Topology of Attractors and Periodic Points
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Abstract

The dynamics of a dissipative and area contracting planar homeomorphism is described in terms of the attractor. This is a subset of the plane defined as the maximal compact invariant set. We prove that the coexistence of a fixed point of saddle type and an N-cycle produces some topological complexity: the attractor cannot be locally connected. The proofs are based on the theory of prime ends.

1 Introduction

In a remarkable paper [11], Levinson introduced the class of planar homeomorphisms which are dissipative and area-contracting. His main motivation came from non-conservative Mechanics but this class of maps is also relevant in other fields, such as population dynamics (see [17, 19] for more details).

A homeomorphism $h : \mathbb{R}^2 \to \mathbb{R}^2$ is called dissipative if the point of infinity is a repeller. The associated attractor $\mathcal{A} \subset \mathbb{R}^2$ is an invariant continuum composed by all the bounded orbits. The set $\mathcal{A}$ has zero measure if $h$ is area-contracting. Understanding the topology of $\mathcal{A}$ is a crucial step in the study of the dynamics of $h$ but this is not an easy task. Many examples have been constructed to show that the set $\mathcal{A}$, as well as the dynamics on it, can be very intricate (see [2, 3, 6, 21]).

In a more recent paper [14] Nakajima used some clever and elementary arguments to prove that the attractor is not arc-wise connected when $h$ has at least two fixed points and one of them is an inverse saddle$^1$. This result seems to be of a new type, showing that some dynamical assumptions (existence of certain fixed points) imply a certain complexity of the attractor (not arc-wise connected). The goal of our paper is to continue this line of research. We will impose some assumptions on the existence of fixed and periodic points and we will conclude that the attractor cannot be locally connected. This is the case if there are a direct saddle and an N-cycle with $N \geq 2$ and also when the map is orientation reversing and there is an N-cycle with $N \geq 3$. The reader will notice that our assumptions are more flexible, since they allow several configurations for fixed and periodic points. On the other hand, our conclusion ($\mathcal{A}$ not locally connected) is weaker than Nakajima’s conclusion ($\mathcal{A}$ not arc-wise connected). We do not know if this difference is essential or just a consequence of the method of our proof. The basic tool in our approach is Caratheodory’s theory of prime ends. The use of this theory in planar dynamics has a long tradition, starting with the work of Cartwright and Littlewood in [4]. Incidentally we notice that these authors were motivated by the study of the attractor of the periodically forced Van der Pol oscillator. Later, Alligood and Yorke analyzed in [1] the connections between the rotation number of a local attractor and the existence of accessible periodic points (see also [16] and [18]). All these papers consider the complement of the attractor in the Riemann sphere $S^2 = \mathbb{R}^2 \cup \{\infty\}$, denoted as $\Omega = S^2 \setminus \mathcal{A}$.

$^1$We recall that a fixed point $p$ of a diffeomorphism $h$ is an inverse saddle if the eigenvalues of the Jacobian matrix $h'(p)$ satisfy $\lambda_1 < -1 < \lambda_2 < 0$. When $0 < \lambda_1 < 1 < \lambda_2$ the fixed point is a direct saddle.
This set is open, simply connected and invariant. The circle of prime ends is a fictitious boundary
attached to $\Omega$ and the map $h$ induces a homeomorphism $h^*$ on this circle. We will follow along
this line and show that there are obstructions for the dynamics of $h^*$ when certain periodic points
exist and $A$ is locally connected. In consequence $A$ will not enjoy this topological property.

In the recent papers [8, 7], Koropecki, Le Calvez and Nassiri have considered a class of planar
homeomorphisms that contains area-preserving maps. This class is disjoint with Levinson’s class
because attractors and trapping regions are excluded. Nevertheless there are common features
with our approach. The main theme in [8, 7] is the analysis of the dynamics on a simply connected
open and invariant set $\Omega \subset S^2$. The boundary $\partial \Omega$ plays a role which is similar to the role of our
attractor $A$. In particular the property of local connectedness is also relevant (see Theorem 1.4 in
[7]). There are also strong differences with our approach. The authors of [8, 7] are mainly interested
in the interaction of the dynamics of $h$ in the boundary and in the interior of the invariant region.
They obtain remarkable results on the dynamics of $h|\partial \Omega$ and $h|\Omega$. In contrast, we only analyze
the dynamics of $h^*$ on the circle of prime ends (the fictitious boundary of $\Omega$). For this reason
our approach is technically simpler and yet sufficient to discover significant properties of the attractor.
This is particularly clear in the short proof of our results on orientation-reversing maps.

The rest of the paper is organized in five sections. In Section 2 we present some basic definitions
and examples. The main results are stated in Section 3. These results cannot be extended to higher
dimensions, at least in a direct way. At the end of the Section we will present examples showing
some differences between the dynamics in two and three dimensions. In Section 4 we prove some
results using Nakajima’s ideas. In Section 5 we sum up some aspects of the theory of prime ends.
This theory is employed in Section 6 to prove the main results.

2 A class of planar homeomorphisms

A homeomorphism of the plane is a continuous and bijective map $h : \mathbb{R}^2 \to \mathbb{R}^2$. The class of
all homeomorphisms will be denoted by $\mathcal{H}(\mathbb{R}^2)$. A map $h \in \mathcal{H}(\mathbb{R}^2)$ is called dissipative if there
exists a closed ball $B \subset \mathbb{R}^2$ attracting all compact sets in a uniform sense. In other words, for each
$p \in \mathbb{R}^2$,

$$\lim_{n \to +\infty} \text{dist}(h^n(p), B) = 0$$

(2.1)

and this limit is uniform in $p \in K$ with $K$ any compact subset of $\mathbb{R}^2$.

The attractor $A \subset \mathbb{R}^2$ is defined as the maximal invariant and compact set. It satisfies the
properties below,

- $A$ is compact and $h(A) = A$,
- $A$ contains any compact set $K \subset \mathbb{R}^2$ satisfying that $h(K) = K$.

With some work it can be proved that $A$ always exists and it is indeed a non-empty continuum
(compact and connected set). Sometimes it is useful to interpret $A$ as the set of the bounded
orbits. More precisely,

$$A = \{ p \in \mathbb{R}^2 : \limsup_{|n| \to +\infty} |h^n(p)| < \infty \}.$$ 

Notice that all orbits are bounded in the future ($n \to +\infty$) and therefore $A$ can be also described
as the set of orbits bounded in the past ($n \to -\infty$). The reader is referred to [5, 11, 19] for a
detailed discussion on attractors and dissipative maps.

Levinson considered in Section 7 of [11] the class of dissipative homeomorphisms contracting
Lebesgue’s measure. In that case the attractor has zero measure and, in particular, the interior
is empty. We will consider the larger class $\mathcal{LH}(\mathbb{R}^2)$ composed by all dissipative homeomorphisms
whose attractor has an empty interior in $\mathbb{R}^2$. The class of Levinson appears very often in the theory of nonlinear oscillations but $\mathcal{LH}(\mathbb{R}^2)$ is more natural from a topological point of view. Notice that $\mathcal{LH}(\mathbb{R}^2)$ is invariant under conjugacy.

