The complex periodic problem for a Riccati equation

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Dedicated to Jean Mawhin, on the occasion of his seventieth birthday

1 Introduction

Complex valued periodic differential equations have been studied by Mawhin in several papers. He already discussed this topic in a course held in Montreal, almost twenty years ago (see [7]). Motivated by his lectures, Campos and I studied the periodic problem for the equation

$$z' + z^2 + Q(t) = 0, \tag{1}$$

where $Q : \mathbb{R} \to \mathbb{C}$ is continuous and T-periodic. Notice that the independent variable t is real but the unknown $z = z(t)$ can take complex values. In [3] we constructed an equation of this class without periodic solutions. A related example had been previously constructed by Lloyd in [5], for another class of Riccati equations. Alternative examples of non-existence were constructed by Miklaszewski in [9] and by Gabdrakhmanov and Filippov in [4]. More advanced results on the equation (1) were obtained later by Campos in [1] and Zoladek in [12] and, more recently, by Wilczyński in [11]. The paper [11] also contains an extensive list of previous works on the complex Riccati equation. It is also interesting to point out that Campos and Mawhin have initiated in [2] the study of the more delicate quaternionic Riccati equation.

*Supported by D.G.I. MTM2011-23652, Spain

The purpose of this paper is to present an elementary approach for the study of the periodic problem associated to (1). As it is well known the change of variables

$$z = \frac{w'}{w}$$

transforms the Riccati equation (1) into the second order linear equation

$$w'' + Q(t)w = 0. \tag{2}$$

The key observation of the paper is that the existence of T-periodic solutions of (1) can be characterized in terms of standard properties of this linear equation when $Q(t)$ is real valued. After this remark, the well developed theory of Hill's equation can be applied to produce many results for the periodic problem associated to (1). The propositions stated below will be derived from classical results on Hill's equations. The function $Q(t)$ is real, but complex solutions $z = z(t)$ will be admissible.[1]

Proposition 1 *Assume that $Q(t)$ is a non-constant, continuous and T-periodic real-valued function. Then there exist two numbers $\lambda_* < \lambda^*$ such that the equation*

$$z' + z^2 + Q(t) + \lambda = 0$$

has no T-periodic solutions if $\lambda \in]\lambda_*, \lambda^*[$.

Proposition 2 *Assume that $Q(t)$ is a continuous and T-periodic real-valued function satisfying*

$$T \int_0^T Q^+(t)dt < 4, \tag{3}$$

with $Q^+(t) = \max\{Q(t), 0\}$. Then the equation (1) has at least one T-periodic solution.

At the end of the paper it will be proved that the number 4 is optimal in this result.

2 Disconjugacy and stability for Hill's equation

From now on the function $Q(t)$ will be real-valued and the theory on Hill's equation developed in the book [6] will be used freely.

[1] $Q : \mathbb{R} \to \mathbb{R}$ and $z : \mathbb{R} \to \mathbb{C}$.

The equation (2) is called *disconjugate* if there exists a non-vanishing, real-valued solution; that is, a solution $w_*(t)$ satisfying

$$w_*(t) \in \mathbb{R} \setminus \{0\} \text{ for each } t \in \mathbb{R}.$$

The same equation is called *stable* if the trivial solution $w = 0$ is stable in the sense of Lyapunov. The families of stable and disconjugate Hill's equations are disjoint.

Theorem 3 *The following conditions are equivalent:*
(i) *The equation (1) has at least one T-periodic solution*
(ii) *The equation (2) is disconjugate or stable.*

To prepare the proof of this result we recall some well-known facts on Hill's equations. A Floquet solution of (2) is a non-trivial solution satisfying

$$w(t+T) = \mu w(t), \quad t \in \mathbb{R},$$

for some $\mu \in \mathbb{C} \setminus \{0\}$. The number μ is a multiplier and there exist at most two of them, μ_1 and μ_2. They satisfy $\mu_1 \cdot \mu_2 = 1$. The equation is elliptic if $\mu_1 = \overline{\mu_2} \in \mathbb{S}^1 \setminus \{\pm 1\}$, parabolic if $\mu_1 = \mu_2 = \pm 1$ and hyperbolic if $\mu_1, \mu_2 \in \mathbb{R} \setminus \{\pm 1\}$. Elliptic equations are stable and hyperbolic equations are unstable. In the parabolic case we must distinguish between the stable-parabolic case (all solutions are periodic) and the unstable-parabolic case (periodic and unbounded solutions coexist). We present a preliminary result on elliptic equations.

