Modulated amplitude waves with non-trivial phase in quasi-1D inhomogeneous Bose-Einstein condensates

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Abstract

We consider a 1D nonlinear Schrödinger equation (NLSE) which describes the mean field dynamics of an elongated Bose-Einstein condensate and prove the existence of modulated amplitude waves with non-trivial phase and minimal spatial period tending to infinite. The proof combines the theory of local continuation of non-degenerate periodic solutions with a property of the Ermakov-Pinney equation.

Keywords: Bose-Einstein condensate; Gross-Pitaevskii equation; modulated amplitude wave; non-degenerate solution; local continuation

1 Introduction and main result

Coherent structures and periodic patterns appear naturally in a variety of models coming from several branches of Physics and Biology [6]. In particular, the analysis of spatiotemporal structures in Bose-Einstein condensates is a problem of primary importance in Matter Physics (see the reviews [2, 18] and the references therein). The objective of this note is to prove the existence of modulated amplitude waves for a nonlinear Schrödinger equation (NLSE) which describes the mean field dynamics of a "cigar shaped" Bose-Einstein condensate (BEC). A BEC is a special state of matter achieved by a trapped cloud of bosons at very (extremely) low temperatures. The existence of BECs was theoretically predicted by Bose and Einstein in 1924-25, while the first experimental realizations were performed in 1995 independently by two teams directed by E. Cornell and C. Wieman at JILA and W. Ketterle at MIT. Cornell, Wieman and Ketterle won the 2001 Nobel Prize in Physics for their achievements. Since then, the interplay between experiments and theory has been continuous.

A Bose-Einstein condensate follows a quasi-one-dimensional (quasi-1D) regime when the transverse dimensions of the cloud are of the order of its healing length and its longitudinal dimension is much larger than its transverse ones. We study a BEC under a quasi-1D

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regime with periodic external trapping and scattering length depending only on the spatial coordinate. The model equation is

$$i\frac{\partial u}{\partial t} = -\frac{1}{2}u_{xx} + V(x)u + g(x)|u|^2 u, \tag{1}$$

where V, g are continuous and L-periodic functions, that is,

$$V(x+L) = V(x), \qquad g(x+L) = g(x)$$

for every x. A NLSE with cubic nonlinearity like (1) is commonly known as a *Gross-Pitaevskii equation* (GPE) in the related literature. I is worth to mention that not only Bose-Einstein condensates, but a large variety of physical phenomena are described by nonlinear Schrödinger equations [24]. In particular, the GPE describes the propagation of electromagnetic waves in optical fibers with variable refractive index [9].

In the BEC context, $V(x)$ models an external magnetic or optical trap that confines the condensate, while $g(x)$ describes the nonlinear particle interaction and it is proportional to the scattering length. Such interaction is tunable (even its sign) by the so-called *Feshbach resonance management* [4, 5]. Attractive (resp. repulsive) interaction between particles corresponds to $g(x) < 0$ (resp. $g(x) > 0$).

We look for coherent structure solutions described by the ansatz

$$u(x,t) = R(x)\exp\left(i\left[\theta(x) - \mu t\right]\right). \tag{2}$$

Here, $R(x)$ describes the amplitude dynamics of the wave function, whereas $\theta(x)$ gives the phase dynamics. The *chemical potential* μ is a free real parameter. Such solutions are always periodic in time with minimal period $T = 2\pi/\mu$ (hence, the chemical potential can be seen as the temporal frequency). When they are also spatially periodic, they are known as *modulated amplitude waves* (MAWs). This type of spatially extended states can be created in experiments [7, 15] and plays a very important role in the understanding of BEC dynamics, because they can be seen as the nonlinear analogues of the Bloch modes that appear in the linear limit, when particle interactions are neglected. In consequence, MAWs have been profusely studied in the latter years [1, 14, 19, 20, 21, 22, 25].

