Prescribing Gaussian curvature on compact surfaces and geodesic curvature on its boundary

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Outline

1. The problem
2. The variational formulation
3. Blow up versus compactness
4. Some ideas of the proof
5. Comments and open problems
A classical problem in geometry is the prescription of the Gaussian curvature on a compact Riemannian surface $\Sigma$ under a conformal change of the metric. Denote by $\tilde{g}$ the original metric and $g = e^{u} \tilde{g}$. The curvature then transforms according to the law:

$$-\Delta u + 2\tilde{K}(x) = 2K(x)e^{u},$$

where $\Delta = \Delta_{\tilde{g}}$ is the Laplace-Beltrami operator and $\tilde{K}, K$ stand for the Gaussian curvatures with respect to $\tilde{g}$ and $g$, respectively.

The solvability of this equation has been studied for several decades: Berger, Kazdan and Warner, Moser, Aubin, Chang-Yang...
Our problem

Let $\Sigma$ be a compact surface with boundary. In this talk we consider the problem of prescribing the Gaussian curvature of $\Sigma$ and the geodesic curvature of $\partial \Sigma$ via a conformal change of the metric.
Our problem

Let \( \Sigma \) be a compact surface with boundary. In this talk we consider the problem of prescribing the Gaussian curvature of \( \Sigma \) and the geodesic curvature of \( \partial \Sigma \) via a conformal change of the metric.

This question leads us to the boundary value problem:

\[
\begin{aligned}
-\Delta u + 2\tilde{K}(x) &= 2K(x)e^u, & x \in \Sigma, \\
\frac{\partial u}{\partial \nu} + 2\tilde{h}(x) &= 2h(x)e^{u/2}, & x \in \partial \Sigma.
\end{aligned}
\]

Here \( e^u \) is the conformal factor, \( \nu \) is the normal exterior vector and

1. \( \tilde{K}, \tilde{h} \) are the original Gaussian and geodesic curvatures, and
2. \( K, h \) are the Gaussian and geodesic curvatures to be prescribed.
Antecedents

The higher order analogue: prescribing scalar curvature $S$ on $\Sigma$ and mean curvature $H$ on $\partial \Sigma$.

The case $S = 0$ and $H = \text{const}$ is the well-known Escobar problem: Ambrosetti-Li-Malchiodi, Escobar, Han-Li, Marques,...

The case $h = 0$: Chang-Yang.

The case $K = 0$: Chang-Liu, Liu-Huang...

The blow-up phenomenon has also been studied: Guo-Liu, Bao-Wang-Zhou, Da Lio-Martinazzi-Rivière...

The case of constants $K, h$:

A parabolic flow converges to constant curvatures (Brendle).
Classification of solutions in the annulus (Jiménez).
Classification of solutions in the half-plane (Li-Zhu, Zhang, Gálvez-Mira).

Our aim is to consider the case of nonconstant $K, h$. The only results we are aware of are due to Cherrier, Hamza.
By the Gauss-Bonnet Theorem,

\[
\int_{\Sigma} Ke^u + \oint_{\partial \Sigma} he^{u/2} = \int_{\Sigma} \tilde{K} + \oint_{\partial \Sigma} \tilde{h} = 2\pi \chi(\Sigma),
\]

where \( \chi(\Sigma) \) is the Euler characteristic of \( \Sigma \).
Preliminaries

By the Gauss-Bonnet Theorem,

\[ \int_{\Sigma} Ke^u + \oint_{\partial \Sigma} he^{u/2} = \int_{\Sigma} \tilde{K} + \oint_{\partial \Sigma} \tilde{h} = 2\pi \chi(\Sigma), \]

where \( \chi(\Sigma) \) is the Euler characteristic of \( \Sigma \).

