Uniformly accelerated motion in General Relativity: completeness of inextensible trajectories

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Abstract

The notion of a uniformly accelerated motion of an observer in a general spacetime is analysed in detail. Such a observer may be seen as a Lorentzian circle, providing a new characterization of a static standard spacetime. The trajectories of uniformly accelerated observers are seen as the projection on the spacetime of the integral curves of a vector field defined on a certain fiber bundle over the spacetime. Using this tool, we find geometric assumptions to ensure that an inextensible uniformly accelerated observer does not disappear in a finite proper time.

2010 MSC: 83C10, 83C75, 53C50.
Keywords: Uniformly accelerated motion, General Relativity, Fermi-Walker connection, third-order differential equation, completeness of inextensible trajectories.

1 Introduction

In a non-Relativistic setting, a particle may be detected as accelerated by using an *accelerometer*. An accelerometer may be intuitively thought as a sphere in whose center there is a small round object which is supported on elastic radii of the sphere surface. If a free falling observer carries such a accelerometer, then it will notice that the small round object remains just at the center. Whereas it will be displaced if the observer obeys an accelerate motion. This argument suggests that a uniformly accelerated motion may be detected from a constant displacement of the small round object. This idea has the advantage that may be used independently if the spacetime where the observer lies is relativistic or not. Now, we need to provide rigour to the assertion "the accelerometer marks a constant value".

[*]The first author is partially supported by Spanish MICINN Grant with FEDER funds MTM2011-23652. The second one by Spanish MICINN Grant with FEDER funds MTM2013-47828-C2-1-P and by the Junta de Andalucía Regional Grant P09-FQM-4496.

The definition of uniformly accelerated motion in General Relativity has been discussed many times over the last 50 years [6]. In the pioneering work by Rindler [16], it was motivated in part by some aspects of intergalactic rocket travel by use of the special relativistic formulas for hyperbolic motion in [11].

The relation between uniformly accelerated motion and Lorentzian circles in Lorentz-Minkowski spacetime described in [17] (see also [12, Sec. 6.2]) was used by Rindler in [16] to define what he named hyperbolic motion in General Relativity, extending uniformly accelerated motion in Lorentz-Minkowsi spacetime (in agreement with one of two proposed notions made by Marder a few years before [9]). This topic is of a current interest [6], [7] and references therein (see also [10]).

The first aim of this paper is to study uniformly accelerated motion in General Relativity. Our approach lies in the realm of modern Lorentzian geometry and, as far as we know, is new. In order to do that, recall that a *particle* of mass $m > 0$ in a spacetime $(M, \langle\, ,\, \rangle)$ is a curve $\gamma : I \longrightarrow M$, such that its velocity $\gamma\,'$ satisfies $\langle \gamma\,', \gamma\,' \rangle = -m^2$ and points to future. A particle with $m = 1$ is called an *observer*. The covariant derivative of $\gamma\,'$, $\dfrac{D\gamma\,'}{dt}$, is its (proper) acceleration, which may be seen as a mathematical translation of the values which measures an accelerometer as named above. Intuitively, the particle obeys a uniformly accelerated motion if its acceleration remains to be unchanged. Mathematically, we need a *connection along* γ which permits to compare spatial directions at different instants of the life of γ. In General Relativity this connection is known as the *Fermi-Walker connection* of γ (see Section 2 for more details). Thus, using the Fermi-Walker covariant derivative $\dfrac{\widehat{D}}{dt}$, we will say that a particle obeys a *uniformly accelerated* (UA) motion if the following third-order differential equation is fulfilled,

$$\frac{\widehat{D}}{dt}\left(\frac{D\gamma\,'}{dt}\right) = 0, \tag{1}$$

i.e., if the acceleration of the observer γ is Fermi-Walker parallel along its world line (Section 2).