Introducing the ansatz (2) into the GP equation (1) and taking real and imaginary parts of the resulting equation, one arrives to the second order equation

$$\ddot{R}(x) = \frac{c^2}{R^3} + 2(V(x) - \mu)R + 2g(x)R^3 \tag{3}$$

where the parameter c is a conserved quantity given by the relation

$$R^2(x)\dot{\theta}(x) = c, \tag{4}$$

in total analogy with the conservation of angular momentum of a particle under a central force field.

If we fix $c = 0$, the phase θ is constant and (3) is a parametrically forced Duffing oscillator. The resulting MAWs are called *standing waves* and have been studied by many authors (see the cited papers [14, 19, 20, 21, 22, 25]) by using different approaches. The case $c \neq 0$ (*rotating waves*) is more difficult because (3) contains a singular term. In [1], MAWs with non-trivial phase have been explicitly constructed by a particular choice of the trapping potential $V(x)$, which depends on the Jacobi elliptic function. On the other hand, the recent papers [11, 12] present an analytical study of the existence of rotating waves by using averaging theory, under a smallness assumption on the functions $V(x), g(x)$.

Given an L-periodic solution $R(x)$ of (3), the phase variable is obtained from (4) by a simple integration

$$\theta(x) = \int_0^x \frac{c}{R^2(s)} ds.$$

However, the solution given by (2) will be in general only quasiperiodic in x. To be a genuine MAW, an additional requirement is necessary. We define

$$\mathrm{rot}(R) = \frac{1}{L}\int_0^L \frac{c}{R^2(s)} ds$$

as the rotation number associated with R. The coherent structure given by (2) can be written as

$$u(x,t) = R(x)\exp\left(i\left[\tilde{\theta}(x) + \mathrm{rot}(R)x - \mu t\right]\right),$$

where $\tilde{\theta}(x) = \theta(x) - \mathrm{rot}(R)x$ is L-periodic. Then, $u(x,t)$ is a MAW (nL-periodic in x) if and only if $\mathrm{rot}(R) = (m/n)(2\pi/L)$ for some relatively prime integers $m \neq 0 \neq n$. In any other case, $u(x,t)$ is quasiperiodic in x with two natural frequencies $(L, \frac{2\pi}{\mathrm{rot}(R)})$.

Our main result is as follows.

Theorem 1 *Let us consider continuous and L-periodic functions V, g. Then, there exists a sequence of positive integer numbers $\{k_n\}_{n\in\mathbb{N}}$ such that $k_n \to +\infty$ as $n \to +\infty$ and for every n the GP equation* (1) *has a MAW with spatial periodicity k_nL.*

From the proof, we will see that the found MAW with spatial periodicity k_nL is embedded in a continuous branch of quasiperiodic solutions. Observe that no sign condition over $g(x)$ is needed. We point out that the result presented above is true even in the trapless case $V(x) \equiv 0$ and generalize the results presented in [11, 12], where a smallness assumption on the coefficients $V(x), g(x)$ is needed.

The rest of the paper is organized as follows: Section 2 provides the main ingredients for the proof, namely a classical continuation result from a non-degenete periodic solution together with a connection between the stability of the Hill's equation and the existence of a non-degenerate solution of the Ermakov-Pinney equation. The proof is developed in Section 3. Finally, Section 4 presents some concluding remarks.

2 Some preliminaries.

This section will provide the main ingredients of the proof.

2.1 A local continuation theorem.

In the context of ODEs, it is more familiar to use t as the indendent variable. Let us consider a general second order equation

$$u'' + f(t,u) = 0 \tag{5}$$

with $f(t,u)$ a T-periodic funciton in t and regular in u.

Definition 1 *A T-periodic solution $\varphi(t)$ of* (5) *is* non-degenerate *if the linearized equation*

$$u'' + f_u(t, \varphi(t))u = 0$$

has no non-trivial T-periodic solutions.

An important property of a non-degenerate solution is that it can be continued forming a local continuous branch of solutions.