It is easy to show that we can prescribe \( h = 0, K = sgn(\chi(\Sigma)) \). Then:

\[
\left\{ \begin{array}{l}
-\Delta u + 2\tilde{K} = 2K(x)e^u, \quad x \in \Sigma, \\
\frac{\partial u}{\partial \nu} = 2h(x)e^{u/2}, \quad x \in \partial \Sigma,
\end{array} \right.
\]

where \( \tilde{K} = sgn(\chi(\Sigma)) \).

We are interested in the case of negative \( K \). For existence of solutions, we will focus on the case \( \chi \leq 0 \).
The variational formulation

The associated energy functional is given by $I : H^1(\Sigma) \to \mathbb{R}$,

$$I(u) = \int_\Sigma \left( \frac{1}{2} |\nabla u|^2 + 2\tilde{K}u + 2|K(x)|e^u \right) - 4 \int_{\partial \Sigma} he^{u/2}.$$ 

For the statement of our results it will be convenient to define the function $\mathcal{D} : \partial \Sigma \to \mathbb{R}$,

$$\mathcal{D}(x) = \frac{h(x)}{\sqrt{|K(x)|}}.$$ 

The function $\mathcal{D}$ is scale invariant.
A trace inequality

Proposition

For any \( \varepsilon > 0 \) there exists \( C > 0 \) such that:

\[
4 \int_{\partial \Sigma} h(x)e^{u/2} \leq \left( \varepsilon + \max_{p \in \partial \Sigma} \mathcal{D}^+(p) \right) \left[ \int_{\Sigma} \frac{1}{2} |\nabla u|^2 + 2|K(x)|e^u \right] + C.
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In particular, if \( \mathcal{D}(p) < 1 \ \forall \ p \in \partial \Sigma \), then \( I \) is bounded from below.
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\]

In particular, if \( \mathcal{D}(p) < 1 \ \forall \ p \in \partial \Sigma \), then \( I \) is bounded from below.

Assume that \( h > 0 \) is constant, and take \( N \) a vector field in \( \Sigma \) such that \( N(x) = \nu(x) \) on the boundary, \( |N(x)| \leq 1 \). Then,

\[
4 \int_{\partial \Sigma} e^{u/2} = 4 \int_{\partial \Sigma} e^{u/2} N(x) \cdot \nu(x)
\]

\[
= 4 \int_{\Sigma} e^{u/2} \left[ \text{div} N + \frac{1}{2} \nabla u \cdot N \right] \leq C \int_{\Sigma} e^{u/2} + 2 \int_{\Sigma} e^{u/2} |\nabla u|
\]

\[
\leq C \int_{\Sigma} e^{u/2} + 2 \int_{\Sigma} h^2 e^u + \frac{1}{2} \int_{\Sigma} |\nabla u|^2.
\]
The case $\chi(\Sigma) < 0$

**Theorem**

Assume that $\chi(\Sigma) < 0$. Let $K, h$ be continuous functions such that $K < 0$ and $\mathcal{D}(p) < 1$ for all $p \in \partial \Sigma$. Then the functional $I$ is coercive and attains its infimum.

By the trace inequality, 

$$I(u) \geq \int_{\Sigma} \varepsilon |\nabla u|^2 + 2\varepsilon |K(x)|e^u + 2\tilde{K}u - C.$$ 

Since $\tilde{K} < 0$, $\lim_{u \to \pm\infty} 2\delta e^u + 2\tilde{K}u = +\infty$, so $I$ is coercive.
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Since $\tilde{K} < 0$, $\lim_{u \to \pm \infty} 2\delta e^u + 2\tilde{K}u = +\infty$, so $I$ is coercive.

If $\chi(\Sigma) = \tilde{K} = 0$, $I$ is bounded from below but not coercive! The reason is that $\int_\Sigma u_n$ could go to $-\infty$ for a minimizing sequence $u_n$. 
Minimizers for $\chi(\Sigma) = 0$.

**Theorem**

Assume that $\chi(\Sigma) = 0$. Let $K$, $h$ be continuous functions such that $K < 0$ and:

1. $D(p) < 1$ for all $p \in \partial\Sigma$.
2. $\int_{\partial\Sigma} h > 0$.