The next result is an almost direct consequence of the definition of $\mathcal{LH}(\mathbb{R}^2)$ and will be repeatedly applied throughout the paper.

**Proposition 2.1.** If $h \in \mathcal{LH}(\mathbb{R}^2)$, then $\mathcal{A}$ does not contain Jordan curves.

**Proof.** By contradiction assume that $\Gamma \subset \mathbb{R}^2$ is a Jordan curve inside the attractor. Let $R_i(\Gamma)$ be the bounded component of $\mathbb{R}^2 \setminus \Gamma$. We will prove that $R_i(\Gamma)$ is also contained in $\mathcal{A}$, but this is impossible if $\mathcal{A}$ has empty interior.

Since $\mathcal{A}$ is invariant, every iterate of the curve is also contained in $\mathcal{A}$, that is,

$$\bigcup_{n \in \mathbb{Z}} h^n(\Gamma) \subset \mathcal{A}.$$ 

Let $B$ be a large ball containing $\mathcal{A}$. Then

$$\bigcup_{n \in \mathbb{Z}} h^n(R_i(\Gamma)) = \bigcup_{n \in \mathbb{Z}} R_i(h^n(\Gamma)) \subset B$$

and all the orbits starting at $R_i(\Gamma)$ remain bounded. This implies that $R_i(\Gamma) \subset \mathcal{A}$. \qed

It is well known that many continua in the plane have an intricate topology and they can be realized as attractors. Some examples will be shown in the next Section. In contrast to graphs or other simple continua, they lack some nice connection properties. For future references we recall the definitions of arc-wise connected and locally connected continua.

A continuum $C \subset \mathbb{R}^2$ is called arcwise connected if every two different points in $C$ can be joined by an arc in $C$. The continuum is called locally connected if the previous property can be localized in the following sense: for each $p \in C$ and $\varepsilon > 0$, there exists $\delta > 0$ such that every point $q \in B(p, \delta) \cap C$ can be connected to $p$ by an arc contained in $B(p, \varepsilon) \cap C$.

This is not the standard definition of locally connected continua but we refer to page 184 of the book [10] for the equivalence. It is also well known that locally connected continua are always arc-wise connected. The converse is false.

A very interesting discussion on possible attractors for maps in $\mathcal{LH}(\mathbb{R}^2)$ can be found in Section 7 of [11]. The simplest possible attractor is a singleton, corresponding to a globally asymptotically stable fixed point. After this case, Levinson presented three figures of increasing complexity. In Figure 1 of [11] the attractor is an arc and two dynamics are considered: three fixed points or one fixed point and a 2-cycle. In Figure 2 the attractor is a triode with one fixed point and two 3-cycles. Finally, in Figure 3, the three branches of the triode wrap themselves infinitely many times around three arcs. The corresponding continuum is not arc-wise connected. The dynamics is also more involved and a 6-cycles appears.

Another possibility, also mentioned by Levinson, is the Cantorian Sun. This continuum is obtained by drawing all the rays connecting the origin to the points of a Cantor set lying in $S^1$. See the Figure A.

It was proved in [6], see also [21], that this set is the attractor of a map $h$ in $\mathcal{LH}(\mathbb{R}^2)$. Moreover the dynamics on the Cantor set is recurrent (non-periodic) and the only periodic point is the origin, $Fix(h) = Fix(h^n) = \{(0,0)\}$ for each $n \geq 1$. In this case $\mathcal{A}$ is arcwise connected but it is not locally connected.
3 Main Results

A point $p \in \mathbb{R}^2$ is a direct saddle for $h \in \mathcal{H}(\mathbb{R}^2)$ if there exist a topological disk $D$ with $p \in \text{Int}(D)$ and a homeomorphism $\Psi : [-1,1]^2 \rightarrow D$ so that $\Psi(0) = p$ and

$$\Psi \circ L_d = h \circ \Psi \quad \text{on} \quad \left[-\frac{1}{2}, \frac{1}{2}\right] \times [-1,1]$$

(3.1)

with

$$L_d(x_1,x_2) = \left(2x_1, \frac{1}{2}x_2\right).$$

The definition of an inverse saddle is the same with the only difference that the linear map in (3.1) is

$$L_i(x_1,x_2) = \left(-2x_1, -\frac{1}{2}x_2\right).$$

The following result can be obtained with the ideas in [14].

**Theorem 3.1.** If $h \in \mathcal{LH}(\mathbb{R}^2)$ has at least two fixed points $p$ and $q$ with $p$ an inverse saddle, then $\mathcal{A}$ is not arc-wise connected.

Some examples of maps where this result is applicable were discussed in [14]. The statement of the previous theorem is slightly different from the result in [14]. We will present a complete proof in the next Section.

It is worth to mention that the inverse saddle cannot be replaced by a direct saddle. This is shown by the first Levinson example. The attractor is arc-wise connected, in fact, $\mathcal{A}$ is an arc, and the map has a direct saddle and two additional fixed points. This observation leads us to the following:

**Theorem 3.2.** If $h \in \mathcal{LH}(\mathbb{R}^2)$ has a direct saddle and an $N$-cycle with $N \geq 2$, then $\mathcal{A}$ cannot be locally connected.

By an $N$-cycle we understand a periodic orbit whose minimal period is $N$. We do not know if the assumption on the existence of a direct saddle can be replaced by the existence of a fixed point.
with topological index $< 1$.

An inverse saddle for the map $h$ becomes a direct saddle for the second iterate $h^2 = h \circ h$. This more or less elementary observation leads to the following result:

**Corollary 3.1.** The conclusion of the previous theorem still holds if $h \in \mathcal{LH}(\mathbb{R}^2)$ has an inverse saddle and an $N$-cycle with $N \geq 3$.

The condition $N \geq 3$ is essential. We construct a map

$$h(x_1, x_2) = \left( f(x_1), -\frac{1}{2}x_2 \right)$$

in the class $\mathcal{LH}(\mathbb{R}^2)$ having an inverse saddle, a 2-cycle and so that the attractor is an arc. To this end we assume that $f : \mathbb{R} \to \mathbb{R}$ is a decreasing homeomorphism with $f(0) = 0$, $f(\pm 1) = \mp 1$ and such that all the non-trivial solutions of the difference equation

$$\xi_{n+1} = f(\xi_n)$$

are attracted by the two cycle $\{-1, 1\}$. Then, the attractor of $h$ is the segment connecting the two cycle $\{(-1, 0), (1, 0)\}$.

In the next result we strengthen the result when we focus on the class of orientation reversing homeomorphisms.

**Theorem 3.3.** Assume that $h \in \mathcal{LH}(\mathbb{R}^2)$ is an orientation reversing homeomorphism and there exists an $N$-cycle with $N \geq 3$. Then, $A$ cannot be locally connected.

