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Existence and multiplicity of entire radial spacelike graphs with prescribed mean curvature function in certain Friedmann–Lemaître–Robertson–Walker spacetimes

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We provide sufficient conditions for the existence of a uniparametric family of entire spacelike graphs with prescribed mean curvature in a Friedmann–Lemaître–Robertson–Walker spacetime with flat fiber. The proof is based on the analysis of the associated homogeneous Dirichlet problem on a Euclidean ball together with suitable bounds for the gradient which permit the prolongability of the solution to the whole space.

Keywords: Entire spacelike graph; quasilinear elliptic equation; Dirichlet boundary condition; prescribed mean curvature function; Friedmann–Lemaître–Robertson–Walker spacetime; singular ϕ-Laplacian.

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1. Introduction

This paper studies the following quasilinear elliptic equation

$$\operatorname{div}\left(\frac{\nabla u}{f(u)\sqrt{f(u)^2-|\nabla u|^2}}\right)+\frac{f'(u)}{\sqrt{f(u)^2-|\nabla u|^2}}\left(n+\frac{|\nabla u|^2}{f(u)^2}\right)=nH(u,x), \tag{E1}$$

$$|\nabla u|<f(u), \tag{E2}$$

where $f\in C^\infty(I)$ is a positive function, I is an open interval in $\mathbb{R}$ with $0\in I$, $H: I\times\mathbb{R}^n\to\mathbb{R}$ is a given smooth radially symmetric function and u satisfies $u(\mathbb{R}^n)\subset I$. This PDE has a clear geometric interpretation which lies in the realm of Lorentzian Geometry. Namely, each solution of (E) defines, in a natural way, a spacelike graph of the fiber on the Friedmann–Lemaître–Robertson–Walker (FLRW) spacetime $\mathcal{M} = I \times_f \mathbb{R}^n$ (see next section for details) where the function H prescribes the mean curvature of the spacelike graph.

A spacelike hypersurface in a spacetime is a hypersurface which inherits a Riemannian metric from the ambient Lorentzian one. Intuitively, a spacelike hypersurface is the spatial universe at one instant of proper time of a family of observers. In fact, a spacelike hypersurface defines the family of *normal observers*: each geodesic in the ambient spacetime determined by a point of the spacelike hypersurface and the future pointing unit normal vector at this point. The corresponding mean curvature function measures *how these observers get away or coming together with respect to a given one*. Indeed, these observers can be locally collected as the integral curves of a reference frame in spacetime and the sign of its divergence (i.e. the measure of expansion/contraction for the observers in the reference frame, [26, 29]) is the same of the sign of the mean curvature function. Precisely, we are interested here in prescribing the mean curvature function for the case these observers get away in an FLRW cosmological model.

On the other hand, a spacelike hypersurface is a suitable subset in spacetime where the initial value problem for each of the classical equations in General Relativity (matter equations, Maxwell equations and Einstein equations) is well posed. In particular, spacelike hypersurfaces with constant mean curvature have shown to be an interesting tool in the study of Einstein equations. Concretely, they have been used to state and solve the constraint equations (see, for instance, [2, 16]). Geometrically, spacelike hypersurfaces with constant mean curvature in a (general) Lorentzian manifold appear as the critical points of the "area" functional under certain "volume constraints" [10, 13, 14]. The existence results for spacelike hypersurfaces with constant mean curvature is a classical and important problem (see [11] and references therein). Consequently, it has been useful to prove satisfactory uniqueness results. Among the uniqueness results, the seminal paper by Cheng and Yau [14] where the proof of the Calabi–Bernstein conjecture for any n-dimensional Lorentz–Minkowski spacetime was given, also introduced a new type of elliptic problems which have been developed in several different spacetimes, see for instance [10, 14, 28].

In the latter years, many researchers have worked on the prescribed mean curvature problem on spacelike hypersurfaces in Lorentzian manifolds. Mainly, the efforts have focused for the case of the Lorentz–Minkowski spacetime $\mathbb{L}^{n+1}$. In this context, we mention the paper of Bartnik and Simon [4], where a kind of "universal existence result" is proved for the Dirichlet problem. More recently, many authors paid attention to the existence of positive solutions by using a combination of variational techniques, critical point theory, sub-supersolutions and topological degree (see for instance [5–7, 17–19] and the references therein). The Dirichlet problem in a more general spacetime was considered by Gerhardt [22].

In comparison with the Dirichlet problem, the number of references devoted to the study of entire spacelike graphs in the Lorentz–Minkowski spacetime with constant or prescribed mean curvature is appreciably lower. In this setting, the study of entire constant mean curvature spacelike graphs developed in [31] is motivated by the remarkable Calabi–Bernstein property in the maximal case, i.e. when mean curvature identically vanishes. Namely, Calabi [12] showed for $n \leq 4$, and latter Cheng and Yau [14] for all n, that an entire maximal graph in $\mathbb{L}^{n+1}$ must be a spacelike hyperplane. Treibergs proved $\mathbb{L}^{n+1}$ the existence of entire graphs of constant mean curvature with certain asymptotic conditions. Later, Bartnik and Simon [4, Theorem 4.4] extended this result to a more general mean curvature function, but related references concerning the prescribed curvature problem for entire graphs are rare. Up to our knowledge, in the latter years only [3, 9] treat this problem by using a variational approach for very concrete prescribed mean curvature. On the other hand, it is natural to wonder for the existence problem of prescribed mean curvature entire spacelike graphs with radial symmetry in spacetimes where they are expected, like in FLRW spacetimes. This is the main aim of this paper, whose main goals are the two following results.