Then $I$ attains its infimum.

Observe that if $u_n = -n$, then: $I(u_n) = \int_\Sigma 2|K(x)|e^{-n} - 4 \int_{\partial\Sigma} he^{-n/2} \to 0$. 
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\[ \inf I \]
Min-max for $\chi(\Sigma) = 0$.

**Theorem**

Assume that $\chi(\Sigma) = 0$. Let $K, h$ be continuous functions such that $K < 0$ and:

1. $\mathcal{D}(p) > 1$ for some $p \in \partial \Sigma$.
2. $\oint_{\partial \Sigma} h < 0$.

Then $I$ has a mountain-pass geometry.
Blow-up versus compactness

Here the (PS) condition is not known to hold. By using the monotonicity trick of Struwe, we can obtain solutions of perturbed problems.

The question of compactness or blow-up for this kind of problems has attracted a lot of attention since the works of Brezis-Merle, Li-Shafrir, etc.
Blow-up versus compactness

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Let \( u_n \) be a blowing-up sequence (namely, \( \sup\{u_n(x)\} \to +\infty \)) of solutions to the problem:

\[
\begin{align*}
-\Delta u_n + 2\tilde{K}_n(x) &= 2K_n(x)e^{u_n}, & \text{in } \Sigma, \\
\frac{\partial u_n}{\partial \nu} + 2\tilde{h}_n(x) &= 2h_n(x)e^{u_n/2}, & \text{on } \partial \Sigma.
\end{align*}
\]

Here \( \tilde{K}_n \to \tilde{K}, \tilde{h}_n \to \tilde{h}, K_n \to K, h_n \to h \) in \( C^1 \) sense, with \( K < 0 \). By integrating:

\[
\int_{\Sigma} K_n e^{u_n} + \int_{\partial \Sigma} h_n e^{u_n/2} = \int_{\Sigma} K_n + \int_{\partial \Sigma} h_n \to \chi_0 = 2\pi \chi(\Sigma).
\]

Hence there could be compensation of diverging masses!!
A blow-up analysis

Theorem

Assume that $u_n$ is unbounded from above and define its singular set:

$$S = \{ p \in \Sigma : \exists x_n \to p \text{ such that } u_n(x_n) \to +\infty \}. \quad (2)$$

$$S \subset \{ p \in \partial \Sigma : \mathcal{D}(p) \geq 1 \}. \quad (1)$$
A blow-up analysis

Theorem

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$$S = \{ p \in \Sigma : \exists x_n \to p \text{ such that } u_n(x_n) \to +\infty \}. \quad (2)$$

1. $S \subset \{ p \in \partial \Sigma : \mathcal{O}(p) \geq 1 \}$.

2. If $\Sigma e^{u_n}$ is bounded, then there exists $m \in \mathbb{N}$ such that

$$S = \{ p_1, \ldots, p_m \} \subset \{ \mathcal{O}(p) > 1 \}.$$

In this case $|K_n e^{u_n} \to \sum_{i=1}^m \beta_i \delta_{p_i}, h_n e^{u_n/2} \to \sum_{i=1}^m (\beta_i + 2\pi) \delta_{p_i}$ for some $\beta_i > 0$. In particular, $\chi_0 = 2\pi m$. 
The infinite mass case

If \( \int_{\Sigma} e^{u_n} \) is unbounded, there exists a unit positive measure \( \sigma \) on \( \Sigma \) such that:

a) \[
\frac{|K_n| e^{u_n}}{\int_{\Sigma} |K_n| e^{u_n}} \rightharpoonup \sigma, \quad \frac{h_n e^{u_n}/2}{\int_{\partial \Sigma} h_n e^{u_n}/2} \rightharpoonup \sigma|_{\partial \Sigma}.
\]

b) \( \text{supp } \sigma \subset \{ p \in \partial \Sigma : \mathcal{D}(p) \geq 1, \mathcal{D}_\tau(p) = 0 \} \).
The infinite mass case