Theorem 2 *Let us consider a parametric family of equations*

$$u'' + f(t, u) + \epsilon h(t, u, \epsilon) = 0 \tag{6}$$

with $h(t, u, \epsilon)$ T-periodic in t and regular in u, ϵ. If $\varphi(t)$ is a non-degenerate T-periodic solution of (5)*, then for small $|\epsilon|$ the equation* (6) *has a T-periodic solution $\varphi(t, \epsilon)$ continuous in ϵ and tending uniformly to $\varphi(t)$ as $\epsilon \to 0$.*

This result is well-known and can be found in many classical books (see for instance [3, Chapter 14]). The proof is a basic application of the Implicit Function Theorem.

2.2 A relation between Hill's and Ermakov-Pinney equations

Along this subsection, $a(t)$ is a T-periodic function. The equation

$$\ddot{u} + a(t)u = \frac{1}{u^3} \tag{7}$$

is commonly known as the *Ermakov-Pinney* and has played an important role in many areas of Physics and Mathematics for a long time. We refer to the nice review [23] for an historical account of this equation.

The main feature of eq. (7) is that its general solution can be written explicitly in terms of a fundamental system of the associated Hill's equation

$$\ddot{x} + a(t)x = 0 \tag{8}$$

through a *nonlinear superposition principle* [17]. As a consequence, the Ermakov-Pinney equation inherits the resonant character of the associated Hill's equation. In other words, if (8) is unstable, all the solutions of (7) experience unbounded oscillations. This fact was exploited in [8, 16] to derive conditions for resonance of the GPE with time-dependent coefficients. The paper [10] presents the main lines of the theory exposed in [8, 16] as well as many other interesting ideas and connections.

A less known property of Ermakov-Pinney equation is the following stability result, proved by M. Zhang in [27].

Definition 2 *Let λ_1, λ_2 be the Floquet multipliers of eq.* (8) *(that is, the eigenvalues of its monodromy matrix). Equation* (8) *is called* elliptic, hyperbolic *or* parabolic, *if $|\lambda_{1,2}| = 1$ but $\lambda_{1,2} \neq \pm 1$, $|\lambda_{1,2}| \neq 1$ or $\lambda_1 = \lambda_2 = \pm 1$, respectively.*

Theorem 3 *[27] The following assertions are equivalent*

(i) The Ermakov-Pinney equation (7) *has a (positive) T-periodic solution.*

(ii) Hill's equation (8) *is either elliptic or parabolic. In the second case, the solutions are all T-periodic or all T-anti-periodic.*

(iii) Hill's equation (8) *is stable in the sense of Lyapunov.*

Moreover, if (8) *is elliptic, the T-periodic solution of* (7) *is unique and non-degenerate.*

A second connection between the Hill's and the Ermakov-Pinney equation is related to the rotation number. Every Hill's equation has an associated rotation number, which is classically defined as follows. Let $x_1(t), x_2(t)$ a fundamental system of equation (8) and write in polar coordinates the complex-valued solution $x(t) = x_1(t) + ix_2(t) = R(t)e^{\theta(t)i}$. Then, the rotation number associated to (8) is given by

$$\rho = \lim_{t \to +\infty} \frac{\theta(t)}{t}.$$

It is well known that this limit does exist and does not depend on the choice of $x(t)$. For a general exposition of Floquet Theory and Hill's equations, see [13].

Lemma 1 *Let us assume that the Hill's equation* (8) *is elliptic. Then, its rotation number is given by*

$$\rho = rot(R) = \frac{1}{T}\int_0^T \frac{1}{R^2(s)}ds, \tag{9}$$

where $R(t)$ is the unique T-periodic solution of of the Ermakov-Pinney equation (7)*.*

Proof. By hypothesis, there exists a Floquet multiplier $\lambda \neq \pm 1$ such that $|\lambda| = 1$. Therefore, there exists a (complex valued) solution of (8) such that

$$x(t+T) = \lambda x(t), \qquad \text{for all } t. \tag{10}$$

Writing in polar coordinates $x(t) = R(t)e^{\theta(t)i}$, taking norms in (10) we deduce that $R(t)$ is T-periodic. Besides, it follows from (8) that $R(t), \theta(t)$ satisfy

$$\ddot{R} + a(t)R - R\dot{\theta}^2 = 0, \qquad R\ddot{\theta} + 2\dot{R}\theta = 0.$$

From the second equation, we get the invariant $R^2\dot{\theta} = c$ constant. After a rescalling of $x(t)$ if necessary, we can take $c = 1$ without loss of generality. Substituting into the first equation, one finds out that $R(t)$ verifies (7) and it is indeed its unique T-periodic solution.

