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Non-resonant boundary value problems with singular ϕ-Laplacian operators

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Abstract. In this paper, using Leray–Schauder degree arguments, critical point theory for lower semicontinuous functionals and the method of lower and upper solutions, we give existence results for periodic problems involving the relativistic operator $u \mapsto \left(\frac{u'}{\sqrt{1-u'^2}}\right)' + r(t)u$ with $\int_0^T r\,dt \neq 0$. In particular we show that in this case we have non-resonance, that is periodic problem

$$\left(\frac{u'}{\sqrt{1-u'^2}}\right)' + r(t)u = e(t), \quad u(0) - u(T) = 0 = u'(0) - u'(T),$$

has at least one solution for any continuous function $e : [0,T] \to \mathbb{R}$. Then, we consider Brillouin and Mathieu-Duffing type equations for which $r(t) \equiv b_1 + b_2 \cos t$ and $b_1, b_2 \in \mathbb{R}$.

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1. Introduction

Consider a particle which moves on a straight line, subject to a restoring force F. Physically, we are assuming the following basic principle: *the mass varies with the speed of the particle according to the familiar formula given by the theory of special relativity.* Accordingly (see e.g. [8, 9, 11]), we have the differential equation of motion

$$\left(\frac{m_0 u'}{\sqrt{1-\frac{u'^2}{c^2}}}\right)' = F,$$

where m_0 is the rest-mass of the particle and c is the speed of light in the vacuum. In the sequel we normalize, assuming that $m_0 = c = 1$.

In this paper we consider restoring forces of the form

$$F(t, u, u') = -r(t)u + f(t, u, u')$$

with r being a continuous function having a non zero mean value on $[0, T]$.

In the first main result (Theorem 1) we prove, using Leray–Schauder degree, that the periodic boundary value problem

$$\left(\frac{u'}{\sqrt{1-u'^2}}\right)' + r(t)u = f(t, u, u'), \quad u(0) - u(T) = 0 = u'(0) - u'(T),$$

has at least one solution when $r : [0, T] \to \mathbb{R}$ and $f : [0, T] \times \mathbb{R}^2 \to \mathbb{R}$ are continuous, $\int_0^T r \neq 0$ and f is bounded on $[0, T] \times \mathbb{R} \times (-1, 1)$. In the special case r constant, this result has been proved in [4].

Then, combining Theorem 1 with a cutting method, we show (Example *3*) that the Brillouin beam-focusing equation with relativistic effects

$$\left(\frac{u'}{\sqrt{1-u'^2}}\right)' + (b_1 + b_2 \cos t)u = \frac{1}{u^\nu}, \quad u(0) - u(2\pi) = 0 = u'(0) - u'(2\pi),$$

has at least one solution if $0 < \nu < 1$ (the weak singularity case), $0 < b_1 < \frac{1}{\pi^{1+\nu}}$ and $b_2 \in [-b_1, b_1]$. The case $\nu \geq 1$ is discussed in [1] using the upper and lower solution method. The corresponding result in the Newtonian case is proved in [14] using a totaly different approach based upon Krasnoselskii fixed point theorem on compression and expansion in cones.

On the other hand, using Szulkin's critical point theory for lower semi-continuous functionals, we show that the following Mathieu-Duffing equation with relativistic effects

$$\left(\frac{u'}{\sqrt{1-u'^2}}\right)' + (b_1 + b_2 \cos t)u + cu^3 = 0, \quad u(0) - u(2\pi) = 0 = u'(0) - u'(2\pi),$$

has at least one nontrivial solution provided that $c < 0 < b_1, b_2 \in \mathbb{R}$ or $b_1 < b_2 < 0 < c$. In the first situation we use a minimization procedure and in the second one the Mountain Pass Theorem. The classical case is discussed in [14].

The paper is organized as follows. In Sect. 2 we introduce some notation and auxiliary results, in Sect. 3 we deal with bounded and sub-superlinear perturbations and in the last section we prove existence of nontrivial periodic solutions of Mathieu-Duffing type periodic problems.