Theorem 1.1. *Let $I \times_f \mathbb{R}^n$ be an* FLRW *spacetime, and let $R > 0$ be such that*

$$I_f(R) \subset I, \quad \varphi^{-1}(\mathbb{R}^-) \subset I,$$

where

$$I_f(R) := \left[-\int_{-R}^{0} f(\varphi^{-1}(s))ds, \int_{0}^{R} f(\varphi^{-1}(s))ds \right] \quad \textit{and} \quad \varphi(t) = \int_0^t \frac{dt}{f(t)}.$$

Then, for each radially symmetric smooth function $H : I \times \mathbb{R}^n \to \mathbb{R}$ such that

$$H(t, r) \leq \frac{f'}{f}(t) \quad \textit{and} \quad f'(t) \geq 0, \quad \textit{for any}\, r \in]0, R[, \ \ t \in I_f(R),$$

there exists an entire radially symmetric spacelike graph with mean curvature function H. In addition, the spacelike slice $t = 0$ intersects the graph in a ball with radius R. In the particular case that $\inf I$ is finite, the entire graph approaches to an hyperplane.

Note that this result specializes to the particular but important case $H = 0$, providing entire maximal graphs in the FLRW spacetime $I \times_f \mathbb{R}^n$.

In order to prove Theorem 1.1, the key point is an existence result for the associated Dirichlet problem in a ball that has its own interest.

Theorem 1.2. *Let $I \times_f \mathbb{R}^n$ be an* FLRW *spacetime, and let B be the Euclidean ball in $\mathbb{R}^n$ with radius R centered at zero. Assume that $I_f(R) \subset I$. Then, for each radially symmetric smooth function $H : I \times \overline{B} \to \mathbb{R}$ such that*

$$H(t, r) \leq \frac{f'}{f}(t) \quad \textit{and} \quad f'(t) \geq 0, \quad \textit{for any}\, r \in]0, R[, \ \ t \in I_f(R),$$

there exists a radially symmetric spacelike graph with mean curvature function H defined on $\overline{B}$, supported on the spacelike slice $t = 0$ and only touching it on the *boundary* $\{0\} \times \partial B$ *and defining a non-zero hyperbolic angle with* ∂_t. *Moreover*, *if the function* H *is increasing in the second variable*, *every spacelike graph satisfying the previous assumptions must be radially symmetric.*

This result extends the main theorem of [20], where a suitable bound for the radius R is required. To remove such assumption, we have to use a different method to achieve the proof. While [20] relies on a basic application of Schauder's fixed point theorem, here we will need a more sophisticated approach. When passing to polar coordinates, we obtain a problem with a double singularity: the first singularity is on the independent variable at the value $r = 0$ and it is the usual singularity that appears at the origin in any radially symmetric problem defined on a ball; the second singularity is not standard on the related literature since it is a singularity on the dependent variable (see the second term of the left-hand side of Eq. (5)). To handle the first singularity, we use an approximation method through family of truncated problems, which is a classical approach for radial problems defined on a ball (see for example [27, Chap. 9] and the references therein), although in this context it is essentially new. On this sequence of approximated problems, the second singularity is handled by an adequate manipulation of the equation (see the first step of the proof of Theorem 4.1) that leads to a sequence of approximated solutions. To prove the convergence of this sequence, the key point is a delicate estimate of an *a priori* bound for the derivative of the solutions on the boundary (see Proposition 3.5). Once the Dirichlet problem is solved, the existence of an entire solution is obtained by extension of the solution of the Dirichlet problem. In performing this program, the paper advances on the application of techniques of Nonlinear Analysis to the problem of prescribed curvature in relativistic spacetimes under a new perspective.

The structure of the paper is detailed in the following. In Sec. 2 we expose the necessary preliminaries. Sections 3 and 4 are devoted to study the Dirichlet problem and to prepare the proof of Theorem 1.1, which is briefly shown in Sec. 5. We finish in Sec. 6 with some conclusions and several explicit examples of special interest from the physical point of view.

2. Preliminaries

First of all, we are going to introduce the ambient spacetimes where our spacelike graphs are embedded. We consider the Euclidean space $(\mathbb{R}^n, \langle\, , \rangle)$, and let I be an open interval in the real line $\mathbb{R}$ endowed with the metric $-dt^2$. Throughout this paper we will denote by $\mathcal{M}$ the $(n+1)$-dimensional product manifold $I \times \mathbb{R}^n$ endowed with the Lorentzian metric

$$g := \pi_I^*(-dt^2) + f^2(\pi_I)\pi_F^*(\langle\, , \rangle) \equiv -dt^2 + f^2(t)\langle\, , \rangle, \tag{1}$$

where $f > 0$ is a smooth function on I, and π_I and π_F denote the projections onto I and $\mathbb{R}^n$ respectively. Thus, $\mathcal{M}$ is a Lorentzian warped product with base I, fiber

$\mathbb{R}^n$ and warping function f. We will denote $\mathcal{M}$ by $I \times_f M$ and refer it as an FLRW (RW) spacetime.

Given an n-dimensional (connected) manifold S, an immersion $\phi : S \to \mathcal{M}$ is said to be *spacelike* if the Lorentzian metric (1) induces, via ϕ, a Riemannian metric g_S on S. In this case, S is called a spacelike hypersurface. Observe that $\partial_t := \partial/\partial t \in \mathfrak{X}(\mathcal{M})$ is a unit timelike vector field which determines a time orientation on $\mathcal{M}$. Thus, if $\phi : S \to \mathcal{M}$ is a spacelike hypersurface in $\mathcal{M}$, we may define $N \in \mathfrak{X}^{\perp}(S)$ as the only globally defined, unit timelike vector field normal to S in the time orientation of ∂_t.

