ATTRACTORS OF DISSIPATIVE HOMEOMORPHISMS OF THE INFINITE SURFACE HOMEOMORPHIC TO A PUNCTURED SPHERE

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Abstract. A class of dissipative orientation preserving homeomorphisms of the infinite annulus, pairs of pants, and generally infinite surface homeomorphic to a punctured sphere is considered. We prove that in some isotopy classes the local behavior of such homeomorphisms at a fixed point, namely the existence of so-called inverse saddle, impacts the topology of the attractor - it cannot be arcwise connected.

1. Introduction

Forced nonlinear oscillators with friction are widely studied for their rich dynamical behavior and because they arise in many physical phenomena [1, 2, 7, 8, 9, 10, 14, 15]. A remarkable property of these models is that they admit a global attractor. The importance of the global attractor stems from the fact that it captures the relevant long-term dynamics. In particular, it contains all nontrivial periodic orbits, invariant closed curves and chaotic patterns. Informally speaking, we can focus our attention only on the global attractor to describe the dynamics of the model. This is a folkloric result in physics with deep repercussions. For example, the damped pendulum with torque,

$$\varphi'' + G\varphi' + \sin\varphi = q(t), \tag{1.1}$$

can be studied via a circle map when G is large and $q(t)$ is periodic.

To visualize this simplification, we first realize that (1.1) is invariant under $\varphi \to \varphi + 2k\pi$ with $k \in \mathbb{N}$. Thus, the cylinder is the natural phase space of (1.1). Next we check that the global attractor of (1.1) is homeomorphic to a circle. See [7] for the detailed mathematical justification of this fact.

Understanding the structure of the attractors in dissipative systems is of critical importance to analyze their behaviour and employ the above type of reductions. However, our knowledge on this problem is still under development. Several results in the literature [1, 2, 7, 9, 15] have provided sufficient conditions to guarantee a simple structure of the attractor. In the opposite side, Nakajima [11] gave an interesting result that related the local behavior of a dynamical system with the complexity of the attractor. More in detail, his framework was a smooth planar dynamical system f having two distinct fixed points, one of which is an inverse saddle. By an inverse

2010 *Mathematics Subject Classification.* Primary: 37E30, 37C25.

Key words and phrases. Attractors, dissipative homeomorphisms, inverse saddle, fixed point index.

Grzegorz Graff was supported by the National Science Centre, Poland within the grant Sheng 1 UMO-2018/30/Q/ST1/00228.

saddle p we understand a fixed point such that the eigenvalues of $Df(p)$, λ_1 and λ_2, satisfy $\lambda_1 < -1 < \lambda_2 < 0$. Nakajima proved that the existence of an inverse saddle (under some additional assumptions) implies that the attractor is not arcwise connected.

In this paper we generalize Nakajima's result, proving its counterpart for surfaces homeomorphic to a punctured sphere. Moreover, we consider the problem in a more general setting. Specifically, we drop the assumption of smoothness and examine dissipative maps that have an attractor with empty interior. Our results provide isotopy classes of homeomorphisms for which the existence of an inverse saddle forces that the attractor is not arcwise connected. Furthermore, we show that our results are optimal in the case of cylinder (Example 3.7) and pairs of pants (Example 4.9), i.e. for homeomorphisms that do not satisfy the proposed conditions one can find arcwise connected attractors. It is worth emphasizing that the study of a dynamical system on a cylinder naturally arises when the models satisfy certain periodicity (e.g. model (1.1)). The possible applications of systems defined on general punctured sphere is an issue that possibly will deserve more attention in the future.

From a more applied point of view, this paper offers new dynamical insights on the global attractor in some classical models coming from non-conservative mechanics. This perspective of our results is related to Martins' work [9]. He considers isolated fixed points of a homeomorphism h called *inverse unstable*. These are the points p satisfying the conditions $\mathrm{ind}(h,p) = 1$ and $\mathrm{ind}(h^2,p) = -1$, where $\mathrm{ind}(h,p)$ denotes fixed point index of h at p. The main result of [9] claims that the attractor of the Poincaré map associated with the pendulum equation with friction (1.1) is not homeomorphic to $\mathbb{S}^1$ provided there exists an inverse unstable fixed point. Under slightly more restrictive assumptions to those in [9], this paper guarantees a stronger property, namely that the attractor is not arcwise connected.

The paper is organized in the following way. In Section 2 we introduce some basic definitions and some preliminary results that will be used in the next parts of the paper. Section 3 is devoted to model the situation of the cylinder, while in Section 4 we deal with the general case of surfaces homeomorphic to many-punctured sphere. In this section we also consider in more details the case of infinite pair of pants surface.

2. The Levinson class of maps on manifolds

In this section we introduce the basic mathematical framework of the paper.

Let M_s be an infinite surface homeomorphic to the punctured sphere $M_s \approx \mathbb{S}^2 \setminus \{p_1, \dots, p_s\}$, i.e. M_2 is an infinite cylinder, M_3 is a pair of pants surfaces and so on.

We will call a homeomorphism of M_s dissipative if there exists a compact set $\tilde{M}_s \subset M_s$ satisfying the condition:

$$\tilde{M}_s \text{ attracts uniformly all compact sets.} \tag{2.1}$$

For a dissipative homeomorphism $h : M_s \to M_s$ we will consider the attractor $\mathcal{A} \subset M_s$, defined as the maximal compact invariant set. This set always exists and is a non-empty continuum (i.e. compact and connected set). In the giving setting the attractor $\mathcal{A}$ may also be equivalently defined as the set of all bounded (forward and backward) orbits (cf. [6]).

Remark 2.1. It is known that for dissipative homeomorphisms the Čech cohomology of the attractor and M_s are isomorphic cf. [13].

Definition 2.2. We will call a homeomorphism $h : M_s \to M_s$ a Levinson homeomorphism if it is dissipative and $\mathcal{A}$ has an empty interior. We will denote by $\mathcal{LH}(M_s)$ the class of all Levinson homeomorphisms of M_s and by $\mathcal{H}(M_s)$ the class of all homeomorphisms of M_s.

Let us remark that $\mathcal{LH}(M_s)$ contains the important class of dissipative homeomorphisms contracting some Borel measure on M_s. This measure must enjoy that the topological disks are of positive and finite measure.