2. Some notations and auxiliary results

Let C be the Banach space of continuous functions on $[0, T]$ endowed with the uniform norm $||\cdot||_\infty$, C^1 denote the Banach space of continuously differentiable functions on $[0, T]$ equipped with the norm

$$||u|| = ||u||_\infty + ||u'||_\infty \quad (u \in C^1).$$

The corresponding open ball (in C or $C^!$) with center in zero and radius ρ is denoted by B_ρ. Let $P, Q : C \to C$ be the continuous projectors defined by

$$Pu(t) = u(0), \quad \overline{u} = Qu(t) = \frac{1}{T}\int_0^T u(\tau)d\tau \quad (t \in [0,T]),$$

and define the continuous linear operator $H : C \to C^1$ by

$$Hu(t) = \int_0^t u(\tau)\, d\tau \quad (t \in [0,T]).$$

If $u \in C$ we write

$$\widetilde{u} = u - \overline{u}, \quad u_L = \min_{[0,T]} u, \quad u_M = \max_{[0,T]} u,$$

and we shall consider the following closed subspace of C^1 :

$$C^1_\sharp = \{u \in C^1 : u(0) - u(T) = 0 = u'(0) - u'(T)\}.$$

Notice that

$$u_M - u_L \leq \frac{T}{2}||u'||_\infty \quad \text{for all } u \in C^1_\sharp \tag{1}$$

holds true (see [1]).

The following assumption upon ϕ (called *singular*) is made throughout the paper:
$(H_\phi)\phi : (-a, a) \to \mathbb{R}$ *is an increasing homeomorphism such that* $\phi(0) = 0$ *and* $0 < a < \infty$.

The model example is

$$\phi(s) = \frac{s}{\sqrt{1-s^2}} \quad (s \in (-1,1)).$$

We recall the following technical result given as Lemma 1 from [4].

Lemma 1. *For each* $h \in C$ *there exists a unique* $Q_\phi(h) \in \mathbb{R}$ *such that*

$$Q \circ \phi^{-1} \circ (I - Q_\phi) \circ h = 0.$$

Moreover, the function $Q_\phi : C \to \mathbb{R}$ *is continuous.*

We recall also the following fixed point result introduced in [4].

Lemma 2. *Let* $F : C^1 \to C$ *be a continuous operator which takes bounded sets into bounded sets and consider the abstract periodic problem*

$$(\phi(u'))' = F(u), \quad u(0) - u(T) = 0 = u'(0) - u'(T). \tag{2}$$

A function u *is solution of* (2) *if and only if* $u \in C^1_\sharp$ *is a fixed point of the completely continuous operator* $M_F : C^1_\sharp \to C^1_\sharp$ *defined by*

$$M_F = P + QF + H \circ \phi^{-1} \circ (I - Q_\phi) \circ [H(I - Q)F].$$

Furthermore, $||(M_F(u))'||_\infty < a$ *for all* $u \in C^1_\sharp$ *and*

$$||\widetilde{u}||_\infty < aT, \tag{3}$$

for any solution u *of* (2).

To each continuous function $f : [0,T] \times \mathbb{R}^2 \to \mathbb{R}$, we associated its Nemytskii operator $N_f : C^1 \to C$ given by

$$N_f(u) = f(\cdot, u(\cdot), u'(\cdot)) \quad (u \in C^1).$$

One has that N_f is continuous and takes bounded sets into bounded sets.

Next, consider the periodic boundary value problem

$$(\phi(u'))' = N_f(u), \quad u(0) - u(T) = 0 = u'(0) - u'(T). \tag{4}$$

If $u, v \in C$ are such that $u(t) \leq v(t)$ for all $t \in [0,T]$, we write $u \leq v$. One has the following (see Definition 1 [4])

Definition 1. A *lower solution* α (resp. *upper solution* β) of (4) is a function $\alpha \in C^1$ such that $||\alpha'||_\infty < a,\ \phi(\alpha') \in C^1,\ \alpha(0) = \alpha(T),\ \alpha'(0) \geq \alpha'(T)$ (resp. $\beta \in C^1,\ ||\beta'||_\infty < a,\ \phi(\beta') \in C^1,\ \beta(0) = \beta(T),\ \beta'(0) \leq \beta'(T)$) and

$$(\phi(\alpha'))' \geq N_f(\alpha) \quad (\textit{resp. } (\phi(\beta'))' \geq N_f(\beta)). \tag{5}$$

We need also the following result (see Theorem 3 from [4] and Theorem 1 from [1]).