Among all the spacelike hypersurfaces in the FLRW spacetime $\mathcal{M}$, there is a remarkable family. Namely, the so-called *spacelike slices*. In the terminology of [1], a spacelike hypersurface in $\mathcal{M}$ is called a *spacelike slice* if the function $\pi_I \circ \phi : S \to I$ is constant. The mean curvature of the spacelike slice $t = t_0$, with respect to the chosen normal vector field, is $f'(t_0)/f(t_0)$. The embedded spacelike slice $t = t_0$ is clearly a graph on the whole fiber. More generally, given $u \in C^\infty(U)$, U a domain in $\mathbb{R}^n$, such that $u(U) \subseteq I$, the graph of u is defined as follows, $\Sigma_u = \{(u(x), x) : x \in U\}$. The graph is spacelike whenever

$$|\nabla u| < f(u) \quad \text{on } U. \tag{2}$$

For a spacelike graph Σ_u, the unit timelike normal vector field in the same time orientation of ∂_t is given by

$$N = \frac{f(u)}{\sqrt{f(u) - |\nabla u|^2}} \left(\frac{1}{f^2(u)} \nabla u + \partial_t \right),$$

and the corresponding mean curvature associated to N, is

$$\frac{1}{n} \left\{ \operatorname{div} \left(\frac{\nabla u}{f(u)\sqrt{f(u)^2 - |\nabla u|^2}} \right) + \frac{f'(u)}{\sqrt{f(u)^2 - |\nabla u|^2}} \left(n + \frac{|\nabla u|^2}{f(u)^2} \right) \right\},$$

which may be seen as a quasilinear elliptic operator, because of (2).

In order to state our problem, the first step is to perform a suitable variable change in (E) to make it easier. Indeed, consider

$$v = \varphi(u), \quad \text{where } \varphi(t) = \int_0^t \frac{ds}{f(s)}.$$

Clearly, φ is a diffeomorphism from I to another open interval J in $\mathbb{R}$. Consequently, it follows that $\nabla v = \frac{1}{f(u)} \nabla u$. Therefore, $|\nabla u| < f(u)$ holds if and only if $|\nabla v| < 1$. It is clear that u is radially symmetric if and only if v is also radially symmetric.

After routine computations, our equation is transformed into

$$\operatorname{div} \left(\frac{\nabla v}{\sqrt{1 - |\nabla v|^2}} \right) + \frac{n f'(\varphi^{-1}(v))}{\sqrt{1 - |\nabla v|^2}} = n f(\varphi^{-1}(v)) H(\varphi^{-1}(v), x). \tag{3}$$

Actually, the previous variable change is equivalent to consider the following conformal map

$$\varphi \times \mathrm{Id} : I \times_f \mathbb{R}^n \to (J \times \mathbb{R}^n, -ds^2 + g)$$
$$(t, p) \mapsto (\varphi(t), p),$$

which has conformal factor $\frac{1}{f(t)}$. The Lorentzian product spacetime in the codomain of previous map is in fact an open subset of Lorentz–Minkowski spacetime $\mathbb{L}^{n+1}$. Note that the mean curvature function of the spacelike graph of v in $\mathbb{L}^{n+1}$ is

$$\frac{1}{n}\operatorname{div}\left(\frac{\nabla v}{\sqrt{1 - |\nabla v|^2}}\right).$$

We will deal next with Eq. (3), under the conditions $|\nabla v| < 1$ on a ball B, centered in 0 of radius R, and $v = 0$ on ∂B. From the boundedness of the length of the gradient of v (the spacelike condition) it follows that $|v| < R$ on $\overline{B}$, i.e. the image of v lies in the interval $[-R, R]$ or, equivalently, the image of the original function $u = \varphi^{-1}(v)$ is contained in $\varphi^{-1}([-R, R])$. Hence, we have an *a priori* upper bound of the spacelike graph. Thus, the first assumption on the interval I in our FLRW spacetime is

(A) $[-R, R] \subset \varphi(I)$, i.e

$$I_f(R) := \left[-\int_{-R}^{0} f(\varphi^{-1}(s))ds,\ \int_{0}^{R} f(\varphi^{-1}(s))ds\right] \subset I.$$

Basically, (A) means that the interval I must be big enough to contain the highest or lowest possible spacelike graph.

Summarizing, in the following sections we will take care of the problem

$$\begin{aligned} &\operatorname{div}\left(\frac{\nabla v}{\sqrt{1 - |\nabla v|^2}}\right) + \frac{nf'(\varphi^{-1}(v))}{\sqrt{1 - |\nabla v|^2}} = nf(\varphi^{-1}(v))H(\varphi^{-1}(v), x) \quad \text{in } B, \\ &v = 0 \quad \text{in } \partial B. \end{aligned} \tag{4}$$

We may observe that the last term in the left-hand side goes to infinity when $|\nabla v|$ approaches to 1. The main difficulty of the problem comes from this singularity of the gradient. For nonlinearities not depending on the gradient, we mentioned in Sec. 1 that Bartnik and Simon proved a kind of general existence result, later generalized to continuous nonlinearities with possible dependence on the gradient in [5, Theorem 2.1]. The presence of the singular term prevents from a direct application of such results.

3. The Associated Dirichlet Problem: *A Priori* Results

The aim of this section is to show several *a priori* properties of the solutions of the associate Dirichlet problem, i.e. we pretend to find out certain results about solutions of our prescription problem, supposing that they exist. These properties are related with the radial symmetry, and the strictly spacelike character of the graphs.

