin\left(\sqrt{w^2 - a^2} t\right) M_1\, N_1\right], \\
h(t) &= \frac{1}{\sqrt{w^2 - a^2}}\left[wN_1 L_1 + a\cos\left(\sqrt{w^2 - a^2} t\right) N_1\, M_1 + a\sin\left(\sqrt{w^2 - a^2} t\right) N_1^2\right].
\end{aligned}$$

Then, the previous system is written as

$$\begin{aligned}
L_k'(t) &= -f(t)\Gamma_{11}^k(\gamma(t)), \\
M_k'(t) &= -g(t)\Gamma_{11}^k(\gamma(t)), \\
N_k'(t) &= -h(t)\Gamma_{11}^k(\gamma(t)).
\end{aligned} \tag{38}$$

The key point is to analyse the particular form of the Christoffel symbols $\Gamma_{11}^k(\gamma(t))$, $k = 3, 4$. Considering that $H$ is defined by (33), we have

$$\Gamma_{11}^3(\gamma) = -\frac{1}{2}\frac{\partial H}{\partial x}(\gamma(t)) = -A(\gamma_1)\gamma_3 - \frac{1}{2}C(\gamma_1)\gamma_4 - \frac{1}{2}D(\gamma_1),$$

and

$$\Gamma^4_{11}(\gamma) = -\frac{1}{2}\frac{\partial H}{\partial y}(\gamma(t)) = -B(\gamma_1)\gamma_4 - \frac{1}{2}C(\gamma_1)\gamma_3 - \frac{1}{2}E(\gamma_1),$$

where $\gamma_1(t)$ is explicitly given by (37). Since

$$\gamma_k(t) = p_k +$$
$$\int_0^t \left[\tfrac{w}{\sqrt{w^2-a^2}}\, L_k(s) + \tfrac{a}{\sqrt{w^2-a^2}}\left[\cos\left(\sqrt{w^2-a^2}\,t\right) M_k(s) + \sin\left(\sqrt{w^2-a^2}\,t\right) N_k(s)\right]\right] ds,$$

then system (38) (with $k = 3, 4$) can be seen as an integro-differential system of six equations. To pass to a standard system of differential equations, we define the new variables

$$\mathcal{L}_k = \frac{w}{\sqrt{w^2-a^2}} \int_0^t L_k(s)ds,$$

$$\mathcal{M}_k = \frac{a}{\sqrt{w^2-a^2}} \int_0^t \cos\left(\sqrt{w^2-a^2}\,s\right) M_k(s)ds,$$
$$\mathcal{N}_k = \frac{a}{\sqrt{w^2-a^2}} \int_0^t \sin\left(\sqrt{w^2-a^2}\,s\right) N_k(s)ds,$$

for $k = 3, 4$. With the new variables,

$$\begin{aligned} \mathcal{L}'_k(t) &= \tfrac{w}{\sqrt{w^2-a^2}} L_k(t), \\ \mathcal{M}'_k(t) &= \tfrac{a}{\sqrt{w^2-a^2}} \cos\left(\sqrt{w^2-a^2}\,t\right) M_k(t), \\ \mathcal{N}'_k(t) &= \tfrac{a}{\sqrt{w^2-a^2}} \sin\left(\sqrt{w^2-a^2}\,t\right) N_k(t), \end{aligned} \tag{39}$$

for $k = 3, 4$. Besides,

$$\gamma_k(t) = \mathcal{L}_k + \mathcal{M}_k + \mathcal{N}_k + p_k \qquad (k = 3, 4).$$

Remember that $\gamma_1(t)$ is known explicitly, see (37). Therefore, attending to the expression of the Christoffel symbols computed before, equations (38) are linear on the variables $\mathcal{L}_k$, $\mathcal{M}_k$, $\mathcal{N}_k$. Summing up, equations (38)-(39) compose a linear system of 12 equations on the involved variables $L_k$, $M_k$, $N_k$, $\mathcal{L}_k$, $\mathcal{M}_k$, $\mathcal{N}_k$ ($k = 3, 4$). The basic theory of linear systems states that any solution of a linear system is globally defined on the whole real line, closing the proof.

□

We end the manuscript with the following result.

**Theorem 7.3** *Every UC inextensible trajectory in a Plane Wave spacetime admitting a global Brinkmann chart is complete.*

*Proof.* Up to now, we have proved that $L_k, M_k, N_k$ with $k = 1, 3, 4$ are defined on the whole $\mathbb{R}$. To finish the proof, it remains to prove the completeness of $L_2(t), M_2(t), N_2(t)$. The equation (32) for $L_2$ is

$$L'_2(t) = \sum_{i,j} \frac{-\Gamma^2_{ij}}{\sqrt{w^2-a^2}} \left[ wL_iL_j + a\cos\left(\sqrt{w^2-a^2}t\right) L_i\, M_j + a\sin\left(\sqrt{w^2-a^2}t\right) L_i\, N_j \right],$$

but note that $\Gamma^2_{ij} = 0$ if $i = 2$ or $j = 2$, and moreover $H$ does not depend on the second variable. This implies that the right-hand side part of the latter equation depends only on functions $L_k(t), M_k(t), N_k(t)$ ($k$=1,3,4), which we have proved that are globally defined, but not on $L_2, M_2, N_2$. Thus, $L_2'(t)$ is defined for every $t$, and a simple integration leads to the conclusion. An analogous argument serves for $M_2(t), N_2(t)$.

$\square$

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