There are several physical situations where UA motions appear. For instance, when an electric charged particle $(\gamma(t), m, q)$ is considered, in presence of a electromagnetic field F, the dynamics of the particle is totally described by the well-known Lorentz force equation,

$$m\frac{D\gamma\,'}{dt} = q\widetilde{F}(\gamma\,'),$$

where $\widetilde{F}$ is the (1,1)-tensor field metrically equivalent to the closed 2-form F. The vector field $\widetilde{F}(\gamma\,')$ along γ is called the electric field relative to γ, [19, p. 75]. The Fermi-Walker conection of γ allows to define that γ perceives a “constant” electric field if

$$\frac{\widehat{D}}{dt}\left(\widetilde{F}(\gamma\,')\right) = 0.$$

Thus, if a particle γ is moving in presence of a electromagnetic field F and its relative electric vector field satisfies previous equation, then γ obeys a UA motion.

The family of UA observers in the Lorentz-Minkowski spacetime $\mathbb{L}^n$ was completely determined long time ago [17] (see [6] and references therein for a historical approach). It consists of timelike geodesics and Lorentzian circles. For instance, in $\mathbb{L}^2$, using the usual coordinates (x,t), the UA observer $\gamma(\tau) = \big(x(\tau), t(\tau)\big)$ throughout $(0,0)$ with zero velocity relative to certain family of inertial observers (the integral curves of vector field ∂_t) and proper acceleration a is given by,

$$x(\tau) = \frac{c^2}{a}\left[\cosh\left(\frac{a\tau}{c}\right) - 1\right], \qquad t(\tau) = \frac{c}{a}\sinh\left(\frac{a\tau}{c}\right),$$

where $\tau \in \mathbb{R}$ is the proper time of γ, and c is the light speed in vacuum.

The wordline of γ described by the inertial observer is given by (see, for instance, [17]),

$$x(t) = \frac{c^2}{a}\Big[\sqrt{1 + \left(\frac{at}{c}\right)^2} - 1\Big],$$

which reduce to the classical one $x(t) \approx \frac{1}{2}at^2$. In fact, according to the spirit of [19, Prop. 0.2.1], γ may be thought, in certain sense, as the *relativistic trajectory associated to the Newtonian one* $x(t)$. Also note that the *radius* of this Lorentzian circle is $1/a$, so if the acceleration approached to zero, then γ becomes a unit timelike geodesic.

The content of this paper is organized as follows. In Section 3 we expose in detail how UA observers can be seen as Lorentzian circles in any general spacetime (Proposition 3.2), in particular, assertion (d) in this result corresponds to the original notion proposed by Rindler in [16]. Moreover, static standard spacetimes are characterized as those 1-connected and geodesically complete Lorentzian manifolds which admit a rigid and locally sincronizable reference frame whose integral curves are UA observers (Theorem 3.4).

The last section of this paper is devoted to characterize UA observers as the projection on the spacetime of the integral curves of a vector field defined on a certain fiber bundle over the spacetime. Using this vector field, the completeness of inextensible UA motions is analysed in the search of geometric assumptions which assure that inextensible UA observers do not disappear in a finite proper time (in particular, the absence of timelike singularities). In particular, any inextensible UA observer is complete under the assumption of compactness of the spacetime and that it admits a conformal and closed timelike vector field (Theorem 4.5).

2 Uniformly accelerated motion

A spacetime is a time orientable $n(\geq 2)-$dimensional Lorentzian manifold $(M, \langle\, ,\, \rangle)$[1], endowed with a fixed time orientation. Along this paper we will denote a spacetime by M. As usual, we will refer the points of M as events and we will consider an observer in M as a (smooth) curve $\gamma : I \longrightarrow M$, I an open interval of the real line $\mathbb{R}$, such that $\langle \gamma'(t), \gamma'(t)\rangle = -1$ and $\gamma'(t)$ is future pointing for any proper time t of γ. At each event $\gamma(t)$ the tangent space $T_{\gamma(t)}M$ splits as

$$T_{\gamma(t)}M = T_t \oplus R_t,$$

[1] Along this paper the signature a Lorentian metric is $(-,+,...,+)$.