It is natural to ask if the previous results can be extended to $\mathbb{R}^3$. The class $\mathcal{LH}(\mathbb{R}^3)$ can be defined following along the lines of Section 2, but there are several interpretations for the notion of a saddle. This will depend on the chosen conjugacy class of linear maps. As an example we take the isomorphism

$$L(x_1, x_2, x_3) = (-2x_1, -\frac{1}{2}x_2, -\frac{1}{2}x_3)$$

and consider the map

$$h(x_1, x_2, x_3) = (f(x_1), -\frac{1}{2}x_2, -\frac{1}{2}x_3),$$

where $f$ is the function introduced after Corollary 3.1. Then $h \in \mathcal{LH}(\mathbb{R}^3)$ and the attractor is the segment connecting $(1, 0, 0)$ and $(-1, 0, 0)$. This is in contrast with Theorem 3.2.

We construct a second example showing that Theorem 3.3 cannot be extended to $\mathbb{R}^3$. The second example by Levinson, discussed previously, will be denoted by $h_*$. This map belongs to $\mathcal{LH}(\mathbb{R}^2)$, it is orientation preserving and it has two 3-cycles. Since the attractor is a triode, in particular a locally connected set, Theorem 3.3 is not valid for orientation preserving maps. This example is also useful to show that Theorem 3.3 cannot be extended to $\mathbb{R}^3$. Consider $h(x_1, x_2, x_3) = (h_*(x_1, x_2), -\frac{1}{2}x_3)$. This map is an orientation reversing homeomorphism in $\mathcal{LH}(\mathbb{R}^3)$ with a 3-cycle and the same attractor as $h_*$. 

### 4 Proofs of the Theorems

#### 4.1 Proof of Theorems 3.1

First we give an useful property on arcs in $\mathbb{R}^2$. For the reader’s convenience, we recall that an arc $\gamma \subset \mathbb{R}^2$ is a set that can be expressed as the image of a continuous and one-to-one map of the form $[0, 1] \to \mathbb{R}^2$. We stress that the definition of arc excludes the presence of loops in $\gamma$. 

Lemma 4.1. Let $\alpha, \beta \subset \mathbb{R}^2$ be two arcs joining $p$ and $q$ with $p \neq q$. If $\alpha \neq \beta$, then $\alpha \cup \beta$ contains a Jordan curve.

Proof. Since $\alpha \neq \beta$, there exists a point $r \in \alpha$ that does not belong to $\beta$. This guarantees the existence of an open sub-arc of $\alpha$, say $\alpha'$, such that $r \in \alpha'$ and $\alpha' \cap \beta = \emptyset$. Let $\prec$ be the natural order on $\alpha$ that satisfies $p \prec q$. We stress that this order is meaningful because the definition of arc excludes the possibility of having loops. Take

$$p_1 = \inf \{ s \in \alpha : s \prec r \text{ and } \hat{s} \cap \beta = \emptyset \}$$

and

$$p_2 = \sup \{ t \in \alpha : r \prec t \text{ and } \hat{r} \cap \beta = \emptyset \}.$$ 

In the previous expressions, $\hat{s}$ refers to the sub-arc in $\alpha$ with ends at $s$ and $r$ (analogously for $\hat{r}$). By the existence of $\alpha'$, $r \neq p_1$ and $r \neq p_2$. Next we prove that $p_1 \in \beta$. If $p_1 = p$, then the conclusion is clear by assumptions. Now, we assume, by contradiction, that $p_1 \notin \beta$ and $p_1 \neq p$. In such a case, there exists $\tilde{\alpha}$ an open sub-arc of $\alpha$ with $p_1 \in \tilde{\alpha}$ and $\tilde{\alpha} \cap \beta = \emptyset$. This is a contradiction with the definition of $p_1$. Analogously, we can deduce that $p_2 \in \beta$. By construction, $\tilde{p}_1 \tilde{p}_2 \cup \beta'$ is a Jordan curve with $\beta'$ the sub-arc in $\beta$ joining $p_1$ and $p_2$. \hfill \Box

It is intuitively clear that there are no invariant arcs emanating from an inverse saddle with the additional property of being invariant for the future or for the past. The reason of this fact is that the orientation is reversed on the invariant manifolds. Next we present a detailed proof.

Lemma 4.2. Let $h : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ be a homeomorphism with $p$ an inverse saddle. If $\gamma \subset \mathbb{R}^2$ is an arc emanating from $p$, then $h(\gamma) \nsubseteq \gamma$ and $\gamma \nsubseteq h(\gamma)$.

Proof. We assume, by contradiction, that either $h(\gamma) \subset \gamma$ or $\gamma \subset h(\gamma)$. Then any sub-arc $\beta \subset \gamma$ with one end point at $p$ will enjoy the same property, either $h(\beta) \subset \beta$ or $\beta \subset h(\beta)$. Note that the inclusions for $\gamma$ and $\beta$ are not necessarily the same. For instance, $h(\gamma) \subset \gamma$ and $\beta \subset h(\beta)$ is an admissible situation. The reduced disk will be defined by

$$\hat{D} = \Psi(\left[-\frac{1}{2}, \frac{1}{2}\right] \times [-1, 1]).$$

Let us take a sub-arc of $\gamma$ ending at $p$, say $\gamma_s = \hat{p}\tau$, such that $\gamma_s \subset \hat{D}$. Then, either $h(\gamma_s) \subset \gamma_s$ or $h(\gamma_s) \subset \gamma_s$, depending on the way $r$ and $h(r)$ are ordered in $\gamma$. We observe that $L_i = \Psi^{-1} \circ h \circ \Psi$ on $\left[-\frac{1}{2}, \frac{1}{2}\right] \times [-1, 1]$ and define $\Gamma = \Psi^{-1}(\gamma_s)$. Since $\Gamma$ is contained in this reduced rectangle,

$$L_i(\Gamma) = \Psi^{-1} \circ h \circ \Psi(\Gamma) = \Psi^{-1}h(\gamma_s).$$

Then either $L_i(\Gamma) \subset \Gamma$ or $L_i(\Gamma) \subset \Gamma$. Let us consider the first case. The iterates $L_i^n(\Gamma) \subset \Gamma$ remain in a bounded set for $n \geq 0$. From this, we deduce that $\Gamma$ is contained in the stable manifold of the linear map $L_i$. The set $\Gamma \backslash \{0\}$ is connected and lies in $(\mathbb{R} \backslash \{0\}) \times \{0\}$. Since the components of this set are permuted by $L_i$, the open arc $\Gamma \backslash \{0\}$ must satisfy

$$L_i(\Gamma \backslash \{0\}) \cap (\Gamma \backslash \{0\}) = \emptyset.$$ 

This is the searched contradiction. \hfill \Box

Proof of Theorem 3.1. We assume, by contradiction, that $A$ is an arc-wise connected set. Then, there is an arc $\gamma \subset A$ that joins $p$ and $q$. Using that $p$ and $q$ are fixed points, $h(\gamma)$ is also an arc in $A$ joining $p$ and $q$. If $\gamma \neq h(\gamma)$, then $\gamma \cup h(\gamma)$ contains a Jordan curve by Lemma 4.1. This is a contradiction with Proposition 2.1. If $\gamma = h(\gamma)$, then we also obtain a contradiction with Lemma 4.2. \hfill \Box
5 Background on Prime Ends