Lemma 3. *Assume that* (4) *has a lower solution α and an upper solution β. If $\alpha \leq \beta$, then* (4) *has a solution u such that $\alpha \leq u \leq \beta$. If there exists $\tau \in [0,T]$ such that $\alpha(\tau) > \beta(\tau)$, then* (4) *has a solution u such that*

$$\min\{\alpha(t_u), \beta(t_u)\} \leq u(t_u) \leq \max\{\alpha(t_u), \beta(t_u)\},$$

for some $t_u \in [0,T]$.

The following result, proved in [1], gives a method to construct lower solutions for problems of the type

$$(\phi(u'))' = g_0(t,u), \quad u(0) - u(T) = 0 = u'(0) - u'(T) \tag{6}$$

where $g_0 : [0,T] \times \mathbb{R} \to \mathbb{R}$ is a continuous function.

Lemma 4. *Let us assume that there exists $x_1 \in \mathbb{R}$ and $c \in C$ such that*

$$g_0(t,x) \leq c(t), \quad \textit{for } (t,x) \in [0,T] \times \left[x_1, x_1 + \frac{aT}{2}\right]. \tag{7}$$

If $\overline{c} \leq 0$, then (6) *has a lower solution α such that $x_1 \leq \alpha < x_1 + \frac{aT}{2}$.*

3. Bounded and super-sub linear perturbations

In this section we will study problems of the type

$$(\phi(u'))' + r(t)u = f(t,u,u'), \quad u(0) - u(T) = 0 = u'(0) - u'(T) \tag{8}$$

where $r \in C$ and $f : [0,T] \times \mathbb{R}^2 \to \mathbb{R}$ is a continuous nonlinearity.

In the following theorem we prove that if $\overline{r} \neq 0$ and f is bounded on $[0,T] \times \mathbb{R} \times (-a,a)$, then (8) has at least one solution. So, resonance occurs only when $\overline{r} = 0$. The case r constant is considered in [4]. We use the strategy introduced in the proof of Theorem 3.1 from [12].

Theorem 1. *If $\overline{r} \neq 0$ and f is bounded on $[0,T] \times \mathbb{R} \times (-a,a)$, then* (8) *has at least one solution.*

Proof. Let $p > 0$ be a constant such that

$$|f(t,x,y)| \le p \quad \text{for all } (t,x,y) \in [0,T] \times \mathbb{R} \times (-a,a).$$

For any $\lambda \in [0,1]$ let us consider the periodic problem

$$\begin{aligned}&(\phi(u'))' = \lambda[N_f(u) - r(t)u] + (1-\lambda)[QN_f(u) - Q(ru)],\\&u(0) - u(T) = 0 = u'(0) - u'(T).\end{aligned} \tag{9}$$

Let $\mathcal{M}(\lambda,\cdot) : C^1_\sharp \to C^1_\sharp$ be the fixed point operator associated to (9) by Lemma 2. Notice that if $u \in C^1_\sharp$ is such that $u = \mathcal{M}(\lambda,u)$, then (3) is satisfied and

$$QN_f(u) = Q(ru),$$

implying that

$$\overline{u} = \frac{1}{\overline{r}} Q[N_f(u) - \widetilde{r}\widetilde{u}].$$

So, one has that

$$|\overline{u}| < \frac{p + aT||\widetilde{r}||_\infty}{\overline{r}}.$$

Then, for any $\rho > 0$ sufficiently large, one has that

$$u \ne \mathcal{M}(\lambda,u) \quad \text{for all } (\lambda,u) \in [0,1] \times \partial B_\rho.$$