where $T_t = \mathrm{Span}\{\gamma'(t)\}$ and $R_t = T_t^{\perp}$. Endowed with the restriction of $\langle\, ,\, \rangle$, R_t is a spacelike hyperplane of $T_{\gamma(t)}M$. It is interpreted as the instantaneous physical space observed by γ at t. Clearly, the observer γ is able to compare spatial directions at t. In order to compare $v_1 \in R_{t_1}$ with $v_2 \in R_{t_2}$, $t_1 < t_2$ and $|v_1| = |v_2|$, the observer γ could use, as a first attempt, the parallel transport along γ defined by the Levi-Civita covariant derivative along γ,

$$P^{\gamma}_{t_1,t_2} : T_{\gamma(t_1)}M \longrightarrow T_{\gamma(t_2)}M.$$

Unfortunately, this linear isometry satisfies $P^{\gamma}_{t_1,t_2}(R_{t_1}) = R_{t_2}$ if γ is free falling (i.e., γ is a geodesic) but this property does not remain true for any general observer. In order to solve this difficulty, recall that the Levi-Civita connection ∇ of M induces a connection along $\gamma : I \longrightarrow M$ such that the corresponding covariant derivative is the well-known covariant derivative of vector fields $Y \in \mathfrak{X}(\gamma)$ (the vector fields along γ), namely, $\dfrac{DY}{dt} = \nabla_{\partial_t} Y \in \mathfrak{X}(\gamma)$.

Now, for each $Y \in \mathfrak{X}(\gamma)$ put Y_t^T, Y_t^R the orthogonal projections of Y_t on T_t and R_t, respectively, i.e., $Y_t^T = -\langle Y_t, \gamma'(t)\rangle\, \gamma'(t)$ and $Y_t^R = Y_t - Y_t^T$. In this way, we define $Y^T, Y^R \in \mathfrak{X}(\gamma)$. We have, [19, Prop. 2.2.1],

Proposition 2.1 *There exists a unique connection* $\widehat{\nabla}$ *along* γ *such that*

$$\widehat{\nabla}_X Y = \left(\nabla_X Y^T\right)^T + \left(\nabla_X Y^R\right)^R,$$

for any $X \in \mathfrak{X}(I)$ *and* $Y \in \mathfrak{X}(\gamma)$.

This connection $\widehat{\nabla}$ is called the *Fermi-Walker connection of* γ. It shows the suggestive property that if $Y \in \mathfrak{X}(\gamma)$ satisfies $Y = Y^R$ (i.e., Y_t may be observed by γ at any t) then $\left(\widehat{\nabla}_X Y\right)_t \in R_t$ for any t.

Denote by $\widehat{D}/dt$ the covariant derivative corresponding to $\widehat{\nabla}$. Then, we have [19, Prop. 2.2.2],

$$\frac{\widehat{D}Y}{dt} = \frac{DY}{dt} + \langle \gamma', Y\rangle \frac{D\gamma'}{dt} - \langle \frac{D\gamma'}{dt}, Y\rangle\, \gamma', \tag{2}$$

for any $Y \in \mathfrak{X}(\gamma)$. Note that $\dfrac{\widehat{D}}{dt} = \dfrac{D}{dt}$ if and only if γ is free falling.

Associated to the Fermi-Walker connection on γ there exist a parallel transport

$$\widehat{P}^{\gamma}_{t_1,t_2} : T_{\gamma(t_1)}M \longrightarrow T_{\gamma(t_1)}M,$$

which is a lineal isometry and satisfies $\widehat{P}^{\gamma}_{t_1,t_2}(R_{t_1}) = R_{t_2}$. Therefore, given $v_1 \in R_{t_1}$ and $v_2 \in R_{t_2}$, with $t_1 < t_2$ and $|v_1| = |v_2|$, the observer γ may consider $\widehat{P}^{\gamma}_{t_1,t_2}(v_1)$ instead v_1, with the advantage to wonder if $\widehat{P}^{\gamma}_{t_1,t_2}(v_1)$ is equal to v_2 or not (compare with [12, Sec. 6.5]).

The acceleration $\dfrac{D\gamma'}{dt}$ satisfies $\dfrac{D\gamma'}{dt}(t) \in R_t$, for any t. Therefore, it may be observed by γ whereas the velocity γ' is not observable by γ.

