Standing waves for a Gauged Nonlinear Schrödinger equation

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Outline

1. The problem
2. The limit functional
3. Main results
The problem

Consider a planar gauged Nonlinear Schrödinger Equation:

\[ iD_0\phi + (D_1D_1 + D_2D_2)\phi + |\phi|^{p-1}\phi = 0. \]

Here \( t \in \mathbb{R} \), \( x = (x_1, x_2) \in \mathbb{R}^2 \), \( \phi : \mathbb{R} \times \mathbb{R}^2 \to \mathbb{C} \) is the scalar field, \( A_\mu : \mathbb{R} \times \mathbb{R}^2 \to \mathbb{R} \) are the components of the gauge potential and \( D_\mu = \partial_\mu + iA_\mu \) is the covariant derivative (\( \mu = 0, 1, 2 \)).
The problem

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In Chern-Simons theory, a modified gauge field equation has been introduced [Hagen, Jackiw, Schonfeld, Templeton, in the ’80s]; see also [Tarantello, PNLDE 2007.]

\[ \partial_\mu F^{\mu\nu} + \frac{1}{2} \kappa \epsilon^{\nu\alpha\beta} F_{\alpha\beta} = j^\nu, \quad F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu. \]

Here \( \kappa \in \mathbb{R} \) is the Chern-Simons constant and \( \epsilon^{\nu\alpha\beta} \) is the Levi-Civita tensor. Moreover, \( j^\mu \) is the conserved matter current,

\[ j^0 = |\phi|^2, \quad j^i = 2\text{Im} (\bar{\phi} D_i \phi). \]
At low energies, the Maxwell term becomes negligible and can be dropped, giving rise to:

\[ \frac{1}{2} \kappa \epsilon^{\nu \alpha \beta} F_{\alpha \beta} = j^\nu. \]

See [Jackiw & Pi, '90s].

Taking for simplicity \( \kappa = 2 \), we arrive to the system

\[
\begin{align*}
\partial_0 A_1 - \partial_1 A_0 &= \text{Im}(\bar{\phi} D_2 \phi), \\
\partial_0 A_2 - \partial_2 A_0 &= -\text{Im}(\bar{\phi} D_1 \phi), \\
\partial_1 A_2 - \partial_2 A_1 &= \frac{1}{2} |\phi|^2,
\end{align*}
\]

\[ (1) \]
As usual in Chern-Simons theory, problem (1) is invariant under gauge transformation,

$$\phi \to \phi e^{i\chi}, \quad A_\mu \to A_\mu - \partial_\mu \chi,$$

for any arbitrary $C^\infty$-function $\chi$.

The initial value problem for $p = 3$, as well as global existence and blow-up, has been addressed in [Bergé, de Bouard & Saut, 1995; Huh, 2009-2013; Liu-Smith-Tataru 2013; Oh-Pusateri, preprint; Liu-Smith, preprint; Chen-Smith, preprint].

The existence of standing waves for (1) and general $p > 1$ has been studied in [Byeon, Huh & Seok, 2012 and preprint]. They look for vortex solutions, i.e., solutions in the form:
\[ \phi(t, x) = u(r)e^{i(N\theta + \omega t)}, \quad A_0(x) = A_0(|x|), \]
\[ A_1(t, x) = -\frac{x_2}{|x|^2}h(|x|), \quad A_2(t, x) = \frac{x_1}{|x|^2}h(|x|). \]

Here \((r, \theta)\) are polar coordinates, \(h\) is a positive function and \(N \in \mathbb{N}\) is the order of the vortex at 0.
\[
\phi(t, x) = u(r)e^{i(N\theta+\omega t)}, \quad A_0(x) = A_0(|x|),
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\]

Here \((r, \theta)\) are polar coordinates, \(h\) is a positive function and \(N \in \mathbb{N}\) is the order of the vortex at 0.