Definition 2.3. A fixed point $p \in M_s$ is an inverse saddle for $h \in \mathcal{H}(M_s)$ if there exists a topological disk D with $p \in \mathrm{int} D$ and a homeomorphism $\psi : [-1,1]\times[-1,1] \to D$ with the following properties:

- $\psi(0) = p$.
- For each $(x_1, x_2) \in [-\frac{1}{2}, \frac{1}{2}] \times [-1, 1]$,

$$\psi(L_d(x_1, x_2)) = h(\psi(x_1, x_2)), \tag{2.2}$$

  with

$$L_d(x_1, x_2) = (-2x_1, -\frac{1}{2}x_2).$$

For an inverse saddle p we define the sets $U(p) = \psi([-1, 1] \times \{0\})$, $S(p) = \psi(\{0\} \times [-1, 1])$. Notice that for each $x \in U(p)$ $h^{-n}(x) \overset{n\to+\infty}{\longrightarrow} p$. Analogously, for each $x \in S(p)$ $h^n(x) \overset{n\to+\infty}{\longrightarrow} p$. Thus, they are subsets of the unstable and stable manifolds at p, i.e., $U(p) \subset W^u(p)$ and $S(p) \subset W^s(p)$.

Remark 2.4. Notice that since L_d in Definition 2.3 is orientetion preserving, any homeomorphism with an inverse saddle is orientatation preserving.

Lemma 2.5. *Consider $h \in \mathcal{H}(M_s)$ with p being an inverse saddle for h. If $\gamma \subset M_s$ is an arc emanaiting from p then $h(\gamma) \not\subset \gamma$ and $\gamma \not\subset h(\gamma)$.*

Proof. We assume, by contradiction, that either $h(\gamma) \subset \gamma$ or $\gamma \subset h(\gamma)$. Then any sub-arc $\beta \subset \gamma$ with one end point at p will enjoy the same property, either $h(\beta) \subset \beta$ or $\beta \subset h(\beta)$. Note that the inclusions for γ and β are not necessarily the same. For instance, $h(\gamma) \subset \gamma$ and $\beta \subset h(\beta)$ is an admissible situation. The reduced disk will be defined by

$$\widehat{D} = \psi([-\frac{1}{2}, \frac{1}{2}] \times [-1, 1]). \tag{2.3}$$

Let us take a sub-arc of γ ending at p, say $\gamma_* = \overset{\frown}{pr}$, such that $\gamma_* \subset \widehat{D}$. Then, either $h(\gamma_*) \subset \gamma_*$ or $h(\gamma_*) \supset \gamma_*$, depending on the way r and $h(r)$ are ordered in γ. We observe that $L_d = \psi^{-1} \circ h \circ \psi$ on $[-\frac{1}{2}, \frac{1}{2}] \times [-1, 1]$ and define $\Gamma = \psi^{-1}(\gamma_*)$. Since Γ is contained in this reduced rectangle,

$$L_d(\Gamma) = \psi^{-1} \circ h \circ \psi(\Gamma) = \psi^{-1} h(\gamma_*).$$

Then either $L_d(\Gamma) \subset \Gamma$ or $L_d(\Gamma) \supset \Gamma$. Let us consider the first case (the second is analogous). The iterates $L_d^n(\Gamma) \subset \Gamma$ remain in a bounded set for $n \geq 0$. From this, we deduce that Γ is contained in the stable manifold of the linear map L_d. The set

$\Gamma\backslash\{0\}$ is connected and lies in $\{0\} \times (\mathbb{R}\backslash\{0\})$. Since the components of this set are permuted by L_d, the open arc $\Gamma\backslash\{0\}$ must satisfy

$$L_d(\Gamma\backslash\{0\}) \cap (\Gamma\backslash\{0\}) = \emptyset.$$

This is the searched contradiction.

$\square$

The next lemma will be needed in the proof of the main results in the forthcoming sections. Notice that it is valid for any surface, as it deals with local properties of the plane.

Lemma 2.6. *Let $\alpha, \beta \subset M_s$ be two arcs that join points p and q with $p \neq q$.*
Assume, for every neighborhood $U \subset M_s$ of p,

$$\alpha \cap U \neq \beta \cap U. \tag{2.4}$$

Then, one of the following conditions holds:

(a) *There exists a Jordan curve Γ in M_s satisfying*

$$\Gamma \subset \alpha \cup \beta, p \in \Gamma. \tag{2.5}$$

(b) *There exists a sequence of Jordan curves in M_s, $\{\Gamma_n\}_{n\geq 0}$, satisfying*

$$\Gamma_n \subset \alpha \cup \beta, p \notin \Gamma_n, \Gamma_n \overset{n\to\infty}{\longrightarrow} \{p\}, \tag{2.6}$$

with the convergence in respect to the Hausdorff distance.

Proof. We start the proof with considering the case in which there exists an neighborhood $U \subset M_s$ such that

$$\alpha \cap \beta \cap U = \{p\}. \tag{2.7}$$

We use the natural order $\prec$ in α in which $p \prec q$.

Let us define $r = \sup\{t \in \alpha\colon \ \overset{\frown}{pt} \cap \beta = \{p\}\}$. If $r = q$ then we have our Jordan curve $\Gamma = \alpha \cup \beta$. Otherwise, let us note that $r \in \beta$, which follows from the definition of r. Now, let α_1 be the sub-arc of α with the end points p and r and similarly, β_1 be the sub-arc of β with the end points p and r. Finally, $\Gamma = \alpha_1 \cup \beta_1$ is the Jordan curve that satisfies the condition (a).

Assume now that the condition (2.7) is not satisfied. Then there exists a sequence of points $(p_n)_n$, $p_n \to p$, $p_n \neq p$ with $p_n \in \alpha \cap \beta$.

