Communications in Contemporary Mathematics
Vol. 15, No. 4 (2013) 1250063 (22 pages)

DOI: 10.1142/S0219199712500630

PERIODIC SOLUTIONS FOR SINGULAR PERTURBATIONS OF THE SINGULAR ϕ-LAPLACIAN OPERATOR

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Received 5 March 2012
Revised 22 August 2012
Accepted 25 August 2012
Published 18 January 2013

In this paper, using Leray–Schauder degree arguments and the method of lower and upper solutions, we give existence and multiplicity results for periodic problems with singular nonlinearities of the type

$$\left(\frac{u'}{\sqrt{1-u'^2}}\right)' + r(t)u + \frac{n(t)}{u^\lambda} = e(t), \quad u(0) - u(T) = 0 = u'(0) - u'(T),$$

where $r, n, e : [0, T] \to \mathbb{R}$ are continuous functions and $\lambda > 0$. We also consider some singular nonlinearities arising in nonlinear elasticity or of Rayleigh–Plesset type.

Keywords: ϕ-Laplacian; periodic solutions; singular nonlinearities; Leray–Schauder degree; upper and lower solutions.

Mathematics Subject Classification 2010: 34B15, 34B16, 34C25

1. Introduction

In [1] it is proven that the periodic problem with an attractive nonlinearity

$$\left(\frac{u'}{\sqrt{1-u'^2}}\right)' + \frac{1}{u^\lambda} = e(t), \qquad u(0) - u(T) = 0 = u'(0) - u'(T),$$

where $e \in C([0, T])$ and $\lambda > 0$, has at least one solution if and only if $\overline{e} := \frac{1}{T}\int_0^T e > 0$. Assuming that $\lambda \geq 1$, in the same paper [1] it is shown that problem

$$\left(\frac{u'}{\sqrt{1-u'^2}}\right)' - \frac{1}{u^\lambda} = e(t), \quad u(0) - u(T) = 0 = u'(0) - u'(T),$$

has at least one solution if and only if $\overline{e} < 0$. The corresponding classical results (for the operator $u \mapsto u''$) are obtained in the seminal paper of Lazer and Solimini [13].

On the other hand, in [6] the authors give a Fredholm alternative-like result for the periodic problem

$$u'' + ru - \frac{1}{u^{\lambda}} = e(t), \quad u(0) - u(T) = 0 = u'(0) - u'(T), \tag{1}$$

where r is a constant, $e \in C([0,T])$ and $\lambda \geq 1$. More precisely, they show, using Leray–Schauder degree theory, that (1) has at least one solution if $r \neq (k\pi/T)^2$ for all $k \in \mathbb{Z}$. The first existence result for $r \in (0,(\pi/T)^2]$ and $\lambda > 0$ (including also the weak case $0 < \lambda < 1$) appears in [16]. Under this assumptions, in the mentioned paper, it is proved, using the method of lower and upper solutions, that (1) has at least one solution if

$$\min_{[0,T]} e > -(1+\lambda)\left(\frac{\pi^2 - rT^2}{\lambda T^2}\right)^{\frac{\lambda}{\lambda+1}}.$$

In case $r \in (0,(\pi/T)^2)$, the main result in [18] provides the alternative condition

$$\min_{[0,T]} e < 0, \quad \max_{[0,T]} e \leq \frac{\min_{[0,T]} e}{\cos^{\lambda}\left(\frac{\sqrt{r}T}{2}\right)} + \frac{\sqrt{r}}{T}\sin(\sqrt{r}T)[\min_{[0,T]} e]^{-\frac{1}{\lambda}}.$$

The main tool used in [18] is Krasnoselskii fixed point theorem on compressions and expansions of cones. In the weak case $0 < \lambda < 1$, if $r \in (0,(\pi/T)^2]$, and

$$\max_{[0,T]} e < 0, \quad \min_{[0,T]} e \geq (\lambda^2 - 1)[r^{\lambda}\lambda^{\frac{2\lambda^2}{1-\lambda}}]^{\frac{1}{1+\lambda}},$$

then, it is shown in [19], using Schauder fixed point theorem, that (1) has at least one solution.

It is interesting to remark that, in contrast to the classical case, the periodic problem with relativistic acceleration

$$\left(\frac{u'}{\sqrt{1-u'^2}}\right)' + ru = e(t), \quad u(0) - u(T) = 0 = u'(0) - u'(T),$$

has at least one solution for any $r \neq 0$ and any continuous forcing term e (see [1, Corollary 3]). For this type of problems see, e.g., [12]. We will show that, in some sense, the same situation occurs also if we add a singular nonlinearity.

In order to explain the main results of the paper, let us introduce some notation. If $x \in \mathbb{R}$, then we write $x^+ = \max\{x,0\}$ and $x^- = \max\{-x,0\}$. For $e \in C([0,T])$ we put

$$E = \int_0^T e(t)dt, \quad E_{\pm} = \int_0^T e^{\pm}(t)dt,$$

and note that $E = E_+ - E_-$.

Motivated by the above results from [1, 6, 16, 18, 19], we consider the periodic problem

$$\left(\frac{u'}{\sqrt{1-u'^2}}\right)' + r(t)u - \frac{1}{u^\lambda} = e(t), \quad u(0)-u(T)=0=u'(0)-u'(T),$$

where $r, e \in C([0,T])$ and $\lambda \geq 1$. If either $\overline{r} > 0$ or $\overline{r} = 0$ and $\overline{e} < -\frac{R_-}{2}$, then we prove that the above problem has at least one solution (see Example 2). In case $\overline{r} < 0$, we show (see Example 4) that the periodic problem

$$\left(\frac{u'}{\sqrt{1-u'^2}}\right)' + r(t)u - \frac{m(t)}{u^\lambda} = e(t), \quad u(0)-u(T)=0=u'(0)-u'(T),$$

with $\lambda > 0$ (so, the weak case is included) and $m \in C([0,T])$ such that $m \geq 0$, is solvable provided that

$$-E > (1+\lambda)\left[\frac{|R|^\lambda M}{\lambda^\lambda}\right]^{\frac{1}{1+\lambda}} + \frac{T}{2}R_-.$$

On the other hand, in the attractive case, we consider the problem

$$\left(\frac{u'}{\sqrt{1-u'^2}}\right)' + r(t)u + \frac{m(t)}{u^\lambda} = e(t), \quad u(0)-u(T)=0=u'(0)-u'(T), \tag{2}$$

where $r, m, e \in C([0,T])$ with $m \geq 0$ and $\lambda > 0$. If either $\overline{r} < 0$ or $r = 0$, and in both cases,

$$E < M\left(\frac{2}{T}\right)^\lambda - \frac{T}{2}R_-,$$

then, we show (see Example 5 and Proposition 1) that the above problem has at least one solution. Moreover, in the pure attractive case, that is $m > 0$, one has that (2) is solvable if either $\overline{r} < 0$ or $\overline{r} = 0$ and $E > \frac{T}{2}R_+$ (Proposition 2).