Now, we are in a position to give rigorously the notion of UA observer. An observer $\gamma : I \longrightarrow M$ is said to obey a uniformly accelerated motion if

$$\widehat{P}^{\gamma}_{t_1,t_2}\left(\frac{D\gamma'}{dt}(t_1)\right) = \frac{D\gamma'}{dt}(t_2), \tag{3}$$

for any $t_1, t_2 \in I$ with $t_1 < t_2$, equivalently, if the equation (1) holds everywhere on I, i.e., $\dfrac{D\gamma'}{dt}$ is Fermi-Walker parallel along γ. Clearly, if γ is free falling, then it is a UA observer.

Since we deal with a third-order ordinary differential equation, the following initial value problem has a unique local solution,

$$\frac{\widehat{D}}{dt}\left(\frac{D\gamma'}{dt}\right) = 0, \tag{4}$$
$$\gamma(0) = p, \quad \gamma'(0) = v, \quad \frac{D\gamma'}{dt}(0) = w,$$

where $p \in M$ and $v, w \in T_pM$ such that $|v|^2 = -1$, $\langle v, w\rangle = 0$, $|w|^2 = a^2$, and a is a positive constant.

For any observer we have a conservation result as a consequence of the slightly more general lemma,

Lemma 2.2 *Let σ be a curve in M, defined on an open interval $I \subset \mathbb{R}$, which satisfies the equation*

$$\frac{D^2\sigma'}{dt^2} = \left|\frac{D\sigma'}{dt}\right|^2 \sigma' - \left\langle \sigma', \frac{D\sigma'}{dt}\right\rangle \frac{D\sigma'}{dt}. \tag{5}$$

Then, $\left|\dfrac{D\sigma'}{dt}\right|^2(t)$ is constant on I.

Proof. Multiplying (5) by $\dfrac{D\sigma'}{dt}$, we directly obtain

$$\left\langle \frac{D^2\sigma'}{dt^2}, \frac{D\sigma'}{dt}\right\rangle = \frac{1}{2}\frac{d}{dt}\left\langle \frac{D\sigma'}{dt}, \frac{D\sigma'}{dt}\right\rangle = 0,$$

and the proof is done. □

Remark 2.3 **(a)** Observe that no assumption is made on the spacetime in previous result. On the other hand, the constant a has a geometrical meaning for a UA observer in terms of its Frenet-Serret formulas (see next section). **(b)** The family of the UA observers lies into a bigger family of observers which has shown to be relevant in the study of the global geometry of spacetimes, the so called bounded acceleration observers. Recall that [1, Def. 6.6] an observer $\gamma : I \longrightarrow M$ is said to have bounded acceleration if there exists a constant $B > 0$ such that $\left|\dfrac{D\gamma'}{dt}\right| \leq B$ for all $t \in I$. **(c)** On the other hand, note that if a UA observer is not free falling, then $\left(1\Big/\left|\dfrac{D\gamma'}{dt}\right|\right) \dfrac{D\gamma'}{dt}$ is also Fermi-Walker parallel along γ.

Taking into account formula (2), an observer γ satisfies equation (1) if and only if

$$\frac{D^2\gamma'}{dt^2} = \left\langle \frac{D\gamma'}{dt}, \frac{D\gamma'}{dt}\right\rangle \gamma', \tag{6}$$

which is a third order equation. Alternatively, γ satisfies (6) if and only if $\dfrac{D^2\gamma'}{dt^2}(t)$ is collinear to $\gamma'(t)$ at any $t \in I$.

Example 2.4 **(a)** In any Generalized Robertson-Walker (GRW) spacetime, each integral curve of the coordinate reference frame is trivially a UA observer. **(b)** Consider now a static standard spacetime $M = S \times I$, with metric $\langle\, ,\, \rangle = g_S - h^2dt^2$, where g_S is a Riemannian metric on S, $h \in C^\infty(S)$, $h > 0$ and I an open interval of the real line $\mathbb{R}$. Let $\gamma = \gamma(s)$ be any integral curve of the reference frame $Q = \dfrac{1}{h}\,\partial_t$. A direct computation gives

$$\frac{D\gamma'}{ds} = \frac{\nabla h}{h} \circ \gamma.$$

On the other hand, taking into account [14, Prop 7.35], we get

$$\frac{D^2\gamma'}{ds^2} = \frac{|\nabla h|^2}{h^2}\,\gamma'.$$

From the two previous formulas and (6) it follows that γ is a UA observer.