3. If $\int_{\Sigma} e^{u_n}$ is unbounded, there exists a unit positive measure $\sigma$ on $\Sigma$ such that:
   a) $\frac{|K_n| e^{u_n}}{\int_{\Sigma} |K_n| e^{u_n}} \rightarrow \sigma$, $\frac{h_n e^{u_n/2}}{\delta_{\partial \Sigma} h_n e^{u_n/2}} \rightarrow \sigma|_{\partial \Sigma}$.
   b) $\text{supp } \sigma \subset \{p \in \partial \Sigma : \mathcal{D}(p) \geq 1, \mathcal{D}_\tau(p) = 0\}$.

4. If there exists $m \in \mathbb{N}$ such that $\text{ind}(u_n) \leq m$ for all $n$, then $S = S_0 \cup S_1$, where:
   
   $S_0 \subset \{p \in \partial \Sigma : \mathcal{D}(p) = 1, \mathcal{D}_\tau(p) = 0\}$,
   
   $S_1 = \{p_1, \ldots, p_k\} \subset \{\mathcal{D}(p) > 1 \text{ and } \Phi(p) = 0\}, \; k \leq m$.

If moreover $\chi_0 \leq 0$, then $S_1$ is empty.
Back to the case $\chi(\Sigma) = 0$.

**Theorem**

Assume that $\chi(\Sigma) = 0$. Let $K, h$ be $C^1$ functions such that $K < 0$ and:

1. $D(p) > 1$ for some $p \in \partial \Sigma$.
2. $\int_{\partial \Sigma} h < 0$. 

We obtain solutions of perturbed problems of mountain-pass type, hence they have Morse index at most 1 ([Fang-Ghoussoub, 94, 99]). Those solutions cannot blow-up so that they converge to a true solution of our problem.
Back to the case $\chi(\Sigma) = 0$.

**Theorem**

Assume that $\chi(\Sigma) = 0$. Let $K, h$ be $C^1$ functions such that $K < 0$ and:

1. $\mathcal{D}(p) > 1$ for some $p \in \partial \Sigma$.
2. $\int_{\partial \Sigma} h < 0$.
3. $\mathcal{D}_\tau(p) \neq 0$ for any $p \in \partial \Sigma$ with $\mathcal{D}(p) = 1$.

Then $I$ has a mountain-pass critical point.

We obtain solutions of perturbed problems of mountain-pass type, hence they have Morse index at most 1 ([Fang-Ghoussoub, 94, 99]).

Those solutions cannot blow-up so that they converge to a true solution of our problem.
Proposition (Jiménez 2012)

If $\Sigma$ is a cylinder and $K = -1$, $h_1$ and $h_2$ are constants, then our problem is solvable iff

1. $h_1 + h_2 > 0$ and both $h_i < 1$ (minima).
2. $h_1 + h_2 < 0$ and some $h_i > 1$ (mountain-pass).
3. $h_1 = 1, h_2 = -1$ or viceversa.
Obstructions to existence

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3. $h_1 = 1$, $h_2 = -1$ or viceversa.

Proposition

Let $\Sigma$ be a compact surface with boundary, and assume that $h(p) > \sqrt{|K-(q)|}$ for all $p \in \partial \Sigma$, $q \in \Sigma$. Then $\Sigma$ is homeomorphic to a disk.
A classification result in the half-plane

Theorem (Gálvez-Mira 2009)

Let $u$ be a solution of:

$$
\begin{cases}
  -\Delta u = 2K_0 e^u & \text{in } \mathbb{R}_+^2, \\
  \frac{\partial u}{\partial \nu} = 2h_0 e^{u/2} & \text{in } \partial \mathbb{R}_+^2,
\end{cases}
\quad \Rightarrow
\quad
\begin{cases}
  -\Delta u = -2e^u & \text{in } \mathbb{R}_+^2, \\
  \frac{\partial u}{\partial \nu} = 2\mathcal{D}_0 e^{u/2} & \text{in } \partial \mathbb{R}_+^2.
\end{cases}

$$

with $\mathcal{D}_0 = \frac{h_0}{\sqrt{|K_0|}}$. Then the following holds:
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\[
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\frac{\partial u}{\partial \nu} &= 2h_0 e^{u/2} \quad \text{in } \partial \mathbb{R}^2_+, \\
\end{align*}
\]