The paper is organized as follows. In Sec. 2 we introduce some notation and auxiliary results (almost all taken from [1]). In Sec. 3 we improve Theorem 4 from [1] and give two applications. In the first one we consider strong repulsive nonlinearities and in the second one we study nonlinearities null at infinity. In Sec. 4 we introduce some methods to construct lower and upper solutions and in the last section we prove the previous results. We also consider some singular nonlinearities arising in nonlinear elasticity or of Rayleigh–Plesset type.

If Ω is an open bounded subset in a Banach space X and $S : \overline{\Omega} \to X$ is compact, with $0 \notin (I-S)(\partial\Omega)$, then $d_{\mathrm{LS}}[I-S, \Omega, 0]$ will denote the Leray–Schauder degree of S with respect to Ω and 0. For the definition and properties of the Leray–Schauder degree we refer the reader to, e.g., [5].

For other results concerning periodic solutions of nonlinear perturbations of the relativistic operator $u \mapsto \left(\frac{u'}{\sqrt{1-u'^2}}\right)'$ see, e.g., [3, 4, 7, 14, 20].

2. Some Notation and Auxiliary Results

Let C^0 denote the Banach space of continuous functions on $[0,T]$ endowed with the uniform norm $\|\cdot\|_\infty$, C^1 denote the Banach space of continuously differentiable functions on $[0,T]$ equipped with the norm

$$\|u\| = \|u\|_\infty + \|u'\|_\infty \quad (u \in C^1).$$

Let $P, Q : C^0 \to C^0$ be the continuous projectors defined by

$$Pu(t) = u(0), \quad \overline{u} = Qu(t) = \frac{1}{T}\int_0^T u(\tau)d\tau \quad (t \in [0,T]),$$

and define the continuous linear operator $H : C^0 \to C^1$ by

$$Hu(t) = \int_0^t u(\tau)d\tau \quad (t \in [0,T]).$$

If $u \in C^0$ we write

$$\widetilde{u} = u - \overline{u},$$

and we shall consider the following closed subspaces of C^1:

$$C^1_\sharp = \{u \in C^1 : u(0) - u(T) = 0 = u'(0) - u'(T)\},$$
$$\widetilde{C}^1_\sharp = \{u \in C^1_\sharp : \overline{u} = 0\}.$$

The following assumption upon ϕ (called *singular*) is made throughout the paper:

(H_ϕ) $\phi : (-a,a) \to \mathbb{R}$ *is an increasing homeomorphism such that* $\phi(0) = 0$ *and* $0 < a < \infty$.

The model example is

$$\phi(s) = \frac{s}{\sqrt{1-s^2}} \quad (s \in (-1,1)).$$

We recall the following technical result given as Lemma 1 from [1].

Lemma 1. *For each* $h \in C^0$ *there exists a unique* $Q_\phi(h) \in \mathbb{R}$ *such that*

$$Q \circ \phi^{-1} \circ (I - Q_\phi) \circ h = 0.$$

Moreover, the function $Q_\phi : C^0 \to \mathbb{R}$ *is continuous.*

We recall also the following fixed point result introduced in [1].

Lemma 2. *Let* $F : C^1 \to C^0$ *be a continuous operator which takes bounded sets into bounded sets and consider the abstract periodic problem*

$$(\phi(u'))' = F(u), \quad u(0) - u(T) = 0 = u'(0) - u'(T). \tag{3}$$

A function u is solution of (3) *if and only if $u \in C^1_\sharp$ is a fixed point of the completely continuous operator $M_F : C^1_\sharp \to C^1_\sharp$ defined by*

$$M_F = P + QF + H \circ \phi^{-1} \circ (I - Q_\phi) \circ [H(I - Q)F].$$

Furthermore, $\|(M_F(u))'\|_\infty < a$ for all $u \in C^1_\sharp$ and

$$\|\widetilde{u}\|_\infty < aT, \tag{4}$$

for any solution u of (3).

To each continuous function $f : [0, T] \times \mathbb{R}^2 \to \mathbb{R}$, we associated its Nemytskii operator $N_f : C^1 \to C^0$ given by

$$N_f(u) = f(\cdot, u(\cdot), u'(\cdot)) \quad (u \in C^1).$$

One has that N_f is continuous and takes bounded sets into bounded sets.

Next, consider the periodic boundary value problem

$$(\phi(u'))' = N_f(u), \quad u(0) - u(T) = 0 = u'(0) - u'(T). \tag{5}$$

We will write M_f instead of M_{N_f}, the fixed point operator associated to (5), given by Lemma 2.

If $u, v \in C^0$ are such that $u(t) \le v(t)$ for all $t \in [0, T]$, we write $u \le v$. Also, we write $u < v$ if $u(t) < v(t)$ for all $t \in [0, T]$. One has the following (see [1, Definition 1]).

Definition 1. A lower solution α (respectively, an upper solution β) of (5) is a function $\alpha \in C^1$ such that $\|\alpha'\|_\infty < a$, $\phi(\alpha') \in C^1$, $\alpha(0) = \alpha(T)$, $\alpha'(0) \ge \alpha'(T)$ (respectively, $\beta \in C^1$, $\|\beta'\|_\infty < a$, $\phi(\beta') \in C^1$, $\beta(0) = \beta(T)$, $\beta'(0) \le \beta'(T)$) and

$$(\phi(\alpha'))' \ge N_f(\alpha) \quad (\text{respectively, } (\phi(\beta'))' \le N_f(\beta)). \tag{6}$$

Such a lower or an upper solution is called strict if the inequality (6) is strict.

We need also the following result given as in [1, Theorem 3].

Lemma 3. *If* (5) *has a lower solution α and an upper solution β such that $\alpha \le \beta$, then* (5) *has a solution u such that $\alpha \le u \le \beta$. Moreover, if α and β are strict, then $\alpha < u < \beta$, and*

$$d_{\mathrm{LS}}[I - M_f, \Omega_{\alpha,\beta}, 0] = -1,$$

where

$$\Omega_{\alpha,\beta} = \{u \in C^1_\sharp : \alpha < u < \beta,\ \|u'\|_\infty < a\}.$$

An easy adaption of the proof of [1, Theorem 3] provides the following useful result.

Lemma 4. *Assume that* (5) *has a lower solution α and an upper solution β such that $\alpha < \beta$ and*

$$u \ne M_f(u) \quad \textit{for all } u \in \partial\Omega_{\alpha,\beta}. \tag{7}$$

Then, one has that

$$d_{\mathrm{LS}}[I - M_f, \Omega_{\alpha,\beta}, 0] = -1.$$

The following result is a particular case of [1, Lemma 4] and is a direct consequence of Schauder's fixed point theorem applied to the equivalent fixed point problem.

Lemma 5. *The periodic problem*

$$(\phi(\widetilde{u}'))' = (I - Q)N_f(\sigma + \widetilde{u}), \quad \widetilde{u} \in \widetilde{C}^1_\sharp, \tag{8}$$

has at least one solution for all $\sigma \in \mathbb{R}$.

The next result is an elementary estimation of the oscillation of a periodic function.