Since (2.4) holds, it is not restrictive to assume the existence of a second sequence $(q_n)_n$, $q_n \to p$, $q_n \neq p$ with $q_n \in \alpha \setminus \beta$. Otherwise we change the roles of α and β. In the relative topology of the arc α, the set $\dot{\alpha} \setminus \beta$ is open and non-empty. Moreover, the sequence q_n is contained in this set but p_n is in the complement. This fact implies that $\dot{\alpha} \setminus \beta$ has infinitely many connected components, that is,

$$\dot{\alpha} \setminus \beta = \bigcup_{i\in\mathbb{N}} \alpha_i. \tag{2.8}$$

Now for each n select $\sigma(n) \in \mathbb{N}$ such that $q_n \in \alpha_{\sigma(n)}$. Let r_n and s_n be the end points of $\alpha_{\sigma(n)}$. Next, denote by $\beta_{\sigma(n)}$ the sub-arc of β connecting r_n with s_n. Then

$$\Gamma_n = \alpha_{\sigma(n)} \cup \beta_{\sigma(n)}$$

is a Jordan curve with $p \notin \Gamma_n$. Since $p_n \to p$ and $q_n \in \alpha_{\sigma(n)}$, we conclude that $r_n, s_n \to p$. Hence, $\alpha_{\sigma(n)} \to \{p\}$. The same applies to $\beta_{\sigma(n)}$, because β is an arc. We get that the conditions of the case (b) are satisfied. $\square$

3. Arcwise-connectedness of attractors of homeomorphisms of the cylinder

In this section we will analyze the case of M_2, the cylinder (denoted as C) in a detailed manner. In the next parts of the paper we will use some results proved below.

We start from the discussion on the maximum number of Jordan curves in the attractor of Levinson homeomorphisms on the cylinder.

Lemma 3.1. *If $h \in \mathcal{LH}(C)$, then $\mathcal{A}$ contains either none Jordan curve or one Jordan curve that is not contractible.*

Proof. Assume, contrary to our claim, that $\mathcal{A}$ contains at least two Jordan curves Γ_1, Γ_2, each of which is not contractible. Then for each $n \in \mathbb{Z}$, $h^n(\Gamma_i) \subset \mathcal{A}$, $i = 1, 2$. Let $R(\Gamma_1, \Gamma_2)$ be the union of the bounded components of $C \setminus (\Gamma_1 \cup \Gamma_2)$. Using the Jordan-Schönflies theorem on $\mathbb{S}^2 = C \cup \{-\infty, \infty\}$ it can be proved that $R(\Gamma_1, \Gamma_2)$ is not empty. Next we take $\hat{C}$ a big enough bounded cylinder containing $\mathcal{A}$. Then, we get

$$\bigcup_{n\in\mathbb{Z}} h^n(R(\Gamma_1, \Gamma_2)) = \bigcup_{n\in\mathbb{Z}} R(h^n(\Gamma_1), h^n(\Gamma_2)) \subset \hat{C}. \tag{3.1}$$

As a consequence, all orbits in $R(\Gamma_1, \Gamma_2)$ are bounded and thus $R(\Gamma_1, \Gamma_2) \subset \mathcal{A}$, which contradicts the fact that $\mathcal{A}$ has empty interior. The similar reasoning applies in case $\mathcal{A}$ contains a Jordan curve that is contractible. $\square$

Now we describe the isotopy classes of the orientation preserving homeomorphisms of the cylinder.

Remark 3.2. (cf. [5] subsection "The twice-punctured sphere".)

There are two classes of isotopy of orientation preserving homeomorphisms in $\mathcal{LH}(C)$ given by the representants:

- $h_1(\theta, r) = (\theta, r)$,
- $h_2(\theta, r) = (-\theta, -r)$.

A homeomorphism h of the cylinder $C = \mathbb{S}^1 \times \mathbb{R}$ can be extended to $\tilde{h}$, the homeomorphism of the sphere $\mathbb{S}^2$, by compactification with two points $+\infty$ and $-\infty$.

Lemma 3.3. *Let $h \in \mathcal{H}(C)$ be an orientation preserving homeomorphism. Assume that there exists an invariant Jordan curve $\Gamma \subset C$, $h(\Gamma) = \Gamma$, such that*

(i) *Γ is not contractible,*
(ii) *the restriction of h to Γ, $h_\Gamma : \Gamma \to \Gamma$, is orientation reversing.*

Then, h interchanges the components of $C \setminus \Gamma$ and $h \sim h_2$.

Proof. Let $R_+ = \{(\theta, r) \in C : r > 0\}$ and $R_- = \{(\theta, r) \in C : r < 0\}$, $\Gamma = \{(\theta, r) \in C : r = 0\}$. Let us take $\mathbb{S}^2 = \{(x, y, z) \in \mathbb{R}^3 : x^2 + y^2 + z^2 = 1\}$, the considered compactification of C, where we identify Γ with $\{(x, y, z) \in \mathbb{S}^2 : z = 0\} = \mathbb{S}^1 \times \{0\}$.

After the reduction we can assume that $h(\theta, 0) = h_\Gamma(\theta, 0)$. We apply the Jordan-Schönflies theorem in $\mathbb{S}^2$ to Γ. The homeomorphism $\Psi(\theta, r) = (h_\Gamma^{-1}(\theta), -r)$ is isotopic to h_2. Therefore, $H = \Psi \circ h$ is an orientation preserving homeomorphism satisfying $H = \mathrm{id}$ on Γ. We can adapt the proof of Lemma 22 in [12] to the cylinder to conclude that H is isotopic to the identity in C (relative to Γ). In particular H preserves the regions R_+ and R_-. The conclusion of Lemma now follows from the fact that $H = \Psi \circ h$ is isotopic to identity and Ψ is isotopic to h_2.

□

Now we are ready to prove the main theorem of this section.

Theorem 3.4. *Let $h \in \mathcal{LH}(C)$. Assume that h is isotopic to identity and has an inverse saddle p. Then the attractor $\mathcal{A}$ is not arcwise connected.*

Proof. We show first that there must be at least one more fixed point of f. Let us consider the cylinder $C_R := \mathbb{S}^1 \times [-R, R]$, where R is large enough.

We will denote the Lefschetz number of f and the fixed point index of f on the set A by $L(f)$ and $\mathrm{ind}(f, A)$ respectively.

Since h is dissipative, $h^n(C_R) \subset \mathrm{int}(C_R)$ for all large enough n.