3.1. *Radial symmetry of positive solutions*

It is possible to state conditions on the prescription function to ensure that any eventual positive solution of (4) must be radially symmetric. In [20] it is exposed the following theorem, whose proof is based in the Alexandroff's Reflection Principle (see [23] for more details).

Theorem 3.1. *Let $I \times_f \mathbb{R}^n$ be an* FLRW *spacetime, and let B a ball of $\mathbb{R}^n$. For each smooth radially symmetric function $H : I \times \overline{B} \to \mathbb{R}, H = H(t,r)$, radially increasing in the second variable and which satisfies $H(0,r) \le \frac{f'(0)}{f(0)}$ on ∂B, any positive solution v of Eq.* (4) *is radially symmetric. Moreover, $\frac{\partial v}{\partial r} < 0$ holds on ∂B.*

Remark 3.2. Geometrically, the last assertion means that the hyperbolic angle between the unit normal vector field N and ∂_t is nowhere zero at the points of the graph corresponding to $\{0\} \times \partial B$.

Theorem 3.1 asserts that, under certain assumptions on the mean curvature function, the problem only has radially symmetric solutions. In this paper, we are going to consider only solutions with radial symmetry.

We take a polar coordinate system centered at $0 \in B(R)$ and write the Euclidean metric as usual as

$$dr^2 + r^2 d\Theta^2,$$

where $d\Theta^2$ is the canonical metric of the $(n-1)$-dimensional unit sphere. In addition, we suppose $H : I \times B(R) \to \mathbb{R}$ will be a radially symmetric smooth function.

Under these considerations, passing to polar coordinates, Eq. (4) is reduced to the following ODE with mixed boundary conditions

$$\begin{gathered} \frac{1}{r^{n-1}}(r^{n-1}\phi(v'))' + \frac{nf'(\varphi^{-1}(v))}{\sqrt{1-v'^2}} = nH(\varphi^{-1}(v), r) f(\varphi^{-1}(v)) \quad \text{in }]0,R[, \\ v'(0) = 0 = v(R), \end{gathered} \tag{5}$$

where $\phi(s) := \frac{s}{\sqrt{1-s^2}}$. By a *solution* we understand a function $v \in C^2]0,R[\cap C^1[0,R]$ with $|v'| < 1$ on $]0,R[$ and satisfying the above mixed boundary value problem. From now on, we will work with this equation.

3.2. *Positivity of the solutions*

In this work, we are interested in spacelike graphs defined on a closed ball of the fiber, whose boundary is supported on the slice $t = 0$. In other words, the function

v, which define the graph, is strictly positive in the open ball, and it is zero at the boundary. In addition, the assertion of Theorem 3.1 suggests the search of conditions which ensure the positivity of the solutions. For the case of radial symmetry (Eq. (5)), we may state the following proposition.

Proposition 3.3. *Assume that*

$$\text{(H)} \qquad H(t,r) \leq \frac{f'}{f}(t) \quad \text{and} \quad f'(t) \geq 0 \quad \text{for all}\, r \in [0,R],\ \ t \in I_f(R).$$

Then, any v not identically zero solution of (5) *verifies $v > 0$ on $[0, R[$.*

Proof. First, note that for all $r \in]0, R[$,

$$v'(r) = -\phi^{-1}\left(\frac{n}{r^{n-1}}\int_0^r \tau^{n-1}\left[-H(\varphi^{-1}(v),\tau)f(\varphi^{-1}(v)) + \frac{f'(\varphi^{-1}(v))}{\sqrt{1-v'^2}}\right] d\tau\right).$$

Taking into account (H) and that ϕ is an odd increasing diffeomorphism, we deduce that v is decreasing. Since $v(R) = 0$, we have $v \geq 0$ on $[0, R]$. If v does not vanished identically, then $v(0) > 0$ and there exists $r_0 \in]0, R[$ where $v'(r_0) < 0$. Then, we get

$$\int_0^{r_0} \tau^{n-1}\left[-H(\varphi^{-1}(v),\tau)f(\varphi^{-1}(v)) + \frac{f'(\varphi^{-1}(v))}{\sqrt{1-v'^2}}\right] d\tau > 0.$$

Since the integrant is positive on $[0, R]$, this implies

$$\int_0^{r} \tau^{n-1}\left[-H(\varphi^{-1}(v),\tau)f(\varphi^{-1}(v)) + \frac{f'(\varphi^{-1}(v))}{\sqrt{1-v'^2}}\right] d\tau > 0, \quad \text{for all } r \geq r_0.$$

We deduce that $v'(r) < 0$ on $[r_0, R]$ and therefore, we conclude that $v > 0$ on $[0, R[$. □

3.3. *Strictly spacelike character and bounds on the derivative of the solutions*

Graphs which are solution of (E) are spacelike on the open ball. However, there could exist solutions which are of light type on the boundary, ∂B. The following lemma ensures *a priori* that each possible solution v of (5) is spacelike on the boundary too, i.e. $|v'| < 1$ on $[0, R]$.