Let Ω be an open and simply connected subset of $S^2$ such that $\infty \in \Omega$, $\Omega \neq S^2$ and $S^2 \setminus \Omega$ is not a singleton. It is well known that $\Omega$ is always homeomorphic to the open unit disk $Int(D)$, where

$$D = \{ z \in \mathbb{C} : |z| \leq 1 \}.$$  

However, the boundary of $\Omega$ in $S^2$, denoted by $\partial S^2 \Omega$, is not necessarily homeomorphic to $\partial D = S^1$. Carathéodory’s theory of prime ends allows to construct an abstract topological space $\Omega^*$ containing $\Omega$ and such that the pairs $(\Omega^*, \Omega)$ and $(D, Int(D))$ are homeomorphic.

The space of prime ends is defined as

$$\mathbb{P} = \Omega^* \setminus \Omega.$$  

The space $\Omega^*$ is not inside $S^2$ but somehow describes the way in which $\Omega$ is embedded in $S^2$. The following property reflects this fact.

Let $g : S^2 \rightarrow S^2$ be a homeomorphism with $g(\infty) = \infty$ and such that $\Omega$ is invariant under $g$, $g(\Omega) = \Omega$. Then there exists another homeomorphism $g^* : \Omega^* \rightarrow \Omega^*$ such that $g = g^*$ on $\Omega$. After conjugation, the restriction of $g^*$ to the space of prime ends, $g^* : \mathbb{P} \rightarrow \mathbb{P}$, can be seen as a homeomorphism of $S^1$.

An end-path is a continuous map $\gamma : [0, 1] \rightarrow S^2$ such that $\gamma(t) \in \Omega$ if $t \in [0, 1]$ and $\gamma(1) \notin \Omega$. End-paths can also be interpreted as paths in $\Omega^*$. More precisely, it is possible to construct a continuous map $\gamma^* : [0, 1] \rightarrow \Omega^*$ such that $\gamma^* = \gamma$ on $[0, 1]$ and $\gamma^*(1) \in \mathbb{P}$ (see Theorem 16 in [12]).

The prime end $\mathcal{P} = \gamma^*(1)$ is called accessible and $\xi = \gamma(1) \in \partial S^2 \Omega$ is the principal point of $\mathcal{P}$. It can be proved that every accessible end has a unique principal point, (Theorem 17.1 in [12]).

Given two end-paths $\gamma_0, \gamma_1 : [0, 1] \rightarrow S^2$, we say that they are homotopic if there is a continuous mapping $\Gamma : [0, 1] \times [0, 1] \rightarrow S^2$ such that $\Gamma(t, 0) = \gamma_0(t)$, $\Gamma(t, 1) = \gamma_1(t)$, $\Gamma(1, s) = \gamma_0(1) = \gamma_1(1)$, $\Gamma(t, s) \in \Omega$ if $t < 1$. This notion is relevant because two end-paths are homotopic if and only if the corresponding accessible prime ends coincide, $\gamma^*_0(1) = \gamma^*_1(1)$ (Theorem 18 in [12]).

For a general domain $\Omega$ not all prime ends are accessible. However, if the boundary of $\Omega$ is locally connected then all prime ends are accessible (Theorem 20 in [12]). From now on we assume that $\partial S^2 \Omega$ is locally connected. Under this assumption the previous discussions are sufficient to construct the space $\mathbb{P}$ and the map $g^*$ in concrete cases. Prime ends with the same principal point $\xi \in \partial S^2 \Omega$ correspond to different ways of approaching $\xi$ from $\Omega$. Given $\mathcal{P} \in \mathbb{P}$, we can have an end-path $\gamma$ such that $\gamma(1) = \xi$ for a suitable point $\xi \in \partial S^2 \Omega$. To construct the topology of $\Omega^*$ one is defined to have a sub-basis of neighborhoods for each $\mathcal{P} \in \mathbb{P}$. Given an end-path $\gamma$ defining $\mathcal{P}$, we take an open ball $B$ centered at the principal point $\xi = \gamma(1)$. Let $B_\gamma$ be the connected component of $B \cap \Omega$ such that $\gamma(t) \in B_\gamma$ if $t < 1$ and $1 - t$ sufficiently small. We define the neighborhood

$$U_B = B_\gamma \cup \{ \lambda^*(1) : \lambda \in \Lambda \}$$  

where $\Lambda$ is the family of end-paths $\lambda : [0, 1] \rightarrow S^2$ satisfying that $\lambda(t) \in B_\gamma$ if $t \in [0, 1]$. After this definition the following result is easily proved.

**Lemma 5.1.** In the previous notations assume that the boundary $\partial S^2 \Omega$ is locally connected and consider the map

$$\Pi : \mathbb{P} \rightarrow \partial S^2 \Omega, \quad \mathcal{P} \mapsto \xi,$$

assigning the corresponding principal point $\xi$ to each prime end $\mathcal{P}$. Then, $\Pi$ is continuous.

Finally, we discuss how to induce homeomorphisms on $\mathbb{P}$. Let $h : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ be a homeomorphism such that $\Omega$ is invariant. Then also $\partial S^2 \Omega$ is invariant. Given an end-path $\gamma$ defining $\mathcal{P}$, the
composition \( h \circ \gamma \) is another end-path inducing a prime end. The homeomorphism \( h^* : \mathbb{P} \rightarrow \mathbb{P} \) satisfies

\[ h^*(\mathcal{P}) = (h \circ \gamma)^*(1). \]

The useful property \((h^*)^n = (h^n)^*\) holds for each \( n \in \mathbb{Z} \).

As an example we consider the domain \( \Omega = \mathbb{S}^2 \backslash A \) where \( A = ([-1,1] \times \{0\}) \cup (\{0\} \times [-1,1]) \).

In this set we distinguish the four end points \( a, b, c, d \) and the origin 0. For each point \( \xi \in A \backslash \{a, b, c, d, 0\} \) there are two prime ends whose principal point is \( \xi \). Four prime ends have the origin as principal point. Finally, there is only one prime end for \( a, b, c, d \). The Figure B illustrates this description.

Consider now the symmetry \( h : \mathbb{S}^2 \rightarrow \mathbb{S}^2, h(z) = \bar{z} \). Then, \( h(\Omega) = \Omega \) and \( h^* : \mathbb{P} \rightarrow \mathbb{P} \) satisfies \( \text{Fix}(h^*) = \{a, c\} \), \( \text{Fix}((h^2)^*) = \mathbb{P} \). In particular, \( h(b) = d, h(O_1) = O_4 \).

