A higher-dimensional Poincaré – Birkhoff theorem without monotone twist

Un théorème de Poincaré – Birkhoff en plusieurs dimensions sans torsion monotone

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Abstract

We provide a simple proof for a higher-dimensional version of the Poincaré–Birkhoff theorem which applies to Poincaré time maps of Hamiltonian systems. These maps are neither required to be close to the identity nor to have a monotone twist.

Résumé

Nous fournissons une preuve simple d'une version en plusieurs dimensions du théorème de Poincaré–Birkhoff qui s'applique aux applications de Poincaré des systèmes hamiltoniens. Ces applications ne sont ni tenues d'être proches de l'identité, ni d'avoir une torsion monotone.

1 Statement of the result

The aim of this short note is to give a simple proof, following the ideas developed in [2, 3], of a higher dimensional version of the Poincaré–Birkhoff theorem which applies to Poincaré time maps of a Hamiltonian system, say

$$(HS) \qquad \dot{z} = J\nabla H(t,z)\,.$$

Here, $J = \begin{pmatrix} 0 & I_N \\ -I_N & 0 \end{pmatrix}$ denotes the standard $2N \times 2N$ symplectic matrix and ∇ stands for the gradient with respect to the z variables. The Hamiltonian function $H : \mathbb{R} \times \mathbb{R}^{2N} \to \mathbb{R}$ is assumed to be T-periodic in its first variable t and C^∞-smooth with respect to all variables.

Consequently, for every initial position $\zeta \in \mathbb{R}^{2N}$, i.e., $\zeta = (\xi, \eta) \in \mathbb{R}^N \times \mathbb{R}^N$, there is a unique solution $\mathcal{Z}(\cdot, \zeta) = \mathcal{Z}(\cdot, \xi, \eta)$ of (HS) satisfying $\mathcal{Z}(0, \zeta) = \zeta$. Let us further assume that, for η in some closed ball $\overline{B} \subset \mathbb{R}^N$ centered at the origin, these solutions can be continued to the whole time interval $[0, T]$. We can then consider the so-called *Poincaré time map*: this is the function $\mathcal{P} : \mathbb{R}^N \times \overline{B} \to \mathbb{R}^N \times \mathbb{R}^N$, defined by

$$\mathcal{P}(\zeta) = \mathcal{Z}(T, \zeta)\,,$$

whose fixed points give rise to T-periodic solutions of (HS).

We use the notation $z = (x, y)$, with $x = (x_1, \dots, x_N) \in \mathbb{R}^N$ and $y = (y_1, \dots, y_N) \in \mathbb{R}^N$, and assume that $H(t, x, y)$ is 2π-periodic in each of the variables $x_1, \dots, x_N$. Then, once a T-periodic solution $z(t) = (x(t), y(t))$ has been found, many others appear by just adding an integer multiple of 2π to some of the components $x_i(t)$; for this reason, we will call *geometrically distinct* two T-periodic solutions of (HS) (or two fixed points of $\mathcal{P}$) which can not be obtained from each other in this way.

The result we want to prove is the following.

Theorem 1.1. *Writing*

$$\mathcal{P}(x, y) = (x + \vartheta(x, y), \rho(x, y)), \qquad (x, y) \in \mathbb{R}^N \times \overline{B},$$

assume that, either

$$\vartheta(x, y) \notin \{\alpha y : \alpha \geq 0\}, \quad \text{for every } (x, y) \in \mathbb{R}^N \times \partial B, \tag{1}$$

or

$$\vartheta(x, y) \notin \{-\alpha y : \alpha \geq 0\}, \quad \text{for every } (x, y) \in \mathbb{R}^N \times \partial B. \tag{2}$$

Then, $\mathcal{P}$ has at least $N+1$ geometrically distinct fixed points in $\mathbb{R}^N \times B$. Moreover, if they are non degenerate, then there are at least 2^N of them.

This is a special case of [3, Theorem 2.1], where a much more general situation was considered. However, we believe that the simple proof proposed below will clarify the main ideas and help the interested reader towards possible further generalizations.