Lemma 3.4. *Let $v \in C^2[0, R]$ be a solution of* (5)*. Then $|v'| < 1$ on $[0, R]$.*

Proof. On $[0, R[$ the solution satisfies $|v'| < 1$. We only have to prove the inequality at $r = R$. Suppose that there exists $\{r_k\} \subset]0, R[$ converging to R, such that

$$\lim_{k\to\infty} |v'(r_k)| = 1 \quad \text{and} \quad \lim_{k\to\infty} |\phi(v'(r_k))| = \infty.$$

For $k \in \mathbb{N}$ sufficiently large, one has for $r = r_k$,

$$\frac{1}{r^{n-1}}(r^{n-1}\phi(v'))' + \frac{nf'(\varphi^{-1}(v))}{v'}\phi(v') = nH(\varphi^{-1}(v), r)f(\varphi^{-1}(v)),$$

implying

$$\frac{[r^{n-1}\phi(v')]'}{[r^{n-1}\phi(v')]} = n\left(\frac{H(\varphi^{-1}(v), r)f(\varphi^{-1}(v))}{\phi(v')} - \frac{f'(\varphi^{-1}(v))}{v'}\right).$$

Let $\overline{r} \in]0, R[$ be such that $|v'(\tau)| > 0$ for any $\tau \in]\overline{r}, R[$. Integrating the last equality, we have

$$\log|r_k^{n-1}\phi(v'(r_k))| - \log|\overline{r}^{n-1}\phi(v'(\overline{r}))| = n\int_{\overline{r}}^{r_k}\left(\frac{H(\varphi^{-1}(v), r)f(\varphi^{-1}(v))}{\phi(v')} - \frac{f'(\varphi^{-1}(v))}{v'}\right)dr.$$

Taking limits, we check that left member tends to infinity while the right one is finite. Therefore, we deduce that $|\phi(v')|$ is bounded and, consequently, $\|v'\|_\infty$ must be strictly lower than 1. □

In the next result, we provide an *a priori* bound of the derivative of the solutions on the boundary R. This fact will play a key role later.

Proposition 3.5. *There exists $0 < \gamma < 1$ such that for any $\varepsilon \in [0, 1]$, one has that any $u \in C^2[R/2, R]$ with $u(R) = 0$ and satisfying on $]R/2, R[$ the equation*

$$\frac{1}{(r+\varepsilon)^{n-1}}((r+\varepsilon)^{n-1}\phi(u'))' + \frac{nf'(\varphi^{-1}(u))}{\sqrt{1-u'^2}} = nH(\varphi^{-1}(u), r)f(\varphi^{-1}(u)),$$

satisfies $|u'(R)| < \gamma$.

Proof. Let $w^+ : [R/2, R] \to \mathbb{R}$ be given by

$$w^+(r) = \int_0^{R-r}\frac{1}{\sqrt{1+\beta(t)}}\,dt,$$

where $\beta(t) = \alpha\, e^{\lambda t}$, with α and λ constants which will be specified later. This type of function was used by Gerhardt in [22] for a similar purpose (see formula (2.9) therein).

Clearly, for all $r \in [R/2, R]$,

$$|(w^+)'(r)| = \frac{1}{\sqrt{1+\beta(R-r)}} < 1.$$

Now, let u be satisfying the hypothesis and consider the elliptic operator depending on u

$$Q_u(v)(r) := -\frac{1}{(r+\varepsilon)^{n-1}}[(r+\varepsilon)^{n-1}\phi(v')]' - \frac{nf'(\varphi^{-1}(u))}{\sqrt{1-v'^2}}.$$

It follows that

$$Q_u(w^+)(r) = \frac{1}{\sqrt{\beta(R-r)}}\left[\frac{n-1}{r+\varepsilon} + \frac{\lambda}{2} - nf'(\varphi^{-1}(u))\sqrt{1+\alpha\, e^{\lambda(R-r)}}\right].$$

Using that $|u| < R/2$ on $[R/2, R]$, we can choose $\lambda > 0$ sufficiently large and $\alpha > 0$ sufficiently small which do not depend on u and $\varepsilon \in [0,1]$ such that

$$\frac{\lambda}{2} + \frac{n-1}{r+\varepsilon} - nf'(\varphi^{-1}(u))\sqrt{1+\alpha\, e^{\lambda(R-r)}} > 0,$$

on $[R/2, R]$. Because of $\varepsilon \in [0,1]$, note that α and λ can be chosen independently of ε. In fact, the choice only depends on functions f and H. Hence, making α smaller if necessary, we can get

$$Q_u(w^+) \geq \max\left\{-nf(t)H(t,r) : r \in \left[\frac{R}{2}, R\right],\ t \in \left[-\frac{R}{2}, \frac{R}{2}\right]\right\},$$

implying that

$$Q_u(w^+) \ \geq\ Q_u(u).$$

We have two situations. In the first one $w^+(R/2) \geq u(R/2)$ and in the second $w^+(R/2) < u(R/2)$. Assume that we are in the second case and take

$$K = \max_{[R/2,R]} |u'|.$$

Observe that $K < 1$ by Lemma 3.4. Then, there exists $r_0 \in]R/2, R[$ satisfying

$$r_0 - \frac{R}{2} > \frac{KR}{2}.$$

So, we can consider $\alpha_u < \alpha$ such that

$$\left[\frac{\left(r_0 - \frac{R}{2}\right)^2}{\left(u\left(\frac{R}{2}\right) - w^+(r_0)\right)^2} - 1\right] e^{-\lambda\frac{R}{2}} > \alpha_u > 0.$$

It follows that, considering the function on $[0, R/2]$ given by

$$\overline{\alpha}(s) = \begin{cases} \alpha & \text{if } s \leq R - r_1, \\ h(s) & \text{if } R - r_1 \leq s \leq R - r_0, \\ \alpha_u & \text{if } R - r_0 < s \leq \dfrac{R}{2}, \end{cases}$$

where $r_1 \in]r_0, R[$, h is a decreasing function that makes $\overline{\alpha}$ differentiable, and

$$w_u^+(r) := \int_0^{R-r} \frac{1}{\sqrt{1+\overline{\alpha}(t)\, e^{\lambda t}}}\, dt, \quad r \in \left[\frac{R}{2}, R\right],$$

one has that $w_u^+(R/2) \geq u(R/2)$. By a simple computation,

$$Q_u(w_u^+) \geq Q_u(w^+).$$

Hence, it follows that $v = w^+$ or $v = w_u^+$ is an upper-solution of the original equation on $[R/2, R]$, that is,

$$Q_u(v) \geq Q_u(u)$$
$$v(R) = u(R) = 0,$$
$$v\left(\frac{R}{2}\right) \geq u\left(\frac{R}{2}\right).$$