3 UA motion and Lorentzian circles

Consider a UA observer $\gamma : I \longrightarrow M$ with $a = \left|\frac{D\gamma'}{dt}\right| > 0$ (constant from Lemma 2.2) and put $e_1(t) = \gamma'(t)$, $e_2(t) = \dfrac{1}{a}\dfrac{D\gamma'}{dt}(t)$. Then, from (6) we have

$$\frac{De_1}{dt} = ae_2(t),$$
$$\frac{De_2}{dt} = ae_1(t).$$

Conversely, assume this system holds true for an observer γ with $a > 0$ constant. Then, equation (6) also holds true. In other words, a (non free falling) UA observer may be seen as a *Lorentzian circle* of constant curvature a and identically zero torsion (see [8]).

Remark 3.1 Circles in a Riemannian manifold were studied by Nomizu and Yano in [13] in order to characterize umbilical submanifolds with parallel mean curvature vector field in an arbitrary Riemannian manifold. They described a circle by a third-order differential equation similar to the previous equation (6). The results in [13] were extended to the Lorentzian case by Ikawa in [8].

The previous results can be summarized as follows,

Proposition 3.2 *For any observer $\gamma : I \longrightarrow M$, the following assertions are equivalent:*

(*a*) *γ is a UA observer.*

(*b*) *γ is a solution of third-order differential equation* (6)*.*

(*c*) *γ is a Lorentzian circle or it is free falling.*

(*d*) *γ has constant curvature and the remaining curvatures equal to zero.*

(*e*) *γ, viewed as an isometric immersion from $(I, -dt^2)$ to M, is totally umbilical with parallel mean curvature vector.*

Now we are in a position to get a converse to previous Example 2.4 (b).

Proposition 3.3 *Consider $M = S \times I$ with a Lorentzian metric of the type $\langle\, ,\, \rangle = g_S - f^2 dt^2$ where $f \in C^\infty(M)$, $f > 0$. Assume each integral curve of the reference frame $\frac{1}{f}\,\partial_t$ is a* UA *observer. Then, there exist $h \in C^\infty(S)$, $h > 0$ and $\phi \in C^\infty(I)$, $\phi > 0$, such that*

$$f(x,t) = h(x)\phi(t),$$

for all $(x,t) \in M$. Therefore, M is a standard static spacetime with $\langle\, ,\, \rangle = g_S - h^2 ds^2$ and $ds = \phi\, dt$.

Proof. We have that the mean curvature vector field of each submanifold $\{x_0\} \times I$, $x_0 \in S$, is

$$-\frac{1}{f}\,\nabla f - \frac{f'}{f^3}\,\partial_t.$$

On the other hand, our assumption means that $-\frac{1}{f}\,\nabla f - \frac{f'}{f^3}\,\partial_t$ is parallel (as a normal vector field to $\{x_0\} \times I$). Now, making use of [4, Prop. 1.2(3)], we get that $f = f(x,t)$ is the product of two positive functions $h = h(x)$ and $\phi = \phi(t)$ on S and I, respectively. $\square$

We end the section with the statement of a characterization of standard static spacetimes in terms of the existence of certain reference frame whose integral curves are UA observers. Before we need to recall some notions to be used later.

A reference frame Q in a spacetime M is said to be *locally sincronizable* if $Q^b \wedge dQ^b = 0$ where Q^b is the 1-form on M metrically equivalent to Q [19, p. 53]. Equivalently, Q is locally sincronizable if and only if the distribution $Q^\perp$ is integrable or, if and only if $\langle \nabla_X Q, Y\rangle = \langle X, \nabla_Y Q\rangle$ for any $X, Y \in Q^\perp$, [14, Prop. 12.30].

On the other hand, a reference frame Q is said to be *rigid* if $\langle \nabla_X Q, Y\rangle + \langle X, \nabla_Y Q\rangle = 0$ for all $X, Y \in Q^\perp$, [19, p. 56].