2 The proof

In order to fix ideas, we assume that B is the open unit ball in $\mathbb{R}^N$ and that (1) holds. As before, for $\zeta = (\xi, \eta) \in \mathbb{R}^N \times \mathbb{R}^N$, we denote by $\mathcal{Z}(t, \zeta)$ the value at time t of the solution z of (HS) with $z(0) = \zeta$. The Hamiltonian $H(t, x, y)$ being 2π-periodic in the variables x_i, the continuous image by $\mathcal{Z}$ of $[0, T] \times (\mathbb{R}^N / 2\pi\mathbb{Z}^N) \times \overline{B}$ will be bounded in the cylinder $(\mathbb{R}^N / 2\pi\mathbb{Z}^N) \times \mathbb{R}^N$ and, after multiplying H by a smooth cutoff function of y, there is no loss of generality in assuming that:

($\bullet$) there is some $R \geq 2$ such that $H(t, x, y) = 0$, if $|y| \geq R$.

In particular, the C^∞-smooth map $\mathcal{Z} : \mathbb{R} \times \mathbb{R}^{2N} \to \mathbb{R}^{2N}$ is now globally defined. For any t, we write $\mathcal{Z}_t := \mathcal{Z}(t, \cdot) : \mathbb{R}^{2N} \to \mathbb{R}^{2N}$, and denote by $\mathcal{X}_t, \mathcal{Y}_t : \mathbb{R}^{2N} \to \mathbb{R}^N$ the corresponding components, i.e., $\mathcal{Z}_t = (\mathcal{X}_t, \mathcal{Y}_t)$. The following assertions are standard consequences from our assumptions.

(i) *$\mathcal{Z}_0$ is the identity map in $\mathbb{R}^{2N}$;*

(ii) *$\mathcal{Z}_t(\zeta + p) = \mathcal{Z}_t(\zeta) + p$, if $p \in 2\pi\mathbb{Z}^N \times \{0\}$;*

(iii) *each $\mathcal{Z}_t$ is a canonical C^∞-diffeomorphism of $\mathbb{R}^{2N}$ on itself;*

(iv) *$\mathcal{Z}(t, \xi, \eta) = (\xi, \eta)$, if $|\eta| \geq R$;*

(v) *there is some constant $\epsilon \in]0, 1[$ such that*

$$\mathcal{X}_T(\xi, \eta) - \xi \notin \{\alpha \eta : \alpha \geq 0\}, \quad \text{if } 1 \leq |\eta| \leq 1 + \epsilon.$$

Choose now a C^∞-function $\gamma : [0, +\infty[\to \mathbb{R}$, with

$$
[\mathbf{h}] \quad \begin{cases} \gamma(s) = 0 \text{ on } [0,1]\,, \quad \gamma'(s) \geq 0 \text{ on }]1, 1+\epsilon[\,, \\ \gamma'(s) \geq 1 \text{ on } [1+\epsilon, 2]\,, \quad \gamma(s) = s^2 \text{ on } [2, +\infty[\,, \end{cases}
$$

and let $\lambda > 0$ be a parameter, to be fixed later. We define the function $\mathfrak{R}_\lambda : \mathbb{R}^{2N} \to \mathbb{R}$ as

$$
\mathfrak{R}_\lambda(\xi, \eta) := -\lambda \gamma(|\eta|)\,,
$$

and the function $R_\lambda : \mathbb{R} \times \mathbb{R}^{2N} \to \mathbb{R}$ by

$$
R_\lambda(t, \cdot) := \mathfrak{R}_\lambda \circ \mathcal{Z}_t^{-1}, \quad \text{if } 0 \leq t < T\,,
$$

extended by T-periodicity in t. Now, set

$$
\widetilde{H}_\lambda(t, z) := H(t, z) + R_\lambda(t, z)\,.
$$

This function $\widetilde{H}_\lambda : \mathbb{R} \times \mathbb{R}^{2N} \to \mathbb{R}$ will be referred to as 'the modified Hamiltonian', and one easily checks that:

(vi) $\widetilde{H}_\lambda(t, z) = \widetilde{H}_\lambda(t + T, z) = \widetilde{H}_\lambda(t, z + p)$, *if* $p \in 2\pi\mathbb{Z}^N \times \{0\}$*;*

(vii) $\widetilde{H}_\lambda(t, x, y) = -\lambda |y|^2$, *if* $|y| \geq R$*;*

(viii) $\widetilde{H}_\lambda$ *and* H *coincide on the open set* $\big\{(t, \mathcal{Z}(t, \xi, \eta)) : 0 < t < T,\ \eta \in B\big\}$.

At this point we would like to apply either [1, Theorem 3], [4, Theorem 4.2], or [5, Theorem 8.1]; the main assumptions of these results are ensured by *(vi)* and *(vii)* above. They would provide the existence of at least $N+1$ geometrically distinct T-periodic solutions of the Hamiltonian system $(\widetilde{HS})_\lambda$ associated to the modified Hamiltonian, and 2^N of them if nondegenerate.

There is, however, a difficulty: these three theorems also assume that the Hamiltonian function is continuous in all variables, but our modified Hamiltonian $\widetilde{H}_\lambda$ will probably be discontinuous when t is an integer multiple of T. Nevertheless, one observes that the restriction of $\widetilde{H}_\lambda$ to $]0, T[\times \mathbb{R}^{2N}$ can be continuously extended to $[0, T] \times \mathbb{R}^{2N}$ (just by the same formula $(t, z) \mapsto H(t, z) + \mathfrak{R}_\lambda \circ \mathcal{Z}_t^{-1}$), and this extension is now C^∞-smooth on $[0, T] \times \mathbb{R}^{2N}$. Under this condition, the proofs of the three results just mentioned keep their validity; hence, we obtain indeed the existence of $N + 1$ geometrically distinct T-periodic solutions of $(\widetilde{HS})_\lambda$. Moreover, if they are nondegenerate, then there are at least 2^N of them.

As a consequence of *(viii)*, the Hamiltonian systems *(HS)* and $(\widetilde{HS})_\lambda$ have the same T-periodic solutions $z(t) = (x(t), y(t))$ departing with $y(0) \in B$. Thus, in order to complete the proof of Theorem 1.1, it will suffice to check the following

Proposition. If $\lambda > 0$ is large enough, then $(\widetilde{HS})_\lambda$ does not have T-periodic solutions $z(t) = (x(t), y(t))$ departing with $y(0) \notin B$.

Proof. In view of ($\bullet$), we may choose some constant $c > 0$ such that

$$
\left| \frac{\partial H}{\partial y}(t, x, y) \right| \leq c\,, \qquad \text{for every} \quad (t, x, y) \in \mathbb{R} \times \mathbb{R}^N \times \mathbb{R}^N\,,
$$

and observe that, consequently,

$$|\mathcal{X}_T(\xi,\eta)-\xi| \le cT\,, \qquad \text{for any } \ \xi,\eta\in\mathbb{R}^N\,. \tag{3}$$

It will be shown that, if

$$\lambda > c\,, \tag{4}$$

then the conclusion holds. We prove this result by a contradiction argument and assume instead that $z=(x,y):[0,T]\to\mathbb{R}^N\times\mathbb{R}^N$ is a solution of $(\widetilde{HS})_\lambda$ for such a value of λ, with $z(0)=z(T)$, departing with $|y(0)|\ge 1$. We consider the C^∞-function $\zeta:[0,T]\to\mathbb{R}^{2N}$, defined by

$$\zeta(t) := \mathcal{Z}_t^{-1}(z(t))\,.$$

Claim. $\dot\zeta = J\nabla\mathfrak{R}_\lambda(\zeta)$.