Now,

$$\rho = \lim_{t \to +\infty} \frac{\theta(t)}{t} = \lim_{t \to +\infty} \frac{1}{t}\left[\int_0^t \dot{\theta}(s)ds + \theta(0)\right] = \lim_{t \to +\infty} \frac{1}{t}\int_0^t \frac{1}{R^2(s)}ds$$

To compute this last limit, we observe that the periodicity of $R(t)$ implies that the function $f(t) = \int_0^t \frac{1}{R^2(s)}ds - \frac{t}{T}\int_0^T \frac{1}{R^2(s)}ds$ is continuous and T-periodic, hence bounded. Then

$$\rho = \lim_{t \to +\infty} \frac{f(t)}{t} + \frac{1}{T}\int_0^T \frac{1}{R^2(s)}ds = \frac{1}{T}\int_0^T \frac{1}{R^2(s)}ds.$$

□

In the next section, after a rescalling we will construct a local branch of solutions starting from a non-degenerate solution of the Ermakov-Pinney equation.

3 Proof of the main result.

In view of the considerations exposed in Section 2, we observe that the Hill's equation

$$\ddot{S} + 2(\mu - V(x))S = 0 \tag{11}$$

has a well-defined rotation number as a function of μ, say $\rho(\mu)$. It is a known fact (see for instance [26, Theorem 2.1]), that $\rho(\mu)$ is a continuous and non-decreasing function such that

$$\lim_{\mu\to-\infty} \rho(\mu) = 0, \qquad \lim_{\mu\to+\infty} \rho(\mu) = +\infty. \tag{12}$$

Fix a positive integer $n_0 > \frac{4\pi}{L}$. In view of (12), there exists a sequence $\{\mu_n\}_{n\geq n_0}$ such that $\rho(\mu_n) = \frac{2\pi}{nL}$ for all $n \geq n_0$. By the choice of n_0, we have

$$0 < \rho(\mu_n) \leq \frac{2\pi}{n_0 L} < \frac{1}{2}.$$

This means that (11) with $\mu = \mu_n$ is in the first stability zone, therefore it is elliptic. Then, by Theorem 3, the Ermakov-Pinney equation

$$\ddot{S} + 2(\mu_n - V(x))S = \frac{1}{S^3} \tag{13}$$

has a positive non-degenerate L-periodic solution, denoted by $S_{0,n}(x)$.

The second step is to introduce the change $R = \sqrt{c}S$ into (3). The equivalent equation is

$$\ddot{S} + 2(\mu_n - V(x))S = \frac{1}{S^3} + 2g(x)cS^3. \tag{14}$$

Taking $\epsilon = c$ as a small parameter, (14) is a perturbation of the Ermakov-Pinney equation (13), which has a non-degenerate solution $S_{0,n}$. As a consequence of Theorem 2, for every $n \geq n_0$ there exists $c_n > 0$ such that for every $0 < c < c_n$ (14) has a positive L-periodic solution $S_{c,n}(x)$ such that

$$S_{c,n}(x) \to S_{0,n}(x)$$

uniformly in x when $c \to 0$. Clearly, once $n \geq n_0$ is fixed, equation (3) has a continuous branch $R_{c,n}(x) = \sqrt{c}S_{c,n}(x)$ for $0 < c < c_n$, which corresponds to a branch of quasi-MAWs of (1). Now, the delicate point is to evaluate the rotation number of such branch of solutions and identify the periodicity in the spatial variable.