Therefore, from Maximum Principle (see the Comparison Principle in [24, Theorem 4.4]) we conclude that

$$v(r) \geq u(r), \quad r \in \left[\left(\frac{R}{2}\right), R\right].$$

Since $v(R) = u(R)$ and taking into account that $v'(R)$ does not depend on u and ε, we deduce that

$$u'(R) \geq v'(R) =: \gamma^+ > -1, \quad |\gamma^+| < 1.$$

Analogously, taking

$$w^-(r) := -\int_0^{R-r} \frac{1}{\sqrt{1+\widehat{\beta}(t)}}\, dt,$$

where $\widehat{\beta}(t) = \widehat{\alpha}\, e^{\widehat{\lambda} t}$, we have

$$u'(R) \leq v'(R) =: \gamma^- < 1, \quad |\gamma^-| < 1,$$

where $v = w^-$ or $v = w_u^-$

Consequently, taking $\gamma := \max\{|\gamma^+|, |\gamma^-|\}$, we conclude that

$$|u'(R)| < \gamma < 1. \qquad \square$$

4. The Associated Dirichlet Problem: Existence Result

In this section we give sufficient conditions for the existence of positive and radially symmetric solutions of problem (5).

Throughout the section $C[0, R]$ denotes the Banach space of the real continuous functions in $[0, R]$, endowed with the maximum norm, and $C^1[a, b]$ the Banach space of continuously differentiable functions in $[a, b]$ endowed with the usual norm.

Our strategy consists on a truncation of the singular term, obtaining a family of problems tending to the original one, that can be solved through degree techniques. Then, we take the limit of the solutions of the truncated equations, and we have to prove that this limit is really a solution of the singular problem. Some arguments in our proof come from [27, Chap. 9] (see also the references therein), nevertheless the computations are essentially different because [27] only considers the case of a regular ϕ-Laplacian defined on the whole real line, whereas in our case the ϕ-Laplacian is singular.

The main existence result goes as follows.

Theorem 4.1. *If* (A) *and* (H) *hold true, then there exists at least one positive solution of problem* (5).

Proof. The proof is organized in three steps.

- First step: *Truncation*

First of all, we embed the initial problem into the family of mixed boundary value problems

$$\begin{gathered}\frac{1}{(r+\varepsilon)^{n-1}}((r+\varepsilon)^{n-1}\phi(v'))' + \frac{nf'(\varphi^{-1}(v))}{\sqrt{1-v'^2}} = nH(\varphi^{-1}(v),r)f(\varphi^{-1}(v)),\\ v'(0)=0=v(R),\end{gathered} \tag{6}$$

where $\varepsilon \in [0,1]$. Expanding the left member of the truncated equation and multiplying by $\sqrt{1-v'^2}$, we get

$$\frac{v''}{1-v'^2} = -(n-1)\,\frac{v'}{r+\varepsilon} + nf(\varphi^{-1}(v))H(v,r)\sqrt{1-v'^2} - nf'(\varphi^{-1}(v)). \tag{7}$$

Since

$$\frac{1}{1-v'^2} = \frac{1}{2}\left(\frac{1}{1+v'} + \frac{1}{1-v'}\right),$$

we may rewrite the previous expression as follows

$$\begin{aligned}\left[\frac{1}{2}\log\left(\frac{1+v'}{1-v'}\right)\right]' &= -(n-1)\,\frac{v'}{r+\varepsilon}\\ &\quad + nH(\varphi^{-1}(v),r)f(\varphi^{-1}(v))\sqrt{1-v'^2} - nf'(\varphi^{-1}(v)).\end{aligned}$$

We define

$$\psi\,:\,]-1,1[\,\to \mathbb{R}, \quad \psi(s) = \frac{1}{2}\log\left(\frac{1+s}{1-s}\right),$$

which is an increasing diffeomorphism satisfying $\psi(0) = 0$. So, we have transformed the initial family of ϕ-Laplacians problems into the following ψ-Laplacians equations

$$\begin{gathered}(\psi(v'))' = -(n-1)\,\frac{v'}{r+\varepsilon} + nH(\varphi^{-1}(v),r)f(\varphi^{-1}(v))\sqrt{1-v'^2} - nf'(\varphi^{-1}(v)),\\ v'(0)=0=v(R).\end{gathered}$$

Note that our problem, corresponding to $\varepsilon = 0$, has now a singular term in zero, but the singularity on the derivative has disappeared.

We denote by

$$\begin{gathered}G\colon]0,R]\times[-R,R]\times[-1,1]\to\mathbb{R}\\ G(r,s,y) := -(n-1)\frac{y}{r} + nH(\varphi^{-1}(s),r)f(\varphi^{-1}(s))\sqrt{1-y^2} - nf'(\varphi^{-1}(s)),\end{gathered}$$

and we define the family of functions depending on $\varepsilon > 0$,

$$G_\varepsilon : [0, R] \times [-R, R] \times [-1, 1] \to \mathbb{R}$$

$$G_\varepsilon(r, s, y) = -(n-1)\frac{y}{r+\varepsilon} + nH(\varphi^{-1}(s), r) f(\varphi^{-1}(s))\sqrt{1-y^2} - nf'(\varphi^{-1}(s)).$$

One clearly has

$$G_\varepsilon \to G \quad \text{pointwise.}$$

On the other hand, for each $\varepsilon > 0$,

$$|G_\varepsilon| \le \frac{n-1}{\varepsilon} + nf^* H^* + nf'^* =: \Lambda,$$

where

$$f^* = \max_{[-R,R]} f, \quad f'^* = \max_{[-R,R]} |f'| \quad \text{and}$$

$$H^* = \max\{|H(\varphi^{-1}(s), r)| : r \in [0, R],\ s \in [-R, R]\}.$$

From [8], for any $\varepsilon > 0$, the problem

$$(\psi(v'))' = G_\varepsilon(r, v, v'), \quad v'(0) = 0 = v(R),$$

has at least one solution $v_\varepsilon \in C^\infty[0, R]$. This is an immediate consequence of Schauder's fixed point theorem.