Theorem 3.4 *Let M be a simply connected and geodesically complete spacetime. If M admits a rigid and locally sincronizable reference frame Q such that any integral curve of Q is a* UA *observer, then M is a static standard spacetime.*

Proof. Since Q is also assumed to be rigid, each leaf of the foliation $R = Q^\perp$ is in fact totally geodesic. Therefore, any inextensible leaf of R is geodesically complete (with respect to the induced Riemannian metric). On the other hand, any leaf of $T = \mathrm{Span}\{Q\}$ is totally umbilical since T is 1-dimensional. Even more, if any integral curve of Q is a UA observer, then the leaves of T are extrinsic spheres. The conclusion follows now from [15, Cor. 1]. $\square$

4 Completeness of the inextensible UA trajectories

This section is devoted to the study of the completeness of the inextensible solutions of equation (6). First of all, we are going to relate the solutions of equation (6) with the integral curves of a certain vector field on a Stiefel bundle type on M (compare with [5, p. 6]).

Given a Lorentzian linear space E and $a \in \mathbb{R}$, $a > 0$, denote by $V^a_{n,2}(E)$ the (n,2)-Stiefel manifold over E, defined by

$$V^a_{n,2}(E) = \big\{(v, w) \in E \times E \ : \ |v|^2 = -1, \, |w|^2 = a^2, \ \ \langle v, w\rangle = 0\big\}.$$

The (n,2)-Stiefel bundle over the spacetime M is then defined as follows,

$$V^a_{n,2}(M) = \bigcup_{\mathrm{p}\in \mathrm{M}} \{p\} \times V^a_{n,2}(T_pM).$$

Note that $V^a_{n,2}(M)$ is a bundle on M with dimension $3(n-1)$ and fiber diffeomorphic to $\mathbb{S}_a\big(\mathbb{H}^{n-1}\big)$, the spherical fiber bundle on the hyperbolic space $(n-1)$-dimensional and fiber $\mathbb{S}^{n-2}$ of radius a.

First we construct a vector field $G \in \mathfrak{X}\big(V^a_{n,2}(M)\big)$, which is the key tool in the study of completeness,

Lemma 4.1 *Let $\sigma : I \longrightarrow M$ be a curve satisfying* (5) *with initial conditions*

$$\sigma'(0) = v, \qquad \frac{D\sigma'}{dt}(0) = w,$$

where v is a unitary timelike vector and w is orthogonal to v. Then, we have $|\sigma'(t)|^2 = -1$ for all $t \in I$, and therefore, $\big\langle \sigma'(t), \dfrac{D\sigma'}{dt}(t)\big\rangle = 0$ holds everywhere on I.

Proof. Multiplying (5) by σ', and after easy computations, we arrive to the following ordinary differential equation

$$\frac{1}{2}\, x'' + \frac{1}{4}\, (x')^2 - a^2 x = a^2,$$

where $a := \big|\dfrac{D\sigma'}{dt}\big|$ is constant, and $x(t) := |\sigma'(t)|^2$. From the assumption, we know that $x(t)$ satisfying the initial conditions

$$x(0) = -1,$$

and

$$x'(0) = 2\left\langle \sigma'(0)\,, \frac{D\sigma'}{dt}(0)\right\rangle = 2\langle v, w\rangle = 0.$$

Since $x(t) = -1$ is a solution of this initial value problem, the result is a direct consequence of the existence and uniqueness of solutions to second order differential equations. □

Now, we are in a position to define the announced vector field G. Let (p, v, w) be a point of $V^a_{n,2}(M)$, and $f \in C^\infty\big(V^a_{n,2}(M)\big)$. Let σ be the unique inextensible curve solution of (5) satisfying the initial conditions

$$\sigma(0) = p, \quad \sigma'(0) = v, \quad \frac{D\sigma'}{dt}(0) = w.$$

So, we define

$$G_{(p,v,w)}(f) := \frac{d}{dt}\Big|_{t=0} f\big(\sigma(t), \sigma'(t), \frac{D\sigma'}{dt}(t)\big).$$

From Lemma 2.2 and Lemma 4.1, we have $\big(\sigma(t), \sigma'(t), \dfrac{D\sigma'}{dt}(t)\big) \in V^a_{n,2}(M)$ and G is well defined.