Therefore,

$$\mathrm{ind}(h^n, C_R) = L(h^n_{|C_R}) = \chi(\mathbb{S}^1) = 0, \tag{3.2}$$

where χ denotes the Euler characteristic.

On the other hand, for each prime number p, the congruences

$$\mathrm{ind}(h^p, C_R) \equiv \mathrm{ind}(h, C_R) \pmod{p}. \tag{3.3}$$

are a particular case of the more general Dold relations (cf. [3, 4]). As a consequence of the formulas (3.2) and (3.3) we obtain that for infinitely many primes p, $p | \mathrm{ind}(h, C_R)$. Thus $\mathrm{ind}(h, C_R) = 0$.

The point p is an isolated fixed point because it is a saddle. Hence, by the additivity property of the fixed point index for $h_R := h_{|C_R}$, we obtain that

$$\mathrm{ind}(h, C_R) = \mathrm{ind}(h_R, p) + \mathrm{ind}(h_R, \mathrm{Fix}(h_R) \setminus \{p\}) = 0. \tag{3.4}$$

On the other hand, since p is an inverse saddle, $\mathrm{ind}(h_R, p) = 1$. As a consequence, there must be at least one more fixed point of h_R, say q.

Now we will show that the fixed points p and q cannot be connected by an arc $\gamma \subset \mathcal{A}$. By a contradiction argument let us take an arc $\gamma \subset \mathcal{A}$ joining the points p and q. It is not restrictive to assume that $\gamma \cap \mathrm{Fix}(h) = \{p, q\}$ (otherwise we take the fixed point closest to p on γ as q).

We distinguish two cases:

(1) $h(\gamma) = \gamma$.

(2) $h(\gamma) \neq \gamma$.

The case (1) could not hold because there are no invariant (in the past or in the future) arcs emanating from the inverse saddle p by Lemma 2.5.

Let us now assume that the case (2) holds.

We are going to apply Lemma 2.6 with $\alpha = \gamma$ and $\beta = h(\gamma)$. Notice that Lemma 2.5 implies that assumption (2.4) holds. To see this claim, observe that if $\alpha \cap U = \beta \cap U$ for some neighborhood of p, then there exists a subarc of arc γ emanating from p

which is invariant under h, either in the past or in the future. However, this is not possible because p is an inverse saddle (see Lemma 2.5 again).

Now we know that one of the alternatives of Lemma 2.6 holds. On the other hand, the alternative (b) is impossible because $\mathcal{A}$ can contain only one Jordan curve (see Lemma 3.1). In consequence, (a) must hold.

Let $\Gamma \subset \gamma \cup h(\gamma)$ be the Jordan curve with $p \in \Gamma$. Since $\Gamma \subset \mathcal{A}$, Γ is non-contractible. Moreover, Γ must be invariant under h. (If $h(\Gamma) \neq \Gamma$, $\mathcal{A}$ would contain two Jordan curves, namely $\Gamma, h(\Gamma)$, a contradiction with Lemma 3.1). Next, we distinguish two cases:

(2i) $q \notin \Gamma$,

(2ii) $q \in \Gamma$.

In the case (2i) p is the only fixed point of the map $h_\Gamma : \Gamma \to \Gamma$. Then all orbits in Γ must converge to p in the future and in the past. This, however, is impossible because Γ would be simultaneously a branch of the stable and unstable manifold of p, but such type of homoclinic loop cannot occur for an inverse saddle.

Finally, assume that (2ii) holds. Then h_Γ has exactly two fixed points (p and q) and the components of $\Gamma \setminus \{p, q\}$ must be interchanged (as $h(\gamma) \neq \gamma$). This implies that h_Γ is orientation reversing. We can now apply Lemma 3.3 to conclude that h is isotopic to h_2, in contradiction to our assumption. □

As mentioned in Introduction, Theorem 3.4 is useful for the study of some classical models in non-conservative mechanics. A particular example is

$$x'' + h(x)x' + g(t, x) = 0, \tag{3.5}$$

where h, g are smooth functions, 2π-periodic on x and T-periodic on t with $h(x) > 0$ for all $x \in \mathbb{R}$. Since h and g are bounded, all solutions of (3.5) are globally defined. The dynamical behavior of (3.5) is determined by the Poincaré map

$$P : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$$

$$(x_0, v_0) \mapsto (x(T; (x_0, v_0)), x'(T; (x_0, v_0))),$$

where $x(t; (x_0, v_0))$ is the unique solution of (3.5) that satisfies $x(0; (x_0, v_0)) = x_0$ and $x'(0; (x_0, v_0)) = v_0$, (see Chapter 1 in [12]).

The periodicity of the state variables in (3.5) implies that P satisfies

$$P(x_0 + 2\pi, v_0) = P(x_0, v_0) + (2\pi, 0).$$

Then P naturally induces a homeomorphism of the cylinder denoted by $\bar{P} : C \to C$ (see Introduction and Section 1 in [9]). The map $\bar{P}$ is dissipative (see [10] for more details) and area contracting. This last property is a consequence of the Liouville formula:

$$\det P'(x_0, v_0) = e^{-\int_0^T h(x(t;(x_0,v_0)))dt} < 1.$$

We also recall that the Poincaré map of a differential equation in $\mathbb{R}^2$ is always isotopic to identity (see Chapter 1 in [12]). This isotopy can be chosen with periodicity properties so that it induces an isotopy beteween $\bar{P}$ and identity on C

Collecting the above information, we can deduce the following result as a direct consequence of Theorem 3.4.

Theorem 3.5. *If the Poincaré map P of (3.5) admits an inverse saddle, then the attractor (associated with $\bar{P}$) is not arcwise connected.*

The reader can consult Section 3 in [9] for sufficient conditions on the existence of inverse saddles in (3.5). We emphasize that the definition of inverse unstable fixed point in [9] allows the eigenvalues 1 or -1. However, the unstable fixed points deduced in [9] are indeed inverse saddles (according to our definition). Observe that condition (10) page 362 in [9] excludes the case of an eigenvalue with modulus equal to 1. It is worth mentioning that the conclusion in [9], namely the attractor is not homeomorphic to S^1, is weaker than our conclusion.

Under the additional assumption that there is another fixed point in addition to an inverse saddle, the counterpart of Theorem 3.4 is also true for the plane (with the appropriate definition of dissipative homeomorphism). Actually, this is the non-smooth version of Nakajima theorem mentioned in Introduction.