The invariance under homotopy of the Leray–Schauder degree implies that

$$d_{LS}[I - \mathcal{M}(0,\cdot), B_\rho, 0] = d_{LS}[I - \mathcal{M}(1,\cdot), B_\rho, 0].$$

Notice that from $Q^2 = Q$, it follows that

$$\mathcal{M}(0,u) = Pu + Q[N_f(u) - ru] \quad (u \in C^1_\sharp).$$

So, the range of the operator $\mathcal{M}(0,\cdot)$ is contained in the space of constant functions which is isomorphic to $\mathbb{R}$. Hence, using the reduction property of the Leray–Schauder degree we deduce that, for ρ sufficiently large,

$$d_{LS}[I - \mathcal{M}(0,\cdot), B_\rho, 0] = d_B[I - \mathcal{M}(0,\cdot)|_{\mathbb{R}}, (-\rho,\rho), 0],$$

which together with the fact that f is bounded and

$$[I - \mathcal{M}(0,\cdot)|_{\mathbb{R}}](x) = \overline{r}x - \frac{1}{T}\int_0^T f(t,x,0)\,dt \quad (x \in \mathbb{R}),$$

imply that

$$d_{LS}[I - \mathcal{M}(0,\cdot), B_\rho, 0] = \operatorname{sign} \overline{r}.$$

We infer that

$$d_{LS}[I - \mathcal{M}(1,\cdot), B_\rho, 0] \ne 0,$$

and the existence property of the Leray–Schauder degree implies that $\mathcal{M}(1,\cdot)$ has at least one fixed point u which is also a solution of (8). $\square$

Example 1. Let $g : \mathbb{R} \to \mathbb{R}$ be a continuous function and $r, e \in C$ with $\overline{r} \neq 0$. Then, problem

$$\left(\frac{u'}{\sqrt{1-u'^2}}\right)' + g(u') + r(t)u = e(t), \quad u(0) - u(T) = 0 = u'(0) - u'(T),$$

has at least one solution.

Remark 1. Consider the non-homogeneous Neumann problem with a singular ϕ-Laplacian

$$\left(\frac{tu'}{\sqrt{1-u'^2}}\right)' = t(\kappa u + \lambda), \quad u'(0) = 0,\ u'(T) = \gamma,$$

where $T > 0,\ \kappa, \lambda \in \mathbb{R}$ and $\gamma \in (-1, 1)$. If $\kappa \neq 0$ (the case $\kappa = 0$ follows immediately by a direct integration), then it is proved, by a shooting argument, in Theorems 4.2 and 6.9 from [10] that the above Neumann problem has at least one solution. Now, using the fixed point operator given in Lemma 3 from [2] and a similar strategy like in the proof of Theorem 1, one can show that the Neumann problem

$$\left(\frac{t^{N-1}u'}{\sqrt{1-u'^2}}\right)' = t^{N-1}(\kappa(t)u + h(t, u, u')), \quad u'(0) = 0,\ u'(T) = \gamma,$$

has at least one solution if $N \geq 1$ is an integer, $\kappa \in C$ satisfies $\int_0^T t^{N-1}\kappa(t)\,dt \neq 0$ and the continuous perturbation $h : [0, T] \times \mathbb{R}^2 \to \mathbb{R}$ is bounded on $[0, T] \times \mathbb{R} \times (-1, 1)$. For the geometric motivation of the above problems see the paper [10].

In the following theorem we assume for example that f is superlinear at zero and sublinear at infinity and prove that (8) has at least one nontrivial solution if $\overline{r} > 0$. The corresponding result in the classical case ($u \mapsto u''$) is proved in [14] (see also [6,7] for Sturm–Liouville boundary value problems).

Theorem 2. *Assume that in* (8), *f does not depends on u'. If one has that $\overline{r} > 0$ and*

$$\liminf_{x \to 0_+} \frac{f(t, x)}{x} > r_M \geq \overline{r} > \limsup_{x \to +\infty} \frac{f(t, x)}{x},$$

uniformly in $t \in [0, T]$, then (8) *has at least one nontrivial solution.*

Proof. First of all, our assumption implies that there exists $\beta > 0$ such that

$$r(t)\beta \leq f(t, \beta) \quad \text{for all } t \in [0, T].$$

This means that β is an upper solution of (8).