- Second step: *Convergence of* v_ε

Firstly, because $\|v_\varepsilon\|_\infty < R$ and $\|v'_\varepsilon\|_\infty < 1$, using Ascoli–Arzela Theorem, passing if necessary to a subsequence, there exists $v \in C[0, R]$ such that

AQ: Ascoli-Arzela (or) Arzela-Ascoli? Please check globally.

$$\|v - v_\varepsilon\|_\infty \to 0.$$

Note that

$$v(R) = 0.$$

Consider $0 < a \le R$. Looking to the expanded problem, we have for any $r \in [a, R]$,

$$|v''_\varepsilon(r)| \le \frac{(n-1)}{a} + nf^* H^* + nf'^*,$$

implying that the family $\{v'_\varepsilon\}_{\varepsilon > 0}$ is equicontinuous on $[a, R]$. Since $\|v'_\varepsilon\|_\infty < 1$, it follows from the Ascoli–Arzela Theorem that there exists $w \in C[a, R]$ such that

$$v'_\varepsilon \to w \quad \text{in } C[a, R].$$

It follows that $v \in C^1[a, R]$ and $\{v_\varepsilon\}$ converges to v in $C^1[a, R]$.

- Third step: *The limit is a solution*

Clearly, from the previous steps we deduce that

$$\lim_{\varepsilon \to 0^+} G_\varepsilon(r, v_\varepsilon(r), v'_\varepsilon(r)) = G(r, v(r), v'(r)) \quad \text{for each } r \in]0, R].$$

Now, choose an arbitrary $r \in]0, R[$, and notice that

$$(\psi(v_\varepsilon'))' = G_\varepsilon(\tau, v_\varepsilon, v_\varepsilon') \quad \text{in } [r, R].$$

Integrating between r and R, we infer that

$$\psi(v_\varepsilon'(R)) - \psi(v_\varepsilon'(r)) = \int_r^R G_\varepsilon(\tau, v_\varepsilon(\tau), v_\varepsilon'(\tau))\, d\tau.$$

Then, the Lebesgue Dominated Convergence Theorem and Proposition 3.5 imply that $|v'| < 1$ on $]0, R]$ and

$$\psi(v'(R)) - \psi(v'(r)) = \int_r^R G(\tau, v(\tau), v'(\tau))\, d\tau, \quad r \in]0, R].$$

It follows that

$$(\psi(v'))' = G(r, v, v') \quad \text{in }]0, R]. \tag{8}$$

Moreover,

$$\int_0^R G_\varepsilon(\tau, v_\varepsilon(\tau), v_\varepsilon'(\tau))\, d\tau = \psi(v_\varepsilon'(R)).$$

Making use of the Proposition 3.5, there exists $\gamma \in (0, 1)$ such that

$$|\psi(v_\varepsilon'(R))| < |\psi(\gamma)| \quad \text{for all } \varepsilon > 0.$$

Then, we rewrite

$$G_\varepsilon(r, s, t) = -(n-1)\, \frac{t}{r+\varepsilon} + g(r, s, t),$$

where

$$g(r, s, t) := nH(\varphi^{-1}(s), r) f(\varphi^{-1}(s)) \sqrt{1-t^2} - nf'(\varphi^{-1}(s)).$$

It is clear that the function $r \mapsto g(r, v_\varepsilon(r), v_\varepsilon'(r))$ is integrable on $[0, R]$. Moreover, we have

$$|g(r, v_\varepsilon(r), v_\varepsilon'(r))| < nf^* H^* + nf'^* =: K \quad \text{for any } \varepsilon > 0.$$

Hence,

$$(n-1) \left| \int_0^R \frac{v_\varepsilon'(\tau)}{\tau+\varepsilon}\, d\tau \right| < RK + |\psi(\gamma)|.$$

On the other hand, from (6), we get

$$v_\varepsilon'(r) = -\phi^{-1}\left[\frac{n}{(r+\varepsilon)^{n-1}} \int_0^r (\tau+\varepsilon)^{n-1} F(\tau, v_\varepsilon(\tau), v_\varepsilon'(\tau))\, d\tau\right],$$

where

$$F(r, s, t) := H(\varphi^{-1}(s), r) f(\varphi^{-1}(s)) - \frac{f'(\varphi^{-1}(s))}{\sqrt{1-t^2}}.$$

Now, using (H), one has that the integrand is positive and, therefore, v'_ε is nonpositive for all $\varepsilon > 0$. Thus,

$$(n-1)\int_0^R \frac{|v'_\varepsilon(\tau)|}{\tau+\varepsilon}\,d\tau = (n-1)\left|\int_0^R \frac{v'_\varepsilon(\tau)}{\tau+\varepsilon}\,d\tau\right| < RK + |\psi(\gamma)|. \tag{9}$$