With this ansatz they obtain the nonlocal equation:

\[
-\Delta u + \left(\omega + \frac{(h_u(|x|) - N)^2}{|x|^2} + A_0(|x|)\right)u = |u|^{p-1}u, \quad (\mathcal{P})
\]

with

\[
h_u(r) = \int_0^r \frac{s}{2}u^2(s) \, ds, \quad A_0(r) = \int_r^{+\infty} \frac{h(s) - N}{s}u^2(s) \, ds.
\]

Moreover, any solution satisfies that \(u(|x|) \sim |x|^N\) around the origin.
In [Byeon, Huh & Seok, 2012 and preprint] it is shown that \((\mathcal{P})\) is indeed the Euler-Lagrange equation of the energy functional:

\[
I_\omega(u) = \frac{1}{2} \int_{\mathbb{R}^2} (|\nabla u|^2 + \omega u^2) \, dx - \frac{1}{p+1} \int_{\mathbb{R}^2} |u|^{p+1} \, dx
\]

\[
+ \frac{1}{8} \int_{\mathbb{R}^2} \frac{u^2(x)}{|x|^2} \left( \int_0^{|x|} u^2(s) \, ds - 2N \right)^2 \, dx
\]

That functional is defined in the space:

\[
\mathcal{H} = \left\{ u \in H^1_r(\mathbb{R}^2) : \int_{\mathbb{R}^2} \frac{u^2(x)}{|x|^2} \, dx < +\infty \right\}.
\]

It can be proved that \(I_\omega\) is well-defined and \(C^1\).
A useful inequality

In [Byeon, Huh & Seok 2012 and preprint], it is proved that, for any \( u \in \mathcal{H} \),

\[
\int_{\mathbb{R}^2} |u(x)|^4 \, dx \leq 2 \left( \int_{\mathbb{R}^2} |\nabla u(x)|^2 \, dx \right)^{\frac{1}{2}} \left( \int_{\mathbb{R}^2} \frac{u^2}{|x|^2} \left( \int_0^{|x|} u^2(s) s \, ds - 2N \right)^2 \, dx \right)^{\frac{1}{2}}.
\]

Furthermore, the equality is attained by the family of functions:

\[
\left\{ u_\lambda = \frac{\sqrt{8\lambda}(N+1)|\lambda x|^N}{1 + |\lambda x|^{2(N+1)}} \in \mathcal{H} : \lambda \in (0, +\infty) \right\}.
\]
Byeon-Huh-Seok results

- If $p > 3$, $I_\omega$ is unbounded from below and exhibits a mountain-pass geometry.
- The case $p = 3$ is special: static solutions can be found via the minimizers of the previous inequality. Alternatively, one can pass, via a self-dual equation, to a singular Liouville equation in $\mathbb{R}^2$.
- If $1 < p < 3$ solutions are found as minimizers on a $L^2$-sphere if $N = 0$. Hence, $\omega$ comes out as a Lagrange multiplier, and it is not controlled.

In general, the global behavior of the energy functional $I_\omega$ is not studied for $1 < p < 3$. This is the main purpose of this talk.
On the boundedness from below of $I_\omega$

**Theorem**

Let $N \in \mathbb{N}$, $p \in (1, 3)$. There exists $\omega_0(p) > 0$ such that:

- If $0 < \omega < \omega_0$, then $I_\omega$ is unbounded from below.
- If $\omega > \omega_0$, then $I_\omega$ is bounded from below and coercive.
- If $\omega = \omega_0$, then $I_{\omega_0}$ is bounded from below, not coercive and $\inf I_{\omega_0} < 0$.

The threshold value $\omega_0$ has an explicit expression, and it is independent of $N$.

Let $u$ a fixed function, and define $u_\rho(r) = u(r - \rho)$. Let us now estimate $I_\omega(u_\rho)$ as $\rho \to +\infty$.

\[
(2\pi)^{-1}I_\omega(u_\rho) = \frac{1}{2} \int_0^{+\infty} (|u'_\rho|^2 + \omega u^2_\rho) r \, dr \\
+ \frac{1}{8} \int_0^{+\infty} \frac{u^2_\rho(r)}{r} \left( \int_0^r s u^2_\rho(s) \, ds - 2N \right)^2 \, dr \\
- \frac{1}{p+1} \int_0^{+\infty} |u_\rho|^{p+1} r \, dr.
\]
The limit functional