Theorem 3.6. *Let $h \in \mathcal{LH}(\mathbb{R}^2)$. Assume that h has an inverse saddle p and there is another fixed point $q \neq p$ of h. Then the attractor $\mathcal{A}$ is not arcwise connected.*

Proof. We assume, by contradiction, that $\mathcal{A}$ is an arcwise connected set. Then, there is an arc γ that joins p and q. Using that p and q are fixed points, $h(\gamma)$ is also an arc in $\mathcal{A}$ joining p and q. As $\gamma \neq h(\gamma)$ by Lemma 2.5, we get that $\gamma \cup h(\gamma)$ contains a Jordan curve by Lemma 2.6. On the other hand, we may prove in the same way as in Lemma 3.1 that $\mathcal{A}$ contains no Jordan curves. $\square$

Example 3.7. Theorem 3.4 is no longer true if we drop the assumption that h is isotopic to identity. We illustrate it by the following counterexample. Let $C = [-4,4] \times \mathbb{R}$ be the vertical strip where the points $(-4, x_2)$ and $(4, x_2)$ are identified. We consider a map $G : C \longrightarrow C$ such that $p = (0,0)$, $q = (4,0)$ are fixed points of G, q is a sink and p is a saddle. For instance we may take $G(x_1, x_2) = (f(x_1), \frac{x_2}{2})$ with $f : [-4,4] \longrightarrow [-4,4]$ a strictly increasing and smooth function with $f(0) = 0$, $f(-4) = -4$, $f(4) = 4$, $f(x_1) > x_1$ for all $x_1 \in (0,4)$ and $f(x_1) < x_1$ for all $x_1 \in (-4,0)$. In addition, $f(x_1) = -f(x_1)$ if $x_1 \in [-4,4]$.

The non-contractible Jordan curve $\gamma = [-4,4] \times \{0\}$ joins p and q and is the unstable manifold for p. Finally, consider the map $g := G \circ h_2$. Notice that p is an inverse saddle for g. Due to the symmetry of f, G and h_2 commute. Therefore $g^{2n} = G^{2n}$ and $g^{2n+1} = G^{2n+1} \circ h_2$. It is now easy to conclude that the attractor $\mathcal{A}$ is γ, an arcwise connected set.

Example 3.8. We will provide now an example of homeomorphism h in the class of $\mathcal{LH}(C)$ with an inverse saddle in the isotopy class of identity.

Let C be the vertical strip mentioned above. Let B_r denote the closed ball centered at the origin of radius r.

We will first construct a homeomorphism $\tilde{h} : C \to C$ such that $\tilde{h} = \mathrm{id}$ outside B_3 and $\tilde{h} = -\mathrm{id}$ in B_1. Namely,

$$\tilde{h}(x) = \begin{cases} -x & \text{if} \quad ||x|| \leq 1, \\ R_{\pi ||x||}(x) & \text{if} \quad 1 \leq ||x|| \leq 2, \\ x & \text{if} \quad 2 \leq ||x|| < 3, \end{cases}$$

where $R_\alpha : B \to B$ denotes the rotation of the angle α with the center at the origin and $|| \cdot ||$ is the Euclidean norm.

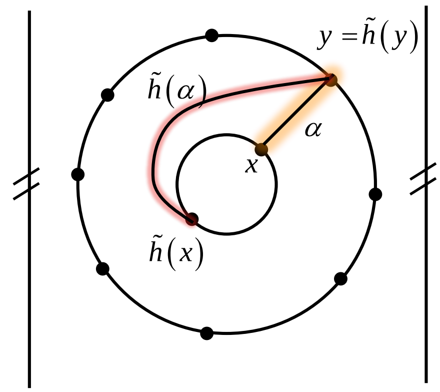

FIGURE 1. The construction of the homeomorphism $\tilde{h}$ in Example 3.8, the parallel lines are identified to obtain a cylinder.

Consider now the map $G : C \to C$ constructed in Example 3.7, with the fixed points in $q = (4, 0)$ (sink) and $p = (0, 0)$ (saddle). Let us define the map $g := G \circ \tilde{h}$. This is a homeomorphism in the homotopy class of identity with an inverse saddle in p.

We observe that G and g coincide outside a compact set, and so the diffeomorphism g is dissipative. After imposing the additional conditions: $\max_{x_1 \in [-4,4]} |f'(x_1)| < \frac{1}{2}$ to the function f, we notice that $\det G'(x_1, x_2) < 1$ for every (x_1, x_2). Since $\tilde{h}$ is area preserving, we conclude that g is area contracting. This allows us to say that g belongs to $\mathcal{LH}(C)$.

4. ARCWISE-CONNECTEDNESS OF ATTRACTORS OF HOMEOMORPHISMS OF M_s FOR $s > 2$

In this section we prove the counterpart of Theorem 3.4 for the space M_s, where $s > 2$.

Let us denote by $\mathrm{Mod}(M_s)$ the mapping class group of M_s i.e. the group of isotopy classes of orientation-preserving homeomorphisms of M_s.

It is convenient to interpret M_s as the sphere $\mathbb{S}^2$ surface with punctures, say $M_s = \mathbb{S}^2 \backslash \mathcal{P}_s$ with $\mathcal{P}_s = \{z_1, \ldots, z_s\}$. Here $z_1, \ldots, z_s$ are s prescribed different points in $\mathbb{S}^2$. Every homeomorphism h of M_s admits an extension to a homeomorphism $\widetilde{h}$ of $\mathbb{S}^2$. Then $\mathrm{Mod}(M_s)$ is the group of homeomorphisms $\tilde{h} : \mathbb{S}^2 \to \mathbb{S}^2$ that leave the set of marked points invariant, modulo isotopies that leave the set of marked points invariant [5].

Notice that two homeomorphisms h_1, h_2 belong to the same isotopy class if the behaviour of $\widetilde{h}_1, \widetilde{h}_2$ coincides on the punctures. In other words, $\widetilde{h}_1$ permutes the punctures in the same manner as $\widetilde{h}_2$.