Lemma 4 *Assume that the equation (2) is elliptic, then Floquet solutions do not vanish.*

Proof. Assume by contradiction that $w(t)$ is a Floquet solution vanishing at some $t_0 \in \mathbb{R}$. Then, by uniqueness, $w(t_0) = 0$ and $w'(t_0) \neq 0$. The solution $\omega(t) = \frac{w(t)}{w'(t_0)}$ satisfies the initial conditions $\omega(t_0) = 0$ and $\omega'(t_0) = 1$. Since $Q(t)$ is real-valued, the same happens to $\omega(t)$. This is not possible because $\omega(t)$ is a Floquet solution associated to a non-real multiplier.■

Remark. In the stable-parabolic case some Floquet solutions can vanish but there always exist a non-vanishing Floquet solution. To justify this consider the solutions $\phi_1(t)$ and $\phi_2(t)$ of (2) satisfying the initial conditions

$$\phi_1(0) = \phi_2'(0) = 1, \quad \phi_1'(0) = \phi_2(0) = 0.$$

Since we are in the stable-parabolic case both of them must be Floquet solutions with the same multiplier $\mu = \pm 1$. Moreover, the Wronskian satisfies

$$W(t) = \phi_1(t)\phi_2'(t) - \phi_1'(t)\phi_2(t) = 1$$

for each t. Then ϕ_1 and ϕ_2 cannot vanish simultaneously and $w(t) = \phi_1(t) + i\phi_2(t)$ is a Floquet solution that never vanishes.

Proof of Theorem 3. $(ii) \Rightarrow (i)$. We will prove that there exists a Floquet solution $w(t)$ that does not vanish. Then the proof is easily completed because $z(t) = \frac{w'(t)}{w(t)}$ is a T-periodic solution of (1). Our assumption is that (2) is either stable or disconjugate. If it is stable then it is either elliptic or stable-parabolic and the previous discussions guarantee the existence of $w(t)$. Assume now that (2) is disconjugate. The multipliers are real and we can find a real-valued Floquet solution $w(t)$. From the definition we know that if t_0 is a zero of $w(t)$ then the same is true for $t_0 + T$. Due to Sturm theory and the definition of disconjugacy we know that real-valued solutions can vanish at most once and so $w(t)$ never vanishes.

$(i) \Rightarrow (ii)$ Assume that $z(t)$ is a T-periodic solution of (1). The primitive $Z(t) = \int_0^t z(s)ds$ can be expressed as

$$Z(t) = \tilde{Z}(t) + At$$

where $\tilde{Z}(t)$ is T-periodic and $A = \frac{1}{T}\int_0^T z(t)dt$ is the average of the solution. By a direct substitution we can verify that the function $w(t) = e^{Z(t)}$ is a solution of (2). Moreover $w(t)$ is a Floquet solution that never vanishes. Notice that the associated Floquet multiplier is $\mu = e^{TA}$. If the equation (2) is elliptic or stable-parabolic then this equation is stable and the proof is complete. From now on we will assume that (2) is hyperbolic or unstable-parabolic and we will prove that then it is also disconjugate. In these assumptions we know that the Floquet multiplier μ is real. Moreover, the space of solutions satisfying $\omega(t+T) = \mu\omega(t)$ has dimension one. In principle this dimension must be considered in the complex sense but we also know that this space contains real-valued Floquet solutions. In consequence there exists a number $\sigma \in \mathbb{C} \setminus \{0\}$ such that $\omega_*(t) = \sigma w(t)$ is a real-valued solution. This solution never vanishes and so we have proved the disconjugacy of the equation (2).■