Lemma 6. *If* $u : \mathbb{R} \to \mathbb{R}$ *is a continuously differentiable and* T*-periodic function, then*

$$\max_{[0,T]} u - \min_{[0,T]} u \leq \frac{T}{2}\|u'\|_\infty.$$

Proof. Let $t_* \in [0, T)$ be such that $u(t_*) = \min_{[0,T]} u$ and $t^* \in [t_*, t_* + T]$ be such that $u(t^*) = \max_{[0,T]} u$. One has that

$$u(t^*) - u(t_*) = \int_{t_*}^{t^*} u'(s)ds \leq \|u'\|_\infty(t^* - t_*),$$

$$u(t^*) - u(t_*) = -\int_{t^*}^{t_*+T} u'(s)ds \leq \|u'\|_\infty(t_* + T - t^*).$$

Then, multiplying both inequalities and using that $xy \leq (x+y)^2/4$ for all $x, y \in \mathbb{R}$, it follows that

$$(u(t^*) - u(t_*))^2 \leq \frac{(\|u'\|_\infty T)^2}{4},$$

and the proof is completed. □

3. Non-Well-Ordered Lower and Upper Solution and Applications

In [1] it is proved that problem (5) has at least one solution if it has a lower and an upper solution. In the following result we prove some additional information concerning the location of the solution. In particular we have some *a posteriori* estimations which will be very useful in the sequel (see Remark 1). We use some ideas from the proof of [15, Theorem 8.10].

Theorem 1. *Assume that* (5) *has a lower solution* α *and an upper solution* β *such that*

$$\exists\, \tau \in [0,T]: \quad \alpha(\tau) > \beta(\tau). \tag{9}$$

Then, (5) *has at least one solution* u *such that*

$$\min\{\alpha(t_u), \beta(t_u)\} \leq u(t_u) \leq \max\{\alpha(t_u), \beta(t_u)\}, \tag{10}$$

for some $t_u \in [0,T]$.

Proof. Consider

$$u^* = \|\alpha\|_\infty + \|\beta\|_\infty + aT,$$

$$m = \max\{|f(t,u,v)| + 1 : (t,u,v) \in [0,T] \times [-u^* - 2, u^* + 2] \times [-a,a]\},$$

and define the continuous function $h : [0,T] \times \mathbb{R}^2 \to \mathbb{R}$ by

$$h(t,u,v) = \begin{cases} -m-1, & u \leq -u^* - 1, \\ f(t,u,v) + (u+u^*)(m+1+f(t,u,v)), & -u^* - 1 < u < -u^*, \\ f(t,u,v), & -u^* \leq u \leq u^*, \\ f(t,u,v) + (u-u^*)m, & u^* < u < u^* + 1, \\ f(t,u,v) + m, & u \geq u^* + 1. \end{cases}$$

Next, consider the modified periodic problem

$$(\phi(u'))' = h(t,u,u'), \quad u(0) - u(T) = 0 = u'(0) - u'(T), \tag{11}$$

and let M_h be the fixed point operator associated to (11).

One has that α is a lower solution and β is an upper solution of the modified problem (11). Moreover, $\alpha_1 = -u^* - 2$ is a lower solution of (11) and $\beta_1 = u^* + 2$ is an upper solution of (11). Notice that

$$\alpha_1 < \min\{\alpha, \beta\} \leq \max\{\alpha, \beta\} < \beta_1,$$

which together with (9) imply that

$$\Omega_{\alpha_1,\beta} \cup \Omega_{\alpha,\beta_1} \subset \Omega_{\alpha_1,\beta_1}, \quad \Omega_{\alpha_1,\beta} \cap \Omega_{\alpha,\beta_1} = \emptyset.$$

So, we can consider the open bounded set

$$\Omega = \Omega_{\alpha_1,\beta_1} \backslash [\overline{\Omega_{\alpha_1,\beta}} \cup \overline{\Omega_{\alpha,\beta_1}}].$$

It follows that

$$\Omega = \{u \in \Omega_{\alpha_1,\beta_1} : u(t_u) > \beta(t_u), u(s_u) < \alpha(s_u) \text{ for some } t_u, s_u \in [0,T]\}$$

and

$$\partial\Omega = \partial\Omega_{\alpha_1,\beta_1} \cup \partial\Omega_{\alpha_1,\beta} \cup \partial\Omega_{\alpha,\beta_1}.$$

One has that any constant function between $\beta(\tau)$ and $\alpha(\tau)$ is contained in Ω, so Ω is a non-empty set.

Next, let us consider $u \in \partial\Omega$ such that $M_h(u) = u$ and $\|u\|_\infty = u^* + 2$. Notice that one has $\|u'\|_\infty < a$. This implies that there exists $t_0 \in [0, T]$ such that $u(t_0) = \max_{[0,T]} u = u^* + 2$ or $u(t_0) = \min_{[0,T]} u = -u^* - 2$. In the first case we can assume that $t_0 \in [0, T)$ and then $u'(t_0) = 0$ and there exists $\varepsilon > 0$ such that $u(t) > u^* + 1$ for all $t \in [t_0, t_0 + \varepsilon]$. So,

$$(\phi(u'(t)))' = f(t, u(t), u'(t)) + m > 0 \quad \text{for all } t \in [t_0, t_0 + \varepsilon],$$

implying that u' is strictly increasing on $[t_0, t_0 + \varepsilon]$ and then $u'(t) > 0$ for all $t \in (t_0, t_0 + \varepsilon]$. It follows that u is strictly increasing on $[t_0, t_0 + \varepsilon]$, which is a contradiction. Analogously, one can obtain a contradiction in the second case. Consequently,

$$[u \in \partial\Omega,\ M_h(u) = u] \Rightarrow \|u\|_\infty < u^* + 2. \tag{12}$$

Now, let $u \in \partial\Omega$ be such that $M_h(u) = u$. It follows from (12) that $\|u\|_\infty < u^* + 2$, $\|u'\|_\infty < a$, and $u \in \partial\Omega_{\alpha_1,\beta} \cup \partial\Omega_{\alpha,\beta_1}$. We infer that there exists $t_0 \in [0, T]$ such that $u(t_0) = \alpha(t_0)$ or $u(t_0) = \beta(t_0)$, implying that $|u(t_0)| \leq \|\alpha\|_\infty + \|\beta\|_\infty$. Then,

$$|u(t)| \leq |u(t_0)| + \int_0^T |u'(t)|dt < u^* \quad \text{for all } t \in [0, T],$$

and, consequently,

$$[u \in \partial\Omega,\ M_h(u) = u] \Rightarrow \|u\|_\infty < u^*. \tag{13}$$

We have two cases.

Case 1. Assume that there exists $u \in \partial\Omega$ such that $M_h(u) = u$. Using (13), we deduce that $\|u\|_\infty < u^*$, implying that u is a solution of (5) and (10) holds true. Actually, in this case there exists $t_u \in [0, T]$ such that $u(t_u) = \alpha(t_u)$ or $u(t_u) = \beta(t_u)$.