Let $u$ a fixed function, and define $u_{\rho}(r) = u(r - \rho)$. Let us now estimate $I_{\omega}(u_{\rho})$ as $\rho \to +\infty$.

$$(2\pi)^{-1} I_{\omega}(u_{\rho}) \sim \frac{1}{2} \int_{-\infty}^{+\infty} (|u'|^2 + \omega u^2)(r + \rho) \, dr$$

$$+ \frac{1}{8} \int_{-\infty}^{+\infty} \frac{u^2(r)}{r + \rho} \left( \int_{-\infty}^{r} (s + \rho)u^2(s) \, ds - 2N \right)^2 \, dr$$

$$- \frac{1}{p+1} \int_{-\infty}^{+\infty} |u|^{p+1}(r + \rho) \, dr.$$
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+ \frac{1}{8} \int_{-\infty}^{+\infty} \frac{u^2(r)}{\rho} \left( \int_{-\infty}^{r} \rho u^2(s) \, ds - 2N \right)^2 \, dr \\
- \frac{1}{p+1} \int_{-\infty}^{+\infty} |u|^{p+1} \rho \, dr.
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The limit functional

Let \( u \) a fixed function, and define \( u_\rho(r) = u(r - \rho) \). Let us now estimate \( I_\omega(u_\rho) \) as \( \rho \to +\infty \).

\[
(2\pi)^{-1}I_\omega(u_\rho) \sim \rho \left[ \frac{1}{2} \int_{-\infty}^{+\infty} (|u'|^2 + \omega u^2) \, dr \\
+ \frac{1}{8} \int_{-\infty}^{+\infty} u^2(r) \left( \int_{-\infty}^{r} u^2(s) \, ds \right)^2 \, dr \\
- \frac{1}{p + 1} \int_{-\infty}^{+\infty} |u|^{p+1} \, dr \right].
\]
The limit functional

Let $u$ a fixed function, and define $u_\rho(r) = u(r - \rho)$. Let us now estimate $I_\omega(u_\rho)$ as $\rho \to +\infty$.

$$
(2\pi)^{-1}I_\omega(u_\rho) \sim \rho \left[ \frac{1}{2} \int_{-\infty}^{+\infty}(|u'|^2 + \omega u^2) \, dr \right. \\
+ \frac{1}{24} \left( \int_{-\infty}^{+\infty} u^2(r) \, dr \right)^3 \\
- \frac{1}{p + 1} \int_{-\infty}^{+\infty} |u|^{p+1} \, dr \right].
$$
It is natural to define the limit functional $J_\omega : H^1(\mathbb{R}) \to \mathbb{R}$,

$$J_\omega(u) = \frac{1}{2} \int_{-\infty}^{+\infty} (|u'|^2 + \omega u^2) \, dr + \frac{1}{24} \left( \int_{-\infty}^{+\infty} u^2 \, dr \right)^3$$

$$- \frac{1}{p + 1} \int_{-\infty}^{+\infty} |u|^{p+1} \, dr.$$

We have

$$I_\omega(u_\rho) \sim 2\pi \rho J_\omega(u), \quad \text{as} \ \rho \to +\infty.$$ 

Then,

$$\inf J_\omega < 0 \Rightarrow \inf I_\omega = -\infty.$$
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We have

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We will actually show that

$$\inf J_\omega < 0 \iff \inf I_\omega = -\infty.$$
The limit functional

**Proposition**

Let $p \in (1, 3)$ and $\omega > 0$. Then $J_\omega$ is coercive and attains its infimum.