3 Proof of Proposition 1

Consider the equation with one parameter

$$w'' + (Q(t) + \lambda)w = 0. \tag{4}$$

As it is well known there exist sequences $\{\lambda_n\}_{n\geq 0}$ and $\{\lambda'_n\}_{n\geq 1}$ satisfying

$$\lambda_0 < \lambda'_1 \leq \lambda'_2 < \lambda_1 \leq \lambda_2 < \lambda'_3 \leq \lambda'_4 < \lambda_3 \leq \lambda_4 < \cdots$$

and such that the equation is hyperbolic if λ belongs to one of the instability intervals, namely $I_0 =]-\infty, \lambda_0[$, $I_n =]\lambda_{2n-1}, \lambda_{2n}[$ or $I'_n =]\lambda'_{2n-1}, \lambda'_{2n}[$ with $n \geq 1$. Notice that I_0 is never empty but the other intervals will be empty as soon as two consecutive eigenvalues coincide. It is also well known that the equation is disconjugate if and only if λ belongs to the closure of I_0. In view of Theorem 3 we can say that the equation

$$z' + z^2 + Q(t) + \lambda = 0 \tag{5}$$

has no T-periodic solutions if λ belongs to some instability interval different from I_0. A deep result due to Borg says that if $Q(t)$ is non-constant then the set $I_n \cup I'_n$ is non-empty for some $n \geq 1$. A proof of this result can be found in [6]. The proof of the Proposition is now completed if we define the interval $]\lambda_*, \lambda^*[$ as a non-empty instability interval.■

Remark. For a generic $Q(t)$ all intervals of instability are non-empty. In such situation the equation (2) is hyperbolic when the parameter λ lies in an instability interval, unstable-parabolic at the boundary of I_n or I'_n and elliptic otherwise. From Theorem 3 we deduce that the Riccati equation (5) has a T-periodic solution if and only if λ is in the interval $E_0 =]-\infty, \lambda'_1[$ or in one of the intervals $E_n =]\lambda'_{2n}, \lambda_{2n-1}[$, $E'_n =]\lambda_{2n}, \lambda'_{2n+1}[$.

4 Proof of Proposition 2

In view of Theorem 3 it is enough to prove that if (3) holds then the equation (2) is disconjugate if it is unstable. Unstable equations are either hyperbolic or parabolic and so they have a real valued Floquet solution $w(t)$. We will prove by a contradiction argument that $w(t)$ never vanishes. Assume that t_0 is a zero of $w(t)$, from the definition of Floquet solution we deduce that $w(t_0) = 0$ implies $w(t_0 + T) = 0$. We are going to prove that this cannot happen if the condition (3) holds. To this end we apply Lemma 3.4 in [10] and deduce that the distance between two consecutive zeros $t_0 < t_1$ of any real-valued solution of (2) satisfies

$$(t_1 - t_0) \int_{t_0}^{t_1} Q^+(t)dt \geq 4.$$

From (3) we deduce that t_0 and $t_0 + T$ cannot be two zeros of $w(t)$.■

To finish the paper we construct an example showing that the number 4 in condition (3) is optimal. Let us fix any number $M > 4$. We consider any sequence of functions $Q_n : \mathbb{R} \to \mathbb{R}$, $n \geq 2$, satisfying

- Q_n is continuous and T-periodic
- $Q_n(t) \geq 0$ if $t \in [0, \frac{T}{n}]$ and $Q_n(t) = 0$ if $t \in [\frac{T}{n}, T]$
- $T \int_0^T Q_n(t)dt = M$.

We will prove that the equation

$$z' + z^2 + Q_n(t) = 0$$

has no T-periodic solutions if n is large enough. Notice that $T \int_0^T Q_n(t)^+ dt = M$ and so the number 4 cannot be replaced by M in Proposition 2.