To this aim, let us define

$$I_n = \{\text{rot}(R_{c,n}) \,:\, 0 < c < c_n\}.$$

Observe that

$$\text{rot}(R_{c,n}) = \frac{1}{L}\int_0^L \frac{c}{R_{c,n}^2(s)}ds = \frac{1}{L}\int_0^L \frac{1}{S_{c,n}^2(s)}ds \to \frac{1}{L}\int_0^L \frac{1}{S_{0,n}^2(s)}ds = \rho(\mu_n) = \frac{2\pi}{nL}$$

as $c \to 0^+$, where we have applied Lemma 1. Therefore, $\frac{2\pi}{nL}$ belongs to the closure of I_n . At this moment, we distinguish two possibilities:

- *Case 1:* $I_n = \{\frac{2\pi}{nL}\}$ for every $n \geq n_0$. In this case it is easy to arrive at the conclusion, because $\mathrm{rot}(R_{c,n}) = \frac{2\pi}{nL}$ for any $0 < c < c_n$ and the coherent structure generated by $R_{c,n}(x)$ is a MAW with spatial periodicity nL. The result is proved by taking $k_n = n$.

- *Case 2:* there exists at least one $n_1 \geq n_0$ such that the interval I_{n_1} is open. Let us fix such n_1. Since $\frac{2\pi}{n_1 L}$ is in the closure of I_{n_1}, at least one of the sequences $\left(\frac{1}{n_1} - \frac{1}{n}\right)\frac{2\pi}{L}$ or $\left(\frac{1}{n_1} + \frac{1}{n}\right)\frac{2\pi}{L}$ belongs to I_{n_1} for n sufficiently large, say $n > n_*$. Suppose the first option holds, the second one being completely analogous. Then,

$$\left(\frac{1}{n_1} - \frac{1}{n}\right)\frac{2\pi}{L} = \left(\frac{n - n_1}{n_1 n}\right)\frac{2\pi}{L} \in I_{n_1}$$

  for $n > n_*$. Let us define the sequence $k_n = n_1 n$ of positive integer numbers. Accordingly, for any $n > n_*$, there exists $\tilde{c}_n \in I_{n_1}$ such that $\mathrm{rot}(R_{\tilde{c}_n,n_1}) = \left(\frac{n-n_1}{n_1 n}\right)\frac{2\pi}{L}$, which implies that the coherent structure generated by $R_{\tilde{c}_n,n_1}$ is a MAW with spatial periodicity $k_n L$. The proof is completed.

4 Summary and final remarks.

For the first time in the literature, we have given a rigorous proof of the existence of modulated amplitude waves with non-trivial phase (angular momentum) in a quasi-1D collisionally inhomogeneous Bose-Einstein condensate, not assuming any kind of condition on the sign or the magnitude of the trapping potential or the scattering length. This result indicates that, at least in principle, the observation of such spatially extended structures is experimentally feasible. Inside a whole continuous branch of MAWs with quasiperiod dependence of the spatial variable, we are able to find a sequence of periodic solutions through a synchronization of the rotation with the amplitude period. We emphasize the generality of the result, in special it is not required any sign or size condition over the coefficients $V(x), g(x)$. With our approach, it is shown that the system under study is robust, in the sense that the existence of MAWs does not depend on the particular choice of the coefficients. For simplicity, we assumed continuity on the functions $V(x), g(x)$, but the arguments remain valid for more general bounded functions, say piece-wise continuous functions. Piece-wise contant coefficents are natural in the optical interpretation of GPE.

A closer inspection of the proof provides additional information on the rotating waves, for instance they have small (but non-trivial) angular momentum and the amplitude is close to zero. Thinking on a complementary numerical or experimental study, this fact may suppose a difficulty to locate this type of coherent structures. In future works, it would be interesting to perform a subsequent study of the stability of the solutions. Even stability in the linear sense with a suitable analysis of eigenvalues may be a non-trivial task.

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