On the other hand from our assumption it follows also that there exists $\varepsilon > 0$ and $x_1 > \max\{aT\overline{r^+}/2\varepsilon, \beta\}$ such that

$$f(t, x) \leq (\overline{r} - \varepsilon)x \quad \text{for all } t \in [0, T],\ x \geq x_1.$$

We will apply Lemma 4 with $g_0(t, x) = f(t, x) - r(t)x$ and

$$c(t) = -r(t)x_1 + \frac{aT}{2}r^-(t) + \max_{[x_1, x_1 + \frac{aT}{2}]} f(t, \cdot),$$

for all $t \in [0,T], x \in \mathbb{R}$. Notice that

$$\begin{aligned} -r(t)x &= r^-(t)x - r^+(t)x \\ &\leq r^-(t)\left(x_1 + \frac{aT}{2}\right) - r^+(t)x_1 \\ &= -r(t)x_1 + \frac{aT}{2}r^-(t), \end{aligned}$$

for all $(t,x) \in [0,T] \times [x_1, x_1 + \frac{aT}{2}]$, implying that (7) holds true. Next, we have

$$\overline{c} \leq -x_1\overline{r} + \frac{aT}{2}\overline{r^-} + (\overline{r} - \varepsilon)\left(x_1 + \frac{aT}{2}\right) \leq 0.$$

Hence, from Lemma 4 we deduce that (8) has a lower solution α such that $x_1 \leq \alpha < x_1 + \frac{aT}{2}$. In particular $\beta \leq \alpha$, and using Lemma 3, we infer that (8) has at least one solution u such that $\beta \leq u(t_u)$, for some $t_u \in [0,T]$, which is also nontrivial. □

Corollary 1. *If one has that $\overline{r} > 0$ and*

$$\lim_{x\to 0_+} \frac{f(t,x)}{x} = +\infty, \quad \lim_{x\to +\infty} \frac{f(t,x)}{x} = 0,$$

uniformly in $t \in [0,T]$, then (8) has at least one nontrivial solution.

Example 2. If $\overline{r} > 0$ and $\nu \in (0,1)$, then problem

$$\left(\frac{u'}{\sqrt{1-u'^2}}\right)' + r(t)u = |u|^\nu, \quad u(0) - u(T) = 0 = u'(0) - u'(T),$$

has at least one nontrivial solution.

4. Singular perturbations

In this section we will apply Theorem 1 to study singular problems of type

$$(\phi(u'))' + r(t)u - \frac{m(t)}{u^\nu} = e(t), \quad u(0) - u(T) = 0 = u'(0) - u'(T), \quad (10)$$

where $r, m, e \in C$ and $\nu > 0$. Applications are given to Brillouin beam-focusing type equations with relativistic effects.

Theorem 3. *Assume that $\overline{r} > 0$, $m \geq 0$ with $m \neq 0$ and*

$$\overline{e} > \frac{aT}{2}\overline{r^+} - \overline{m}\left(\frac{2}{aT}\right)^\nu. \quad (11)$$

Then, (10) has at least one positive solution.

Proof. Let us define the (auxiliary) continuous and increasing functions

$$\Psi_1(x) = \frac{aT}{2}\overline{r^+} - \frac{\overline{m}}{x^\nu}, \quad \Psi_2(x) = x\overline{r^+} - \frac{\overline{m}}{x^\nu} \quad (x > 0).$$

From (11) it follows that $\overline{e} > \Psi_2(\frac{aT}{2})$, and there exists $\varepsilon > 0$ such that

$$\overline{e} > \Psi_2\left(\frac{aT}{2} + \varepsilon\right). \tag{12}$$

Now, consider the continuous function $g : [0, T] \times \mathbb{R} \to \mathbb{R}$ given by

$$g(t, x) = \begin{cases} \frac{m(t)}{x^\nu}, & x \geq \varepsilon, \\ \frac{m(t)}{\varepsilon^\nu}, & x < \varepsilon, \end{cases}$$

and consider the modified problem

$$(\phi(u'))' + r(t)u - g(t, u) = e(t), \quad u(0) - u(T) = 0 = u'(0) - u'(T). \tag{13}$$

Using that g is bounded and $\overline{r} \neq 0$, it follows from Theorem 1 that (13) has a solution u.