In view of Theorem 3 it will be enough to prove that the second order equation

$$w'' + Q_n(t)w = 0 \tag{6}$$

is neither stable nor disconjugate. Indeed we will prove that the discriminant Δ_n satisfies

$$\Delta_n < -2$$

if n is large enough. If this condition holds then the equation is hyperbolic and all real solutions have infinitely many zeros. We claim that

$$\lim_{n \to \infty} \Delta_n = 2 - M.$$

To compute this limit we observe that the discriminant can be expressed as

$$\Delta_n = \phi_{1,n}(T) + \phi'_{2,n}(T),$$

where $\phi_{1,n}(t)$, $\phi_{2,n}(t)$ are the solutions of (6) with initial conditions

$$\phi_{1,n}(0) = \phi'_{2,n}(0) = 1, \ \ \phi'_{1,n}(0) = \phi_{2,n}(0) = 0.$$

The claim on the limit is a consequence of the following auxiliary result.

Lemma 5 *Let us fix numbers $a, b \in \mathbb{R}$ and let $w_n(t)$ be the solution of (6) with initial conditions $w_n(0) = a$, $w_n'(0) = b$. Then $w_n(T) \to (1-M)a + Tb$ and $w_n'(T) \to b - \frac{M}{T}a$ as $n \to \infty$.*

Proof. The function $w_n(t)$ satisfies the integral equation

$$w_n(t) = a + bt - \int_0^t (t-s)Q_n(s)w_n(s)ds.$$

For each $t \in [0, T]$,

$$|w_n(t)| \leq |a| + |b|T + T\int_0^t Q_n(s)|w_n(s)|ds$$

and Gronwall's inequality implies that

$$|w_n(t)| \leq (|a| + |b|T)e^M := B_0.$$

The identity

$$w_n'(t) = b - \int_0^t Q_n(s)w_n(s)ds \tag{7}$$

leads to the estimate

$$|w_n'(t)| \leq |b| + \frac{M}{T}B_0 =: B_1.$$

Then

$$w_n(t) = a + \int_0^t w_n'(s)ds = a + O(\frac{1}{n}) \text{ if } t \in [0, \frac{T}{n}].$$

From now on all asymptotic formulas will be uniform in n. Evaluating (7) at $t = \frac{T}{n}$ and using the conditions imposed on the support of Q_n,

$$w_n'(\frac{T}{n}) = b - \int_0^{\frac{T}{n}} Q_n(s)w_n(s)ds = b - \frac{M}{T}a + O(\frac{1}{n}).$$

Since w_n'' vanishes on $[\frac{T}{n}, T]$ we conclude that

$$w_n'(T) = w_n'(\frac{T}{n}) = b - \frac{M}{T}a + O(\frac{1}{n})$$

and

$$w_n(T) = w_n(\frac{T}{n}) + w_n'(\frac{T}{n})T(1 - \frac{1}{n}) = a + Tb - Ma + O(\frac{1}{n}). \blacksquare$$

The previous construction is based on the following observations:

- The sequence Q_n converges to a periodic measure μ.

  This measure is defined as a functional by

  $$\langle \mu, \phi \rangle = \frac{M}{T} \sum_{n \in \mathbb{Z}} \phi(nT), \quad \phi \in C_0(\mathbb{R}),$$

  where $C_0(\mathbb{R})$ is the space of continuous functions $\phi : \mathbb{R} \to \mathbb{R}$ with compact support. The convergence is in the weak sense, that is

  $$\int_{\mathbb{R}} Q_n \phi \to \langle \mu, \phi \rangle$$

  for each $\phi \in C_0(\mathbb{R})$.

- The generalized differential equation

  $$w'' + \mu(t) w = 0$$

  has a well posed Cauchy problem.

  In the interval $]0, T]$ the solution is

  $$w(t) = w(0) + w'(0^+)t, \; t \in]0, T], \quad w'(0^+) = w'(0) - \frac{M}{T} w(0).$$

  From this formula we can compute the value of the associated discriminant, $\Delta_\infty = 2 - M$.

- The properties of continuous dependence of the generalized Hill's equation imply that $\Delta_n \to \Delta_\infty$.

The previous claims can be made precise using the paper [8] by Meng and Zhang. This paper contains a general theory for a generalized equation of Hill type whose coefficient $Q(t)$ is a periodic measure.

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