Case 2. Assume that $M_h(u) \neq u$ for all $u \in \partial\Omega$. Then, from Lemma 4 applied to h, it follows that

$$\begin{aligned} d_{\mathrm{LS}}[I - M_h, \Omega_{\alpha_1,\beta_1}, 0] &= d_{\mathrm{LS}}[I - M_h, \Omega_{\alpha_1,\beta}, 0] \\ &= d_{\mathrm{LS}}[I - M_h, \Omega_{\alpha,\beta_1}, 0] \\ &= -1. \end{aligned}$$

This together with the additivity property of the Leray–Schauder degree imply that

$$d_{\mathrm{LS}}[I - M_h, \Omega, 0] = 1,$$

which together with the existence property of the Leray–Schauder degree imply that there exists $u \in \Omega$ such that $M_h(u) = u$. It follows that there exists $t_1, t_2 \in [0, T]$ such that $u(t_1) < \alpha(t_1)$ and $u(t_2) > \beta(t_2)$. Then, using once again that $\|u'\|_\infty < a$, it follows that $\|u\|_\infty < u^*$, and u is a solution of (5). Moreover, from $u \in \Omega$ it follows that (10) holds true. □

Remark 1. Assume that (5) has a lower solution α and an upper solution β. From Lemma 3 and Theorem 1, we deduce that (5) has at least one solution u satisfying (10). In particular,

$$\|u\|_\infty < \|\alpha\|_\infty + \|\beta\|_\infty + aT.$$

3.1. *Lower and upper solutions method for strong singular problems*

In our first application of the previous theorem we deal with singular strong nonlinearities. Consider the periodic problem

$$(\phi(u'))' + h(u)u' = g(u) + f(t, u, u'), \quad u(0) - u(T) = 0 = u'(0) - u'(T), \quad (14)$$

where $f : [0, T] \times \mathbb{R}^2 \to \mathbb{R}$ and $h, g : (0, \infty) \to \mathbb{R}$ are continuous functions such that the following *strong force condition* is satisfied

$$\int_0^1 g(t)dt = +\infty, \quad (15)$$

and assume that $h \geq 0$. Under those assumptions we have the following theorem.

Theorem 2. *Assume that* (14) *has a lower solution* $\alpha > 0$ *and an upper solution* $\beta > 0$. *Then* (14) *has at least one solution* u *which satisfies* (10).

Proof. If $\alpha \leq \beta$ then the result follows from Lemma 3 and [1, Remark 8] (without any additional assumption). Assume now that (9) holds true and define

$$\delta = \min_{[0,T]} \min\{\alpha, \beta\}, \quad B = \|\alpha\|_\infty + \|\beta\|_\infty + aT,$$
$$m = \max_{[0,T]\times[-B,B]\times[-a,a]} |f|, \quad K = mTa + \int_\delta^B |g(s)|ds.$$

From (15) it follows that

$$\exists\, \varepsilon \in (0, \delta)\colon \quad g(\epsilon) > 0 \quad \text{and} \quad \int_\varepsilon^\delta g(s)ds > K. \quad (16)$$

Let $\widehat{g}, \widehat{h} : \mathbb{R} \to \mathbb{R}$ be the continuous functions given by

$$\widehat{g}(u) = \begin{cases} g(u), & u \geq \varepsilon, \\ g(\varepsilon), & u \leq \varepsilon, \end{cases} \quad \widehat{h}(u) = \begin{cases} h(u), & u \geq \varepsilon, \\ h(\varepsilon), & u \leq \varepsilon, \end{cases}$$

and consider the modified periodic problem

$$(\phi(u'))' + \widehat{h}(u)u' = \widehat{g}(u) + f(t, u, u'), \quad u(0) - u(T) = 0 = u'(0) - u'(T). \quad (17)$$

From $\varepsilon < \delta$ it follows that α and β are lower and upper solutions of (17), respectively. Then, using Theorem 1 and Remark 1 it follows that (17) has a solution u

which satisfies (10) and such that

$$-B \leq u \leq B, \quad \|u'\|_\infty < a. \tag{18}$$

We shall prove that $u > \varepsilon$. Consider $t_0, t_1 \in [0,T]$ such that $u(t_0) = \min_{[0,T]} u$ and $u(t_1) = \max_{[0,T]} u$. Assume by contradiction that $u(t_0) \leq \varepsilon$. From Theorem 1 one has that

$$u(t_1) \geq \delta. \tag{19}$$

Assume that $t_0 \leq t_1$ and notice that $u'(t_0) = 0 = u'(t_1)$. Putting $x = \phi \circ u'$, one has that $u' = \phi^{-1} \circ x$, $x(t_0) = 0 = x(t_1)$ and

$$\begin{aligned}\int_{t_0}^{t_1} (\phi(u'(t)))'u'(t)dt &= \int_{t_0}^{t_1} x'(t)\phi^{-1}(x(t))dt \\ &= \int_{x(t_0)}^{x(t_1)} \phi^{-1}(y)dy \\ &= 0.\end{aligned}$$

Using that $x(0) = x(T)$ and a similar computation, we infer that

$$\int_0^T (\phi(u'(t)))'u'(t)dt = 0.$$

So, multiplying (17) by u' and integrating on $[0,T]\backslash[t_0,t_1]$, it follows that

$$\int_{[0,T]\backslash[t_0,t_1]} \widehat{h}(u)u'^2 dt = -\int_{t_0}^{t_1} \widehat{g}(u)u'dt + \int_{[0,T]\backslash[t_0,t_1]} f(t,u,u')u'dt,$$

which together with (18) and the positivity of h imply that

$$\int_{u(t_0)}^{u(t_1)} \widehat{g}(t)dt \leq mTa. \tag{20}$$

From (16), (19) and (20) we deduce that

$$\int_\varepsilon^\delta g(t)dt = \int_{u(t_0)}^{u(t_1)} \widehat{g}(t)dt - g(\varepsilon)(\varepsilon - u(t_0)) - \int_\delta^{u(t_1)} g(t)dt \leq K,$$

which is a contradiction with (16). Similar considerations hold also when $t_1 \leq t_0$ using integration on $[t_1, t_0]$. Consequently, $u > \varepsilon$, implying that u is also a solution of (14). □

Remark 2. The above result holds also with similar arguments when $h \leq 0$.

Actually, it can be assumed that $h : (0,\infty) \to \mathbb{R}$ is a continuous function having limit (finite or not) at 0. It suffices to remark that (14) can be written as

$$\begin{aligned}(\phi(u'))' + h^+(u)u' - h^-(u)u' &= g(u) + f(t,u,u'), \\ u(0) - u(T) = 0 &= u'(0) - u'(T),\end{aligned}$$

and, in this case h^+ or h^- has no singularity at 0.

3.2. *Nonlinearities null at infinity*

Next application deal with nonlinearities null at infinity. This type of nonlinearities has been introduced in [8]. We consider the periodic problem

$$(\phi(u'))' + f(t,u) = s + \widetilde{e}(t), \quad u(0) - u(T) = 0 = u'(0) - u'(T), \tag{21}$$

where $f : [0,T] \times \mathbb{R} \to \mathbb{R}$ is a continuous function, $\widetilde{e} \in C^0$ with $\int_0^T \widetilde{e}(t)dt = 0$ and $s \in \mathbb{R}$ is a parameter. We have the following theorem.