Then the following holds:

- If $\mathcal{D}_0 < 1$ there is no solution.
- If $\mathcal{D}_0 = 1$ the only solutions are:

\[
u(s, t) = 2 \log \left( \frac{\lambda}{1 + \lambda t} \right), \quad \lambda > 0, \ s \in \mathbb{R}, \ t \geq 0.
\]
A classification result in the half-plane

If $\mathcal{D}_0 > 1$, then:

$$u(z) = 2 \log \left( \frac{2|g'(z)|}{1 - |g(z)|^2} \right),$$

where $g$ is locally injective holomorphic map from $\mathbb{R}^2_+$ to a disk of geodesic curvature $\mathcal{D}_0$ in the Poincaré disk $\mathbb{H}^2$. For instance, to $B(0, R)$ with $\mathcal{D}_0 = \frac{1+R^2}{2R}$. 

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Moreover, $g$ is a Möbius transform if and only if

$$\text{either } \int_{\mathbb{R}^2_+} e^u < +\infty \text{ and/or } \int_{\partial \mathbb{R}^2_+} e^{u/2} < +\infty.$$ 

In such case $u$ can be written as:

$$u(s, t) = 2 \log \left( \frac{2\lambda}{(s - s_0)^2 + (t + t_0)^2 - \lambda^2}, t \geq 0, \right),$$

where $\lambda > 0$, $s_0 \in \mathbb{R}$, $t_0 = \mathcal{D}_0 \lambda$. We call these solutions “bubbles”.
Passing to a limit problem in the half-plane

Let us recall the definition of the singular set:

\[ S = \{ p \in \Sigma : \exists y_n \in \Sigma, \ y_n \to p, \ u_n(y_n) \to +\infty \} . \]

**Proposition**

*Let* \( p \in S \). *Then there exist* \( x_n \in \Sigma \), \( x_n \to p \) *such that, after a suitable rescaling, we obtain a solution of the problem in the half-plane in the limit. In particular* \( S \subset \{ p \in \partial \Sigma : \mathcal{D}(p) \geq 1 \} \).

In the Liouville equation, if the mass is finite, then a key integral estimate ([Brezis-Merle, 1991]) implies that \( S \) is finite. Hence one can take \( x_n \) as local maxima ([Li-Shafrir, 1994]).

Here the idea is to choose a **good sequence** \( x_n \), even if they are not local maxima!
Choosing good sequences

Let us fix \( p \in S \). Via a conformal map we can pass to either \( B_0(r) \) or \( B_0^+(r) \).

By definition there exist \( y_n \in \Sigma \) with \( y_n \to p \) and \( u_n(y_n) \to +\infty \). Define:

\[
\varphi_n = e^{-\frac{u_n}{2}}, \quad \varepsilon_n = e^{-\frac{u_n(y_n)}{2}} \to 0.
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\]

By Ekeland variational principle there exists a sequence \( x_n \) such that

- \( e^{-\frac{u_n(x_n)}{2}} \leq e^{-\frac{u_n(y_n)}{2}} \),
- \( |x_n - y_n| \leq \sqrt{\varepsilon_n} \),
- \( e^{-\frac{u_n(x_n)}{2}} \leq e^{-\frac{u_n(z)}{2}} + \sqrt{\varepsilon_n} \ |x_n - z| \) for every \( z \in B \).

The last conditions implies that, when we rescale, the rescaled functions are bounded from above, so we can pass to a limit.
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The last conditions implies that, when we rescale, the rescaled functions are bounded from above, so we can pass to a limit.