The proof of the coercivity is based on the Gagliardo-Nirenberg inequality:

$$\|u\|_{L^4(\mathbb{R})} \leq C\|u'\|_{L^2(\mathbb{R})}^{1/4} \|u\|_{L^2(\mathbb{R})}^{3/4}.$$ 

Hence

$$\int_{-\infty}^{+\infty} u^4 \, dr \leq \frac{C}{2} \left[ \int_{-\infty}^{+\infty} |u'|^2 \, dr + \left( \int_{-\infty}^{+\infty} u^2 \, dr \right)^3 \right].$$
The limit problem

The Euler-Lagrange equation of the functional $J_\omega$ is:

$$- u'' + \left[ \omega + \frac{1}{4} \left( \int_{-\infty}^{+\infty} u^2(s) \, ds \right)^2 \right] u = |u|^{p-1} u, \quad \text{in } \mathbb{R}. \quad (2)$$
The limit problem

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Then $u = \pm w_k$ up to translations, where

$$w_k(r) = k^{\frac{1}{p-1}} w_1(\sqrt{kr}),$$

and

$$w_1(r) = \left( \frac{2}{p+1} \cosh^2 \left( \frac{p-1}{2} r \right) \right)^{\frac{1}{1-p}}.$$
Therefore,

\[ k = \omega + \frac{1}{4} \left( \int_{-\infty}^{+\infty} w_k(r)^2 \, dr \right)^2 = \omega + \frac{1}{4} m^2 k^{\frac{5-p}{p-1}}, \]

where

\[ m = \int_{-\infty}^{+\infty} w_1(r)^2 \, dr. \]
Proposition

\( u \) is a nontrivial solution of (2) if and only if \( u(r) = \pm w_k(r - \xi) \)
for some \( \xi \in \mathbb{R} \) and \( k \) is a root of the equation

\[
k = \omega + \frac{1}{4} m^2 k^{\frac{5-p}{p-1}}, \quad k > 0. \tag{3}
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    k = \omega + \frac{1}{4} m^2 k^{\frac{5-p}{p-1}}, \quad k > 0. \tag{3}
\]

Moreover, there exists \( \omega_1 > 0 \) such that:

- If \( \omega > \omega_1 \), (3) has no solution.
- If \( \omega = \omega_1 \), (3) has only one solution \( k_0 \).
- If \( \omega \in (0, \omega_1) \), (3) has two solutions \( k_1(\omega) < k_2(\omega) \).

Moreover,

\[
    \omega_1 = \left( \frac{(5-p)m^2}{4(p-1)} \right)^{-\frac{p-1}{2(3-p)}} - \frac{m^2}{4} \left( \frac{(5-p)m^2}{4(p-1)} \right)^{-\frac{(5-p)}{2(3-p)}}.
\]
The threshold value $\omega_0$

Hence, for $\omega \in (0, \omega_1)$ there are three solutions: 0, $w_{k_1}$ and $w_{k_2}$. By evaluating $J_\omega$, we obtain that $J_\omega(0) = 0$, $J_\omega(w_{k_1}) > 0$ and

$$J_\omega(w_{k_2}) < 0 \iff \omega < \omega_0,$$

with

$$\omega_0 = \frac{3-p}{3+p} \cdot 3^{\frac{p-1}{2(3-p)}} \cdot 2^{\frac{2}{3-p}} \left( \frac{m^2(3+p)}{p-1} \right)^{-\frac{p-1}{2(3-p)}}.$$

Moreover $J_{\omega_0}(w_{k_2}) = 0$. 
For some values of $p$, $m$ can be computed, and hence $\omega_0$. For instance, if $p = 2$, $m = 6$ and $\omega_0 = \frac{2}{5\sqrt{15}}$.

**Figura:** The value $\omega_0(p)$ for $p \in (1, 3)$. 
Theorem

Let $p \in (1, 3)$. We have:

- if $\omega \in (0, \omega_0)$, then $I_\omega$ is unbounded from below;
- if $\omega = \omega_0$, then $I_{\omega_0}$ is bounded from below, not coercive and $\inf I_{\omega_0} < 0$;
- if $\omega > \omega_0$, then $I_\omega$ is bounded from below and coercive.
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- if \( \omega = \omega_0 \), then \( I_{\omega_0} \) is bounded from below, not coercive and \( \inf I_{\omega_0} < 0 \);
- if \( \omega > \omega_0 \), then \( I_\omega \) is bounded from below and coercive.