We deduce that, $\{-(n-1)\,\frac{v'_\varepsilon(r)}{r+\varepsilon}\}_{\varepsilon>0}$ is a set of positive integrable functions, satisfying (9) and pointwise convergent to the function $-(n-1)\frac{v'(r)}{r}$. Applying Fatou Lemma, we conclude that the limit is integrable on $[0,R]$ and

$$r \mapsto G(r, v(r), v'(r)) \quad \text{is integrable on } [0,R].$$

Now we are in a position to prove that $\lim_{r\to 0} v'(r) = 0$. From integrability of $r \mapsto \frac{v'(r)}{r}$, it is clear that, if the limit exists, it should be 0. So, it suffices to prove the existence of $\lim_{r\to 0} v'(r)$. From (8), integrating from r to R, we obtain

$$\psi(v'(r)) = \psi(v'(R)) - \int_r^R G(\tau, v(\tau), v'(\tau))d\tau.$$

Since $\tau \mapsto G(\tau, v(\tau), v'(\tau))$ is integrable on $[0,R]$, the limit of the right member exists when r tends to 0. Therefore, by using that ψ is a diffeomorphism, we deduce the existence of $\lim_{r\to 0} v'(r)$. The proof is done. □

5. Proof of the Main Result

Theorem 1.2 is a direct consequence of Theorems 3.1 and 4.1, which were proved in previous sections. To prove Theorem 1.1, once R is fixed, Theorem 4.1 provides a solution v of problem (5). Then, it suffices to guarantee that v can be continued until $+\infty$ as a strictly decreasing solution. First, we can rewrite Eq. (7), with $\epsilon = 0$, as a system of two ordinary differential equations of first order

$$v' = z,$$
$$z' = (1-z^2)\left(-(n-1)\,\frac{z}{r} + n(f(\varphi^{-1}(v))\,H(\varphi^{-1}(v), r)\sqrt{1-z^2} - n(f'(\varphi^{-1}(v))\right),$$

which we can abbreviate

$$\begin{bmatrix} v' \\ z' \end{bmatrix} = \mathcal{F}(r, (v,z)),$$

where $\mathcal{F} : \mathbb{R}^+ \times J \times \,]-1\,,1[\to \mathbb{R}^2$.

Let $[0, b[$ be the maximal interval of definition of v. Suppose that $b < +\infty$. By the standard prolongability theorem of ordinary differential equations (see for instance [30, Sec. 2.5]), we have that the graph $\{(r, v(r), v'(r)) : r \in [R/2, b[\}$ goes out of any compact subset of $\mathbb{R}^+ \times J \times \,]-1, 1[$. However $|v(r)| < b$ then, since $\mathbb{R}^- \subset J$ and v is decreasing, we know that $v(r) \in [-b, R]$. Moreover, by Lemma 3.4, $|v'(r)| < \rho < 1$. Therefore, the graph cannot go out of the compact subset $[R/2\,, b] \times [-b, R] \times [-\rho, \rho]$ contained in the domain of $\mathcal{F}$. This is a contradiction, then $b = +\infty$.

From $\mathbb{R}^- \subset \varphi(I)$ we have that $f(t)$ tends to 0 when t goes to $\inf I$. Then u' tends to 0 and, taking into account that u is strictly decreasing, we obtain the conclusion.

6. Final Remarks and Applications

It should be pointed out that the assumptions of the main result have a reasonable physical interpretation. In fact, the inequality $f'(t) \geq 0$ means that the divergence in the spacetime $I \times_f \mathbb{R}^n$ of the reference frame ∂_t is non-negative, which indicates that the comoving observers are on average spreading apart [29, p. 121] and so, for these observers, the universe is really expanding whenever $f'(t) > 0$. On the other hand, the inequality $H(t,r) \leq (f'/f)(t)$ expresses an above control of the prescription function by the Hubble function f'/f of the spacetime $I \times_f \mathbb{R}^n$. This kind of inequality has been used to characterize the spacelike slices of some $I \times_f \mathbb{R}^n$ when $n = 2$ [28].

Moreover, the family of FLRW spacetimes where the result may be applied is very wide, and it contains relevant relativistic spacetimes. Indeed, it includes the Lorentz–Minkowski spacetime ($f = 1$, $I = \mathbb{R}$), the Einstein–De Sitter spacetime ($I =]-t_0, +\infty[$, $f(t) = (t+t_0)^{2/3}$, with $t_0 > 0$), and the steady state spacetime ($I = \mathbb{R}$, $f(t) = e^t$), which is an open subset of the De Sitter spacetime.

Computing the interval $I_f(R)$ in the two previous cases, we obtain respectively,

$$]-\infty, -\log(1-R)[\quad \text{and} \quad \left]-t_0 + \left(t_0^{\frac{1}{3}} - \frac{R}{3}\right)^3, \left(\frac{R}{3} + t_0^{\frac{1}{3}}\right) - t_0\right[,$$

and for the interval $J = \varphi(I)$,

$$]-\infty, 1[\quad \text{and} \quad]-3t_0^{\frac{1}{3}}, \infty[.$$

Observe that we can ensure the existence of radially symmetric spacelike graphs with prescribed mean curvature (under the hypotheses of Theorem 1.2) on a ball when the radius R is less than 1 and $3t_0^{1/3}$ respectively.

Finally, note that for the steady state spacetime such a graph can be extended to the whole fiber $\mathbb{R}^n$, because $\int_{-\infty}^0 e^{-s} ds = \infty$. It is very easy to construct explicit examples of FLRW spacetimes leading to entire graphs tending to a hyperplane. For instance, $I =]-t_0, +\infty[$ and $f(t) = (t+t_0)^\alpha$, with $t_0 > 0$ and $\alpha \geq 1$.

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