- Since \( K(p) < 0 \), there is no entire solution of \(-\Delta u = 2K(p)e^u\) in \( \mathbb{R}^2 \).
- Hence \( p \) in \( \partial \Sigma \), the limit problem is posed in a half-plane and \( \mathcal{D}(p) \geq 1 \).
Infinite mass

Proposition

Assume that \( \rho_n = \int_\Sigma |K_n| e^{u_n} \to +\infty \), \( \oint_{\partial \Sigma} |h| e^{u_n/2} \to +\infty \). Then there exists a positive unit measure \( \sigma \) on \( \partial \Sigma \) such that:

\[
\frac{|K_n| e^{u_n}}{\rho_n} \to \sigma, \quad \frac{h_n e^{u_n/2}}{\rho_n} \to \sigma.
\]

Multiplying the equation by \( \phi \in C^2(\Sigma) \) and integrating:

\[
2 \oint_{\partial \Sigma} h_n e^{u_n/2} \phi - 2 \int_{\Sigma} |K_n| e^{u_n} \phi = O(1) + \int_{\Sigma} u_n \Delta \phi + \oint_{\partial \Sigma} \frac{\partial \phi}{\partial \nu} u_n. + o(\rho_n)
\]

We use a Kato-type inequality to estimate \( u_n^- \).
On the support of $\sigma$

Clearly $\text{supp } \sigma \subset S \subset \{ p \in \partial \Sigma : \mathcal{D}(p) \geq 1 \}$. Moreover, we have:

**Proposition**

*The support of $\sigma$ is contained in the set* $\{ p \in \partial \Sigma : \mathcal{D}_\tau(p) = 0 \}$.

Let $\Lambda_0$ be a connected component of $\partial \Sigma$. Via a conformal map, we can pass to a problem in an annulus.
Multiply the equation by $\nabla u_n \cdot F$, where $F$ is a tangential vector field, to obtain:

$$
\iint_{\Lambda_0} (4h_n e^{u_n/2} - 4\tilde{h}_n)(\nabla u_n \cdot F)
$$

$$
= \int_{\Sigma} [4\tilde{K}_n \nabla u_n \cdot F + 4e^{u_n} (\nabla K_n \cdot F + K_n \nabla \cdot F) + 2 DF(\nabla u_n, \nabla u_n) - \nabla \cdot F |\nabla u_n|^2].
$$
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$$\int_{\Lambda_0} (4h_n e^{u_n/2} - 4\tilde{h}_n) (\nabla u_n \cdot F)$$

$$= \int_{\Sigma} [4\tilde{K}_n \nabla u_n \cdot F + 4e^{u_n} (\nabla K_n \cdot F + K_n \nabla \cdot F) + 2DF(\nabla u_n, \nabla u_n) - \nabla \cdot F|\nabla u_n|^2].$$

We get rid of the Dirichlet term by using holomorphic functions $F$. Integrating by parts and passing to the limit, we obtain:

$$\oint_{\Lambda_0} \frac{\partial f}{\partial \tau} d\sigma = 0,$$

where $f = (F \cdot \tau)$. But $f$ can be any arbitrary analytic function, and then $\mathcal{D}_{\tau} \sigma = 0$ as a measure.
Morse index

This is all the information that we can obtain without further assumptions on $u_n$.

From now on we assume that the sequence of solutions $u_n$ has bounded Morse index.

If $u_n$ has bounded Morse index, the solutions of the limit problem obtained by rescaling have finite Morse index.
Morse index of the limit problem

Theorem

Let $u$ be a solution of the problem:

$$
\begin{cases}
-\Delta u = -2e^u & \text{in } \mathbb{R}^2_+, \\
\frac{\partial u}{\partial \nu} = 2\mathcal{D}_0e^{u/2} & \text{in } \partial \mathbb{R}^2_+.
\end{cases}
$$