Proof of the Claim. Differentiating in the equality $z(t)=\mathcal{Z}(t,\zeta(t))$, we find

$$\dot z = \frac{\partial\mathcal{Z}}{\partial t}(t,\zeta)+\frac{\partial\mathcal{Z}}{\partial\zeta}(t,\zeta)\dot\zeta\,,$$

so that

$$\frac{\partial\mathcal{Z}}{\partial\zeta}(t,\zeta)\dot\zeta = J\nabla\widetilde{H}_\lambda(t,z)-J\nabla H(t,z) = J\nabla R_\lambda(t,z)\,. \tag{5}$$

By (*iii*) above, $\mathcal{Z}_t$ is canonical, so that

$$\frac{\partial\mathcal{Z}}{\partial\zeta}(t,\zeta(t))^* J\,\frac{\partial\mathcal{Z}}{\partial\zeta}(t,\zeta(t)) = J\,, \quad \text{for every } t\in[0,T]\,.$$

Hence, if we multiply both sides of (5) by $-J(\partial\mathcal{Z}/\partial\zeta)^*J$, we get

$$\dot\zeta = J\frac{\partial\mathcal{Z}}{\partial\zeta}(t,\zeta)^*\,\nabla R_\lambda(t,z) = J\nabla\mathfrak{R}_\lambda(\zeta)\,,$$

the last equality coming from the fact that $R_\lambda(t,\mathcal{Z}(t,\zeta))=\mathfrak{R}_\lambda(\zeta)$. This finishes the proof of the Claim. □

Let us now complete the proof of our Proposition. We write $\zeta(t)=(\xi(t),\eta(t))$; combining the Claim and the definition of $\mathfrak{R}_\lambda$, we have

$$\dot\xi = -\lambda\gamma'(|\eta|)\frac{\eta}{|\eta|}\,, \qquad \dot\eta = 0\,,$$

and consequently, recalling *(i)*,

$$\eta(t)=\eta(0)=y(0)\,, \qquad \xi(t)=x(0)-t\lambda\gamma'(|y(0)|)\frac{y(0)}{|y(0)|}\,,$$

for every $t\in[0,T]$. In particular,

$$x(T)=\mathcal{X}_T\big(\xi(T),\eta(T)\big)=\mathcal{X}_T\left(x(0)-T\lambda\gamma'(|y(0)|)\frac{y(0)}{|y(0)|}\,,\,y(0)\right). \tag{6}$$

In order to obtain the desired contradiction, we shall show that $x(T)\neq x(0)$. We distinguish three cases:

Case I: $1 \leq |y(0)| < 1+\epsilon$. Since $\gamma'(|y(0)|) \geq 0$, by [**h**], the combination of (6) and *(v)* gives

$$x(T) - x(0) + T\lambda\gamma'(|y(0)|)\frac{y(0)}{|y(0)|} \notin \{\alpha y(0) : \alpha \geq 0\}\,,$$

implying that $x(T) \neq x(0)$.

Case II: $1+\epsilon \leq |y(0)| \leq R$. By the triangle inequality,

$$|x(T) - x(0)| \geq T\lambda\gamma'(|y(0)|) - \left| x(T) - x(0) + T\lambda\gamma'(|y(0)|)\frac{y(0)}{|y(0)|} \right|\,,$$

and remembering that $\gamma'(|y(0)|) \geq 1$, by [**h**], the joint action of (3), (4) and (6) gives

$$|x(T) - x(0)| \geq T\lambda - Tc > 0\,,$$

implying again that $x(T) \neq x(0)$.

Case III: $|y(0)| > R$. Now $\gamma'(|y(0)|) = 2|y(0)|$, by [**h**]; combining (6) and *(iv)* we have that $x(T) = x(0) - 2T\lambda y(0)$. In particular, $x(T) \neq x(0)$ also in this case.

The proof is complete. □

Remark. Even though we have always assumed, for the sake of simplicity, that H is C^∞-smooth with respect to all variables, everything in the proof works just the same by assuming that this dependence is merely of class C^3 with respect to the state variable z.

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