Theorem 3. *Assume that*

$$f(t,u) \to 0 \quad \text{if } |u| \to \infty \text{ uniformly with } t \in [0,T], \tag{22}$$

and that there exists $\mu \in C^0$ *with* $\overline{\mu} > 0$ *such that*

$$\liminf_{|u|\to\infty} uf(t,u) > \mu(t) \quad \text{uniformly with } t \in [0,T]. \tag{23}$$

Then, there exists $\varepsilon_1 < 0 < \varepsilon_2$ *such that* (21) *has no solutions if* $s \notin [\varepsilon_1, \varepsilon_2]$ *and at least one solution if* $s \in [\varepsilon_1, \varepsilon_2]$. *Moreover, if* $s \in (\varepsilon_1, \varepsilon_2)$ *and* $s \neq 0$, *then* (21) *has at least two solutions.*

Proof. For every fixed integer $k \in \mathbb{Z}$, let us consider the periodic problem

$$\begin{gathered}(\phi(\widetilde{u}'))' + f(t, k + \widetilde{u}) - \widetilde{e}(t) = \frac{1}{T}\int_0^T f(\tau, k + \widetilde{u})d\tau \\ \widetilde{u}(0) - \widetilde{u}(T) = 0 = \widetilde{u}'(0) - \widetilde{u}'(T).\end{gathered} \tag{24}$$

Then, taking into account that $\int_0^T \widetilde{e}(t)dt = 0$ it follows from Lemma 5 that (24) has at least one solution $\widetilde{u}_k \in \widetilde{C}^1_\sharp$. Notice that $u_k := k + \widetilde{u}_k$ is a solution of (21) for $s = \frac{1}{T}\int_0^T f(\tau, u_k)d\tau$. So, in particular, there exists at least one $s \in \mathbb{R}$ such that (21) has at least one solution.

Next, let us define

$$S_j = \{s \in \mathbb{R} : (21) \text{ has at least } j \text{ solutions}\} \quad (j = 1, 2)$$

and $\varepsilon_1 = \inf S_1$, $\varepsilon_2 = \sup S_1$. Using that f is bounded on $[0,T] \times \mathbb{R}^2$ and $\frac{1}{T}\int_0^T f(\tau,u)d\tau = s$ for any solution u of (21), we infer that $\varepsilon_1, \varepsilon_2$ are finite.

Now, we will prove that $\varepsilon_1 < 0 < \varepsilon_2$. It suffices to prove that there exists $\delta > 0$ such that $[-\delta, \delta] \subset S_1$. One has that

$$\exists\, k_0 \geq 1,\ \forall\, s \leq \frac{\overline{\mu}}{4k_0}\colon \quad \frac{1}{T}\int_0^T f(\tau, u_{k_0})d\tau \geq s. \tag{25}$$

Assume by contradiction that

$$\forall\, k \geq 1,\ \exists\, s_k \leq \frac{\overline{\mu}}{4k}\colon \quad \frac{1}{T}\int_0^T f(\tau, u_k)d\tau < s_k.$$

Using (22), (23) and the fact that $\|\widetilde{u}_k\|_\infty < aT$ for all $k \in \mathbb{Z}$, it follows that there exists $K \geq 1$ such that

$$\frac{1}{T}\int_0^T f(\tau, u_k)u_k d\tau \geq \overline{\mu}$$

and

$$\frac{1}{T}\int_0^T f(\tau, u_k)\widetilde{u}_k d\tau \leq \frac{\overline{\mu}}{4}$$

for all $k \geq K$. It follows that

$$\begin{aligned} 0 &> \frac{k}{T}\int_0^T f(\tau, u_k)d\tau - ks_k \\ &= \frac{1}{T}\int_0^T f(\tau, u_k)u_k d\tau - \frac{1}{T}\int_0^T f(\tau, u_k)\widetilde{u}_k d\tau - ks_k \\ &\geq \frac{\overline{\mu}}{2}, \quad \text{for all } k \geq K, \end{aligned}$$

which is a contradiction with the assumption $\overline{\mu} > 0$. So, (25) holds true. This implies that u_{k_0} is a lower solution of (21) for all $s \leq \frac{\overline{\mu}}{4k_0}$. Analogously, it follows that there exists $k_1 \leq -1$ such that u_{k_1} is an upper solution of (21) for all $s \geq \frac{\overline{\mu}}{4k_1}$. Then, $[-\delta, \delta] \subset S_1$, just taking δ sufficiently small and applying Theorem 4 of [1].

Next, let us prove that $(0, \varepsilon_2) \subset S_2$. Consider $s \in (0, \varepsilon_2)$. It follows that there exists $\widehat{s} > s$ such that $\widehat{s} \in S_1$, so, (21) has at least one solution α for $s = \widehat{s}$. Then, α is a strict lower solution of (21). Using once again (22) and the fact that $\|\widetilde{u}_k\|_\infty < aT$ for all $k \in \mathbb{Z}$, it follows that there exists $k \geq 1$ sufficiently large such that $u_{-k} < \alpha < u_k$ and

$$\frac{1}{T}\int_0^T f(\tau, u_j)d\tau < s \quad (j = -k, k).$$

It follows that u_{-k}, u_k are strict upper solutions for (21). Then, from Lemma 3 we infer that (21) has a solution v_1 such that $\alpha < v_1 < u_k$. On the other hand, from Theorem 1, it follows that (21) has a solution v_2 such that $u_{-k}(t) \leq v_2(t) \leq \alpha(t)$ for some $t \in [0, T]$. Hence, $v_1 \neq v_2$ and $s \in S_2$. Analogously, one has that $(\varepsilon_2, 0) \subset S_2$.

Finally, let us prove that $\varepsilon_2 \in S_1$. Consider a sequence $\{s_n\}$ in $(0, \varepsilon_2)$ converging to ε_2 and u_n a solution of (21) with $s = s_n$. Notice that

$$\frac{1}{T}\int_0^T f(\tau, u_n)d\tau = s_n \quad (n \in \mathbb{N}),$$

which together with $\|\widetilde{u}_n\|_\infty < aT$ for all $n \in \mathbb{N}$, $\varepsilon_2 > 0$ and (22) imply that $\{\overline{u}_n\}$ is a bounded sequence. Consequently, $\{u_n\}$ is a bounded sequence in C^1 and a simple application of the Arzela–Ascoli's theorem implies that $\{u_n\}$ has a subsequence converging uniformly to some $u \in C^0$ which is a solution of (21) with $s = \varepsilon_2$. Analogously, one has that $\varepsilon_1 \in S_1$. $\square$

Example 1. Let $\widetilde{e} \in C^0$ with $\int_0^T \widetilde{e}(t)dt = 0$ and consider the periodic problem

$$\left(\frac{u'}{\sqrt{1-u'^2}}\right)' + \frac{u}{1+u^2} = s + \widetilde{e}(t), \quad u(0) - u(T) = 0 = u'(0) - u'(T). \tag{26}$$

There exists $\varepsilon_1 < 0 < \varepsilon_2$ such that (26) has no solutions if $s \notin [\varepsilon_1, \varepsilon_2]$ and at least one solution if $s \in [\varepsilon_1, \varepsilon_2]$. Moreover, if $s \in (\varepsilon_1, \varepsilon_2)$ and $s \neq 0$, then (26) has at least two solutions.