In the sequel, given h a homeomorphism on $\mathbb{S}^2 \backslash \mathcal{P}_s$, $T(h)$ denotes the permutation of the punctures mentioned above.

To simplify notation, we will denote by the same letter a homeomorphism on an unbounded space M_s and its conjugated counterpart on the sphere with s punctures.

Remark 4.1. Let Σ_s denote the group of permutation of s elements. Notice that $T : \mathrm{Mod}(M_s) \to \Sigma_s$ is an injective group homomorphism.

With each non-contractible Jordan curve $\Gamma \subset M_s$, we can associate a splitting of the set of punctures. Jordan theorem implies that $\mathbb{S}^2 \backslash \Gamma$ has two connected components C_1 and C_2. Then the sets

$$X_1 = \mathcal{P}_s \cap C_1, X_2 = \mathcal{P}_s \cap C_2$$

are non-empty, for otherwise the curve Γ would be contractible. The splitting $\{X_1, X_2\}$ is a partition of $\mathcal{P}_s$.

Given a homeomorphism h of M_s, the splitting of the curve $\widehat{\Gamma} = h(\Gamma)$ will be induced by h. In fact, it can be described as $\{\widehat{X}_1, \widehat{X}_2\}$ with $\widehat{X}_i = \widetilde{h}(X_i)$. Also note that $\widehat{X}_i = \sigma(X_i)$ where σ is the permutation of $\mathcal{P}_s$ induced by $\widetilde{h}$. This fact will be used without further mention.

Given $h \in \mathcal{LH}(M_s)$, the attractor $\mathcal{A} \subset M_s$ can only contain a finite number of Jordan curves and they can be labelled by their splitting. This fact will follow from the next lemma, whose proof is analogous to the proof of Lemma 3.1.

Lemma 4.2. *If $h \in \mathcal{LH}(M_s)$, then for any $k \in \mathbb{N}$ there are no Jordan curves $\Gamma_1, \ldots, \Gamma_k$ in $\mathcal{A}$ such that some connected component of $\mathbb{S}^2 \setminus (\Gamma_1 \cup \ldots \cup \Gamma_k)$ is contractible in M_s*

Assume that Γ_1 and Γ_2 are two different Jordan curves contained in $\mathcal{A}$. From the previous result, we know that they are non-contractible and now we claim that the corresponding splittings are different. By a contradiction argument assume that both curves have the same splitting $\{X_1, X_2\}$. Then $\mathbb{S}^2 \backslash (\Gamma_1 \cup \Gamma_2)$ must have a connected component without punctures. This component would be contractible and this would be a contradiction with Lemma 4.2.

Now we may formulate the main theorem in this section.

Theorem 4.3. *Let $h \in \mathcal{LH}(M_s)$, $s > 2$. Assume that h has an inverse saddle p and that there is another fixed point $q \neq p$ of h. Assume also that $T(h)$ is a product of disjoint odd cycles. Then the attractor $\mathcal{A}$ is not arcwise connected.*

Before we give the justification of this theorem, we first prove some lemmas.

Definition 4.4. Let $\sigma \in \Sigma_s$. We say that σ has the *even property* if there exists an even integer $N \geq 2$ and a proper subset X of $\{1, \ldots, s\}$ such that

$$\sigma^N(X) = X, \;\; \sigma^k(X) \neq X \text{ for } 1 \leq k < N. \tag{4.1}$$

Lemma 4.5. *A permutation $\sigma \in \Sigma_m$ that is a product of odd cycles does not have the even property.*

We postpone the proof of this combinatorial lemma to the Appendix (Section 5).

Lemma 4.6. *In the conditions in Theorem 4.3 assume that* $\Gamma \subset \mathcal{A}$ *is a Jordan curve. Then there exists an odd integer* $N = N(\Gamma) \geq 1$ *such that*

$$h^k(\Gamma) \neq \Gamma \text{ if } 1 \leq k < N,\ h^N(\Gamma) = \Gamma. \tag{4.2}$$

Moreover, the components of $M_s \setminus \Gamma$ *are preserved by* h^N*.*

Proof. The Jordan curves in the sequence $\Gamma, h(\Gamma), h^2(\Gamma), \ldots, h^n(\Gamma), \ldots$ are contained in the attractor. Since $\mathcal{A}$ may contain only a finite number of Jordan curves, $h^n(\Gamma) = h^m(\Gamma)$ for some $n > m$.

The existence of the integer $N = N(\Gamma) \geq 1$ follows from the fact that h is one-to-one. Let us now prove that N is odd. The splitting associated to Γ is denoted by $\{X_1, X_2\}$. For each k with $1 \leq k < N$, the curves Γ and $h^k(\Gamma)$ are different and so the corresponding splittings cannot coincide (see Lemma 4.2). In particular, $h^k(X_1) \neq X_1$. From $h^N(\Gamma) = \Gamma$, we deduce that either $h^N(X_1) = X_1$ or $h^N(X_1) = X_2$. Let us prove by contradiction that the second part of this alternative is impossible. Otherwise we have: $h^{r+N}(X_1) = h^r(h^N(X_1)) = h^r(X_2)$. Notice that for each $1 \leq r < N$ the curves Γ and $h^r(\Gamma)$ cannot induce the same splitting. In particular, $h^r(X_2) \neq X_1$.

Summing up, the assumption $h^N(X_1) = X_2$ leads to

$$h^k(X_1) \neq X_1 \text{ if } 1 \leq k < 2N,\ h^{2N}(X_1) = X_1. \tag{4.3}$$

This, however, is impossible because we know that $\sigma = T(h)$ does not enjoy the even property (Lemma 4.5).

Once we know that $h^N(X_1) = X_1$, we observe that σ satisfies (4.1) with $X = X_1$. In consequence N is odd.

The components of $M_s \setminus \Gamma$ are invariant under h^N due to the fact that $h^N(X_1) = X_1$, $h^N(X_2) = X_2$.

∎

Proof of Theorem 4.3 Assume, contrary to our claim, that $\mathcal{A}$ is arcwise connected.