We will show that $u_L > \varepsilon$.

Integrating (13) on $[0, T]$ we deduce that

$$T\overline{e} = \int_0^T r^+(t)u\, dt - \int_0^T r^-(t)u\, dt - \int_0^T g(t, u)\, dt, \tag{14}$$

which together with (1) imply that

$$\overline{e} \leq \frac{aT}{2}\overline{r^+} + \overline{r}u_L - \frac{1}{T}\int_0^T g(t, u)\, dt. \tag{15}$$

On the other hand, using that $\overline{m} > 0$, one has that

$$\Psi_2\left(\frac{aT}{2} + \varepsilon\right) \geq \frac{aT}{2}\overline{r^+} + \varepsilon\overline{r} - \frac{\overline{m}}{\varepsilon^\nu}. \tag{16}$$

Let us assume that $u_M \leq \varepsilon$. Then, using (15) and $\overline{r} > 0$, we infer that

$$\overline{e} \leq \frac{aT}{2}\overline{r^+} + \overline{r}\varepsilon - \frac{\overline{m}}{\varepsilon^\nu},$$

contradicting (12) and (16). So, $u_M > \varepsilon$.

Next, using (15), (11), (1) and $m \geq 0$, it follows that

$$\begin{aligned} 0 &\leq \Psi_1(u_M) - \overline{e} + \overline{r}u_L \\ &\leq \Psi_1(u_M) - \Psi_1\left(\frac{aT}{2}\right) + \overline{r}u_L \\ &< \Psi_1\left(u_L + \frac{aT}{2}\right) - \Psi_1\left(\frac{aT}{2}\right) + \overline{r}u_L, \end{aligned}$$

which together with $\overline{r} > 0$, imply that $u_L > 0$.

Using then (12) and (14) we deduce that

$$\Psi_2\left(\frac{aT}{2} + \varepsilon\right) < \overline{e} \leq \Psi_2(u_M),$$

implying that $\frac{aT}{2} + \varepsilon < u_M$. This together with (1) imply that $u_L > \varepsilon$, and our claim is proved. Consequently, u is also a solution of (10) and the proof is completed. $\square$

Example 3. Consider the Brillouin beam-focusing equation with relativistic effects

$$\left(\frac{u'}{\sqrt{1-u'^2}}\right)' + (b_1 + b_2 \cos t)u = \frac{1}{u^\nu}, \quad u(0) - u(2\pi) = 0 = u'(0) - u'(2\pi),$$

where $b_1 > 0$ and $b_2 \in \mathbb{R}$. We have shown in [1] that the above problem has at least one solution provided that $\nu \geq 1$. Now, in the weak case, $0 < \nu < 1$, using Theorem 3, it follows that the above problem has at least one solution if the additional conditions $b_1 < \frac{1}{\pi^{1+\nu}}$ and $b_2 \in [-b_1, b_1]$ hold true.

5. Mathieu-Duffing type perturbations

In this section we consider Mathieu-Duffing type equations

$$(\phi(u'))' + (b_1 + b_2 \cos t)u + cu^3 = 0, \quad u(0) - u(2\pi) = 0 = u'(0) - u'(2\pi), \tag{17}$$

where $b_1, b_2, c \in \mathbb{R}$. We assume that (H_ϕ) holds true and

(H_Φ) *there exists* $\Phi : [-a, a] \to \mathbb{R}$ *such that* $\Phi(0) = 0$, Φ *is continuous, of class* C^1 *on* $(-a, a)$, *with* $\Phi' = \phi$.

Clearly, $\Phi(x) \geq 0$ for all $x \in [-a, a]$.