Define:

$$Q(\psi) = \int_{\mathbb{R}^2_+} |\nabla \psi|^2 + 2 \int_{\mathbb{R}^2_+} e^u \psi^2 - \mathcal{D}_0 \int_{\partial \mathbb{R}^2_+} e^{u/2} \psi^2, \quad \text{and}$$

$$\text{ind}(v) = \sup\{\dim(E) : E \subset C^\infty_0(\mathbb{R}^2_+) \text{ vector space, } Q(\psi) < 0 \ \forall \ \psi \in E\}.$$
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\end{aligned}
\]  

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\]

\[
\text{ind}(\psi) = \sup \{ \dim(E) : E \subset C_0^\infty(\mathbb{R}^2_+) \text{ vector space, } Q(\psi) < 0 \ \forall \ \psi \in E \}.
\]

1. If \( \mathcal{D}_0 = 1 \), then \( \text{ind}(u) = 0 \), that is, \( u \) is stable.
2. If \( \mathcal{D}_0 > 1 \) and \( u \) is a bubble, then \( \text{ind}(u) = 1 \). Otherwise, \( \text{ind}(u) = +\infty \).

This theorem implies that infinite mass blow-up with bounded Morse index occurs only if \( \mathcal{D}(p) = 1 \), and the number of bubbles is limited.
Morse index of the limit problem

If \( D_0 = 1 \), \( \psi(s, t) = \frac{1}{1+t} \) is a positive solution of the linearization.
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If \( D_0 > 1 \), then we pass to the problem posed in \( B(0, R) \subset \mathbb{H}^2 \):

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\begin{aligned}
-\Delta \gamma + 2\gamma &= 0, & \text{in } B(0, R), \\
\frac{\partial \gamma}{\partial \nu} &= \lambda \gamma, & \text{in } \partial B(0, R). \\
\end{aligned}
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The Morse index is the number of eigenvalues \( \lambda \) smaller than \( D_0 \).

- The functions \( \gamma_i(z) = \frac{z_i}{1-|z|^2} \) solve (4) with \( \lambda = D_0 \).
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- For a convenient cut-off $\phi$, $\psi = \phi(g \circ \gamma)$ satisfies $Q(\psi) < 0$.
- If moreover $\int_{\partial \mathbb{R}^2_+} e^{\mu/2} = +\infty$ we can choose $\phi$ to be 0 outside any arbitrary compact set.
Explicit examples of blow-up

Let us consider the problem:

\[
\begin{align*}
-\Delta u &= -2e^u, & \text{in } A(0; r, 1), \\
\frac{\partial u}{\partial \nu} + 2 &= 2h_1e^{u/2}, & \text{on } |x| = 1, \\
\frac{\partial u}{\partial \nu} - \frac{2}{r} &= 2h_2e^{u/2}, & \text{on } |x| = r.
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Here $K = -1$ and $h$ is constant on each component of the boundary. All solutions of this problem have been classified ([Jiménez, 2012]).
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For example, the function:

\[
u(x) = \log \left( \frac{4}{|x|^2(\lambda + 2 \log |x|)^2} \right), \quad \text{for any } \lambda < 0,
\]

is a solution with \( h_1 = 1 \) and \( h_2 = -1 \). Observe that if \( \lambda \) tends to 0 then \( u \) blows up at a whole component of the boundary.

The singular set \( S = \{|x| = 1\} \) is not finite.
A second example

Given any $h_1 > 1$, $\gamma \in \mathbb{N}$, there exists a explicit solution:

$$u_\gamma(z) = 2 \log \left( \frac{\gamma |z|^{\gamma - 1}}{h_1 + \text{Re}(z^\gamma)} \right),$$

where $h_2 = -h_1 r^{-\gamma}$. 
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The asymptotic profile is:

$$u(s, t) = 2 \log \left( \frac{e^{-t}}{h_1 + e^{-t} \cos s} \right),$$

defined in the half-plane $\{t \geq 0\}$. This is indeed a solution to the limit problem in the half-space with $K = -1$ and $h_1 > 1$, with infinite Morse index.
Open problems

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5. Higher-order analogue.
Muito obrigado pela sua atenção!