Let us define the arc $\gamma := \overset{\frown}{pq} \subset \mathcal{A}$, where q is another fixed point of h. Now we observe that the condition (2.4) holds for the arcs $\alpha := \gamma$ and $\beta = h(\gamma)$. This is a consequence of Lemma 2.5. The alternative (b) in Lemma 2.6 can be excluded because we know that $\mathcal{A}$ only contains a finite number of Jordan curves. Let $\Gamma \subset \gamma \cup h(\gamma)$ be a Jordan curve with $p \in \Gamma$. Since Γ is contained in the attractor, we know by Lemma 4.6 that there is a first $N \geq 1$ such that $h^N(\Gamma) = \Gamma$ and N is odd. We consider $\tilde{h} = h^N$ and observe that $\tilde{h}$ is orientation preserving and belongs to $\mathcal{LH}(M_s)$, because it is dissipative and has the same attractor as h. Moreover, p is an inverse saddle for $\tilde{h}$. As a consequence, we may repeat the same reasoning as in the proof of Theorem 3.4 (part (2i) and (2ii)) and we conclude that Γ must contain either a stable of unstable manifold of p. Then, by the analog of Lemma 3.3 applied to $\tilde{h} : \mathbb{S}^2 \to \mathbb{S}^2$, $\tilde{h}$ changes the components of $M_s \setminus \Gamma$ which contradicts Lemma 4.6. This completes the proof. ∎

Now we describe in more details the case of the space M_3, called the pair of pants surface. We start with the description of the mapping class group of M_3.

Let us denote by Σ_3 the group of permutation of a set of three elements i.e. marked points $\mathcal{P}_3 = \{z_1, z_2, z_3\}$. Then Σ_3 consists of the following cycles: $\Sigma_3 = \{id, (z_1, z_2), (z_1, z_3), (z_2, z_3), (z_1, z_2, z_3), (z_2, z_1, z_3)\}$.

Lemma 4.7. (Proposition 2.3. in [5])

$$\mathrm{Mod}(\mathrm{M}_3) \equiv \Sigma_3.$$

Theorem 4.8. *Let $h \in \mathcal{LH}(M_3)$. Assume that h has an inverse saddle p and that $T(h)$ is either identity or an 3-cycle in Σ_3. Then the attractor $\mathcal{A}$ is not arcwise connected.*

Proof. Part (I). First we prove that (under our assumption that $T(h) \in \{id, (z_1, z_2, z_3), (z_2, z_1, z_3)\}$) there exists another fixed point, say q, $q \neq p$ for a map h.

First of all, analogously to the case of cylinder, we find a compact set $G \subset M_3$, (where G is obtained as $\mathbb{S}^3$ with three open disks removed, satisfies (2.1)) and is such that for large enough n, $h^n(G) \subset G$. Define $f := h_{|G}$.

If $T(h) = id$, then $L(f^n) = \chi(G) = -1$, for n large enough, and repeating the reasoning for cylinder in the proof of Theorem 3.4, we end with counterpart of the formula (3.4):

$$\mathrm{ind}(f, G) = \mathrm{ind}(f, p) + \mathrm{ind}(f, \mathrm{Fix}(f) \setminus \{p\}), \tag{4.4}$$

which forces the existence of another fixed point in addition to p.

Now we deal with the case $T(h) \in \{(z_1, z_2, z_3), (z_2, z_1, z_3)\}$. The first homology group $H_1(G, \mathbb{Q})$ is equal to $\mathbb{Z} \oplus \mathbb{Z}$. Let us consider $\gamma_1, \gamma_2, \gamma_3$ - the boundaries of the removed disks having the same orientation. Let us identify γ_1 and γ_2 with generators of $H_1(G, \mathbb{Q})$. Assume for instance that $T(h) = (z_1, z_2, z_3)$, then the following equalities hold for maps induced by f on the first homology group:

$$f_{*1}(\gamma_1) = \gamma_2, \quad f_{*1}(\gamma_2) = -(\gamma_1 + \gamma_2). \tag{4.5}$$

Identifying linear map f_{*1} with its matrix, we get that

$$f_{*1} = \begin{bmatrix} 0 & -1 \\ 1 & -1 \end{bmatrix}. \tag{4.6}$$

As a consequence, $L(f) = 1 - \mathrm{tr} f_{*1} = 2$.

Furthermore, we have $L(f^{3k+1}) = L(f)$ for $k = 0, 1, 2, \ldots$.

By Dold relations for a prime p

$$\mathrm{ind}(f^p, G) \equiv \mathrm{ind}(f, G) \pmod{p}, \tag{4.7}$$

On the other hand, in the arithmetical progression $a_k = 3k + 1$ there are infinitely many prime numbers by Dirichlet theorem. Thus the equality (4.7) holds for infinitely many primes, which implies $\mathrm{ind}(f, G) = L(f^{3k+1}) = 2$. Finally, we get 2 in the right hand-side of the formula (4.4), which proves the existence of another fixed point.

Part (II). Now we observe that $T(h)$ as identity or 3-cycle is a product of odd disjoint cycles, and thus we may apply Theorem 4.3 for $s = 3$, and get the thesis. $\square$

Remark 4.9. Theorem 4.3 does not hold if the assumption that $T(h)$ is a product of odd cycles is dropped.

Example 4.10. Let us take $\mathbb{S}^2 = \mathbb{R}^2 \cup \{\infty\}$, $z_1 = (1,0)$, $z_2 = (-1,0)$, $z_3 = \infty$. We interpret M_3 as $\mathbb{R}^2 \smallsetminus \{z_1, z_2\}$. Consider the permutation $\sigma \in \Sigma_3$ such that $\sigma(z_1) = z_2$, $\sigma(z_2) = z_1$, $\sigma(z_3) = z_3$. Let $\{\Phi_t\}_{t\in\mathbb{R}}$ be the flow given in Figure 4. We assume that Φ_t commutes with R - the rotation of 180^o around p, i.e. $R \circ \Phi_t = \Phi_t \circ R$.

We define $h = R\circ\Phi_T$ for some fixed $T > 0$. Then h is a dissipative homeomorphism of M_3 and p is an inverse saddle. However, the attractor is an arcwise connected set (a "pair of glasses").