Following [3], one has that the action functional $I : C \to (-\infty, +\infty]$ associated to (17) is given by $I = \Psi + \mathcal{G}$, where

$$\Psi(v) = \begin{cases} \displaystyle\int_0^{2\pi} \Phi(v')\, dt, & v \in K, \\ +\infty, & \text{otherwise}, \end{cases}$$

with

$$K = \{v \in W^{1,\infty} : \ ||v'||_\infty \leq a, \ v(0) = v(2\pi)\}$$

and

$$\mathcal{G}(v) = -\int_0^{2\pi} \left[\frac{1}{2}(b_1 + b_2 \cos t)v^2 + \frac{c}{4}v^4\right] dt \quad (v \in C).$$

Then, from [3] (see also [5]) it holds that Ψ is proper (with a closed domain K), convex and lower semi-continuous and $\mathcal{G}$ is of class C^1. So, the functional I has the structure required by Szulkin's critical point theory (see [13]). In this context, a critical point of I means a function $u \in K$ such that

$$\int_0^{2\pi} [\Phi(v') - \Phi(u')]\, dt - \int_0^{2\pi} [(b_1 + b_2 \cos t)u + cu^3][v - u]\, dt \geq 0 \quad \text{for all } v \in K.$$

From Proposition 2 in [3] one has that

Lemma 5. *If* $u \in K$ *is a critical point of* I, *then* u *is a solution of* (17).

In the proof of the first main result of this section we will use the following minimization procedure which is a substitute, in this context, of the direct method of the calculus of variations in reflexive spaces.

Lemma 6. *Assume that there is some $\rho > 0$ such that $\inf_{K_\rho} I = \inf_K I$, where $K_\rho = \{v \in K : \ |\overline{v}| \leq \rho\}$. Then I is bounded from below on C and attains its infimum at some $u \in K_\rho$, which is a critical point of I and solves (17).*

The first main result of this section is the following

Theorem 4. *If $c < 0 < b_1$, then (17) has at least one nontrivial solution for any $b_2 \in \mathbb{R}$.*

Proof. First of all, notice that

$$I(v) = \int_0^{2\pi} \Phi(v')\, dt - \int_0^{2\pi} \left[\frac{1}{2}(b_1 + b_2 \cos t)v^2 + \frac{c}{4}v^4\right] dt \quad (v \in K).$$

Then, using that Φ is positive on $[-a, a]$ and

$$||\widetilde{v}||_\infty \leq 2\pi a \quad \text{for all } v \in K, \tag{18}$$

we infer that

$$I(v) \geq -\frac{c\pi}{2}\overline{v}^4 + p(|\overline{v}|) \quad \text{for all } v \in K,$$

where p is a polynomial with $deg\ p = 3$. On the other hand, from (18) one has that $||v||_\infty \to \infty$, $v \in K$ if and only if $|\overline{v}| \to \infty$, $v \in K$. So,

$$I(v) \to +\infty \quad \text{as } ||v||_\infty \to \infty, v \in K,$$

implying that there is some $\rho > 0$ such that $\inf_{K_\rho} I = \inf_K I$. Then, by Lemma 6, (17) has a solution u such that $I(u) = \inf_K I$.

Let us show that u is nontrivial. Using $\Phi(0) = 0$, one has that

$$I(x) = -x^2\pi \left[\frac{c}{2}x^2 + b_1\right] \quad (x \in \mathbb{R}),$$

implying that $I(x) < 0$ for all sufficiently small $x \in \mathbb{R}$, which together with $I(0) = 0$, imply that $u \neq 0$ and the proof is completed. □

The main tool used in the proof of the second main result of this section is the Non-smooth Mountain Pass Theorem due to Szulkin (Theorem 3.2 in [13]). Towards the application of this minimax result to our action functional I, we have to know when I satisfies the compactness Palais-Smale (in short (PS)) condition.