6 Proofs of Theorems 3.2 and 3.3

Let us assume that \( h \in \mathcal{LH}(\mathbb{R}^2) \) is such that the attractor is not reduced to a singleton. The next result shows that \( \Omega = \mathbb{S}^2 \backslash A \) is a simply connected domain in the conditions of Section 5.

**Lemma 6.1.** Given \( h \in \mathcal{LH}(\mathbb{R}^2) \), the set \( \Omega = \mathbb{S}^2 \backslash A \) is a simply connected domain with \( \partial_{\mathbb{S}^2} \Omega = A \).

**Proof.** Let \( \{\Omega_i\}_{i \in I} \) be the family of connected components of \( \Omega \). Since \( h \) is a homeomorphism and \( \Omega \) is invariant, components must be mapped onto components. This means that \( h(\Omega_i) = \Omega_{\sigma(i)} \), where \( \sigma \) is a permutation of \( I \). Let \( \Omega_\infty \) denote the component containing \( \infty \). From \( h(\infty) = \infty \), we deduce that \( h(\Omega_\infty) = \Omega_\infty \). In consequence \( K = \mathbb{S}^2 \backslash \Omega_\infty \) is a compact and invariant subset of \( \mathbb{R}^2 \).

The definition of attractor implies that \( K \) is contained in \( A \). Therefore, \( \Omega = \mathbb{S}^2 \backslash A \) is contained in \( \mathbb{S}^2 \backslash K = \Omega_\infty \). Once we know that \( \Omega = \Omega_\infty \) is connected, it is easy to prove that it is also simply connected. Indeed, it is sufficient to observe that the complement \( \mathbb{S}^2 \backslash \Omega = A \) is connected.
Finally, we observe that $\mathcal{A} \subset cl_Ω(\Omega)$ because $\mathcal{A}$ has empty interior. This allows us to conclude that $\partial_Ω \Omega = \mathcal{A}$.

\[\Box\]

Remark 6.1. Proposition 2.1 can be obtained as a corollary of the previous result.

In the above conditions let $\mathcal{P}$ denote the space of prime ends associated to $\Omega = S^2 \setminus \mathcal{A}$. Since $\mathcal{A}$ is invariant under $h$, the same is true for $\Omega$ and we can construct the map $h^* : \Omega^* \rightarrow \Omega^*$. The point of infinity is fixed under $h^*$.

**Proof of Theorem 3.3.** We proceed by contradiction, assuming that $\mathcal{A}$ is locally connected. The map $h$ is orientation reversing and so the map $h^* : \Omega^* \rightarrow \Omega^*$ is also orientation reversing. Here we are using that $h$ and $h^*$ coincide on the open set $\Omega$. Since $\mathcal{P}$ is in the boundary of the manifold $\Omega^*$ we conclude that the restriction $h^* : \mathcal{P} \rightarrow \mathcal{P}$ is orientation-reversing (see for instance, page 304 of [20]). Then $h^*$ has exactly two fixed points in $\mathcal{P}$ and every $\omega$-limit set $L_\omega(\mathcal{P}, h^*)$ is either one of these fixed points or a 2 cycle. Let us take a point $q \in \mathbb{R}^2$ such that the sequence $\{h^n(q)\}_{n \in \mathbb{Z}}$ is an $N$-cycle with $N \geq 3$. Then $q \in \mathcal{A} = \partial_\infty \Omega$ and we can find a prime end $\mathcal{P}$ with $\Pi(\mathcal{P}) = q$. The sequence $\mathcal{P}_n = (h^*)^{2n}(\mathcal{P})$ must be convergent. In consequence also $\Pi(\mathcal{P}_n)$ is convergent. But we know that

$$\Pi(\mathcal{P}_n) \in \mathcal{O} = \{g, h(g), \ldots, h^{N-1}(g)\}.$$ 

Therefore, $\Pi(\mathcal{P}_n)$ is eventually constant, say

$$\Pi(\mathcal{P}_n) = h^k(q)$$

if $n \geq n_0$, where $k \in \{0, 1, \ldots, N - 1\}$ is fixed. We conclude that

$$2n \equiv k \mod N$$

for each $n \geq n_0$. This is impossible if $N \geq 3$.

Next we prove that the presence of fixed points of $h^*$ is guaranteed when $h$ has a direct saddle and the attractor is locally connected. To understand in depth this result, it is convenient to keep in mind the horseshoe map as described in Section 3.5 of [2]. This map is in Levinson’s class but the attractor is not locally connected. Moreover there exist many hyperbolic periodic points that are not accessible (from outside the attractor) and that do not produce periodic points of $h^*$.

**Proposition 6.1.** Assume that $h \in \mathcal{LH}(\mathbb{R}^2)$ has a direct saddle at the point $p$. If $\mathcal{A}$ is locally connected, then every prime end $\mathcal{P}$ whose principal point is $p$ satisfies $h^*(\mathcal{P}) = \mathcal{P}$. In particular, $\text{Fix}(h^*) \cap \mathcal{P} \neq \emptyset$.

This result is crucial for the proofs of Theorem 3.2. We will prove it at the end of the Section. First we show how to employ it.

**Proof of Theorem 3.2.** By a contradiction argument we assume that $\mathcal{A}$ is locally connected. The map $h$ is orientation preserving because it has a direct saddle. The same reasoning of the previous proof implies now that $h^*$ is orientation preserving. From Proposition 6.1, we know that $\text{Fix}(h^*) \cap \mathcal{P} \neq \emptyset$ and therefore the rotation number of $h^*$ is zero. Well-known facts on the dynamics of $S^1$ imply that every orbit in $\mathcal{P}$, $P_n = (h^*)^n(\mathcal{P})$, is convergent as $n \rightarrow \infty$.

Assume that $q \in \mathbb{R}^2$ is such that $\{h^n(q)\}_{n \in \mathbb{Z}}$ is an $N$-cycle. This periodic orbit is contained in $\mathcal{A} = \partial_\infty \Omega$ and we can find an end-path $\gamma(t)$ with $\gamma(1) = q$. Let $\mathcal{P}$ be a prime end so that $\Pi(\mathcal{P}) = q$. Each $P_n = (h^*)^n(\mathcal{P})$ is defined by the end-path $h^n \circ \gamma(t)$ and has principal point $\Pi(P_n) = h^n(q)$. We have found a contradiction with the continuity of the map $\Pi$ defined in Lemma 5.1. The sequence $\{P_n\}_{n \geq 0}$ is convergent but $\{\Pi(P_n)\}_{n \geq 0}$ oscillates periodically. 

$\Box$
Figure C: We insert the disk $B$ in the topological disk $D$.