We estimate \( I_\omega (w_{k_2} (\cdot - \rho)) \), obtaining:

\[
I_\omega (w_{k_2} (\cdot - \rho)) = 2\pi \rho J_\omega (w_{k_2}) - C + o_\rho (1), \text{ as } \rho \to +\infty, \ C > 0.
\]

Since \( J_\omega (w_{k_2}) < 0 \) for \( \omega \in (0, \omega_0) \) the first part is proved.

Moreover, \( J_{\omega_0} (w_{k_2}) = 0 \), so \( I_{\omega_0} \) is not coercive and \( \inf I_{\omega_0} < 0 \).
$I_\omega$ bounded from below if $\omega \geq \omega_0$.

By using BHS inequality,

$$(2\pi)^{-1}I_\omega(u) \geq \frac{1}{4}\|u\|^2 + \frac{1}{16} \int_0^{+\infty} \frac{u^2(r)}{r} \left( \int_0^r su^2(s) \, ds - 2N \right)^2 \, dr$$

$$+ \int_0^{+\infty} f(u)r \, dr. \quad (4)$$

Here $\| \cdot \|$ is the $H^1_r(\mathbb{R}^2)$ norm and $f(u) = \omega \frac{u^2}{2} + \frac{u^4}{4} - \frac{u^{p+1}}{p+1}$. 
Define

\[ A(u) = \{ x \in \mathbb{R}^2 : u(x) \in (\alpha, \beta) \}, \quad \rho(u) = \sup \{ |x| : x \in A(u) \} \].

Then we obtain:

\[ \frac{I_\omega(u)}{2\pi} \geq \frac{1}{4} \|u\|^2 + \frac{1}{16} \int_0^{+\infty} \frac{u^2(r)}{r} \left( \int_0^r su^2(s) ds - 2N \right)^2 dr - m|A(u)|. \]
Define

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In particular, \( I_\omega \) is coercive when restricted to \( H^1_0(B(0,n)) \). Take \( u_n \) a minimizer, and observe that

\[ I_\omega(u_n) \to \inf I_\omega, \text{ as } n \to +\infty. \]

If \( u_n \) is bounded we are done, so let us assume that \( \|u_n\| \) diverges. In particular \( |A_n| \) must diverge, and hence \( \rho_n \).
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In particular, \(I_\omega\) is coercive when restricted to \(H^1_0(B(0,n))\). Take \(u_n\) a minimizer, and observe that
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I_\omega(u_n) \to \inf I_\omega, \quad \text{as } n \to +\infty.
\]

If \(u_n\) is bounded we are done, so let us assume that \(\|u_n\|\) diverges. In particular \(|A_n|\) must diverge, and hence \(\rho_n\).

It can be proved that indeed \(\rho_n \sim |A_n| \sim \|u_n\|^2\).
By concentration-compactness, we can prove the existence of $\xi_n \sim \rho_n$ such that

$$0 < c < \int_{\xi_n-1}^{\xi_n+1} (u_n^2 + u_n')^2 \, dr < C.$$
By concentration-compactness, we can prove the existence of \( \xi_n \sim \rho_n \) such that

\[
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\]

Take a cut-off function \( \psi_n \) such that

\[
\psi_n(r) = \begin{cases} 
0, & \text{if } r \leq \xi_n - 3\|u_n\|, \\
1, & \text{if } r \geq \xi_n - 2\|u_n\|.
\end{cases}
\]

We now split the expression of \( I_\omega \), but an extra term comes due to its non-local character:
\[ I_\omega(u_n) \geq I_\omega(u_n \psi_n) + I_\omega(u_n(1 - \psi_n)) \\
+ c\|u_n(1 - \psi_n)\|_{L^2(\mathbb{R}^2)}^2 + O(\|u_n\|). \]
\[ I_\omega(u_n) \geq 2\pi \xi_n I_\omega(u_n \psi_n) + I_\omega(u_n(1 - \psi_n)) \]
\[ + c \|u_n(1 - \psi_n)\|_{L^2(\mathbb{R}^2)}^2 + O(\|u_n\|). \]
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I_\omega(u_n) \geq 2\pi \xi_n I_\omega(u_n \psi_n) + I_\omega(u_n (1 - \psi_n)) \\
+ c \|u_n (1 - \psi_n)\|_{L^2(\mathbb{R}^2)}^2 + O(\|u_n\|).
\]

Since \(\|u_n \psi_n\|_{H^1(\mathbb{R})} \to 0\), for \(\omega > \omega_0\), we can prove that \(I_\omega(u_n \psi_n) \to c > 0\).