Viewing the definition of I and following Szulkin, we say that a sequence $(u_n) \subset K$ is a (PS)-sequence if $I(u_n) \to \delta \in \mathbb{R}$ and

$$\int_0^{2\pi} [\Phi(v') - \Phi(u_n')]\, \mathrm{d}t - \int_0^{2\pi} [(b_1 + b_2 \cos t)u_n + cu_n^3][v - u_n]\, \mathrm{d}t \\ \geq -\varepsilon_n ||v - u_n||_\infty \quad \text{for all } v \in K,$$

where $\varepsilon_n \to 0_+$. The functional I is said to satisfy the (PS)-condition if any (PS)-sequence has a convergent subsequence in C.

Notice that from Lemma 3 (ii) in [3] one has that if $(u_n) \subset K$ is a (PS)-sequence with $(\overline{u}_n)$ bounded, then (u_n) has a convergent subsequence in C. We recall for the convenience of the reader the following

Mountain Pass Theorem. *Suppose that $I(0) = 0$, I satisfies (PS)-condition and*

(i) *there exists $\delta, \rho > 0$ such that $I|_{\partial B_\rho} \geq \delta$,*
(ii) *$I(e) \leq 0$ for some $e \notin \overline{B}_\rho$.*

Then I has at least one nontrivial critical point.

The second main result of this section is the following

Theorem 5. *If $b_1 < b_2 < 0 < c$, then (17) has at least one nontrivial solution.*

Proof. First of all, recall that

$$I(x) = -x^2\pi\left[\frac{c}{2}x^2 + b_1\right] \quad (x \in \mathbb{R}),$$

which together with $c > 0$ imply that

$$I(x) \to -\infty \quad as\ |x| \to \infty. \tag{19}$$

Next, consider $\rho > 0$ and $(u_n) \subset \partial B_\rho \cap K$ such that $I(u_n) \to \inf_{\partial B_\rho} I$. Clearly, (u_n) is bounded in $W^{1,\infty}$ which is compactly embedded in C. So, passing if necessarily to a subsequence, we can assume that there exists $u \in \partial B_\rho \cap K$ (recall that K is closed in C) such that $u_n \to u$ in C and $I(u) \leq \liminf I(u_n)$. Then, one has that $I(u) = \inf_{\partial B_\rho} I$. Now, using that $b_2 < 0 < c$, we infer that

$$\int_0^{2\pi} u^2\left(b_1 + b_2\cos t + \frac{c}{2}u^2\right)dt \leq \int_0^{2\pi} u^2\left(b_1 - b_2 + \frac{c}{2}\rho^2\right)dt,$$

which together with $b_1 < b_2$ imply that

$$\int_0^{2\pi} u^2\left(b_1 + b_2\cos t + \frac{c}{2}u^2\right)dt < 0,$$

if ρ is small enough. Hence,

$$I(u) \geq -\frac{1}{2}\int_0^{2\pi} u^2\left(b_1 + b_2\cos t + \frac{c}{2}u^2\right)dt > 0,$$

implying that

$$\exists\delta > 0 :\ I|_{\partial B_\rho} \geq \delta. \tag{20}$$

So, from (19) and (20) it follows that I satisfies condition $(i), (ii)$ from Mountain Pass Theorem. Now, let us check that I satisfies (PS)-condition. Let (u_n) be a (PS)-sequence. Using that

$$I(u_n) \leq -\frac{c\pi}{2}\overline{u}_n^4 + p(|\overline{u}_n|),$$

where p is a polynomial of degree three, and $c > 0$, it follows that $(\overline{u}_n)$ is bounded. So, (u_n) has a convergent subsequence in C and I satisfies (PS)-condition. Then, by Mountain Pass Theorem, I has a nontrivial critical point which is also a solution of (17). $\square$

Example 4. Taking

$$\Phi(x) = 1 - \sqrt{1 - x^2} \qquad (x \in [-1, 1]),$$

it follows from Theorems 4 and 5 that problem

$$\left(\frac{u'}{\sqrt{1 - u'^2}}\right)' + (b_1 + b_2 \cos t)u + cu^3 = 0, \quad u(0) - u(2\pi) = 0 = u'(0) - u'(2\pi),$$

has at least one nontrivial solution provided that $c < 0 < b_1, b_2 \in \mathbb{R}$ or $b_1 < b_2 < 0 < c$.

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