Let $\alpha = \hat{qr}$ be an arc in $\mathbb{R}^2$ with $p \in \hat{\alpha} = \hat{\alpha}\setminus\{q,r\}$ and assume that $\gamma, \tilde{\gamma} : [0,1] \rightarrow \mathbb{R}^2$ are two continuous paths with $\gamma(1) = \tilde{\gamma}(1) = p$ and such that there exists $\tau \in ]0,1[$ with
\[ \gamma(t), \tilde{\gamma}(t) \not\in \alpha \quad if \ t \in ]\tau,1[. \] (6.1)

We will say that $\gamma$ and $\tilde{\gamma}$ reach $p$ from the same side of $\mathbb{R}^2\setminus \alpha$ if given a topological disk $D \subset \mathbb{R}^2$ with $p \in \text{Int}(D)$ and $q,r \not\in D$, there exists $\tau_\ast \in ]\tau,1[$ such that $\gamma(t)$ and $\tilde{\gamma}(t)$ lie in the same connected component of $D\setminus \alpha$ if $t \in ]\tau_\ast,1[$.

It is clear that nothing changes if we replace $\alpha$ by any sub-arc $\beta$ with $p \in \hat{\beta}$. To understand this statement rigorously it is convenient to consider a canonical situation. After conjugacy, it is not restrictive to assume that the arc $\alpha$ is the segment $[-1,1] \times \{0\}$ and the point $p$ is the origin. At this moment we are using that all arcs in two dimensions are tame (see [13]). The paths $\gamma$ and $\tilde{\gamma}$ will reach $p$ from the same side if and only if the second coordinates satisfy
\[ \gamma_2(t) \cdot \tilde{\gamma}_2(t) > 0 \quad if \ t \in ]\tau,1[. \]

Nothing changes if we replace $[-1,1] \times \{0\}$ by $[a,b] \times \{0\}$ with $-1 \leq a < 0 < b \leq 1$.

To get more insights on the previous definition we can consider an arc $\alpha = \hat{pr}$ and a topological disk $D$ with $q,r \not\in \text{Int}(D)$. Then $D\setminus \alpha$ can have many connected components but only two of them, say $G_1$ and $G_2$, will have the property $p \in \text{cl}(G_i)$. The proof of this fact follows easily from Figure C, where a small ball $B \subset D$ has been inserted. Again we are using that all arcs in the plane are tame.

Now we can say that there are two ways of reaching $p$ from $\mathbb{R}^2\setminus \alpha$, we can go through $G_1$ or $G_2$.

The next auxiliary result is very intuitive but the details of the proof are delicate.

**Proposition 6.2.** Assume that $A$ is locally connected and $p \in A$ with $A \neq \{p\}$. Consider two arcs $\gamma_1$ and $\gamma_2$ with ends at $p \in A$ and such that $\gamma_1 \cap A = \gamma_2 \cap A = \{p\}$. If $\gamma_1$ and $\gamma_2$ define different prime ends ($\gamma_1^\ast(1) \neq \gamma_2^\ast(1)$) then there exists a closed ball $B$ centered at $p$ and two arcs $\alpha_1 = \hat{p}q_1$ and $\alpha_2 = \hat{p}q_2$ satisfying that $\alpha_1 \cup \alpha_2 \subset A$, $\alpha_1 \cap \alpha_2 = \{p\}$, $q_i \in \partial B$, $\alpha_i \setminus \{q_i\} \subset \text{Int}(B)$ for $i = 1,2$. Moreover, $\gamma_1$ and $\gamma_2$ will eventually enter in different connected components of $B\setminus(\alpha_1 \cup \alpha_2)$. 
Proof. Let us fix a closed ball $B_1$ centered at $p$ and such that $\mathcal{A}$ is not contained in $B_1$. It is not restrictive to assume that $\gamma_1$ and $\gamma_2$ are contained in $B_1$. We claim that $p$ is isolated in $\gamma_1 \cap \gamma_2$.

This will be proved by a contradiction argument. If $p$ were an accumulation point of the compact set $\gamma_1 \cap \gamma_2$, the two complements $\gamma_i \setminus (\gamma_1 \cap \gamma_2)$, $i = 1, 2$ could be expressed as a countable union of disjoint open sub-arcs of $\gamma_i$ having end points in $\gamma_1 \cap \gamma_2$. More precisely,

$$\gamma_i \setminus (\gamma_1 \cap \gamma_2) = \bigcup_{j \in \mathbb{N}} \tilde{\Gamma}_{ij}$$

where $\tilde{\Gamma}_{ij}$ is a sub-arc of $\gamma_i$ and both arcs $\Gamma_{ij}$ and $\Gamma_{2j}$ have the same end points. Notice that $\gamma_1$ and $\gamma_2$ cannot coincide in a neighborhood of $p$ because they define different prime ends. The Jordan curve $\Gamma_{ij} \cup \Gamma_{2j}$ is disjoint with $\mathcal{A}$. Since $\mathcal{A}$ is connected it must be contained in one of the connected components of $\mathbb{S}^2 \setminus (\Gamma_{ij} \cup \Gamma_{2j})$, say $\mathcal{R}_i$ and $\mathcal{R}_e$, with $\infty \in R_e$. The inclusion $\mathcal{A} \subset \mathcal{R}_e$ can be discarded because $\mathcal{R}_i \subset B_1$ and $\mathcal{A} \not\subset B_1$. Therefore, $\mathcal{A} \subset \mathcal{R}_e$. In view of the Jordan-Schönflies Theorem, we know that $\mathcal{R}_i \cup \Gamma_{ij} \cup \Gamma_{2j}$ is homeomorphic to $\mathbb{D}$. Then, we can deform isotopically $\Gamma_{ij}$ onto $\Gamma_{2j}$ inside the disk $\mathcal{R}_i \cup \Gamma_{ij} \cup \Gamma_{2j}$. Reiterating this argument, we can construct a homotopy between $\gamma_1$ and $\gamma_2$. This would imply that $\gamma_1$ and $\gamma_2$ define the same prime end, a contradiction with the assumptions.

Once we know that $p$ is isolated in $\gamma_1 \cap \gamma_2$, it is not restrictive to assume that $\gamma_1 \cap \gamma_2 = \{p\}$. Let us select a new closed ball $B_2 \subset B_1$, centered at $p$ and such that both arcs $\gamma_1$ and $\gamma_2$ get out of $B_2$. We can select two sub-arcs $\tilde{\gamma}_i = \tilde{r}_i \tilde{p}$ ending at $p$ and emanating from $\partial B_2$. More precisely,

$$\tilde{\gamma}_i \setminus \{r_i\} \subset \text{Int}(B_2), \quad i = 1, 2 \quad \text{and} \quad \tilde{\gamma}_1 \cap \tilde{\gamma}_2 = \{p\}.$$ 

Notice that $r_1 \neq r_2$ and the prime end defined by $\gamma_1$ and $\tilde{\gamma}_i$ is the same. Since $\mathcal{A}$ is locally connected, we can find a smaller ball $B_3 \subset B_2$ such that any two points in $B_3 \cap \mathcal{A}$ can be connected by an arc lying in $B_2 \cap \mathcal{A}$. The connected components of $B_2 \setminus (\tilde{\gamma}_1 \cup \tilde{\gamma}_2)$ will be denoted by $G_1$ and $G_2$. We claim that

$$G_i \cap \mathcal{A} \cap B_3 \neq \emptyset, \quad i = 1, 2. \quad (6.2)$$

To prove this we first observe that the pairs $(B_2, \tilde{\gamma}_1 \cup \tilde{\gamma}_2)$ and $(\mathbb{D}, [-1, 1] \times \{0\})$ are homeomorphic, see page 174 of [15]. Then, we proceed by contradiction and assume that $G_i \cap \mathcal{A} \cap B_3 = \emptyset$. A small sub-arc of $\tilde{\gamma}_i$ ending at $p$ could be deformed on a small sub-arc of $\tilde{\gamma}_2$ without touching $\mathcal{A} \setminus \{p\}$. We would conclude that $\tilde{\gamma}_1$ and $\tilde{\gamma}_2$ define the same prime end.