The following result follows easily,

Lemma 4.2 *There exists a unique vector field G on $V^a_{n,2}(M)$ such that the curves $t \longmapsto \big(\gamma(t), \gamma'(t), \frac{D\gamma'}{dt}(t)\big)$ are the integral curves of G, for any solution γ of equation* (4).

Once defined G, we will look for assumptions which assert its completeness.

Recall that an integral curve α of a vector field defined on some interval $[0, b)$, $b < +\infty$, can be extended to b (as an integral curve) if and only if there exists a sequence $\{t_n\}_n$, $t_n \nearrow b$, such that $\{\alpha(t_n)\}_n$ converges (see for instance [14, Lemma 1.56]). The following technical result directly follows from this fact and Lemma 4.2.

Lemma 4.3 *Let $\gamma : [0, b) \longrightarrow M$ be a solution of equation* (4) *with $0 < b < \infty$. The curve γ can be extended to b as a solution of* (5) *if and only if there exists a sequence $\big\{\gamma(t_n), \gamma'(t_n), \frac{D\gamma'}{dt}(t_n)\big\}_n$ which is convergent in $V^a_{n,2}(M)$.*

Although we know that $|\gamma'(t)|^2 = -1$ from Lemma 4.1, this is not enough to apply Lemma 4.3 even in the geometrically relevant case of M compact. The reason is similar to the possible geodesic incompleteness of a compact Lorentzian manifold (see for instance [14, Example 7.16]).

However, it is relevant that if a compact Lorentzian manifold admits a timelike conformal vector field, then it must be geodesically complete [18]. Therefore, from a geometric viewpoint, it is natural to assume the existence of such infinitesimal conformal symmetry to deal with the extendibility of the solutions of (4).

Recall that a vector field K on M is called *conformal* if the Lie derivative of the metric with respect to K satisfies

$$L_K\langle\, ,\rangle = 2h\langle\, ,\rangle, \tag{7}$$

for some $h \in C^\infty(M)$, equivalently, the local flows of K are conformal maps. In particular, if holds (7) with $h = 0$, K is called a *Killing* vector field.

Note that for any curve $\gamma : I \longrightarrow M$, the relation (7) implies

$$\frac{d}{dt}\big\langle K, \gamma'\big\rangle = \big\langle K, \frac{D\gamma'}{dt}\big\rangle + h(\gamma)|\,\gamma'|^2. \tag{8}$$

On the other hand, if a vector field K satisfies

$$\nabla_X K = hX \qquad \text{for all} \qquad X \in \mathfrak{X}(M), \tag{9}$$

then clearly we get (7). Moreover, for the 1-form K^b metrically equivalent to K, we have

$$dK^b(X, Y) = \langle \nabla_X K, Y\rangle - \langle \nabla_Y K, X\rangle = 0,$$

for all $X, Y \in \mathfrak{X}(M)$, i.e., K^b is closed. We will call to K which satisfies (9) a *conformal and closed* vector field.

The following result, inspired from [3, Lemma 9], will be decisive to assure that the image of the curve in $V^a_{n,2}(M)$, associated to a UA observer γ, is contained in a compact subset.

Lemma 4.4 *Let M be a spacetime and let Q be a unitary timelike vector field. If $\gamma : I \longrightarrow M$ is a solution of (4) such that $\gamma(I)$ lies in a compact subset of M and $\langle Q, \gamma' \rangle$ is bounded on I, then the image of $t \longmapsto \big(\gamma(t), \gamma'(t), \dfrac{D\gamma'}{dt}\big)$ is contained in a compact subset of $V^a_{n,2}(M)$ where a is the constant $|\frac{D\gamma'}{dt}|$.*