Remark 3. It is interesting to note that in [2], using a completely different strategy based upon Leray–Schauder degree arguments, the authors deal with nonlinearities f null at infinity such that $f > 0$.

4. Constructing Lower and Upper Solutions

We consider the following periodic problem:

$$(\phi(u'))' = g_0(t, u) + e(t), \quad u(0) - u(T) = 0 = u'(0) - u'(T), \tag{27}$$

where $g_0 : [0, T] \times (0, \infty) \to \mathbb{R}$ is a continuous singular nonlinearity and $e \in C^0$.

The following result gives a method to construct a lower solution to (27), getting also control on its localization.

Theorem 4. *Let us assume that there exist $x_1 > 0$ and $c \in C^0$ such that*

$$g_0(t, x) \leq c(t), \quad \forall (t, x) \in [0, T] \times \left[x_1, x_1 + \frac{aT}{2}\right]. \tag{28}$$

If

$$\overline{c} + \overline{e} \leq 0, \tag{29}$$

then (27) *has a lower solution α such that*

$$x_1 \leq \alpha < x_1 + \frac{aT}{2}. \tag{30}$$

Proof. Consider the function $\psi = c + e$. We have two cases.

Case 1. Assume that $\Psi_+ = 0$. Taking $\alpha \equiv x_1$ and using that $c + e \leq 0$, it follows from (28) that α is a lower solution of (27).

Case 2. Assume that $\Psi_+ > 0$. Using

$$\int_0^T [\psi^+(t)\Psi_- - \psi^-(t)\Psi_+]dt = 0$$

and [1, Proposition 1], it follows that there exists w such that

$$(\phi(w'))' = \psi^+(t)\Psi_- - \psi^-(t)\Psi_+, \quad w(0) - w(T) = 0 = w'(0) - w'(T).$$

Let us take $x_0 = 1/\Psi_+$ and

$$\alpha = x_1 + H \circ \phi^{-1} \circ [I - Q_\phi](x_0\phi(w')) - \min_{[0,T]}\{H \circ \phi^{-1} \circ [I - Q_\phi](x_0\phi(w'))\}.$$

The definition of Q_ϕ implies $\alpha(0) = \alpha(T)$. On the other hand, one has that

$$\alpha' = \phi^{-1} \circ [I - Q_\phi](x_0\phi(w')),$$

implying that $\alpha'(0) = \alpha'(T)$. Then, Lemma 6 implies (30). Now, using (29), it follows that $\Psi_+ \leq \Psi_-$, implying that

$$(\phi(\alpha'))' = x_0(\phi(w'))' = x_0[\psi^+\Psi_- - \psi^-\Psi_+] \geq \psi.$$

From (28) and (30) we deduce that

$$g_0(t, \alpha(t)) + e(t) \leq \psi(t), \quad \forall\, t \in [0.T].$$

Consequently

$$(\phi(\alpha'(t)))' \geq g_0(t, \alpha(t)) + e(t), \quad \forall\, t \in [0.T],$$

and the proof is completed. $\square$

Using similar arguments, one can prove the following theorem.

Theorem 5. *Let us assume that there exist $x_2 > 0$ and $d \in C^0$ such that*

$$g_0(t, x) \geq d(t), \quad \forall (t, x) \in [0, T] \times \left[x_2, x_2 + \frac{aT}{2}\right]. \tag{31}$$

If

$$\overline{d} + \overline{e} \geq 0, \tag{32}$$

then (27) *has an upper solution β such that*

$$x_2 \leq \beta < x_2 + \frac{aT}{2}.$$

5. Applications

In this section, combining the method of upper and lower solutions (Lemma 3 and Theorem 2) with the results from the previous section, we give various existence and multiplicity results concerning periodic solutions for singular perturbations of the relativistic operator $u \mapsto \left(\frac{u'}{\sqrt{1-u'^2}}\right)'$.

5.1. *Strong repulsive perturbations*

Consider the problem

$$(\phi(u'))' + r(t)u - g(u) = e(t), \quad u(0) - u(T) = 0 = u'(0) - u'(T), \tag{33}$$

where $r, e \in C^0$ and $g : (0, \infty) \to \mathbb{R}$ is continuous and satisfies

$$\lim_{x\to 0} g(x) = +\infty, \quad \lim_{x\to\infty} g(x) = 0, \quad \int_0^1 g(x)dx = +\infty. \tag{34}$$

The main result of this subsection is the following theorem.

Theorem 6. *Assume that* (34) *holds true. If either*

$$\overline{r} > 0$$

or

$$\overline{r} = 0, \quad \overline{e} < -\frac{aR_-}{2},$$

then problem (33) *has at least one solution.*

Proof. Notice that from (34) it follows that there exists a constant β sufficiently small such that β is an upper solution of (33).

In order to apply Theorem 4 we introduce some notation. Consider the continuous functions $g_0 : [0, T] \times (0, \infty) \to \mathbb{R}$ given by

$$g_0(t, x) = -r(t)x + g(x),$$

$g^* : (0, +\infty) \longrightarrow \mathbb{R}$ defined by

$$g^*(x) = \max_{[x, x+\frac{aT}{2}]} g,$$

and $\gamma^* : (0, \infty) \to \mathbb{R}$ given by

$$\gamma^*(x) = -Rx + \frac{aT}{2}R_- + Tg^*(x).$$

Case 1. Assume that $\overline{r} > 0$. This together with (34) imply that

$$\lim_{x\to\infty} \gamma^*(x) = -\infty,$$

so there exists $x_1 > 0$ such that $\gamma^*(x_1) \leq -E$. In order to apply Theorem 4, let us take

$$c(t) = r^-(t)\left(x_1 + \frac{aT}{2}\right) - r^+(t)x_1 + g^*(x_1) \quad (t \in [0, T]). \tag{35}$$

It follows that $C = \gamma^*(x_1)$ and $C + E \leq 0$, meaning that condition (29) holds true. One has that

$$\begin{aligned} g_0(t,x) &= r^-(t)x - r^+(t)x + g(x) \\ &\leq r^-(t)\left(x_1 + \frac{aT}{2}\right) - r^+(t)x_1 + g^*(x_1), \end{aligned}$$

for all $(t,x) \in [0,T] \times [x_1, x_1 + \frac{aT}{2}]$. So, condition (28) is fulfilled. Then, from Theorem 4 we infer that (33) has a lower solution α. Now the result follows from Theorem 2.

Case 2. Assume that $\overline{r} = 0$ and $\overline{e} < -\frac{aR_-}{2}$. It follows that

$$\gamma^*(x) = \frac{aT}{2}R_- + g^*(x), \quad \lim_{x\to\infty}\gamma^*(x) = \frac{aT}{2}R_-.$$

Then, there exists $x_1 > 0$ such that $\gamma^*(x_1) \leq -E$. The result follows now exactly like in Case 1. □

Remark 4. Theorem 8 in [1] follows from Theorem 6 just taking $r = 0$.