Once we know that $(6.2)$ holds, we pick up points $\eta_i \in G_i \cap \mathcal{A} \cap B_3$. We find arcs $\lambda_1$ and $\lambda_2$ joining $\eta_i$ to $p$ inside $\mathcal{A} \cap B_2$. Finally we take a ball $B_4 \subset B_3$ such that $\eta_1, \eta_2 \not\subset B_4$. Then we can take sub-arcs $\alpha_1$ and $\alpha_2$ of $\lambda_1$ and $\lambda_2$, connecting $\partial B_4$ to $p$. These are the searched arcs but this is not obvious.

The Figure D illustrates the general situation.

Let $G_i^*, \quad i = 1, 2$, be the two connected components of $B_4 \setminus (\tilde{\gamma}_1 \cup \tilde{\gamma}_2)$ so that $p \in cl(G_i^*)$. They are labelled in agreement with $G_1$ and $G_2$, meaning that $p \in cl(G_i \cap G_i^*)$, $i = 1, 2$. Since the connected set $\alpha_i \setminus \{p\}$ is disjoint with $\gamma_1 \cup \gamma_2$, it must be contained in one of the components of $B_4 \setminus (\tilde{\gamma}_1 \cup \tilde{\gamma}_2)$, say $\alpha_i \setminus \{p\} \subset G_i^*$ for some $j \in \{1, 2\}$. In particular, $\alpha_i \setminus \{p\} \subset G_i \cap G_i^*$ and therefore $p \in cl(G_i \cap G_i^*)$. Our previous convention implies that $i = j$ and $\alpha_i \setminus \{p\} \subset G_i^*$. We are ready to prove that $\tilde{\gamma}_1$ and $\tilde{\gamma}_2$ reach $p$ from different sides of $\mathbb{R}^2 \setminus (\alpha_1 \cup \alpha_2)$. After a conjugacy, we can assume that $p$ is the origin and $\alpha_1 \cup \alpha_2 = [-r_4, r_4] \times \{0\}$, where $r_4$ is the radius of $B_4$. By a contradiction argument, assume that $\tilde{\gamma}_1$ and $\tilde{\gamma}_2$ reach $p$ from the same side of $\mathbb{R}^2 \setminus (\alpha_1 \cup \alpha_2)$. Then, $\tilde{\gamma}_1$ and $\tilde{\gamma}_2$ should lie in the upper half plane $\{x_2 > 0\}$ or in the lower half plane $\{x_2 < 0\}$. To fix ideas let us assume that $\tilde{\gamma}_1, \tilde{\gamma}_2 \subset \{x_2 > 0\}$. Then $B_4 \cap \{x_2 \leq 0\}$ is a connected subset of $B_4 \setminus (\tilde{\gamma}_1 \cup \tilde{\gamma}_2)$ and this is against the property $\alpha_i \setminus \{p\} \subset G_i^*$, which was obtained previously. 

\qed
In the next discussions we shall use that \( p \) is a direct saddle. We define two sets \( U \) and \( S \) playing the role of local unstable and stable manifolds. Define \( U = \Psi([-1,1] \times \{0\}) \), (see (3.1) for the definition of \( \Psi \)). As a consequence of the definition of direct saddle we deduce that \( U \subset h(U) \). Then \( h^{-1} \) is an orientation preserving embedding mapping \( U \) into itself. Since \( U \) is homeomorphic to \([-1,1]\) and \( U \cap \text{Fix}(h) = \{p\} \), we deduce that for each \( x \in U \),

\[
h^{-n}(x) \to p
\]

as \( n \to +\infty \). In other words,

\[
U \subset W^u(p),
\]

\( W^u(p) \) is the unstable manifold of the fixed point \( p \). Similarly, if we define \( S = \Psi(\{0\} \times [-1,1]) \) then \( h(S) \subset S \) and

\[
S \subset W^s(p),
\]

where \( W^s(p) \) is the stable manifold of the fixed point \( p \). Unstable manifolds are always contained in the attractor and therefore

\[
U \subset A.
\]

In contrast, the set \( S \) is not necessarily contained in \( A \). Sometimes we will distinguish the two sides of these sets,

\[
U_+ = \Psi([0,1] \times \{0\}), \quad U_- = \Psi([-1,0] \times \{0\})
\]

and similarly for \( S_+ \) and \( S_- \). The set \( D \setminus (U \cup S) \) has four components, denoted by \( \{C_i\}_{1 \leq i \leq 4} \). The reduced disk was previously defined, see (4.1). We observe that \( \Psi \circ L_d \circ \Psi^{-1} = h \) on \( \hat{D} \).

Next we will analyze the dynamics of the arcs ending at \( p \). To obtain a precise statement we employ the Hausdorff distance in the space of compact subsets of \( S^2 \). Given compact sets \( A, B \subset S^2 \), we define

\[
h(A, B) = \max\{\rho(A,B), \rho(B,A)\}
\]

where \( \rho(A, B) = \max_{x \in A} d(x, B) \) and \( d(x, B) \) is the distance in \( S^2 \). Given a sequence \( \{A_n\} \to A \) and a continuous function \( f : S^2 \to S^2 \), we have that \( \{f(A_n)\} \to f(A) \).
Lemma 6.2. Assume that $\gamma = \hat{\gamma}q$ is an arc contained in $\hat{D}$ satisfying

$$(\gamma \setminus \{p\}) \cap U = \emptyset \quad \text{and} \quad \gamma \not\subset S.$$ 

Let $\{\gamma_n\}_{n \geq 0}$ be the sequence of arcs defined recursively in the following way, $\gamma_{n+1}$ is the connected component of $h(\gamma_n) \cap \hat{D}$ with $p \in \gamma_{n+1}$. Then $\{\gamma_n\}$ is convergent and the limit is one of the three compact sets $U \cap \hat{D}, U_+ \cap \hat{D}$ and $U_- \cap \hat{D}$.