Proof. Consider the 1-form Q^b metrically equivalent to Q and the associated Riemannian metric $g_R := \langle\, , \,\rangle + 2\, Q^b \otimes Q^b$. We have,

$$g_R(\gamma', \gamma') = \langle \gamma', \gamma' \rangle + 2\, \langle Q, \gamma' \rangle^2,$$

which, by hypothesis, is bounded on I. Hence, there exists a constant $c > 0$ such that

$$\big(\gamma(I), \gamma'(I), \frac{D\gamma'}{dt}(I)\big) \subset C, \qquad C := \big\{(p, v, w) \in V^a_{n,2}(M) \,:\, p \in C_1, \quad g_R(v, v) \leq c\big\},$$

where C_1 is a compact set on M such that $\gamma(I) \subset C_1$. Hence, C is a compact in $V^a_{n,2}(M)$. $\square$

Now, we are in a position to state the following completeness result (compare with [3, Th. 1] and [2, Th. 1]),

Theorem 4.5 *Let M be a spacetime which admits a timelike conformal and closed vector field K. If $\mathrm{Inf}_M \sqrt{-\langle K, K \rangle} > 0$ then, each solution $\gamma : I \longrightarrow M$ of (4) such that $\gamma(I)$ lies in a compact subset of M can be extended.*

Proof. Let $I = [0, b)$, $0 < b < +\infty$, be the domain of a solution γ of equation (4). Derivating (8), it follows

$$\frac{d^2}{dt^2}\langle K, \gamma' \rangle = \langle \frac{DK}{dt}, \frac{D\gamma'}{dt} \rangle + \langle K, \frac{D^2\gamma'}{dt^2} \rangle - \frac{d}{dt}(h \circ \gamma).$$

The first right term vanishes because K is conformal and closed,

$$\left\langle \frac{DK}{dt}, \frac{D\gamma'}{dt} \right\rangle = h(\gamma)\, \langle \gamma', \frac{D\gamma'}{dt} \rangle = 0.$$

On the other hand, the second right term equals to $a^2\langle K, \gamma' \rangle$. Thus, the function $t \mapsto \langle K, \gamma' \rangle$ satisfies the following differential equation,

$$\frac{d^2}{dt^2}\langle K, \gamma' \rangle - a^2\langle K, \gamma' \rangle = (h \circ \gamma)'(t). \tag{10}$$

Using now that $\gamma(I)$ is contained in a compact of M, the function $h \circ \gamma$ is bounded on I. Moreover, since I is assumed bounded, using (10) there exists a constant $c_1 > 0$ such that

$$|\langle K, \gamma' \rangle| < c_1. \tag{11}$$

Now, if we put $Q := \dfrac{K}{|K|}$, where $|K|^2 = -\langle K, K \rangle > 0$, then Q is a unitary timelike vector field such that, by (11),

$$|\langle Q, \gamma' \rangle| \leq m\, c_1 \qquad \text{on} \quad I,$$

where $m = \mathrm{Sup}_M |K|^{-1} < \infty$. The proof ends making use of Lemmas 4.3 and 4.4. $\square$

Remark 4.6 Note that the previous theorem implies the following result of mathematical interest: *Let M be a compact spacetime which admits a timelike conformal and closed vector field K. Then, each inextensible solution of* (4) *must be complete.* Note that the Lorentzian universal covering of M inherits the completeness of inextensible UA observers form the same fact on M.

Example 4.7 Let $f \in C^\infty(\mathbb{R})$ be a positive periodic function and let (N, g) be a compact Riemannian manifold. The GRW spacetime $\mathbb{R} \times_f N$ is a Lorentzian covering manifold of the compact spacetime $\mathbb{S}^1 \times_{\widetilde{f}} N$ where $\widetilde{f}$ is the induced function from f on $\mathbb{S}^1$. The result in [18] may be applied to $\mathbb{S}^1 \times_{\widetilde{f}} N$ with $K = \widetilde{f}Q$, which is timelike, conformal and closed [20], where Q is the vector field on $\mathbb{S}^1$ induced from ∂_θ. Thus, we have that any inextensible UA observer in the spacetime $\mathbb{R} \times_f N$ must be complete.

Acknowledgments

The authors would like to thank the referees for their comments and corrections to improve the original writing of the manuscript.

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