Example 2. Consider the problem

$$\left(\frac{u'}{\sqrt{1-u'^2}}\right)' + r(t)u - \frac{1}{u^\nu} = e(t), \quad u(0) - u(T) = 0 = u'(0) - u'(T), \tag{36}$$

where $r, e \in C^0$ and $\nu \geq 1$. If either $\overline{r} > 0$ or $\overline{r} = 0$ and $\overline{e} < -\frac{R_-}{2}$, then (36) has at least one solution.

In case $r < 0$ there exists $s_0 < 0$ such that (36) has at least two solutions provided $e \leq s_0$ holds true. Indeed, in this case problem (36) has two strict upper solutions $\beta_1, \beta_2 > 0$ and a strict lower solution $\alpha > 0$ such that $\beta_1 < \alpha < \beta_2$. Hence, the result follows from Lemma 3 and Theorem 2.

If $0 < \nu < 1$ and $r = 0$, then, using similar arguments like in [13], it follows that there exists $e \in C^0$ such that (36) has no solutions.

Example 3. If $\nu \geq 1$, it follows from the previous example that the Brillouin beam-focusing equation with relativistic effects

$$\left(\frac{u'}{\sqrt{1-u'^2}}\right)' + (b_1 + b_2\cos t)u = \frac{1}{u^\nu}, \quad u(0) - u(2\pi) = 0 = u'(0) - u'(2\pi),$$

has at least one solution for any $b_1 > 0$ and $b_2 \in \mathbb{R}$.

In the classical case, it has been proved in [21] that the periodic problem

$$u'' + b_1(1 + \cos t)u = \frac{1}{u^\nu}, \quad u(0) - u(2\pi) = 0 = u'(0) - u'(2\pi),$$

has at least one solution for any $0 < b_1 < 0,16488$. In case $\nu = 1$, it is proven in [17] that the above problem has at least one solution for any $0 < b_1 < 1$. The main tool used in [17, 21] is Mawhin's coincidence degree theory.

5.2. *Mixed singularities*

Consider the periodic problem

$$(\phi(u'))' + r(t)u + g(t,u) = e(t), \quad u(0) - u(T) = 0 = u'(0) - u'(T) \tag{37}$$

where $r, e \in C^0$ and $g : [0,T] \times (0,\infty) \to \mathbb{R}$ is continuous and satisfies

$$\lim_{x\to\infty} g(t,x) = 0 \quad \text{uniformly with } t \in [0,T]. \tag{38}$$

Let us introduce the continuous functions $g_0 : [0,T] \times (0,\infty) \to \mathbb{R}$ given by

$$g_0(t,x) = -r(t)x - g(t,x),$$

$g_*, g^* : [0,T] \times (0,\infty) \to \mathbb{R}$ defined by

$$g_*(t,x) = \min_{[x,x+\frac{aT}{2}]} g(t,\cdot), \quad g^*(t,x) = \max_{[x,x+\frac{aT}{2}]} g(t,\cdot),$$

and $\gamma_* : \left(\frac{aT}{2}, \infty\right) \to \mathbb{R}$, $\gamma^* : (0,\infty) \to \mathbb{R}$, given by

$$\gamma_*(x) = -Rx + \frac{aT}{2}R_+ - \int_0^T g_*\left(t, x - \frac{aT}{2}\right) dt,$$

$$\gamma^*(x) = -Rx - \frac{aT}{2}R_+ - \int_0^T g^*(t,x) dt.$$

The key result of this subsection is the following lemma.

Lemma 7. *Assume that* (38) *holds true and consider* $\gamma_*^m := \inf \gamma_*$. *If* $\overline{r} < 0$ *and* $-E > \gamma_*^m$, *then* (37) *has at least one solution.*

Proof. Since $-E > \gamma_*^m$, there exists $z > \frac{aT}{2}$ such that $\gamma_*(z) \le -E$. Let us define $x_1 = z - \frac{aT}{2} > 0$ and $c \in C^0$ by

$$c(t) = r^-(t)\left(x_1 + \frac{aT}{2}\right) - r^+(t)x_1 - g_*(t,x_1).$$

Then, it follows that conditions (28) and (29) hold true. Hence, from Theorem 4 we infer that (37) has a lower solution α such that $x_1 \le \alpha < x_1 + \frac{aT}{2}$.

One the other hand, using that $\overline{r} < 0$, there exists $x_2 \ge z$ such that $\gamma^*(x_2) \ge -E$. Consider $d \in C^0$ by

$$d(t) = r^-(t)x_2 - r^+(t)\left(x_2 + \frac{aT}{2}\right) - g^*(t,x_2).$$

Then, it follows that conditions (31) and (32) hold true. Hence, from Theorem 5 we infer that (37) has an upper solution β such that $x_2 \le \beta < x_2 + \frac{aT}{2}$.

Consequently, (37) has a lower solution α and an upper solution β such that $\alpha \le \beta$. The result follows now from Lemma 3. $\square$

Consider now the periodic problem

$$(\phi(u'))' + r(t)u + \frac{n(t)}{u^\lambda} = e(t), \quad u(0) - u(T) = 0 = u'(0) - u'(T), \tag{39}$$

where $r, n, e \in C^0$ and $\lambda > 0$. We have the following theorem.

Theorem 7. *If* $\overline{r} < 0$ *and*

$$-E > (1+\lambda)\left[\frac{|R|^\lambda N_-}{\lambda^\lambda}\right]^{\frac{1}{1+\lambda}} + \frac{aT}{2}R_- - N_+\left[\frac{aT}{2} + \left(\frac{\lambda N_-}{|R|}\right)^{\frac{1}{1+\lambda}}\right]^{-\lambda}, \tag{40}$$

then (39) *has at least one solution.*

Proof. We have two cases.

Case 1. Assume that $N_- = 0$. In this case one has that

$$\gamma_*(x) = -Rx + \frac{aT}{2}R_+ - \frac{N_+}{x^\lambda},$$

implying that $\gamma_*^m = \gamma_*(aT/2)$. So, (40) becomes $-E > \gamma_*^m$, and the result follows from Lemma 7.

Case 2. Assume that $N_- > 0$. Notice that the minimum of $x \mapsto -Rx + \frac{aT}{2}R_+ + \frac{N_-}{(x-\frac{aT}{2})^\lambda}$ is attained in $x_0 = \frac{aT}{2} + [\frac{\lambda N_-}{|R|}]^{\frac{1}{\lambda+1}}$ and

$$\gamma_*(x_0) \leq -Rx_0 + \frac{aT}{2}R_+ + \frac{N_-}{\left(x_0 - \frac{aT}{2}\right)^\lambda} - \frac{N_+}{x_0^\lambda}.$$

But, the right-hand side in the previous inequality is just the right-hand side in (40). Hence, from (40) we infer that $-E > \gamma_*^m$, and the result follows again from Lemma 7. □

Example 4. Consider the periodic problem with repulsive singularity (possibly weak!)

$$\left(\frac{u'}{\sqrt{1-u'^2}}\right)' + r(t)u - \frac{m(t)}{u^\lambda} = e(t), \quad u(0) - u(T) = 0 = u'(0) - u'(T),$$

where $r, m, e \in C^0$ with $m \geq 0$ and $\lambda > 0$. If $\overline{r} < 0$ and

$$-E > (1+\lambda)\left[\frac{|R|^\lambda M}{\lambda^\lambda}\right]^{\frac{1}{1+\lambda}} + \frac{T}{2}R_-,$$

then the above problem has at least one solution.