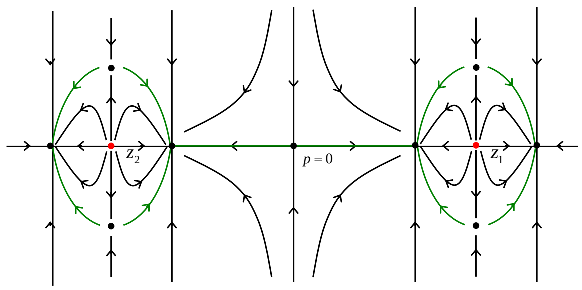

FIGURE 2. The construction of the homeomorphism h in Example 4.10, with the arcwise connected attractor.

5. Appendix: the proof of Lemma 4.5

Given a permutation in a set S of s elements, $\sigma \in \Sigma_s$, we know that it can be decomposed as a product of disjoint cycles

$$\sigma = \widehat{\sigma}_1 \cdots \widehat{\sigma}_m.$$

The set $A_i = \{j \in S : \widehat{\sigma}_i(j) \neq j\}$ will be called the support of $\widehat{\sigma}_i$. We recall that these sets are pairwise disjoint and they do not contain invariant proper subsets. The cardinal of A_i is the length of $\widehat{\sigma}_i$, denoted by l_i. Lemma 4.5 may be expressed in the form of the following Proposition:

Proposition 5.1. *In the previous notations assume that all the lengths l_i are odd. Then σ does not satisfy the even property.*

Before the proof we review some properties of cycles:

- **(i):** Assume that $\widehat{\sigma}$ is a cycle of length l and $k \geq 1$ is an arbitrary integer. Let $\alpha = (l,k)$ be the greatest common divisor of the numbers l and k and $\alpha \cdot \beta = l$. The $\widehat{\sigma}^k$ is a permutation with decomposition $\widehat{\sigma}^k = \widehat{\sigma}_1 \cdots \widehat{\sigma}_\alpha$ where each $\widehat{\sigma}_i$ is a cycle of length β.
- **(ii):** In the previous notations assume that $\alpha \cdot \theta = k$. Then, if $\widehat{\sigma}^\alpha$ has the decomposition $\widehat{\sigma}^\alpha = \tau_1 \cdots \tau_\alpha$ with τ_i a cycle of length β, $\widehat{\sigma}^k$ has the decomposition $\widehat{\sigma}^k = \tau_1^\theta \cdots \tau_\alpha^\theta$. Moreover, since θ and β are relative prime, each τ_i^θ is a cycle of length β. The supports of τ_i^θ and τ_i are the same.

We will employ these properties to prove the following:

Lemma 5.2. *Assume that $\widehat{\sigma}$ is a cycle of length l on the set $\{1, 2, ..., l\}$ and $N \geq 1$ is an arbitrary integer. Let Y be a subset of $\{1, 2, ..., l\}$ such that $\widehat{\sigma}^N(Y) = Y$. Then $\widehat{\sigma}^\alpha(Y) = Y$ where $\alpha = (l, N)$.*

Proof. If $\alpha = 1$, then l and N are relatively prime. In consequence $\widehat{\sigma}^N$ is also a cycle. This implies that either $Y = \emptyset$ or $Y = \{1, 2, ..., l\}$. The conclusion is obvious in this case.
Assume now that $\alpha = l$. Since $\widehat{\sigma}^l = id$ all sets are invariant under $\widehat{\sigma}^\alpha$.
Finally, let us assume that $1 < \alpha < l$. Then $l = \alpha\beta$, $N = \alpha\theta$ with β and θ relative prime. We deduce that $\widehat{\sigma}^\alpha = \tau_1 \cdots \tau_\alpha$ and $\widehat{\sigma}^N = \tau_1^\theta \cdots \tau_\alpha^\theta$. Both τ_i and τ_i^θ are cycles of length β with a common support A_i. For each i,

$$\widehat{\sigma}^N(Y \cap A_i) = \widehat{\sigma}^N(Y) \cap \widehat{\sigma}^N(A_i) = Y \cap A_i.$$

Since τ_i^θ is a cycle over A_i, we conclude that either $Y \cap A_i = \emptyset$ or $Y \cap A_i = A_i$. Define $I = \{i \in \{1, ..., \alpha\} : Y \cap A_i \neq \emptyset\}$. Then, $Y = \bigcup_{i \in I} A_i$ and $\widehat{\sigma}^\alpha(Y) = \bigcup_{i \in I} \widehat{\sigma}^\alpha(A_i) = \bigcup_{i \in I} \tau_i(A_i) = \bigcup_{i \in I} A_i = Y$. □

Proof of Proposition 5.1.
Assume that σ satisfies the even property for the even integer N and the subset X, that is,

$$\sigma^N(X) = X \quad \text{and} \quad \sigma^k(X) \neq X, \;\; 1 \leq k < N.$$

We decompose σ as a product of disjoint cycles

$$\sigma = \widehat{\sigma}_1 \cdots \widehat{\sigma}_m$$

of odd lengths $l_1, ..., l_m$.

Define $\widetilde{N}$ as the lowest common multiple of $\alpha_1 = (l_1, N)$,...,$\alpha_m = (l_m, N)$. We claim that

$$\sigma^{\widetilde{N}}(X) = X \tag{5.1}$$

and this will be a contradiction because $\widetilde{N} < N$. In fact, $\widetilde{N}$ divides N, $\widetilde{N}$ is odd and N is even.
To prove the claim, we denote by A_i the support of $\widehat{\sigma}_i$. Then,

$$\widehat{\sigma}_i^N(X \cap A_i) = \sigma^N(X \cap A_i) = \sigma^N(X) \cap \sigma^N(A_i) = X \cap A_i.$$

Since $\widehat{\sigma}_i$ is a cycle of length l_i we can apply the Lemma with $Y_i = X \cap A_i$. This implies that $\widehat{\sigma}_i^{\alpha_i}(X \cap A_i) = X \cap A_i$. In consequence, $\widehat{\sigma}_i^{\widetilde{N}}(X \cap A_i) = X \cap A_i$ and

$$\sigma^{\widetilde{N}}(X) = \sigma^{\widetilde{N}}(\bigcup_{i=1}^m Y_i) = \bigcup_{i=1}^m \widehat{\sigma}_i^{\widetilde{N}}(Y_i) = \bigcup_{i=1}^m Y_i = X.$$

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