**Proof.** Assume first that $\gamma_0$ is contained in the closure of some $C_i$, say $C_1$. We will prove that $\gamma_n \to U_+ \cap \hat{D}$. To this end we define $\Gamma_n = \Psi^{-1}(\gamma_n)$ and observe that $\Gamma_{n+1} \subset L_d(\Gamma_n)$. Let $\theta > 0$ be an upper bound for the second coordinate on $\Gamma_0$, so that $\Gamma_0 \subset [0, \frac{1}{2}] \times [0, \theta]$. Then $\Gamma_n \subset [0, \frac{1}{2}] \times [0, \frac{\theta}{2\pi}]$ and therefore

$$\rho\left(\Gamma_n, \left[0, \frac{1}{2}\right] \times \{0\}\right) \to 0.$$ 

This is equivalent to $\rho(\gamma_n, U_+ \cap \hat{D}) \to 0$.

Let us fix a point $(x_1, x_2) \in \Gamma_0$ with $x_2 > 0$. For some $n_0 \geq 1$ the point $L_d^n(x_1, x_2)$ is outside $\hat{D}$. Then, $\text{proj}_1(\Gamma_n) = [0, \frac{1}{2}]$ if $n \geq n_0$. Here $\text{proj}_1$ is the projection onto the $x_1$-axis. The Euclidean distance from any point $(x_1, 0)$ to $\Gamma_n$ is less or equal than $\frac{1}{2\pi} \theta$ if $n \geq n_0$. This implies that $\rho([0, \frac{1}{2}] \times \{0\}, \Gamma_n) \to 0$ in $\mathbb{S}^2$, or equivalently,

$$\rho(U_+ \cap \hat{D}, \gamma_n) \to 0.$$ 

Assume now that $\gamma_0$ contains points lying in $\text{Int}(C_1)$ and $\text{Int}(C_2)$. Then we can find $(x_1, x_2), (y_1, y_2) \in \Gamma_0$ with $x_1 < 0 < y_1, x_2 > 0, y_2 > 0$. We prove in a similar way that $\gamma_n \to U \cap \hat{D}$. □

Consider the family of four invariant branches emanating from $p$,

$$\mathcal{F} = \{U^+, U^-, S^+, S^-\}.$$ 

Now we take $\alpha$ an arc obtained by gluing two members of $\mathcal{F}$; that is, $\alpha = \alpha_1 \cup \alpha_2$ with $\alpha_1, \alpha_2 \in \mathcal{F}$. The arc $\alpha$ inherits a property of local invariance under $h^{-1}$. Namely, there exists $\varepsilon > 0$ such that if $x \in \mathbb{S}^2$, $d(x, p) < \varepsilon$ and $h(x) \in \alpha$, then $x \in \alpha$.

Let $\gamma$ be an end-path satisfying

$$\gamma(t) \not\subset \alpha \quad \text{if} \ t \in [0, 1], \quad \gamma(1) = p. \quad (6.3)$$

Then, the couple $\gamma$ and $\tilde{\gamma} = h \circ \gamma$ will satisfy the condition (6.1).

**Lemma 6.3.** Assume that $\alpha$ is in the above conditions and the path $\gamma$ satisfies (6.3). Then, $\gamma$ and $h \circ \gamma$ will reach $p$ from the same side of $\mathbb{R}^2 \setminus \alpha$.

**Proof.** It is not restrictive to assume that both paths $\gamma$ and $\tilde{\gamma} = h \circ \gamma$ are contained in $\hat{D}$. We can transport the configuration to a linear setting with

$$\Gamma = \Psi^{-1}(\gamma), \quad \tilde{\Gamma} = \Psi^{-1}(\tilde{\gamma}) = L_d(\Gamma).$$ 

It is clear that $\Gamma \setminus \{0\}$ and $L_d(\Gamma \setminus \{0\})$ lie in the same components of $[-1, 1]^2 \setminus \Psi^{-1}(\alpha)$. □

**Proof of Proposition 6.1.** Assume by contradiction that there exists a prime end $P \in \mathcal{P}$
with $\Pi(P) = p$ and such that $h^*(P) \neq P$. The point $p$ is fixed under $h$ and so it belongs to the attractor, $p \in A$. Then, we can find an end-path $\gamma$ defining $P$. It is not restrictive to assume that $\gamma$ is one-to-one (see Theorem 17.1 in [12]). From now on the path $\gamma$ and the arc $\gamma([0,1])$ will be denoted by $\gamma$. We are in the conditions of Proposition 6.2 with $\gamma_1 = \gamma$ and $\gamma_2 = h \circ \gamma$. Then, we can find a small ball $B \subset \hat{D}$ and arcs $\alpha_1$, $\alpha_2$ in the conditions of this Proposition. We are going to prove that these arcs are contained (at least locally) in $S \cup U$. The discussion is organized in two claims:

**Claim 1.** If $(\alpha_i \setminus \{p\}) \cap U \neq \emptyset$ then there is a sub-arc $\widehat{\alpha}_i$ ending at $p$ and such that $\widehat{\alpha}_i \subset U$.

Given a point $q \in (\alpha_i \setminus \{p\}) \cap U$, we can join $p$ and $q$ by sub-arcs of $\alpha_i$ and $U$. Since $A$ does not contain Jordan curves, it follows from Lemma 4.1 that these sub-arcs must coincide, $\widehat{\alpha}_i = \widehat{pq} \subset \alpha_i \cap U$.

**Claim 2.** If $(\alpha_i \setminus \{p\}) \cap U = \emptyset$ then $\alpha_i \subset S$.

Assume by contradiction that $\alpha_i \not\subset S$ and let $\{\gamma_n\}$ be the sequence of arcs constructed in Lemma 6.2 for $\gamma = \alpha_i$. Then $\gamma_n$ converges to one of the three sets specified in the Lemma, say, for instance, $\gamma_n \rightarrow U_+ \cap \hat{D}$. Let us fix a point $\xi \in U_+ \cap \hat{D}$ with $d(\xi, p) > \varepsilon$ for some $\varepsilon > 0$. Since $A$ is locally connected, there exists $\delta > 0$ such that every point $\eta \in A$ with $d(\xi, \eta) < \delta$ can be connected to $\xi$ by an arc $\beta$ lying inside $A \cap B(\xi, \frac{\varepsilon}{2})$. Since $\gamma_n$ converges to $U_+ \cap \hat{D}$ we can find $n$ large enough and a point $\xi \in \gamma_n$ with $d(\xi, \eta) < \delta$. The points $\xi$ and $\eta$ can be connected by the arc $\beta$ and also by a sub-arc of $U_+ \cup \gamma_n$. Since these arcs must be different we deduce from Lemma 4.1 that the attractor contains a Jordan curve. The second claim is proved.

We know from Proposition 6.2 that the paths $\gamma$ and $h \circ \gamma$ reach point from different sides of $\mathbb{R}^2 \setminus \alpha$. The previous claims imply that $\alpha = \alpha_1 \cup \alpha_2$ is a sub-arc of two members of the family $\mathcal{F}$. Then, Lemma 6.3 is applicable and $\gamma$ and $h \circ \gamma$ should reach $p$ from the same side. This is the searched contradiction.

\[ \Box \]

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