Example 5. Consider the periodic problem with attractive singularity

$$\left(\frac{u'}{\sqrt{1-u'^2}}\right)' + r(t)u + \frac{m(t)}{u^\lambda} = e(t), \quad u(0)-u(T) = 0 = u'(0)-u'(T), \quad (41)$$

where $r, m, e \in C^0$ with $m \geq 0$ and $\lambda > 0$. If $\overline{r} < 0$ and

$$E < M\left(\frac{2}{T}\right)^\lambda - \frac{T}{2}R_-,$$

then the above problem has at least one solution.

In connection with Example 5, if $r = 0$, then we have the following proposition.

Proposition 1. *Consider the periodic problem with attractive singularity*

$$(\phi(u'))' + \frac{m(t)}{u^\lambda} = e(t), \quad u(0)-u(T) = 0 = u'(0)-u'(T), \quad (42)$$

where $m, e \in C^0$ such that $m \geq 0$ and $\lambda > 0$. If

$$0 < E < M\left(\frac{2}{aT}\right)^\lambda, \quad (43)$$

then (42) *has at least one solution.*

Proof. We will use the same strategy as in the proof of Lemma 7. In this case one has that $g_0(t,x) = -\frac{m(t)}{x^\lambda}$.

Using (43) it follows that there exists $z > \frac{aT}{2}$ such that $E < Mz^{-\lambda}$. Let us define $x_1 = z - \frac{aT}{2} > 0$ and $c \in C^0$ by $c(t) = -m(t)(x_1 + \frac{aT}{2})^{-\lambda}$. Then, it follows that conditions (28) and (29) hold true. Hence, from Theorem 4 we infer that (42) has a lower solution α such that $x_1 \leq \alpha < x_1 + \frac{aT}{2}$.

Using again (43) it follows that there exists $x_2 > z$ such that $E \geq Mx_2^{-\lambda}$. Let us define $d \in C^0$ by $d(t) = -m(t)x_2^{-\lambda}$. Then, it follows that conditions (31) and (32) hold true. Hence, from Theorem 5 we infer that (42) has an upper solution β such that $x_2 \leq \beta < x_2 + \frac{aT}{2}$.

Consequently, (42) has a lower solution α and an upper solution β such that $\alpha \leq \beta$. The result follows now from Lemma 3. □

In the "pure" attractive case we have the following result concerning (37).

Proposition 2. *Assume that* (38) *and*

$$\lim_{x\to 0} g(t,x) = +\infty \quad \textit{uniformly with } t \in [0,T] \quad (44)$$

hold true. Then (37) *has at least one solution provided that either*

$$\overline{r} < 0$$

or

$$\overline{r} = 0, \quad E > \frac{aT}{2}R_+.$$

Proof. Notice that from (44) it follows that any sufficiently small positive constant α is a lower solution for (37). The construction of an upper solution $\beta \geq \alpha$ for (37) is similar as in Lemma 7. The result follows now from Lemma 3. $\square$

Remark 5. Theorem 7 from [1] follows taking $r = 0$ in Proposition 2.

Example 6. Let us consider again problem (41), assuming that $m > 0$. If either $\overline{r} < 0$ or $\overline{r} = 0$ and $E > \frac{T}{2}R_+$, then (41) has at least one solution.

5.3. *A problem in nonlinear elasticity*

The radial oscillations of an elastic spherical membrane made up of a Neo-Hookean material, subjected to an internal continuous pressure $p : \mathbb{R} \to (0, \infty)$ are governed by the scalar equation

$$u'' = p(t)u^2 - u + \frac{1}{u^5}. \tag{45}$$

If relativistic effects are taken into account and looking for T-periodic solutions, then the above equation becomes

$$\left(\frac{u'}{\sqrt{1-u'^2}}\right)' = p(t)u^2 - u + \frac{1}{u^5}, \quad u(0) - u(T) = 0 = u'(0) - u'(T). \tag{46}$$

If

$$\max_{[0,T]} p < \frac{6}{7^{7/6}}, \tag{47}$$

then (46) has at least two positive solutions. Indeed, using (47) it follows that $\alpha = 7^{1/6}$ is a strict lower solution of (46). On the other hand, clearly there exists $\beta_1, \beta_2 > 0$ strict upper solution of (46) such that $\beta_1 < \alpha < \beta_2$. Now the result follows from Lemma 3 and Theorem 2.

Notice that in [9] it is proved, using variational arguments, that (45) has at least two T-periodic solutions provided that (47) is satisfied.

Next, let us consider the periodic problem

$$(\phi(u'))' = p(t)u^\delta - u + \frac{1}{u^\mu}, \quad u(0) - u(T) = 0 = u'(0) - u'(T), \tag{48}$$

where $p \in C^0$, $\delta > 0$ and $\mu \geq 1$. Let $\beta > 0$ be a constant small enough such that β is an upper solution of (48). Putting

$$g_0(t, x) = p(t)x^\delta - x + \frac{1}{x^\mu} \quad ((t, x) \in [0, T] \times (0, \infty))$$

and

$$c(t) = p^+(t)\left(x_1 + \frac{aT}{2}\right)^\delta - p^-(t)x_1^\delta - x_1 + \frac{1}{x_1^\mu} \quad (t \in [0, T]),$$

we deduce that (28) holds true for any $x_1 > 0$. One the other hand, in this particular case, (29) holds true if and only if there exists $x_1 > 0$ such that

$$P_+ \left(x_1 + \frac{aT}{2} \right)^\delta - P_- x_1^\delta - T x_1 + \frac{T}{x_1^\mu} \le 0.$$

So, by virtue of Theorem 4 one has that (48) has a lower solution α. Then, using Theorem 2 we infer that (48) has at least one solution. This is the case if either $\overline{p} < 0$ or $\delta < 1$.

5.4. *Rayleigh–Plesset type problems*

Let $\lambda > 0, \mu \ge 1$ be such that $\mu > \lambda$. Consider also $e \in C^0$ with $e \le 0$. Using Theorem 2 and Remark 2, it follows that the Rayleigh–Plesset type problem with relativistic effects

$$\left(\frac{u'}{\sqrt{1-u'^2}} \right)' + c\frac{u'}{u^{\frac{4}{5}}} + \frac{1}{u^\lambda} - \frac{1}{u^\mu} = e(t)u^\lambda, \quad u(0) - u(T) = 0 = u'(0) - u'(T)$$

has at least one solution for any $c \in \mathbb{R}$.

For corresponding results in the classical case, see [10, 11].

Acknowledgments

C. Bereanu is supported by a GENIL grant YTR-2011-7 (Spain). D. Gheorghe is supported from the grant PN-II-RU-PD-2011-3-0052 (CNCS-Romania). M. Zamora is supported by the Ministerio de Educación y Ciencia, Spain, project MTM2011-23652, and by the Junta de Andalucía, Spain, Project FQM2216. The authors offer their sincere thanks to the referee for his careful reading and valuable suggestions.

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