Hence, \(I_\omega(u_n) > I_\omega(u_n (1 - \psi_n))\), which is a contradiction with the definition of \(u_n\).
\[ I_\omega(u_n) \geq 2\pi \xi_n J_\omega(u_n \psi_n) + I_\omega(u_n(1 - \psi_n)) \\
+ c \|u_n(1 - \psi_n)\|_{L^2(\mathbb{R}^2)}^2 + O(\|u_n\|). \]

Since \( \|u_n \psi_n\|_{H^1(\mathbb{R})} \not\to 0 \), for \( \omega > \omega_0 \), we can prove that \( J_\omega(u_n \psi_n) \to c > 0 \).

Hence, \( I_\omega(u_n) > I_\omega(u_n(1 - \psi_n)) \), which is a contradiction with the definition of \( u_n \).

If \( \omega = \omega_0 \), we reach a contradiction unless \( \psi_n u_n(\cdot - \xi_n) \to w_{k_2} \).

With this extra information, we have a better estimate:

\[ I_{\omega_0}(u_n) \geq 2\pi \xi_n J_{\omega_0}(u_n \psi_n) + I_{\omega_0}(u_n(1 - \psi_n)) \\
+ c \|u_n(1 - \psi_n)\|_{L^2(\mathbb{R}^2)}^2 + O(1). \]

Therefore

\[ I_{\omega_0}(u_n) \geq I_{(\omega_0 + 2c)}(u_n(1 - \psi_n)) + O(1) \geq O(1). \]
On the solutions of $(\mathcal{P})$

**Theorem**

- If $\omega$ is large, then $(\mathcal{P})$ has no solutions different from zero.
- If $\omega > \omega_0$ is close to $\omega_0$, then $(\mathcal{P})$ admits at least two positive solutions.
- For almost every $\omega \in (0, \omega_0)$, $(\mathcal{P})$ admits a positive solution.
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**Theorem**

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- For almost every $\omega \in (0, \omega_0)$, $(\mathcal{P})$ admits a positive solution.

**Non-existence of solutions if $\omega$ large.**

If $N = 0$, the proof is very simple: multiply the equation by $u$, integrate and plug the BHS inequality, to get

$$0 \geq \frac{1}{4} \int_{\mathbb{R}^2} |\nabla u|^2\,dx + \int_{\mathbb{R}^2} \left( \omega u^2 + \frac{3}{4} u^4 - |u|^{p+1} \right)\,dx.$$  

And this is a contradiction for $\omega$ large.

For $N > 0$ this proof becomes delicate, and will be skipped in this talk.
Two solutions if $\omega > \omega_0$ is close to $\omega_0$.

Recall that $\inf I_{\omega_0} < 0$, then $\inf I_{\omega} < 0$ for $\omega$ close to $\omega_0$. Being $I_{\omega}$ coercive, the infimum is attained (at negative level).

Moreover, $I_{\omega}$ satisfies the geometrical assumptions of the Mountain Pass Theorem.

Since $I_{\omega}$ is coercive, (PS) sequences are bounded.

We find a second solution (a mountain-pass solution) which is at a positive energy level.
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Since $I_{\omega}$ is coercive, (PS) sequences are bounded.

We find a second solution (a mountain-pass solution) which is at a positive energy level.

• **For almost every** $\omega \in (0, \omega_0)$ **there is a positive solution.**

If $\omega < \omega_0$, the functional $I_{\omega}$ satisfies the geometric properties of the Mountain-Pass lemma.

However, (PS) sequences could be unbounded. Here we use the so-called monotonicity trick of Struwe. In this way we can obtain solutions, but only for almost every $\omega \in (0, \omega_0)$. 
Thank you for your attention!